8 Recent developments in Kira

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In this section we report on the recent progress made in the development of the Feynman integral reduction program `Kira`. The development is focused on algorithmic improvements that are essential to extend the range of feasible high precision calculations for present and future colliders like the FCC-ee.

8.1 Introduction

`Kira` [1] implements Laporta's algorithm [2] to reduce Feynman integrals to a basis of master integrals. In this approach, large systems of integration-by-parts [3] and Lorentz invariance [4] identities, and symmetry relations, are generated and solved by a variant of Gaussian elimination, systematically expressing complicated integrals in terms of simpler integrals wrt. a given complexity criterion. Though alternative reduction techniques have been proposed and applied to specific problems, see e.g. [5–8], to date programs based on Laporta's algorithm [9–11] pose the only general purpose tools suited for large scale applications. Since these reduction programs constitute one of the bottlenecks of high precision predictions, their continous improvement is crucial to meet the increasing demand for such calculations.

A key element of `Kira` is its equation selector to extract a linearly independent system of equations, discarding equations that are not required to fully reduce all integrals requested by the user. The selector is based on Gaussian elimination using modular arithmetic on the coefficients.

8.2 Improved symmetrisation

The detection of symmetry relations between sectors within and across topologies received a performance boost due to the implementation of the algorithm described in [12]. In this approach, a canonical form of the integrand of each sector is constructed, so that a one-to-one comparison of the representations can be done. Additionally, the combinatorial complexity of the loop momentum shift finder to determine the mapping prescriptions of equivalent sectors has been reduced. Furthermore, the detection of trivial sectors received a significant speed-up by employing `Kira`'s IBP solver instead of the less optimised previous linear solver.

As an example, the "cube topology" shown in Fig. C.6, i.e. the 5-loop vacuum bubble with 12 propagators of equal mass and the symmetry of a cube, can now be analysed in less than 10 minutes on a state-of-the-art desktop computer.

8.3 Parallel simplification algorithms for coefficients

Algebraic simplifications with Fermat

To simplify multivariate rational functions in masses and kinematic quantities, which appear as coefficients in the Gaussian elimination steps, `Kira` relies on the program `Fermat` [13]. In almost all cases, the runtime for the reduction is dominated by those algebraic simplifications. It turns out that, when a new coefficient is constructed from several (often thousands of) known coefficients, combining them naïvely and simplifying them in one step results in an avoidable performance penalty. Instead, `Kira` recursively combines coefficients pairwise, choosing the pairs based on the size of their string representations. Besides the improved performance, this

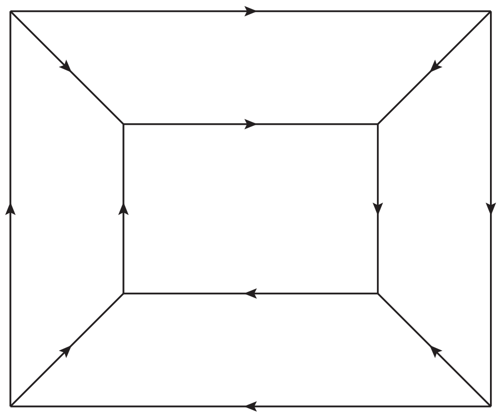

Fig. C.6: The cube topology is the 5-loop vacuum bubble with 12 propagators of equal mass and octahedral symmetry. The high symmetry of 48 equivalent propagator permutations in the top-level sector makes this topology an ideal candidate for symmetrisation benchmarks.

strategy also offers new possibilities for the parallelisation, since the pairwise combinations can be evaluated by different `Fermat` instances.

In the Gaussian back substitution, one can restrict a solver to calculate only the coefficients of a specific master integral. This allows the user to parallelise the reduction across several machines and merge the results in a final step.

Algebraic reconstruction over integers

An alternative algorithm to simplify the coefficients is given by algebraic reconstruction over integers, introduced in [7, 14, 15]. This strategy is based on sampling the rational functions by setting kinematic invariants and masses to integer values repeatedly. Each sample can be evaluated rather quickly, but the number of samples required to reconstruct the simplified result increases with the degree of the numerator and denominator of the rational function, the number of invariants involved, and the number of invariants over which is sampled. Of course, the sample can again be evaluated in parallel, leading to the potential for massive parallelisation on dozens of CPU cores. An implementation of this algorithm is available in `Kira 1.2` and is continuously improved and extended. Furthermore, `Kira` automatically decides which simplification strategy, i.e. algebraic reconstruction or `Fermat`, is expected to be more efficient in each case. The criteria for these decisions are subject to investigation and offer room for future improvements.

Algebraic reconstruction over finite integer fields

Instead of sampling rational functions over integers, it is also possible to reconstruct them from samples over finite integer fields. Mapping coefficients to a finite field limits the size of each coefficient and with that the complexity of each operation. Choosing the module as a word-size prime, numerical operations on coefficients correspond to the native arithmetic capabilities of the employed CPU, allowing for high performance sampling of the coefficients. A reconstruction algorithm for multivariate rational functions was first presented in [16]. Recently, the library `FireFly` [17] became available, implementing a similar algorithm. `FireFly` has been combined with `Kira` to use it for Feynman integral reduction, calculating the samples with `Kira`'s finite integer Gaussian elimination. An independent implementation is available in `FIRE 6` [11].

In the sampling over (arbitrary-size) integers described above, whenever a coefficient is required to proceed with the reduction, the solver needs to wait until that coefficient has been reconstructed. Using finite integers, the entire solver can be parallelised, opening the possibility to distribute solvers over different machines. The reconstructor can then collect the samples from the solvers and finish the calculation when a sufficient number of samples is available. The finite integer reconstruction is expected to become publicly available in a future `Kira` release in combination with `FireFly`.

8.4 Basis choice

It is well known that the reduction time strongly depends on the choice of the master integrals. In a convenient basis, the reduction coefficients tend to become much simpler than e.g. in the basis that follows directly from the integral ordering. In this respect, uniformly transcendental bases [18], finite bases [19], or finite uniformly transcendental bases [20] present interesting candidates to study the impact of the basis choice on the reduction performance. These special choices involve linear combinations of integrals as basis elements that we call "master equations".

In `Kira`, integrals are represented by integer "weights" in such a way that they obey the imposed integral ordering. Choosing a specific basis of master integrals is already possible. To this end, the weights are modified so that the preferred basis integrals are regarded as simpler than all other integrals. In the presence of master equations, a new kind objects must be introduced, representing the master equation instead of a particular integral. With an appropriate bookkeeping, the implementation becomes straight forward and will soon be available in a `Kira` release.

8.5 Conclusions

The complexity of precision calculations needed to match the accuracy of the FCC-ee experiment demands for integral reduction tools beyond the state-of-the-art capabilities. For example, the computation of pseudo observables at the Z-boson resonance, involving reductions of 3-loop Feynman diagrams with up to five scales, will be necessary to reach the accuracy that may be achieved with the FCC-ee [21]. We expect that Feynman integral reduction programs based on Laporta's algorithm will continue to play a key role in such calculations. E.g. by harnessing the potential of rational reconstruction, basis choices, and large-scale parallelisation, we are convinced that `Kira` will keep up with the arising technical challenges.

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