## 5 Theoretical Luminosity Precision for the FCC-ee: Overview of the Path to 0.01%

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We present an overview of the pathways to the required theoretical precision for the luminosity targeted by the FCC-ee precision studies. We put the discussion in context with a brief review of the situation at the time of LEP. We then present the current status and an overview of routes to the desired 0.01% targeted by the FCC-ee (as well as by the ILC).

We use the situation that existed at the end of LEP as our starting point. At the end of LEP, the error budget for the `BHLUMI4.04` MC used by all LEP collaborations to simulate the luminosity process was calculated in Ref. [1]. For reference, we reproduce this result here in Table B.3. In this table, we show the published works upon which the various error estimates are based as they are discussed in Ref. [1].

| | LEP1 | | LEP2 | |
|---|---|---|---|---|
| Type of correction/error | 1996 | 1999 | 1996 | 1999 |
| (a) Missing photonic $\mathcal{O}(\alpha^2)$ [2,3] | 0.10% | 0.027% | 0.20% | 0.04% |
| (b) Missing photonic $\mathcal{O}(\alpha^3 L_e^3)$ [4] | 0.015% | 0.015% | 0.03% | 0.03% |
| (c) Vacuum polarization [5,6] | 0.04% | 0.04% | 0.10% | 0.10% |
| (d) Light pairs [7,8] | 0.03% | 0.03% | 0.05% | 0.05% |
| (e) $Z$ and $s$-channel $\gamma$ [9,10] | 0.015% | 0.015% | 0.0% | 0.0% |
| Total | 0.11% [10] | 0.061% [1] | 0.25% [10] | 0.12% [1] |

Table B.3: Summary of the total (physical+technical) theoretical uncertainty for a typical calorimetric detector. For LEP1, the above estimate is valid for a generic angular range within 1°–3° (18–52 mrads), and for LEP2 energies up to 176 GeV and an angular range within 3°–6°. Total uncertainty is taken in quadrature. Technical precision included in (a).

One way to address the 0.01% precision tag needed for the luminosity theory error for the FCC-ee is to develop the corresponding improved version of the `BHLUMI`. This problem is addressed recently in Ref. [11], wherein the path to 0.01% theory precision for the FCC-ee luminosity is presented in some detail. The results of this latter reference are shown in Table B.4, wherein we also present the current state of the art for completeness, as it is discussed in more detail in Ref. [11].

The key steps in arriving at Table B.4 are as follows. The errors associated with the photonic corrections in lines (a) and (b) in the LEP results in Table B.3 are due to effects which are known from Refs. [2–4] but which were not implemented into `BHLUMI`. In Table B.4 we show what these errors will become after these known results are included in `BHLUMI` as discussed in Ref. [11]. Similarly, in line (c) of Table B.3 the error is due to the uncertainty at the time of LEP on the hadronic contribution to the vacuum polarization for the photon at the respective momentum transfers for the luminosity process; in Table B.4 we show the improvement of this error that is expected for the FCC-ee as discussed in Refs. [12,16].

Continuing in this way, in line (d) in Table B.4 we show the expected [11] improvement, with reference to the LEP time for Table B.3, in the light pairs error for the FCC-ee . As we

| Type of correction / Error | Update 2018 | FCC-ee forecast |
|---|---|---|
| (a) Photonic $[\mathcal{O}(L_e\alpha^2)]$ $\mathcal{O}(L_e^2\alpha^3)$ | 0.027% | $0.1 \times 10^{-4}$ |
| (b) Photonic $[\mathcal{O}(L_e^3\alpha^3)]$ $\mathcal{O}(L_e^4\alpha^4)$ | 0.015% | $0.6 \times 10^{-5}$ |
| (c) Vacuum polariz. | 0.014% [12] | $0.6 \times 10^{-4}$ |
| (d) Light pairs | 0.010% [13, 14] | $0.5 \times 10^{-4}$ |
| (e) $Z$ and $s$-channel $\gamma$ exchange | 0.090% [9] | $0.1 \times 10^{-4}$ |
| (f) Up-down interference | 0.009% [15] | $0.1 \times 10^{-4}$ |
| (f) Technical Precision | (0.027)% | $0.1 \times 10^{-4}$ |
| Total | 0.097% | $1.0 \times 10^{-4}$ |

Table B.4: Anticipated total (physical+technical) theoretical uncertainty for a FCC-ee luminosity calorimetric detector with the angular range being 64–86 mrad (narrow), near the $Z$ peak. Description of photonic corrections in square brackets is related to the 2nd column. The total error is summed in quadrature.

explain in Ref. [11], the complete matrix element for the additional real $e^+e^-$ pair radiation should be used, because non-photonic graphs can contribute as much as 0.01% for the cut-off $z_{\rm cut} \sim 0.7$. This can be done with the MC generators developed for the $e^+e^- \to 4f$ processes for LEP2 physics - see Ref. [11] for further discussion. With known methods [11], the contributions of light quark pairs, muon pairs and non-leading, non-soft additional $e^+e^- + n\gamma$ corrections can be controlled such that the error on the pairs contribution is as given in line (d) for the FCC-ee. As noted, we also show the current state of the art [11] for this error in line(d) of Table B.4.

Turning to line (e) in Table B.4, we show the improvement of the error on the $Z$ and $s$-channel $\gamma$ exchange for the FCC-ee as well as its current state of the art. In Ref. [11], a detailed discussion is presented of all of the six interference and three additional squared modulus terms that result from the $s$-channel $\gamma$, $s$-channel $Z$, and $t$-channel $Z$ exchange contributions to the amplitude for the luminosity process. It is shown that, if the predictions of `BHLUMI` for the luminosity measurement at FCC-ee are combined with the ones from `Bhwide` [17] for this $Z$ and $s$-channel $\gamma$ exchange contribution, then the error in the second column of line (e) of Table B.4 could be reduced to 0.01%. In order to reduce the uncertainty of this contribution practically to zero we would include these $Z$ and $\gamma_s$ exchanges within the CEEX [18] type matrix element at $\mathcal{O}(\alpha^1)$ in `BHLUMI`. Here, CEEX stands for coherent exclusive exponentiation which acts at the level of the amplitudes as compared the original Yennie–Frautschi–Suura [19](YFS) exclusive exponentiation (EEX) that is used in `BHLUMI`4.04 and that acts at the level of the squared amplitudes. It is expected to be enough to add the EW corrections to the LABH process in the form of effective couplings in the Born amplitudes. This leads to the error estimate shown in Table B.4 in line(e) for the FCC-ee.

For completeness, we note that for our discussion of the $Z$ and $s$-channel $\gamma$ exchanges we made in Ref. [11] a numerical study using `Bhwide` for the the calorimetric LCAL-type detector, as described in ref. [20], for the symmetric angular range 64–86 mrad without any cut on acoplanarity. The pure weak corrections were calculated with the `ALIBABA` EW library [21, 22]. The results, shown in Table B.5, were obtained for three values of the centre-of-mass (CM) energy: $E_{\rm CM} = M_Z$, $M_Z \pm 1\,$GeV, where the latter two values have $Z$ contributions that are close to maximal in size. The results in the second column for the total size of the $Z$ and $\gamma_s$ exchanges are consistent with our expectations as explained in ref. [11]: the contribution is positive below the $Z$ peak where it reaches a size $\sim 0.64\%$, is close to zero near the peak, and

| $E_{\mathrm{CM}}$ [GeV] | $\Delta_{\mathrm{tot}}$ [%] | $\delta^{\mathrm{QED}}_{\mathcal{O}(\alpha)}$ [%] | $\delta^{\mathrm{QED}}_{\mathrm{h.o.}}$ [%] | $\delta^{\mathrm{weak}}_{\mathrm{tot}}$ [%] |
|---|---|---|---|---|
| 90.1876 | +0.642 (12) | −0.152 (59) | +0.034 (38) | −0.005 (12) |
| 91.1876 | +0.041 (11) | +0.148 (59) | −0.035 (38) | +0.009 (12) |
| 92.1876 | −0.719 (13) | +0.348 (59) | −0.081 (38) | +0.039 (13) |

Table B.5: Results from `Bhwide` for the $Z$ and $\gamma_s$ exchanges contribution to the FCC-ee luminosity with respect to the $\gamma_t \otimes \gamma_t$ process for the calorimetric LCAL-type detector [20] with the symmetric angular range 64–86 mrad; no acoplanarity cut was applied. MC errors are marked in brackets.

changes sign above the peak where it reaches a size $\sim -0.72\%$. The third column features the fixed-order (non-exponentiated) $\mathcal{O}(\alpha)$ QED correction and shows that it is sizeable and up to a half of the size of the Born level effect, with a sign that is opposite to that of the latter effect. The fourth column shows the size of the higher-order QED effects from YFS exponentiation, which also change their sign near the $Z$-peak, oppositely to the corresponding change of the $\mathcal{O}(\alpha)$ corrections. We see that the size of the former effects is about a quarter of that of the latter. The effects in the fourth column allow us to make a conservative estimate of the size of the missing higher-order QED effects in `Bhwide` using the big log factor $\gamma = \frac{\alpha}{\pi} \ln \frac{|\bar{t}|}{m_e^2} = 0.042$ of Section 4 of Ref. [11] and a safety factor of 2 of Ref. [9] together with the largest higher-order effect in Table B.5, 0.081%, as $0.081\% \times \gamma \times 2 \simeq 0.007\%$. The last column shows that the size of the pure weak corrections, as implemented within the $\mathcal{O}(\alpha)$ YFS exponentiation scheme, is at the level of 0.01% below and at $M_Z$ and increases up to $\sim 0.04\%$ above $M_Z$. We may use the same factor as we did for the higher order corrections to estimate the size of the missing higher order pure weak corrections in `Bhwide` as $\sim 0.003\%$. Altogether, by adding the two estimates of its massing effects, we obtain a conservative estimate of 0.01% for the physical precision of `Bhwide` to justify our remarks above concerning the error in line (e) of Table B.4 that would result from the combination of the prediction of `BHLUMI` and that of `Bhwide` for this contribution.

In line (f) in Table B.4 we show the estimate of the error on the up-down interference between radiation from the $e^-$ and $e^+$ lines. Unlike in LEP1, where it was negligible, for the FCC-ee this effect, calculated in Ref. [15] at $\mathcal{O}(\alpha^1)$, is 10 times larger and has to be included in the upgraded `BHLUMI`. Once this is done, the error estimate shown in line (f) for the FCC-ee obtains [11].

This brings us to the issue of the technical precision. In an ideal situation, in order to get the upgraded `BHLUMI`ś technical precision at the level $10^{-5}$ for the total cross section and $10^{-4}$ for single differential distributions, one would need to compare it with another MC program developed independently, which properly implements the soft-photon resummation, LO corrections up to $\mathcal{O}(\alpha^3 L_e^3)$, and the second-order corrections with the complete $\mathcal{O}(\alpha^2 L_e)$. In principle, an extension of a program like `BabaYaga` [23–25], which is currently exact at NLO with a matched QED shower, to the level of NNLO for the hard process, while keeping the correct soft-photon resummation, would provide the best comparison to the upgraded `BHLUMI` to establish the technical precision of both programs at the $10^{-5}$ precision level[¶]. During the intervening time period, a very good test of the technical precision of the upgraded `BHLUMI` would follow from the comparison between its results with EEX and CEEX matrix elements; for, the basic multi-photon phase space integration module of `BHLUMI` was already well tested

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[¶] The upgrade of the `BHLUMI` distributions will be relatively straightforward because its multi-photon phase space is exact [26] for any number of photons.

in Ref. [27] and such a test can be repeated at an even higher-precision level.

In summary, we conclude that, with the appropriate resources, the path to 0.01% precision for the FCC-ee luminosity (and the ILC luminosity) at the $Z$ peak is open via an upgraded version of `BHLUMI`.

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