CERN-ACC-NOTE-2019-0017

2019-05-16

Giovanni.Iadarola@cern.ch

Properties of the electromagnetic fields generated by a circular-symmetric e-cloud pinch in the ultra-relativistic limit

G. Iadarola

Keywords: electron-cloud, beam-dynamics, Maxwell's equations

Abstract

The electromagnetic fields generated by an electron cloud in the beam pipe of a particle accelerator introduce additional forces in the dynamics of the particle beams, which can be responsible of beam instabilities, beam losses and transverse emittance blow-up.
These effects are studied mainly employing numerical simulations, using macroparticle sets to model the circulating beam and the e-cloud. The fields are computed using the Particle In Cell method (PIC). In most simulation codes, the electrodynamics equations are solved in the quasi-electrostatic approximation.
To understand the validity of this approximation, we investigate analytically the full electromagnetic problem for a circular-symmetric case, identifying the structure of the electromagnetic fields. Expressions for the error introduced by the quasi-electrostatic approximation are derived, showing that this has no effects when calculating the transverse forces exerted by the e-cloud on the particle beam.
The longitudinal component of the field is also investigated and it is related to the power transferred from the beam to the e-cloud.

1 Introduction

To investigate the properties of the interaction of a high energy particle bunch interacting with an electron cloud [1, 2], we consider a cylindrical perfectly conducting pipe having radius R, filled with electrons initially at rest. We call z the longitudinal axis of the pipe and we define a system of cylindrical coordinates (r, ϕ, z).
A circular-symmetric particle bunch is travelling at the speed of light inside the pipe. The bunch generates a circular-symmetric electron pinch propagating at the speed of light along z together with the bunch, with the electrons moving only in the transverse direction:

$$\rho = \rho\left(r, t - \frac{z}{c}\right) \tag{1}$$

$$\mathbf{J} = J_r\left(r, t - \frac{z}{c}\right)\hat{\mathbf{i}}_r \tag{2}$$

The two distributions are related by the equation of continuity of charge:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \tag{3}$$

We want to study the properties of the electromagnetic fields generated by such a source and, in particular, how these fields compare with the solution obtained in the quasi-electrostatic approximation, which is usually made in macroparticle simulation codes used for e-cloud studies [2].

2 The field equations

Due to the symmetry in the geometry of the defined problem, the solution will be independent on ϕ. As the pipe is perfectly conducting and the source travels at the speed of light, we expect the generated electromagnetic field to travel at the same speed:

$$\mathbf{e} = \mathbf{e}\left(r, t - \frac{z}{c}\right) \tag{4}$$

$$\mathbf{h} = \mathbf{h}\left(r, t - \frac{z}{c}\right) \tag{5}$$

The fields are solutions of Maxwell's equations [3]:

$$\nabla \times \mathbf{e} = -\mu_0 \frac{\partial \mathbf{h}}{\partial t} \tag{6}$$

$$\nabla \times \mathbf{h} = \varepsilon_0 \frac{\partial \mathbf{e}}{\partial t} + \mathbf{J} \tag{7}$$

Using the expression of the curl in cylindrical coordinates we can write:

$$\nabla \times \mathbf{e} = \left(\frac{1}{r}\frac{\partial e_z}{\partial \phi} - \frac{\partial e_\phi}{\partial z}\right)\hat{\mathbf{i}}_r + \left(\frac{\partial e_r}{\partial z} - \frac{\partial e_z}{\partial r}\right)\hat{\mathbf{i}}_\phi + \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r e_\phi\right) - \frac{\partial e_r}{\partial \phi}\right)\hat{\mathbf{i}}_z \tag{8}$$

As the fields do not depend on ϕ, all derivatives with respect to ϕ vanish:

$$\nabla \times \mathbf{e} = -\frac{\partial e_\phi}{\partial z}\hat{\mathbf{i}}_r + \left(\frac{\partial e_r}{\partial z} - \frac{\partial e_z}{\partial r}\right)\hat{\mathbf{i}}_\phi + \frac{1}{r}\frac{\partial}{\partial r}\left(r e_\phi\right)\hat{\mathbf{i}}_z \tag{9}$$

Replacing Eq. 9 into Eq. 6 and comparing the individual components we obtain:

$$\frac{\partial e_\phi}{\partial z} = \mu_0 \frac{\partial h_r}{\partial t} \tag{10}$$

$$\frac{\partial e_r}{\partial z} - \frac{\partial e_z}{\partial r} = -\mu_0 \frac{\partial h_\phi}{\partial t} \tag{11}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r e_\phi\right) = -\mu_0 \frac{\partial h_z}{\partial t} \tag{12}$$

Following the same procedure for Eq. 7, we obtain:

$$\frac{\partial h_\phi}{\partial z} = -\varepsilon_0 \frac{\partial e_r}{\partial t} - J_r \tag{13}$$

$$\frac{\partial h_r}{\partial z} - \frac{\partial h_z}{\partial r} = \varepsilon_0 \frac{\partial e_\phi}{\partial t} \tag{14}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r h_\phi\right) = \varepsilon_0 \frac{\partial e_z}{\partial t} \tag{15}$$

Eqs. 10 - 15 constitute a system of six independent scalar partial differential equations in the six unknowns $\left(e_r, e_\phi, e_z, h_r, h_\phi, h_z\right)$.
We define an auxiliary variable τ as:

$$\tau = t - \frac{z}{c} \tag{16}$$

For all electric and magnetic field components, remembering that we are assuming that all our sources and fields depend on t and z in the form given by Eqs. 4 and 5 (i.e. they travel at the speed of light along z), we can write:

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} \tag{17}$$

$$\frac{\partial}{\partial z} = -\frac{1}{c}\frac{\partial}{\partial \tau} \tag{18}$$

Note that $\tau = t$ at $z = 0$, therefore the fields $\mathbf{e}(r, \tau)$ and $\mathbf{h}(r, \tau)$ are those observed at the section z=0.
Using Eqs. 17 and 18, Eq. 10 becomes:

$$\frac{\partial}{\partial \tau}\left(e_\phi + \zeta_0 h_r\right) = 0 \tag{19}$$

where $\zeta_0 = \sqrt{\mu_0/\varepsilon_0}$.
As the electrons are initially at rest at $z = 0$, we have

$$e_\phi(r, \tau = -\infty) = 0 \tag{20}$$

$$h_r(r, \tau = -\infty) = 0 \tag{21}$$

Combining Eq. 19, 20 and 21 we obtain:

$$e_\phi = -\zeta_0 h_r \tag{22}$$

Using Eqs. 17, 18 and 22, Eq. 14 becomes:

$$\frac{\partial h_z}{\partial r} = 0 \tag{23}$$

which means that h_z must be uniform in r. Several considerations (for example the fact that sending the radius of the pipe to infinity, the total magnetic energy would diverge) suggest that in fact have :

$$h_z(r, \tau) = 0 \tag{24}$$

Using Eqs. 17, 18, 22 and 24, one can similarly simplify Eqs.12 and 14 to obtain:

$$e_\phi(r, \tau) = 0 \tag{25}$$
$$h_r(r, \tau) = 0 \tag{26}$$

Equations 24, 25 and 26 mean that **the electromagnetic field has the following form:**

$$\mathbf{e} = e_r \left(r, t - \frac{z}{c}\right) \hat{\mathbf{i}}_r + e_z \left(r, t - \frac{z}{c}\right) \hat{\mathbf{i}}_z \tag{27}$$
$$\mathbf{h} = h_\phi \left(r, t - \frac{z}{c}\right) \hat{\mathbf{i}}_\phi \tag{28}$$

i.e. **the electromagnetic field has a Transverse Magnetic (TM) structure.**
A field in this form automatically satisfies Eqs. 10, 12 and 14. The remaing three unknowns (e_r, e_z, h_ϕ) will have to be found by solving Eqs. 11, 13 and 15, which can be rewritten as:

$$\frac{1}{c}\frac{\partial e_r}{\partial \tau} + \frac{\partial e_z}{\partial r} = \mu_0 \frac{\partial h_\phi}{\partial \tau} \tag{29}$$
$$\frac{1}{c}\frac{\partial h_\phi}{\partial \tau} = \varepsilon_0 \frac{\partial e_r}{\partial \tau} + J_r \tag{30}$$
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r h_\phi\right) = \varepsilon_0 \frac{\partial e_z}{\partial \tau} \tag{31}$$

3 Equation of the magnetic field

Equation 30 can be rewritten as:

$$\frac{\partial e_r}{\partial \tau} = -\frac{J_r}{\varepsilon_0} + \zeta_0 \frac{\partial h_\phi}{\partial \tau} \tag{32}$$

Replacing Eq. 32 into Eq. 29 we obtain:

$$\frac{\partial e_z}{\partial r} = \zeta_0 J_r \tag{33}$$

This equation relates the longitudinal electric field generated by the electrons and the speed of the electron themselves (we will use this in Sec. 5 for some energy consideration).
Now, we derive both sides of Eq. 31 with respect to r:

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(rh_{\phi}\right)\right) = \varepsilon_0 \frac{\partial^2 e_z}{\partial r \partial \tau} \tag{34}$$

Using Schwatz's theorem to swap the derivatives and Eq. 33, we obtain a **second order differential equation relating directly the magnetic field with the assigned electron current distribution**:

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(rh_{\phi}\right)\right) = \frac{1}{c}\frac{\partial J_r}{\partial \tau} \tag{35}$$

4 Comparison against quasi-electrostatic solution

Most simulation codes compute the electric field using a quasi-electrostatic 2D method, which assume that time variations are slow enough to neglect time derivatives [2]. Assuming the pinch is travelling rigidly along z, as done in the previous section, this means:

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} \simeq 0 \tag{36}$$

$$\frac{\partial}{\partial z} = -\frac{1}{c}\frac{\partial}{\partial \tau} \simeq 0 \tag{37}$$

It is possible to investigate the impact of this approximation without solving explicitly Eq. 35.
We call $\mathbf{e}^{\text{qes}}$ the transverse component of the electric field calculated in this approximation. With the assumption in Eq. 36, the Faraday-Neumann's equation (Eq. 7) becomes:

$$\nabla \times \mathbf{e}^{\text{qes}} = 0 \tag{38}$$

which, projected along z, gives:

$$\frac{\partial e_y^{\text{qes}}}{\partial x} - \frac{\partial e_x^{\text{qes}}}{\partial y} = 0 \tag{39}$$

This means (Stokes's theorem) that for any chosen values t_0 and z_0 the 2D field:

$$\mathbf{e}^{\text{qes}}(x, y, z_0, t_0) = e_x^{\text{qes}}(x, y, z_0, t_0)\hat{\mathbf{i}}_x + e_y^{\text{qes}}(x, y, z_0, t_0)\hat{\mathbf{i}}_y \tag{40}$$

can be written as the gradient of a scalar potential $\phi_{z_0,t_0}(x, y)$:

$$e_x^{\text{qes}}(x, y, z_0, t_0) = \frac{\partial \phi_{z_0,t_0}}{\partial x} \tag{41}$$

$$e_y^{\text{qes}}(x, y, z_0, t_0) = \frac{\partial \phi_{z_0,t_0}}{\partial y} \tag{42}$$

In the approximation given by Eq. 37, Gauss's law becomes:

$$\frac{\partial e_x^{\text{qes}}}{\partial x} + \frac{\partial e_y^{\text{qes}}}{\partial y} = \frac{\rho\left(x, y, z, t\right)}{\varepsilon_0} \tag{43}$$

We evaluate Eq. 43 in the plane $z = z_0$, at $t = t_0$:

$$\frac{\partial}{\partial x} e_x^{\text{qes}}\left(x, y, z_0, t_0\right) + \frac{\partial}{\partial y} e_y^{\text{qes}}\left(x, y, z_0, t_0\right) = \frac{\rho\left(x, y, z_0, t_0\right)}{\varepsilon_0} \tag{44}$$

Replacing Eqs. 41 and 42 into into Eq. 44 we obtain a 2D Poisson equation, which provides the electric potential in the plane $z = z_0$ at time $t = t_0$:

$$\frac{\partial^2 \phi_{z_0, t_0}}{\partial x^2} + \frac{\partial^2 \phi_{z_0, t_0}}{\partial y^2} = \frac{\rho\left(x, y, z_0, t_0\right)}{\varepsilon_0} \tag{45}$$

In simulation codes (e.g. PyECLOUD) this equation is solved at each time-step to compute the field generated by the e-cloud at a given section of the machine.
For our circular-symmetric case as defined in Sec. 1 we can rewrite Eq. 45 in cylindrical coordinates:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\phi_{z_0, t_0}\left(r\right)\right) = \frac{\rho\left(r, z_0, t_0\right)}{\varepsilon_0} \tag{46}$$

with:

$$e_r^{\text{qes}}(r, z_0, t_0) = \frac{d\phi_{z_0, t_0}}{dr} \tag{47}$$

By replacing Eq. 47 into Eq. 46 and integrating with respect to r we obtain:

$$\mathbf{e}^{\text{qes}}(r, z_0, t_0) = \frac{1}{2\pi\varepsilon_0}\frac{Q(r, z_0, t_0)}{r}\hat{\mathbf{i}}_r \tag{48}$$

where $Q(r, z_0, t_0)$ is the charge in a circle of radius r :

$$Q(r, z_0, t_0) = \int_0^r \rho(r', z_0, t_0) 2\pi r' dr' \tag{49}$$

As ρ does not depend independently on z_0 and t_0 but only on $\tau_0 = t_0 - z_0/c$, also Q and $\mathbf{e}^{\text{qes}}$ depends only on τ_0. As z_0 and t_0 can be arbitrarily chosen, Eqs. 48 and 49 can be written for any value of τ:

$$\mathbf{e}^{\text{qes}}(r, \tau) = \frac{1}{2\pi\varepsilon_0}\frac{Q(r, \tau)}{r}\hat{\mathbf{i}}_r \tag{50}$$

$$Q(r, \tau) = \int_0^r \rho(r', \tau) 2\pi r' dr' \tag{51}$$

Using the charge continuity equation (Eq. 3) integrating over as circle of radius r, we obtain:

$$J_r = \frac{\partial}{\partial \tau}\left(\frac{Q(r, \tau)}{2\pi r}\right) \tag{52}$$

Comparing Eq. 52 against Eq. 50 we obtain:

$$\frac{\partial e_r^{\text{qes}}}{\partial \tau} = -\frac{J_r}{\varepsilon_0} \tag{53}$$

We call δe_r the error introduced by the quasi-electrostatic approximation with respect to the solution of the complete set of Maxwell's equations:

$$e_r(r,\tau) = e_r^{\text{qes}} + \delta e_r \tag{54}$$

Replacing Eq. 54 into Eq. 30 we obtain:

$$\frac{1}{c}\frac{\partial h_\phi}{\partial \tau} = \varepsilon_0 \frac{\partial \delta e_r}{\partial \tau} + \varepsilon_0 \frac{\partial e_r^{\text{qes}}}{\partial \tau} + J_r \tag{55}$$

The sum of the last two terms on the r.h.s. is zero from Eq. 53. Therefore Eq. 55 simply becomes:

$$\frac{\partial}{\partial \tau}\left(h_\phi - \frac{1}{\zeta_0}\delta e_r\right) = 0 \tag{56}$$

Assuming that the electrons were initially at rest (at that point stationary equations were exact) we can write:

$$h_\phi(r, \tau = -\infty) = 0 \tag{57}$$

$$\delta e_r(r, \tau = -\infty) = 0 \tag{58}$$

Using these initial conditions Eq. 56 becomes:

$$h_\phi = \frac{1}{\zeta_0}\delta e_r \tag{59}$$

which shows that **the magnetic field is proportional to the error on the transverse electric field introduced by the quasi-electrostatic approximation**.
Now we want to compute the effect on the Lorentz force acting on a beam particle travelling at the speed of light along the z direction:

$$\mathbf{F} = q_{\text{part}}\left(\mathbf{e} + \mu_0 \mathbf{v}_{\text{part}} \times \mathbf{h}\right) = q_{\text{part}}\left(e_r^{\text{qes}}\hat{\mathbf{i}}_r + \delta e_r \hat{\mathbf{i}}_r + e_z \hat{\mathbf{i}}_z + \mu_0 \mathbf{v}_{\text{part}} \times \mathbf{h}\right) \tag{60}$$

We can easily compute:

$$\mu_0 \mathbf{v}_{\text{part}} \times \mathbf{h} = \mu_0\left(c\hat{\mathbf{i}}_z\right) \times \left(h_\phi \hat{\mathbf{i}}_\phi\right) = -\zeta_0 h_\phi \hat{\mathbf{i}}_r \tag{61}$$

Then Eq. 60 can be rewritten as:

$$\mathbf{F} = q_{\text{part}}\left(e_r^{\text{qes}}\hat{\mathbf{i}}_r + e_z \hat{\mathbf{i}}_z + \delta e_r \hat{\mathbf{i}}_r - \zeta_0 h_\phi \hat{\mathbf{i}}_r\right) \tag{62}$$

Using Eq. 59 we can see that **the last two terms, i.e. the force introduced by the electric field correction and the force introduced by the magnetic field, exactly cancel each other**, leaving us with:

$$\mathbf{F} = q_{\text{part}}\left(e_r^{\text{qes}}\hat{\mathbf{i}}_r + e_z \hat{\mathbf{i}}_z\right) \tag{63}$$

i.e **in the ultrarelativistic assumption, the transverse kick computed with the quasi-electrostatic approximation is exact**.
A longitudinal component of the Lorentz force is also present, which introduces an acceleration/deceleration on the beam particle.

5 Energy balance

The instantaneous power per unit path length transferred by the electric field generated by the electrons to an ultrarelativistic beam travelling along z is given by:

$$\frac{dP}{dz} = \iint_S e_z^{\text{ele}} J_z^{\text{beam}} dS \tag{64}$$

where S is the section of the pipe.
J_z^{beam} is the beam current density and, as all beam particles travel at the speed of light along z, it is related to the beam charge density by:

$$\rho^{\text{beam}}(r,\tau) = \frac{J_z^{\text{beam}}(r,\tau)}{c} \tag{65}$$

The electric field generated by the beam has only the radial component, which is given by:

$$e_r^{\text{beam}} = \frac{1}{2\pi\varepsilon_0}\frac{1}{r}\int_0^r \rho\left(r',\tau\right) 2\pi r' dr' \tag{66}$$

Using Eq. 65, we can write:

$$\iint_S e_z^{\text{ele}} J_z^{\text{beam}} dS = c\int_0^R e_z^{\text{ele}}\left(r',\tau\right)\rho^{\text{beam}}\left(r',\tau\right) 2\pi r' dr' \tag{67}$$

This integral can be computed by parts taking into account that a primitive of $\left(\rho\left(r',\tau\right)2\pi r'\right)$ is provided by Eq. 66:

$$\iint_S e_z^{\text{ele}} J_z^{\text{beam}} dS = 2\pi\varepsilon_0 c\left[r e_z^{\text{ele}} e_r^{\text{beam}}\right]_0^R - \varepsilon_0 c\int_0^R \frac{\partial e_z^{\text{ele}}\left(r',\tau\right)}{\partial r'} e_r^{\text{beam}}\left(r',\tau\right) 2\pi r' dr' \tag{68}$$

Taking in into account that due to the perfectly conducting pipe

$$e_z^{\text{ele}}\left(r=R,\tau\right) = 0 \tag{69}$$

the first term on the r.h.s of Eq. 68 vanishes. The second term can be rewritten using Eq. 33 obtaining:

$$\iint_S e_z^{\text{ele}} J_z^{\text{beam}} dS = -\int_0^R J_r^{\text{ele}} e_r^{\text{beam}} 2\pi r' dr' \tag{70}$$

which can be read as:

$$-\iint_S e_z^{\text{ele}} J_z^{\text{beam}} dS = \iint_S e_r^{\text{beam}} J_r^{\text{ele}} dS \tag{71}$$

This proves that **the power subtracted from the beam by the electric field generated by the electrons is equal to the power provided to the electrons by the electric field generated by the beam**.

6 Conclusions

In a circular-symmetric geometry, the electromagnetic field generated by an e-cloud pinch following an ultrarelativistic bunch has a Transverse Magnetic (TM) structure.
A second order partial differential equation can be derived, which relates directly the magnetic field to the electron current distribution.
The magnetic field is found to be proportional to the error introduced by the quasi-electrostatic approximation on the transverse component of the electric field. When computing the Lorentz force on an ultrarelativistic particle, these two terms cancel each other. For this reason the transverse force computed with the quasi-electrostatic approximation turns out to be exact.
The longitudinal component of the field is also investigated and it is related to the power transferred from the beam to the e-cloud.

Acknowledgements

The author would like to thank G. Arduini, E. Métral and G. Rumolo for their valuable input to this work.

Bibliography

[1] G. Rumolo, F. Ruggiero, and F. Zimmermann, “Simulation of the electron-cloud build up and its consequences on heat load, beam stability, and diagnostics”, Phys. Rev. ST Accel. Beams 4, 012801, 2001.

[2] G. Iadarola, “Electron cloud studies for CERN particle accelerators and simulation code development”, CERN-THESIS-2014-047, 2014.

[3] J.D. Jackson, “Classical electrodynamics”, Wiley New York, 1999.