NEW LONGITUDINAL BEAM IMPEDANCE FORMULA WITH 8 TERMS

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Abstract

A previous formula for longitudinal beam impedance was developed by S. Heifets, A. Wagner and B. Zotter and consists of 13 terms. The formula by Heifets, Wagner and Zotter is developed up to terms of second order in the transverse offsets. In this report we develop a new formula, which use symmetries from the Lorentz reciprocity principle and the multipolar decomposition of transverse fields to reduce the number of terms from 13 to 8. This new formula is also developed up to second order, but could of course be developed up to any higher order. The transverse beam impedances (for horizontal and vertical planes) can be by obtained by differentiation of this new formula.

INTRODUCTION

The previous formula for the longitudinal impedance by S. Heifets, A. Wagner and B. Zotter in Ref. [1] (See also Ref. [2]) and it gives the longitudinal beam coupling impedance as a function of the position of the transverse positions of the drive and test particles (See Ref. [3] for a definition of the drive and test particles). The new formula shown in this paper is based on additional constrains to Heifets, Wagner and Zotter's formula. These constraints comes from two physical principles: *The Lorentz reciprocity principle* and the *Multipolar expansion of 2D fields*. It is important to note that both the Heifets, Wagner and Zotter formula and the new formula gives the beam coupling impedance only - neither includes direct nor indirect space charge impedance.

The Lorentz reciprocity principle

The Lorentz reciprocity principle says that the longitudinal beam impedance must stays unchanged if the drive and test particles are interchanged. The principle can be illustrated with two antenna. Injecting a current in the first antenna will give a voltage on the second antenna. Reversing the situation, and now injecting the same current in the second antenna, we will get exactly the same voltage in the first antenna as we had in the second antenna (see Fig. 1):

Figure 1: Injecting a current in the antenna to the left, will induce a voltage in the antenna to the right. The Lorentz reciprocity principle says that injecting the same current in the antenna to the right will produce exactly the same voltage in the antenna to the left.

The Lorentz reciprocity principle describes what is also called the mutual impedance. Mutual impedance is best illustrated by transformers. (see Fig. 2). In this example a current I_1 (this is an oscillating current) is injected on the primary winding. This current will induce a flux ÏĘ in the iron core: $\phi = B \cdot A$, where ϕ is the flux, B is the B-field and A is the area of the transformer core. Using Amperes law: $\oint_C B\, dl = \mu_0 \mu_r I_1 \cdot N_1$, where C is the circumference of the transformer; μ_0 and μ_r are the permeability of free space and the relative permeability of the iron in the transformer; I_1 is the current in the primary winding and N_1 is the number of windings on the primary side. Making the approximation that the circumference is everywhere the same, then the flux will be: $\phi = A \cdot \mu_0 \mu_r \cdot \frac{N_1}{C} \cdot I_1$. The induced voltage on the secondary windings from this flux is: $V_2 = -N_2 \cdot \frac{d\phi}{dt} = -A \cdot \mu_0 \mu_r \cdot \frac{N_1 * N_2}{C} \cdot \frac{dI_1}{dt}$. With a similar argument, the voltage on the primary winding can be calculated as a function of the secondary current: $V_1 = -A \cdot \mu_0 \mu_r \cdot \frac{N_1 * N_2}{C} \cdot \frac{dI_2}{dt}$. If I_1 is identical to I_2 then the voltages V_1 and V_2 will also be identical and the Lorentz reciprocity principle is shown for a transformer.

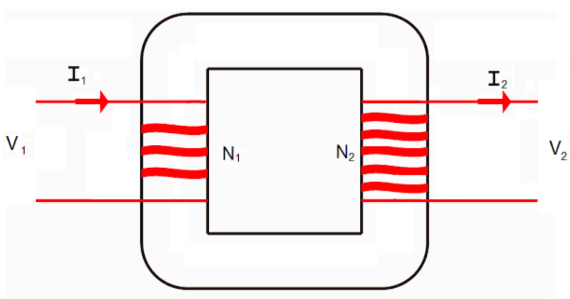

Figure 2: Injecting the current " I_1" in the primary winding, will give a voltage " V_2" on the secondary winding. In the same way, if we would inject the same current in the secondary winding, the voltage on the primary winding would be exactly the same. The Lorentz reciprocity principle is clear if the number of windings would have been the same on both the primary and secondary sides, because then currents and voltages would also be the same on both sides.

It is interesting to note that the beam impedance coming from the wall currents is more precisely called the beam coupling impedance. The reason is that a beam going through a vacuum chamber represents a current. This current will induce currents on the chamber walls, because just like a transformer they are coupled like the currents in the primary (i.e. the the beam current) and secondary windings (i.e the wall currents) of a transformer.

Whether the path of the drive particle works as the transmitting antenna while the path of the test particle works as

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the receiving antenna or vise-versa i.e. The longitudinal beam coupling impedance is the same when the drive and test particle positions are interchanged: $Z_{||}(x_d, x_t, y_d, y_t) = Z_{||}(x_t, x_d, y_t, y_d)$. This feature was already documented by S.Heifets and B.Zotter in Ref. [4].

Multipolar expansion in 2D

Multipolar expansion is described in Ref. [5] and the expansion in cartesian coordiantes is detailed in Ref. [6]. Multipolar expansion is well known for accelerator magnets, where e.g. dipole magnet bends the beam in circular trajectories, and quadrupolar magnets (i.e. 4 poles, 2 North and 2 South) acts as special bending magnets that bends the beam more the further the beam is from the center. Fig. 3 shows the shape of the fields. For each order of multipole there are two types of fields, called Normal and Skew fields. The Normal or Skew types of field depends on the azimuthal angle of the magnet:

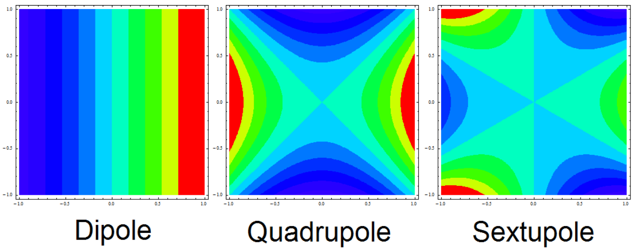

Figure 3: **Normal** field patterns up to third order for respectively a dipole, quadrupole and sextupole magnet. The potentials are:
Dipole: x Quadrupole: $\frac{x^2}{2} - \frac{y^2}{2}$ Sextupole: $\frac{x^3}{3} - xy^2$

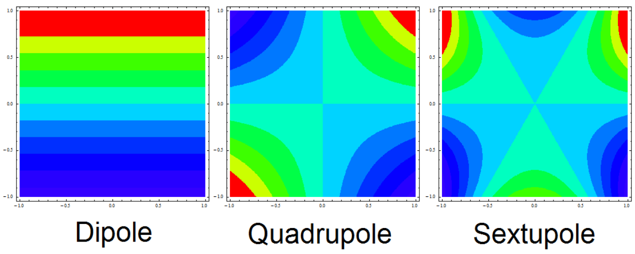

Figure 4: **Skew** field patterns up to third order for respectively a dipole, quadrupole and sextupole magnet. The potentials are:
Dipole: y Quadrupole: $x \cdot y$ Sextupole: $x^2 y - \frac{y^3}{3}$

There is no limit to how high the order can be. In LHC there are correction magnets up to 12'th order (Dodecapole). The idea of multipolar decomposition is the same as in Fourier Transforms, where a function is decomposed into a sum of Sin[] and Cos[] functions. The new formula for longitudinal beam coupling impedance has only terms up to second order i.e. only terms up to quadrupolar order are included. Please note that multipolar decomposition only works for realistic field patterns i.e. field patterns that occurs in nature. In this sense the decomposition is an analytical function (See Ref. [7]).

DERIVATION OF THE NEW FORMULA

Standard Taylor series evaluation, up to second order, gives the longitudinal beam coupling impedance with 15 terms. Each term consist of a function of frequency multiplied with a combination of transverse positions of the drive and test particles:
$Z[x_d, x_t, y_d, y_t] = Z_1[w] + Z_2[w]x_d + Z_3[w]x_t + Z_4[w]y_d + Z_5[w]y_t + Z_6[w]x_d^2 + Z_7[w]x_t^2 + Z_8[w]y_d^2 + Z_9[w]y_t^2 + Z_{10}[w]x_d x_t + Z_{11}[w]x_d y_d + Z_{12}[w]x_d y_t + Z_{13}[w]x_t y_d + Z_{14}[w]x_t y_t + Z_{15}[w]y_d y_t$
Using the multipolar decomposition, the following relations are obtained: $Z_6[w] = Z_7[w] = -Z_8[w] = -Z_9[w]$
The formula for the longitudinal beam coupling impedance is now reduced to 13 terms and these terms are identical to the terms in Heifets, Wagner and Zotter's formula (See Ref. [1] equation (24)). By using the Lorentz reciprocity principle i.e. that the longitudinal beam coupling impedance is the same when the drive and test particle positions are interchanged: $Z_{||}[x_d, x_t, y_d, y_t] = Z_{||}[x_t, x_d, y_t, y_d]$ the number of terms in the formula is reduced from 13 to 8 terms and the new formula is obtained:

$$\begin{aligned} Z_{||}[x_d, x_t, y_d, y_t] = \; & Z_0 \\ & + Z_{1,x}(x_d + x_t) + Z_{1,y}(y_d + y_t) \\ & + Z_{2,A}({x_d}^2 + {x_t}^2 - {y_d}^2 - {y_t}^2) \\ & + Z_{2,B}(x_d y_d + x_t y_t) + Z_{2,C}(x_d y_t + x_t y_d \\ & + Z_{2,D}(x_d x_t) + Z_{2,E}(y_d y_t) \qquad (1) \end{aligned}$$

where x_d and y_d are the transverse positions of the drive particle i.e. generally the beam moves along the this path. x_t and y_t are the transverse positions of the test particle i.e. the induced voltage is measured along this path.

To illustrate the Lorentz reciprocity principle, a simulation with CST (See Ref. [8]) was done on a collimator type structure, see Fig. 5:

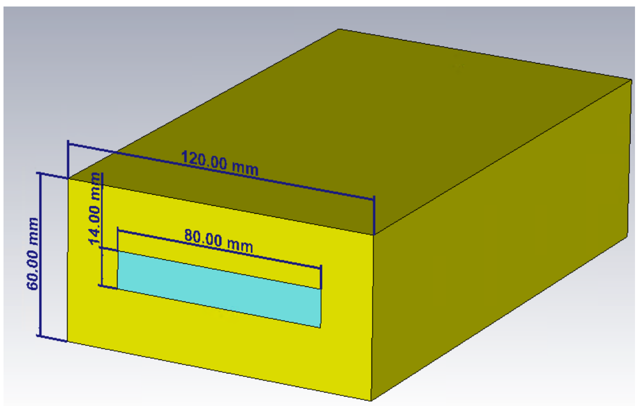

Figure 5: CST simulation of a collimator type structure. The jaws (yellow) have a conductivity of 10^5 [S/m]

where the jaws are made of lossy metal with a conductivity of 10^5 [S/m]. The result is shown in Fig. 6 and clearly demonstrates that the Longitudinal beam coupling impedance is identical when the path of the drive and test particles are interchanged:

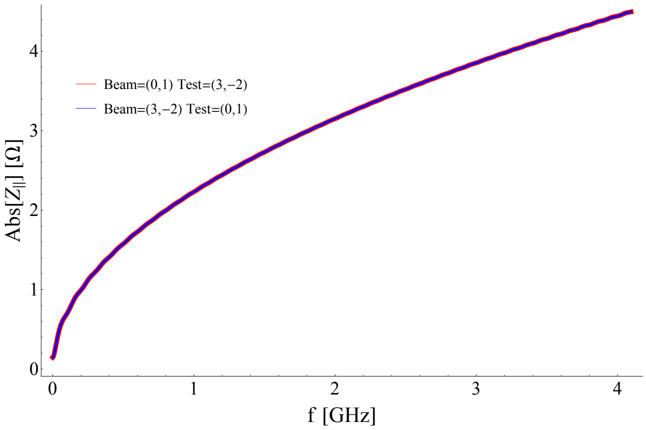

Figure 6: Result of the CST simulations of the collimator in Fig. 5. The longitudinal beam coupling impedance is identical when the path of the drive and test particles are interchanged. Red curve: The drive particle have the transverse coordinates: (x,y)=(0,1) and the test particle has the coordinates: (x,y)=(3,-2). Blue curve: The drive particle have the transverse coordinates: (x,y)=(3,-2) and the test particle has the coordinates: (x,y)=(0,1).

The new formula in equation (1) make several predictions, one of which is that changing the product $x_d y_d$ should give the same effect on the $Z_{||}$ as changing the product $x_t y_t$, i.e. both these product have the same coefficient $Z_{2,B}$. This is again illustrated with a CST simulation of the structure in Fig. 5, but this time the conductivity of the jaws is 10^2 [S/m]. This is done to avoid the classical thick wall regime, where the real and imaginary parts of $Z_{||}$ are always equal to each other. Using a conductivity in the jaws equal to 10^2 [S/m] ensures that the real and imaginary parts are different. The simulation result is shown in Figure 7:

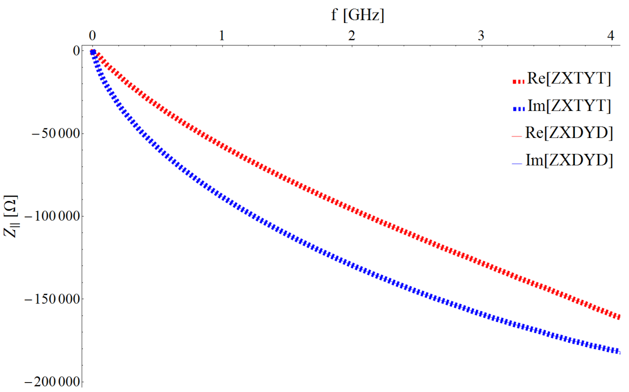

Figure 7: Result of the CST simulations of the collimator in Fig. 5 illustrating that the coefficient $Z_{2,B}$ is the same for both the product of XTYT as the product of XDYD.

The new formula in equation (1) for the longitudinal beam coupling impedance can, in combination with the Panofsky-Wenzel theorem, be used to calculate the transverse impedances: $Z_{\perp,x}=\frac{\beta c}{\omega}\frac{dZ_{||}}{dx_t}$ and $Z_{\perp,y}=\frac{\beta c}{\omega}\frac{dZ_{||}}{dy_t}$.

Looking at the horizontal transverse beam impedance terms: $Z_{\perp,x}=Z_{1,x}+2\frac{\beta c}{\omega}Z_{2,A}x_t+\frac{\beta c}{\omega}Z_{2,B}y_t+\frac{\beta c}{\omega}Z_{2,C}y_d+\frac{\beta c}{\omega}Z_{2,D}x_d$ we see the two well known transverse impedance terms, the dipolar impedance: $\frac{\beta c}{\omega}Z_{2,D}$ and the quadrupolar impedance: $2\frac{\beta c}{\omega}Z_{2,A}$.

Conventionally an equipment is characterized by only five parameters: Z_0 = Longitudinal impedance; $2\frac{\beta c}{\omega}Z_{2,D}$ = Dipolar impedance (horizontal plane); $2\frac{\beta c}{\omega}Z_{2,A}$ = Quadrupolar impedance (horizontal plane); $\frac{\beta c}{\omega}Z_{2,E}$ = Dipolar impedance (vertical plane) and finally $-2\frac{\beta c}{\omega}Z_{2,A}$ = Quadrupolar impedance (vertical plane). However, since the quadrupolar impedances for horizontal and vertical planes have the same amplitude, but opposite signs, there are in fact only four distinct parameters. To characterize an equipment with the above four parameters neglects the additional four other parameters: $Z_{1,x}$; $Z_{1,y}$; $Z_{2,B}$ and $Z_{2,C}$. Even though the parameters $Z_{1,x}$; $Z_{1,y}$; $Z_{2,B}$ and $Z_{2,C}$ are often zero because of symmetries (see next chapter SYMMETRIC VACUUM CHAMBERS AND THEIR EFFECT ON BEAM IMPEDANCE), they should not be forgotten in asymmetric structures.

SYMMETRIC VACUUM CHAMBERS AND THEIR EFFECT ON BEAM IMPEDANCE

Vacuum chamber symmetries will reduce the number of parameters in the longitudinal beam coupling impedance formula (1). A Left/right symmetric vacuum chamber gives $Z_{1,X}=0$, $Z_{2,B}=0$ and $Z_{2,C}=0$. (this results from solving the equation: $Z_{||}[x_d,x_t,y_d,y_t]=Z_{||}[-x_d,-x_t,y_d,y_t]$) Similarly an up/down symmetric structure, see Figure 8, gives $Z_{1,Y}=0$, $Z_{2,B}=0$ and $Z_{2,C}=0$:

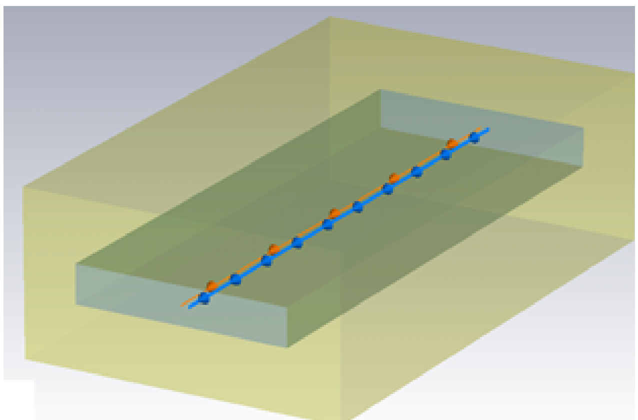

Figure 8: Up/Down symmetric structure. The transverse wake potentials of this structure is simulated with CST and the result of these simulations are shown in Fig. 9 and 10

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For the Up/Down symmetric structure in Fig. 8 above, the transverse horizontal beam impedance is:
$Z_{\perp,x}=Z_{1,x}+2\frac{\beta c}{\omega}Z_{2,A}x_t+\frac{\beta c}{\omega}Z_{2,B}y_t+\frac{\beta c}{\omega}Z_{2,C}y_d+\frac{\beta c}{\omega}Z_{2,D}x_d$

If $Z_{2,B}$ and $Z_{2,C}$ are zero, then the transverse horizontal wake potential should stay constant when the vertical position of either the drive- or the test particles are changed. This is illustrated in Fig. 9 and 10.

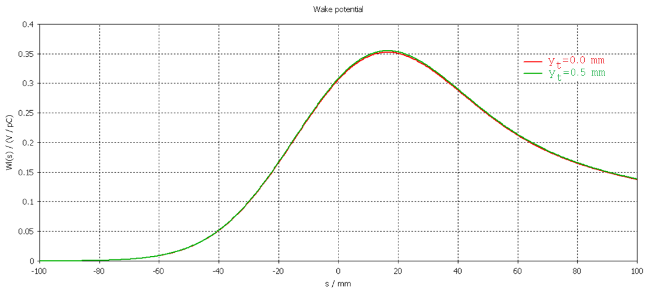

Figure 9: CST simulation of the transverse horizontal wake potential for the Up/Down symmetric structure in Fig. 8. The red curve is the transverse wake potential for y_t = 0.0 mm. The green curve is the transverse wake potential for y_t = 0.5 mm. Since the two curves are identical, $Z_{2,B}$ must be zero.

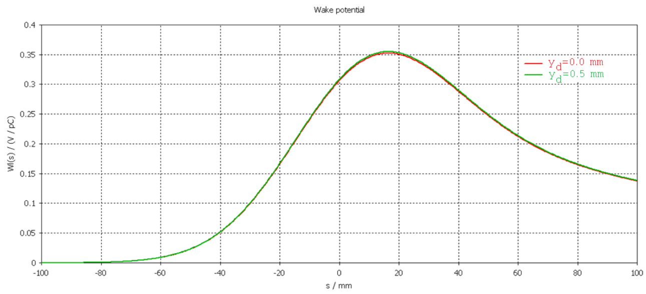

Figure 10: CST simulation of the transverse horizontal wake potential for the Up/Down symmetric structure in Fig. 8. as a function of the vertical position of the drive particle. The red curve is the transverse wake potential for y_d = 0.0 mm. The green curve is the transverse wake potential for y_d = 0.5 mm. Since the two curves are identical, $Z_{2,C}$ must be zero.

A 90 degrees symmetric structure, i.e. a structure which is unchanged when rotated by 90 degrees (see Fig. 11), only have dipolar tranverse impedance.

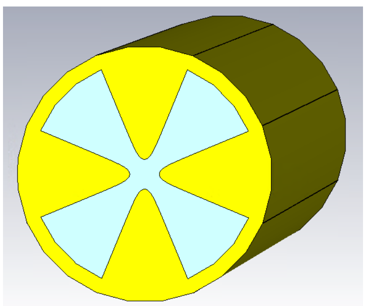

Figure 11: Structure with 90 degrees symmetry. Because of the 90 degrees symmetry, this structure only have dipolar beam impedance.

This means that only the horizontal dipolar term $Z_{2,D}$ and the vertical dipolar term $Z_{2,E}$ are different from zero while all the other transverse parameters:
$Z_{1,x}$, $Z_{1,y}$, $Z_{2,A}$, $Z_{2,B}$ and $Z_{2,C}$ are all zero (See Ref. [9]).
is interesting to note also that the horizontal and vertical dipolar terms are equal for 90 degrees symmetric structures $Z_{2,D} = Z_{2,E}$ - in fact the dipolar impededance is the same in any direction - i.e. independent of the angle that it is measured - see Fig. 12:

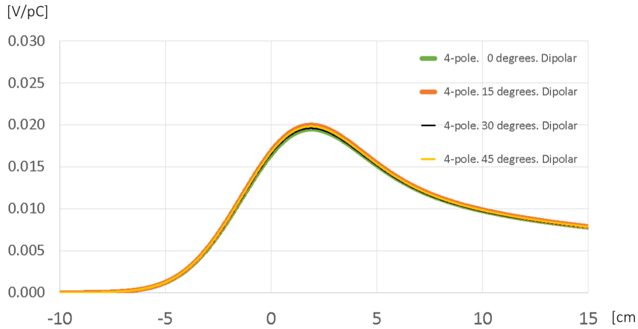

Figure 12: PBCI (See Ref. [10]) simulation of the 90 degrees symmetric structure in Fig. 11. The dipolar wakefields are identical for any rotation angle of the structure in Fig. 11, confirming that structures with 90 degrees symmetry have the same dipolar beam impedance in any direction. *Simulation courtesy of Dr. E. Gjonaj TU Darmstadt*

MORE WORK TO BE DONE

More work needs to be done to better understand the new formula for the longitudinal beam coupling impedance. The formula is presently developed to second order, but should be developed to third order, so that feed-down analysis can be done. Feed-down analysis will show the beam impedance if the beam does not go through the center, but is offset from the center. The principle of feed-down analysis is best known from accelerator magnets and is shown in Ref. [11].

It was shown recently that the quadrupolar beam impedance for circular vacuum chambers is only zero if the beam moves at ultra-relativistic speeds i.e. $\gamma = \infty$ (See Ref. [12]). The formula for the quadrupolar beam impedance per unit length, was gives as: $\frac{dZ_y(quad)}{dz} = \frac{k_0 Z_s}{4\pi b \beta \gamma^2 I_0^2(\frac{k_0 b}{\beta\gamma})}$
and we can indeed see that it will only be zero if $\gamma \longrightarrow \infty$. However, this formula looks very much like it describes a space charge beam impedance, and if this is indeed the case, then the new formula for the beam coupling impedance in (1) could still be true for $\gamma < \infty$, as it only describes beam coupling impedance and not space charge impedance.

Another *very interesting prospect* for multipolar expansion is that possibly one could eliminate all transverse beam impedance in collimators (and other structures) by shaping the collimator as a very high order multipole (See Ref. [13])

ACKNOWLEDGEMENTS

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