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Large Amplitude Stability Criteria for a Beam Loaded Synchrotron R.F. System

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Abstract

For the case of a beam-loaded radio-frequency acceleration system, we have found the equation for coherent oscillation of the bunch centroid; and we have found the equation for incoherent oscillation of an individual particle. Stability of coherent motion depends upon the generator induced partial voltage, whereas stability of incoherent motion depends upon the total cavity voltage including the beam induced voltage component. For the case of single particle motion, stability depends on the synchronous phase ϕ_s. With beam loading, the stability of coherent motion depends upon $\phi_s+\psi+\phi_g$, with tuning angle ψ and generator current phase ϕ_g , and the coherent bucket is deformed. With beam loading, the stability of incoherent motion depends upon $\phi_b+\phi_c$, with beam current phase ϕ_b and perturbed cavity phase ϕ_c .

1 INTRODUCTION

The effect of the sinusoidal rf electric fields in a synchrotron is to accelerate the beam and to longitudinally focus beam particles into bunches contained within buckets. The rf cavities are driven both by the generator current and the beam current. As a result of this 'beam loading', the coherent motion of a bunch taken as a whole may differ considerably from the incoherent motion of an individual charged particle. We shall find the coherent and incoherent buckets.

Linear stability analysis[1, 2] shows the frequency Ω of small amplitude rigid dipole oscillations to scale according to

$$\Omega^2 \;=\; \Omega_s^2[1-I_b\sin 2\psi/(2I_0\cos\phi_s)]\,,$$

where Ω_s is the synchrotron frequency, $I_b/(I_0\cos\phi_s)$ is the beamload ratio. This scaling suggests that the coherent bucket shrinks to zero at some threshold beam current I_b, but begs the questions: "is the bucket merely scaled in height, or is the shape distorted?"

While answering these questions, we shall suppose that the synchrotron period is much greater than the cavity time constant; so that cavity dynamical effects can be ignored and the cavity response is virtually instantaneous. We shall assume that the cavity is detuned in the correct sense to avoid the dynamical Robinson instability.

Throughout this paper steady state vector quantities, are indicated by a superscript zero, while perturbed vectors shall carry no superscripts.

2 COHERENT BUCKET

Let the cavity gap voltage be $\mathbf{V}_c=V_ce^{j\phi_c}$ and the cavity complex impedance be $R\cos\psi e^{j\psi}$ where R is the shunt resistance and ψ is the detuning angle. We adopt the phase convention used by Pedersen[3]: that the steady state voltage phasor $\mathbf{V}_c^0$ is aligned with the positive real axis. Let the generator current driving the cavity be a fixed vector $\mathbf{I}_g^0=I_ge^{j\phi_s}$ which is chosen to produce the correct cavity voltage in the steady state. Let the beam image current be $\mathbf{I}_b=I_be^{-j(\pi/2+\phi_b)}$ where ϕ_b is the phase of the bunch centroid measured from the zero crossing of the voltage waveform.

2.1 Coherent Bucket Stability Criterion

The energy change ΔE of the bunch per cavity is

$$\Delta E(\phi_b)=R\cos\psi\,[\cos\psi\,|I_b|^2-I_gI_b\sin(\psi+\phi_g+\phi_b)](\tau_{\rm rev}/n)\,,$$

where $\tau_{\rm rev}$ is the revolution period. The energy change of a bunch which arrives at the synchronous phase ϕ_s is obtained by putting $\phi_b=\phi_s$. The motion will be unstable if changes in bunch phase produce no change in energy increment; in the limit of small oscillations, this implies the instability condition

$$\frac{d}{d\phi_b}\Delta E(\phi_b)\Big|_{\phi_b=\phi_s}=0\,.$$ Hence arises the condition:

$$\cos(\psi+\phi_g+\phi_s)=0 \quad\text{or}\quad \psi+\phi_g+\phi_s=\pi/2\,,\,3\pi/2\,,\ldots \tag{1}$$

The physical interpretation of this condition is as follows. The cavity voltage can be decomposed into parts due to the generator current and the beam current, that is $\mathbf{V}_c=\mathbf{V}_g+\mathbf{V}_b$ where $\mathbf{V}_g=R\cos\psi e^{j\psi}I_ge^{j\phi_s}$. At the stability limit, the generator induced voltage leads the cavity voltage by $\pi/2-\phi_s$. Now the steady state beam current leads the cavity voltage by $\pi/2-\phi_s$. Thus at the instability limit (1) the bunch sits on the crest of the sinusoidal-shaped generator-induced voltage waveform.

2.2 Equation for Coherent Oscillation

For the coherent motion, it is the phase difference between the bunch centroid and the zero crossing of $\mathbf{V}_g$ which is important. In the absence of feedbacks, ψ and ϕ_g are constants. Accordingly, we define new variables

$$\phi_b' \;=\; \phi_b+\psi+\phi_g\,, \tag{2}$$
$$\phi_s' \;=\; \phi_s+\psi+\phi_g\,. \tag{3}$$

Angle ϕ'_s for coherent motion plays an analogous rôle to the synchronous phase ϕ_s for single particle motion; motion is unstable if either is equal to $\pi/2$.

Let $I_0 = V_0/R$ be the generator current if the cavity were operating on resonance and with no beam. The beam current modulus is given by

$$\frac{I_b}{I_0}\cos\psi = \frac{\sin(\psi+\phi_g)}{\cos(\phi_s+\phi_g)} = \frac{\sin(\phi'_s-\phi_s)}{\cos(\psi-\phi'_s)} . \quad (4)$$

The generator current modulus is given by

$$\frac{I_g}{I_0}\cos\psi = \frac{\cos(\psi-\phi_s)}{\cos(\phi_s+\phi_g)} = \frac{\cos(\psi-\phi_s)}{\cos(\psi-\phi'_s)} . \quad (5)$$

It will be useful to define the factor $U \equiv (I_g/I_0)\cos\psi$. For the normal mode of operation with $0 \le \psi < \pi/2$ and $I_g > 0$ the function U is always greater than zero, but is very small for large tuning angles.

Let the bunch contain N_b particles of charge q. For brevity, let $E_b = E(\phi_b)$ and $E_s = E_s(\phi_s)$ be the bunch centroid energy and the synchronous energy, respectively. The rate of relative energy change obeys

$$\frac{d}{dt}(E_b-E_s) = -\frac{R\cos\psi\, I_g I_b}{N_b}[\sin(\psi+\phi_g+\phi_b)-\sin(\psi+\phi_g+\phi_s)] ,$$

Now the bunch current is $I_b = qN_b/\tau_{\rm rev}$ and the synchrotron has n radio-frequency cavities. Hence the acceleration rate is

$$\tau_{\rm rev}\frac{d}{dt}(E_b-E_s) = -nqR\cos\psi\, I_g[\sin\phi'_b - \sin\phi'_s] . \quad (6)$$

Suppose h is the harmonic number. In the absence of any frequency error, the rate of change of phase is

$$\frac{d}{dt}\phi'_b = \frac{2\pi h}{\tau_{\rm rev}}\eta\frac{(E_b-E_s)}{E_s} \quad \text{where } \eta = \alpha_p - \frac{1}{\gamma_s^2} , \quad (7)$$

and $E_s = \gamma_s m_0 c^2$ and α_p is the momentum compaction factor. Here we have assumed $\beta = v/c = 1$. After combining equations (5), (6) and (7), the equation for coherent oscillations is

$$\tau_{\rm rev}\frac{d}{dt}\left[\frac{E_s\tau_{\rm rev}}{h\eta}\frac{d}{dt}\phi'_b\right] = -nqV_0U[\sin\phi'_b - \sin\phi'_s] . \quad (8)$$

Apart from a prefactor U which alters the bucket height, this is just the equation for synchrotron oscillations about a stable phase angle ϕ'_s . Immediately from (8) we recognize that for given ϕ_s, curves such that $\phi'_s(\psi, I_b/I_g) =$ constant are curves of constant bucket length and (apart from a height scale factor) constant shape.

2.3 Coherent Stability Limits

In general, the beam-loaded coherent bucket behaves like a moving-bucket ($\phi'_s \neq 0$), even when the beam is not accelerating ($\phi_s = 0$). Thus, the phase and energy extent of stable oscillations diminishes as the beam load increases.

2.3.1 Small oscillations To find the limit of stability, we substitute $\phi'_s = \pi/2$ into equations (4) and (5). The threshold beam current value is identical with power-limited instability threshold derived by Robinson[1].

The important consequence of the deformation of the coherent bucket by beam loading is that a beam may be unstable for large amplitude coherent synchrotron oscillations well before the Robinson limit is reached. The nature of the large amplitude motion is determined by ϕ'_s , but is modified by the scale factor $\sqrt{U}$.

The condition $U = 1$, given by the curve of values

$$I_b/I_0 = 2\sin(\psi-\phi_s)/\cos\psi , \quad (9)$$

has a special significance: below it, bucket height falls more rapidly than we should guess from ϕ'_s alone; and above the curve, bucket height is boosted.

The quantities ϕ'_s and U are plotted in Reference[4] as functions of beam load ratio and tuning angle for a variety of synchronous phases. For some applications the coherent bucket length, height and area might be more useful quantities, and these parameters are presented in Reference[4].

2.4 Coherent Bucket Shape

The separatrix for coherent motion is called the coherent bucket. The bucket coordinates are the bunch energy with respect to the synchronous energy (i.e. $E_b - E_s$) and the bunch centroid phase (ϕ'_b) with respect to the zero crossing of $\mathbf{V}_g^0$. The centre of phase motion is ϕ'_s, or ϕ_s with respect to the zero crossing of the cavity waveform $\mathbf{V}_c^0$. The maximum extent of oscillations about ϕ_s is $\pi - 2\phi'_s$.

It is useful to introduce the frequency Ω_s defined by:

$$\Omega_s^2 = q\, n\, V_0\, h\, \eta / \left(\tau_{\rm rev}^2 E_s\right) .$$

Equation (8) is derivable from the function:

$$(\dot\phi'_b)^2 = 2\Omega_s^2 U\left\{\cos\phi'_b + \cos\phi'_s + \sin\phi'_s[\phi'_b + \phi'_s - \pi]\right\} , \quad (10)$$

whose condition also gives the coherent bucket.

3 INCOHERENT BUCKET

We shall derive the bounding bucket for incoherent motions of individual particles. The derivation manifests the fact that under conditions of beam loading the incoherent bucket moves with the bunch centroid.

3.1 Incoherent Bucket Stability Criterion

Let the phase of an individual particle with respect to the bunch centre be ϕ such that the phase with respect to the zero crossing of unperturbed voltage $\mathbf{V}_c^0$ is $\phi_b + \phi$. If the bunch phase is not equal to ϕ_s , then the cavity voltage will deviate from the steady state value; let the new value be $\mathbf{V}_c = V_c e^{j\phi_c}$. Let the individual particle contribute a current phasor $\delta\mathbf{I} = -jqe^{-j(\phi_b+\phi)}/\tau_{\rm rev}$ where q is the charge. Thus, the individual particle energy change at the accelerating cavity is

$$c\Delta E(\phi_b+\phi) = -q\,c\,V_c\sin(\phi_b+\phi_c+\phi)\ .$$

The motion will be unstable if changes in particle phase produce no change in energy increment; in the limit of small oscillations, this implies the instability condition

$\frac{d}{d\phi}\Delta E(\phi_b+\phi)\Big|_{\phi=0} = 0$. Hence arises the condition

$$\cos(\phi_b+\phi_c)=0 \quad \text{or} \quad \phi_b+\phi_c=\pi/2\,,\ 3\pi/2\,,\ldots \quad (11)$$

The incoherent motion is unstable when the perturbed beam current $-\mathbf{I}_b$ and the perturbed cavity voltage $\mathbf{V}_c$ are in phase. This corresponds to the case that the bunch sits on the crest of the perturbed cavity voltage waveform.

In fact, the condition $\phi_b+\phi_c=\pi/2$ is quite hard to arrange, particularly at large beamload values. This is because as ϕ_b increases, so ϕ_c decreases. Given ϕ_c is a function of ϕ_b , so one must find a self-consistent solution of (11). If solutions ϕ_b exist, then they must satisfy

$$-\cos(\psi+\phi_g+\phi_b) = \frac{\sin\psi\sin(\psi+\phi_g)}{\cos(\psi-\phi_s)} \leq 1\ .$$

Clearly $\pi/2<\psi+\phi_g+\phi_b<3\pi/2$ for instability to occur. Moreover, at sufficiently large values of the detuning angle $\psi\geq\hat{\psi}$ no solutions ϕ_b can be found. The angle $\hat{\psi}$ is the solution of the equality $\sin\psi\,\sin(\psi+\phi_g)=\cos(\psi-\phi_s)$.

3.2 Equation for Incoherent Oscillation

For incoherent motion, it is the phase difference between the individual particle and the perturbed cavity voltage $\mathbf{V}_c$ which matters. Accordingly, we define a new variable

$$\tilde{\phi}_s = \phi_b+\phi_c\ . \quad (12)$$

Here $\pi/2-\tilde{\phi}_s$ is the phase difference between the perturbed beam current $-\mathbf{I}_b$ and the perturbed cavity voltage $\mathbf{V}_c$. Angle $\tilde{\phi}_s$ for the incoherent motion plays a similar rôle to ϕ_s in single particle formulations of the phase motion; whenever either is equal to $\pi/2$, the motion is unstable.

The relative energy change of an individual particle with respect to the bunch centroid is

$$\Delta E(\phi_b+\phi)-\Delta E(\phi_b) = -qV_c\left[\sin(\tilde{\phi}_s+\phi)-\sin(\tilde{\phi}_s)\right]\ .$$

So the problem has been reduced to that of finding V_c and $\tilde{\phi}_s$. We find the forms

$$\begin{aligned} V_c\sin\tilde{\phi}_s &= V_0\left\{\sin\phi_s+U\left[\sin\phi_b'-\sin\phi_s'\right]\right\} \\ V_c\cos\tilde{\phi}_s &= V_0\left\{\cos\phi_s+U\left[\cos\phi_b'-\cos\phi_s'\right]\right\}\ . \end{aligned} \quad (13)$$

For brevity we shall write $E_q=E(\phi_b+\phi)$ to mean the energy of a particle with phase ϕ with respect to the bunch centre, and $E_s=E(\phi_s)$ to mean the energy of the particle which always arrives at the synchronous phase. The equation for phase advance is

$$\frac{d}{dt}\phi = \frac{2\pi h}{\tau_{\rm rev}}\,\eta\,\frac{(E_q-E_b)}{E_s}\ .$$

After incrementing the energy through n cavities, the equation for incoherent oscillations is:

$$\frac{d}{dt}\left[\frac{E_s\tau_{\rm rev}}{h\eta}\frac{d}{dt}\phi\right] = -\frac{nqV_c}{\tau_{\rm rev}}\left[\sin(\tilde{\phi}_s+\phi)-\sin\tilde{\phi}_s\right]\ , \quad (14)$$

where $\tilde{\phi}_s$ and V_c are obtained from equations (13). Equation (14) is that for synchrotron oscillations about a stable phase angle $\tilde{\phi}_s$. If $\phi_b=\phi_s$, then $\tilde{\phi}_s=\phi_s$ and $V_c=V_0$; and if also $E_b=E_s$ then equation (14) reduces to that for single particle motion.

Let $U_0=\cos(\psi-\phi_s)/\sin\psi$. For the case $\phi_s'=\pi/2$, and small displacements $|\phi_b-\phi_s|\ll 1$ we find

$$(V_c/V_0)^2\approx 1-(\phi_b-\phi_s)2U_0\cos\phi_s \quad \text{and}$$

$$\tan\tilde{\phi}_s\approx\tan\phi_s[1+(\phi_b-\phi_s)U_0/\cos\phi_s]\ .$$

Hence, even at the limit of stability for coherent motions, the incoherent bucket is only slightly distorted when the bunch moves away from the synchronous phase; and the beam stays bunched.

3.3 Incoherent Bucket Shape

The separatrix for incoherent motion is the incoherent bucket. The bucket coordinates are phase and energy with respect to the bunch centroid, that is ϕ and (E_q-E_b) . The stable point of the motion ($\phi=0$) is located at $\phi_b(t)$ with respect to the zero crossing of the unperturbed waveform $\mathbf{V}_c^0$. The incoherent bucket length and shape, however, depend on $\tilde{\phi}_s$. The maximum extent of oscillations about ϕ_b is $\hat{\phi}=\pi-2\tilde{\phi}_s$.

Equation (14) is derivable from the function:

$$(\dot{\phi})^2\Big/2\Omega_s^2(V_c/V_0) =$$

$$\left\{\cos(\tilde{\phi}_s+\phi)+\cos\tilde{\phi}_s\ +\ \sin\tilde{\phi}_s[\phi+2\tilde{\phi}_s-\pi]\right\}\ , \quad (15)$$

whose condition also gives the incoherent bucket.

4 CONCLUSION

For the case of a beam-loaded radio-frequency acceleration system, the coherent bucket is distorted and large amplitude motions may be unstable well before the Robinson limit is reached. The stability and distortion depend on $\phi_s'=\phi_s+\psi+\phi_g$ and U.

For the case of a beam-loaded rf system, the incoherent bucket is 'pinned' to the bunch central phase ϕ_b. The stability and distortion of the incoherent bucket depend on $\tilde{\phi}_s=\phi_b+\phi_c$ and V_c/V_0. However, provided that and $\phi_b\approx\phi_s$ and $E_b\approx E_s$, the incoherent bucket is found to be only slightly distorted.

5 REFERENCES

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