Tensionless strings and the weak gravity conjecture

Seung-Joo Lee,[a] **Wolfgang Lerche**[a] **and Timo Weigand**[a,b]

[a]*CERN, Theory Department,*
CH-1211 Geneva 23, Switzerland

[b]*Institut für Theoretische Physik, Ruprecht-Karls-Universität,*
Philosophenweg 19, 69120 Heidelberg, Germany

E-mail: seung.joo.lee@cern.ch, wolfgang.lerche@cern.ch,
timo.weigand@cern.ch

Abstract: We test various conjectures about quantum gravity for six-dimensional string compactifications in the framework of F-theory. Starting with a gauge theory coupled to gravity, we analyze the limit in Kähler moduli space where the gauge coupling tends to zero while gravity is kept dynamical. We show that such a limit must be located at infinite distance in the moduli space. As expected, the low-energy effective theory breaks down in this limit due to a tower of charged particles becoming massless. These are the excitations of an asymptotically tensionless string, which is shown to coincide with a critical heterotic string compactified to six dimensions.

For a more quantitative analysis, we focus on a U(1) gauge symmetry and use a chain of dualities and mirror symmetry to determine the elliptic genus of the nearly tensionless string, which is given in terms of certain meromorphic weak Jacobi forms. Their modular properties in turn allow us to determine the charge-to-mass ratios of certain string excitations near the tensionless limit. We then provide evidence that the tower of asymptotically massless charged states satisfies the (sub-)Lattice Weak Gravity Conjecture, the Completeness Conjecture, and the Swampland Distance Conjecture. Quite remarkably, we find that the number theoretic properties of the elliptic genus conspire with the balance of gravitational and scalar forces of extremal black holes, such as to produce a narrowly tuned charge spectrum of superextremal states. As a byproduct, we show how to compute elliptic genera of both critical and non-critical strings, when refined by Mordell-Weil U(1) symmetries in F-theory.

Contents

1 Introduction and summary

Quantum gravitational effects are deeply woven into the fabric of string theory. Effective field theories that derive from string theory should reflect these, and in particular must be consistent with any consistency constraints that quantum gravity may impose. String theory is therefore an ideal framework to investigate and exemplify general ideas about quantum gravity in a reasonably well-controlled and computable setting. The dichotomy between theories which can or which cannot be coupled to gravity manifests itself in the distinction between the so-called landscape of consistent vacua, and the swampland of effective theories without a UV completion [1]. Various, mostly conjectured criteria characterize the boundary between both regions in theory space [2]. Since this boundary is still under considerable debate, it is important to use the opportunity of confronting such conjectures with as quantitative an analysis as possible in the context of string compactifications.

In this work we address the interplay between gauge and gravitational effects in what one may call the open string sector, by analyzing gauge symmetries under which the matter fields are charged. Realizing such a setup in F-theory allows us to relate general arguments about constraints of quantum gravity to questions in algebraic and enumerative geometry, number theoretic properties of modular forms, and conformal field theory. This includes applying the theory of weak Jacobi forms to state counting via the elliptic genus, as well as computations in topological strings and mirror symmetry.

Among the earliest conjectures on general quantum gravity properties is that continuous global symmetries should not exist in the presence of gravity, as pointed out in the context of perturbative string theory in [3]. Rather, what appears as a global symmetry at low energies should derive from a gauge theory in the UV. This is motivated e.g. by the physics of quantum black holes: global continuous symmetries would lead to charged stable remnants that would exceed certain covariant entropy bounds [4].

More generally, according to the Weak Gravity Conjecture [5], gravity should be the weakest force at any point in moduli space. Starting from a gauge theory coupled to gravity and taking a limit in the stringy moduli space where the gauge symmetry becomes a global symmetry should therefore be prohibited within the landscape of consistent theories. At a macroscopic level, this means that the global symmetry limit must lie at an infinite distance in the moduli space. Microscopically, the effective field theory should break down as the problematic limit is approached.

This in turn ties in beautifully with another general expectation about the structure of quantum gravity, which can be summarized in the Swampland Distance Conjecture [6] as follows: the effective field theory formulated at a given point in moduli space is well-defined only within a finite distance from this reference point. As we travel an infinite distance in moduli space, a tower of infinitely many states should become massless at an exponentially fast rate. This leads to a breakdown of the effective field theory and, in turn, explains why such points at infinite distance cannot be attained within the effective field theory [7, 8]. In combination with the rationale that no points with global symmetries can exist in the presence of gravity, this suggests the appearance of infinitely many *charged* states that become massless in the limit where a gauge symmetry becomes global.

This reasoning has been recently verified in the context of Type IIB compactifications to four dimensions with eight supercharges [8]: as infinite distance points in complex structure moduli space are approached, a tower of BPS states from D3-branes wrapping vanishing 3-cycles becomes massless. These states are charged under abelian Ramond-Ramond gauge symmetries which indeed become global symmetries at infinite distance.

If true, the Weak Gravity and Swampland Distance Conjectures would have far-reaching consequences for cosmology and phenomenology, which, together with various refinements, have been the subject of intensive investigations in the recent literature, including [7, 9–35].

In this article we initiate a systematic investigation of quantum gravity constraints for compactifications where the gauge symmetries are localised on branes, in the framework of 6-dimensional F-theory [36–38]. We will address the Swampland Distance Conjecture, the Weak Gravity Conjecture and its refinement as a (Sub)Lattice Weak Gravity Conjecture [17, 39] as well as the Completeness Hypothesis of [40].

Our starting point is the question whether it is possible to go to a point in moduli space where an ('open string') gauge symmetry becomes a global symmetry while gravity is kept dynamical, or rather what prevents us from realising such a limit. This question turns out to have an answer directly from the compactification geometry. Recall that the 6-dimensional Planck mass is set by the volume of the F-theory compactification space. This is a complex Kähler surface, B_2, which serves as the base of an elliptically fibered Calabi-Yau 3-fold, Y_3. The open string gauge symmetry, on the other hand, arises from 7-branes that wrap holomorphic curves C on B_2. Thus we have

$$M_{\text{Pl}}^4 \sim \text{vol}_J(B_2), \qquad \frac{1}{g_{\text{YM}}^2} \sim \text{vol}_J(C), \qquad (1.1)$$

where J denotes a chosen Kähler form with respect to which the volumes are measured. In the special case of abelian gauge groups that we will sometimes restrict our discussion to, C has a description in terms of the so-called height-pairing of a rational section of the elliptic fibration over B_2. The critical limit we are interested in thus corresponds to the limit

$$\text{vol}_J(C) \to \infty \quad \text{with} \quad \text{vol}_J(B_2) \text{ finite.} \qquad (1.2)$$

Our first result is that this necessarily implies that B_2 must contain a rational curve C_0 whose volume goes to zero as $\text{vol}_J(C) \to \infty$. The proof we provide makes use of various properties of Kähler geometry including in particular Mori's cone theorem. As we will see, the limit (1.2) manifestly lies at infinite distance in the Kähler moduli space, which realizes one of the general expectations spelled out above. Technically, this means that the curve C_0 is not contractible in the sense of algebraic geometry, which would require the self-intersection of a contractible curve to be strictly negative. By contrast, the curve C_0 whose volume vanishes asymptotically is shown to have zero self-intersection.

A D3-brane wrapping the curve C_0 gives rise to a solitonic string in the uncompactified six dimensions. Asymptotically, its tension T is controlled by

$$T \sim \text{vol}_J(C_0) \to 0. \qquad (1.3)$$

As it turns out, C_0 necessarily intersects the curve C and hence the strings associated to the C_0 are charged under the 7-brane gauge group. A tower of light charged states thus appear in the effective theory, whose masses vanish exponentially fast as we approach the limit $g_{\mathrm{YM}} \to 0$. This signals the breakdown of the effective theory, which in a sense provides a microscopic censor that forbids the appearance of a global symmetry.

We can be considerably more quantitative. Since the curve C_0 is always a rational curve of vanishing self-intersection, its zero-mode structure [41, 42] coincides with the zero mode structure of a heterotic string. Near the tensionless limit, the string in fact becomes identical to the familiar, critical heterotic string compactified to six dimensions. Note that the physics of the nearly tensionless heterotic string is fundamentally different from the non-critical strings that arise from curves with negative self-intersection. These become tensionless at superconformal points of 6d $N = (1, 0)$ gauge/tensor theories in the absence of gravity [43, 44] (see the recent review [45] for more details). The critical heterotic strings we consider include gravity, however, and may also be weakly coupled.

The appearance of a weakly coupled heterotic string is evident when the base, B_2, of the elliptically fibered Calabi-Yau 3-fold, Y_3, is a Hirzebruch surface. In this case the 6d F-theory vacuum is known [37, 38] to be dual to the perturbative heterotic string compactified on a $K3$ surface. The rational fiber of the Hirzebruch surface plays the role of the curve C_0, and the tensionless limit coincides with the heterotic weak coupling limit [37].

More generally, we will see that in order for the limit (1.2) in Kähler moduli space to exist, the elliptic fibration Y_3 must always admit a $K3$ fibration, whose $K3$ fiber class $\widehat{C}_0$ is exactly the restriction of the elliptic fibration to C_0. The D3-brane that wraps C_0 maps to an M5-brane along $\widehat{C}_0$ in the dual M-theory. Upon S^1-reduction, which takes us to the dual IIA string on Y_3, the M5-brane turns into an NS5-brane wrapped on the $K3$ surface $\widehat{C}_0$. As an NS5-brane on $K3$ is known to be dual to the heterotic string in 6d [46], adiabatically fibering it over the base of the $K3$ fibration again identifies the string in question with the heterotic one, albeit compactified in a less standard way (see the recent work [47] for an account of that more general Type IIA/heterotic duality).

The appearance of a potentially weakly coupled, critical heterotic string from a D3-brane wrapped on a shrinking curve C_0 allows us to determine the charge and mass spectrum in some detail, at least near that limit. However, even for configurations where the heterotic string is non-perturbative and not necessarily weakly coupled, estimates for the spectrum can be made by considering its elliptic genus. Such reasoning is quite analogous to using the elliptic genus to determine properties of non-critical strings in 6 dimensions. This has been a rich and beautiful topic, pioneered in [48] and with dramatic progress also in the recent literature, including [41, 49–59] (see e.g. the review [45] for more references).

The elliptic genus is defined as a weighted trace over the BPS spectrum of winding and Kaluza-Klein momentum modes of the string wrapped on S^1, and counts the index of BPS particles in the dual 5d M-theory. In the limit of infinite radius, it reduces to a trace over the left-moving excitations of the 6d string. As a consequence, their partition function is meromorphic and thus highly constrained by modular invariance. This fact remains true in the presence of background gauge fields whose field strengths enter the partition function as 'fugacities'. The corresponding modular forms are weak Jacobi forms, which

are characterized by a "fugacity index" besides their modular weight. The fugacity index, in turn, is related to the anomaly polynomial of the string [55, 56]. The interplay of modular invariance and anomalies traces back to the earliest applications [60–62] of the elliptic genus in string theory.

A new result that we will find is that for abelian flavour groups, the fugacity index is given by the intersection of the shrinking curve C_0 with the height-pairing of the rational section that is responsible for the appearance of the abelian symmetry. With this input one can then make, for any concrete model, an ansatz for the elliptic genus with a few free parameters. These can in turn be fixed by using a non-trivial duality [50, 52] which relates the elliptic genus to the topological string theory on Y_3.

For given examples, this allows us to determine a characteristic part of the charge/mass spectrum of the nearly tensionless heterotic string explicitly, and to address questions related to the various Quantum Gravity conjectures mentioned above. However, even without explicitly determining the elliptic genus for concrete examples, we can draw some important conclusions just from its general properties. These include the following:

1. The excitation spectrum of the string contains states for every allowed value of the charge under the abelian gauge group. In this sense the full charge lattice is populated, in agreement with the Completeness Hypothesis of [40]. While this is a rather trivial fact for perturbative heterotic strings, it seems less so for non-perturbative variants for which no partition function is known. For example, in F-theory the spectrum of open string excitations in the 7-brane sector alone would spectacularly fail to populate the full charge lattice. Indeed, at the massless level only a few charges are realized explicitly. Their stringy or Kaluza-Klein excitations do not carry higher charges either because the charge is set by the Chan-Paton factors of the open string.

2. At least in the limit of vanishing tension, the states of maximal charge per string excitation level satisfy the Sublattice Weak Gravity Conjecture bound of [39], i.e. the charge-to-mass ratio exceeds that of an extremal (non-BPS) Reissner-Nordström black hole in six dimensions. This condition is required in order for non-BPS extremal black holes to decay, which in turn was argued [5] to be a necessary property of any theory of quantum gravity to evade various entropy bounds.

The conclusions we draw concerning the censorship against global symmetries or concerning the Weak Gravity Conjecture can be read either as statements about six-dimensional F-theory vacua, or as analogous properties of the six-dimensional heterotic string with $N = (1,0)$ supersymmetry as such, even without any reference to an F-theory dual.

There are, however, a few caveats to the above reasoning and we therefore need to spell out our working assumptions concerning them. Foremost is the absence of a BPS property of particle excitations in $(1,0)$ supersymmetric 6d theories. Rather what is BPS protected are strings. Their BPS nature allows one to infer the appearance of asymptotically tensionless strings in certain limits of the moduli space. It is natural to expect that their particle excitations are gapped in terms of the tension, but strictly speaking there

is little control of the mass spectrum. Fortunately some of the geometries lead to weakly coupled, perturbative heterotic strings, for which one can make quantitative statements at least in the asymptotic limit of vanishing coupling. For strongly coupled versions the only tool is, as mentioned above, the elliptic genus, but since it does not involve BPS protected states in 6 dimensions, it is not entirely clear what conclusions can be rigorously drawn. We assume, as our working hypothesis, that the statements about charge-to-mass relationships inferred from the elliptic genus remain at least qualitatively valid. The same applies also to the extremal black hole solutions with which masses and charges of the string states are to be compared, as these are not protected by a BPS property either.

This article is organized as follows: in section 2.1 we begin with a brief recapitulation of 6d F-theory vacua, mainly in order to set up our notation. In section 2.2 we characterize the geometric limit (1.2) in which a generic 7-brane-supported gauge symmetry becomes a global symmetry in the presence of gravity. The technical details of the derivation are relegated to appendix B. As we will see in section 2.3, the limit lies, quite as expected, at infinite distance in Kähler moduli space. In section 2.4 we explain the appearance of a critical 6d heterotic string that becomes tensionless as the 7-brane gauge group becomes global. The main properties of the charge spectrum of this heterotic string are summarized in section 2.5. The results of this section will be derived in the technical section 3, but we present the main points already at this stage in order to give the casual reader a quick overview. In section 2.6 we put these findings into perspective with various conjectured quantum gravity constraints. Apart from verifying at least asymptotically the Sublattice Weak Gravity Conjecture, we give evidence that integrating out the tower of charged states reproduces the vanishing of the gauge coupling in the tensionless limit [7, 8].

The technical section 3 contains a detailed analysis of the tensionless heterotic string with regard to its elliptic genus. In section 3.1 we explain how to infer from the elliptic genus of the perturbative heterotic string the appearance of (a characteristic subset of the) physical charged states that become light in the tensionless limit. We then review some basic properties of the elliptic genus, its relation to weak Jacobi forms and to the topological string in section 3.2. The upshot of this discussion is to relate the elliptic genus of the heterotic string to the genus-zero free energy of the topological string on the F-theory 3-fold Y_3. This allows us to obtain results also for situations where Y_3 is not a fibration over a Hirzebruch surface, in which case the nearly tensionless heterotic string is not necessarily weakly coupled.

In section 3.3 we provide a new result, namely the computation of the U(1) fugacity index via the height pairing of a rational section in F-theory. Subsequently, the general strategy to determine the charge-to-mass ratio of excited states via the elliptic genus is summarized in section 3.4; a proof of some underlying mathematical facts is presented in appendix D.

In section 4 we apply the general machinery of section 3 to various explicit examples of elliptic fibrations with a non-trivial Mordell-Weil group; these correspond to F-theory compactifications with a single unbroken U(1) gauge symmetry. Apart from being interesting by themselves, these computations serve as a check of the predicted charge-to-mass ratio of the string excitations as encoded in the elliptic genus. More specifically, in sec-

tion 4.1 we present models on Hirzebruch bases and recover a U(1)-refined version of the familiar, conformal field theoretic heterotic elliptic genus on $K3$ [60, 61, 63–66]. In section 4.2 we exemplify the more general situation where the dual heterotic string theory involves additional tensor fields and thus is non-perturbative. The elliptic genus then differs from the familiar perturbative version, but is consistent with a tensor transition to a model on a more conventional Hirzebruch base. Details of the computation are relegated to appendix E.

In section 5 we conclude with an outlook and some speculations.

2 Global symmetries and their geometric realizations

In this section we will analyze the "ungauging" of a local gauge symmetry in 6d F-theory compactifications, while keeping gravity dynamical. We begin with an analysis of the geometric realisation of this process as a very special limit in Kähler moduli space, and then study its physical implications.

2.1 F-theory on elliptic Calabi-Yau 3-folds

We will be working in the framework of F-theory [36–38] compactified on a Calabi-Yau 3-fold Y_3 which is elliptically fibered over a compact Kähler surface B_2,

$$\pi: \quad \begin{matrix} \mathbb{E}_\tau & \to & Y_3 \\ & & \downarrow \\ & & B_2 \end{matrix} \tag{2.1}$$

The effective theory in six dimensions is a 6d $N = (1, 0)$ supergravity theory. Background on F-theory in general and its compactifications to six dimensions in particular can be found e.g. in the reviews [67–69]. In units where the string length is $\ell_s = 1$, the Planck mass in the 6d Einstein frame is determined by the volume of the base surface B_2 as follows:

$$M_{\text{Pl}}^4 = 4\pi \, \text{vol}_J(B_2) \,. \tag{2.2}$$

Throughout this article the value of M_{Pl} will be kept fixed by normalizing the volume of the base B_2 as

$$\text{vol}_J(B_2) = 1 \,. \tag{2.3}$$

The volume is defined with respect to a Kähler form J. A summary of our conventions for the effective action in 6d is provided in appendix A.

Non-trivial gauge symmetries arise essentially along the world-volume of 7-branes that wrap holomorphic divisors of the base B_2. A non-abelian gauge symmetry algebra $\mathfrak{g}_I$ is localised on a stack of 7-branes that wrap a divisor Σ_I, which is in itself a component of the discriminant divisor of the elliptic fibration. More specifically, suppose we describe the elliptic fibration Y_3 by a Weierstrass model

$$y^2 = x^3 + fxz^4 + gz^6 \,, \tag{2.4}$$

where $[x:y:z]$ denote the homogeneous coordinates on the fiber ambient space, $\mathbb{P}_{231}$. If the elliptic fibration is to be Calabi-Yau, we must require that

$$f \in H^0(B_2, \mathcal{O}_{B_2}(4\bar{K})), \qquad g \in H^0(B_2, \mathcal{O}_{B_2}(6\bar{K})) \tag{2.5}$$

in terms of the anti-canonical divisor $\bar{K}$ on the base. In order for these sections to exist, $\bar{K}$ must be an effective class on B_2. The vanishing locus of the discriminant

$$\Delta = 4f^3 + 27g^2 \tag{2.6}$$

of the elliptic fibration is a divisor

$$\Sigma = \Sigma_0 \cup \bigcup_I \Sigma_I \,. \tag{2.7}$$

Then to each component Σ_I we associate a non-abelian gauge algebra $\mathfrak{g}_I$, while the residual discriminant divisor Σ_0 carries a trivial gauge algebra. For later purposes note that as a consequence of (2.5) the divisor class of Σ is

$$[\Sigma] = 12\,\bar{K} \,. \tag{2.8}$$

On the other hand, *abelian* gauge symmetries are generated by non-trivial rational sections S of the fibration Y_3 [37], more precisely by the image of such a section under the Shioda homomorphism [70–72]

$$\sigma(S) = S - S_0 - \mathfrak{D} \,. \tag{2.9}$$

Here S_0 is the zero-section of the fibration and $\mathfrak{D} \in \pi^* H^2(B_2)$ is a base divisor class which is chosen such that $\sigma(S)$ satisfies certain transversality conditions. In absence of additional non-abelian gauge algebra factors, the divisor class $\mathfrak{D}$ is given as

$$\mathfrak{D} = \pi^{-1}(\pi_*((S - S_0) \cdot S_0)) \,, \tag{2.10}$$

which we will use later in section 4 when analyzing explicit models. More generally, in presence of non-abelian algebras, $\mathfrak{D}$ contains in addition a linear combination of the blow-up divisors that resolve the singularities in the fiber. For more information we refer to the summary in appendix A of [73] and references therein.

What is important for us is the fact that the gauge coupling g_{YM} associated with both a non-abelian or an abelian gauge symmetry is determined by the Kähler volume of a certain divisor on B_2,

$$\frac{1}{g_{\mathrm{YM}}^2} = \frac{1}{2\pi}\,\mathrm{vol}_J(C) \,, \tag{2.11}$$

where[1]

$$C = \begin{cases} \Sigma_I & \text{if } \mathfrak{g} = \mathfrak{g}_I \\ b := -\pi_*(\sigma(S) \cdot \sigma(S)) & \text{if } \mathfrak{g} = \mathfrak{u}(1) \end{cases} \tag{2.12}$$

---

[1]We are normalizing the gauge coupling as spelled out in appendix A.

The quantity b is called the height pairing of the section S, and in absence of additional non-abelian gauge group factors it is given by

$$b = 2\bar{K} + 2\pi_*(S \cdot S_0)\,. \tag{2.13}$$

In the presence of several abelian gauge group factors, mixing can occur between the kinetic terms of the individual field strengths, which makes the definition of the gauge coupling more complicated. In such a situation, b is the diagonal part of the gauge kinetic function. We will restrict ourselves, for simplicity, to situations with a single abelian gauge group factor to avoid the need to diagonalise the gauge kinetic function.

Massless charged matter fields in the 6d $N = (1, 0)$ supergravity theory arise from strings stretched between the 7-branes. Note that these matter fields transform under a finite number of irreducible representations, or have a finite number of U(1) charges, respectively.

2.2 Global symmetries as a limit in Kähler moduli space

Let us suppose that the coupling of a certain gauge group factor, be it a U(1) or a simple non-abelian one, is parametrically smaller than the gravitational coupling, that is,

$$g_{\mathrm{YM}} \ll M_{\mathrm{Pl}}^{-1}\,. \tag{2.14}$$

In view of (2.2) and (2.11), this can be rephrased in geometric terms as

$$\mathrm{vol}_J(C) \gg (\mathrm{vol}_J(B_2))^{1/2}\,. \tag{2.15}$$

As we will show, in such a geometric setup, there must arise an infinite tower of charged particles which become asymptotically massless. This is because the limit (2.15) drives the base B_2 to a very degenerate regime in Kähler moduli space, where the Kähler form J approaches the boundary of the Kähler cone. More precisely, we are interested in the regime where

$$\mathrm{vol}_J(C) \sim t \to \infty\,, \qquad \mathrm{vol}_J(B_2) \text{ finite}\,. \tag{2.16}$$

This forces the base B_2 to become very anisotropic in the sense that it grows 'long and thin', as specified in technical terms below. In order for the base volume to remain finite while the volume of the curve C goes to infinity, there must exist another curve C_0 which intersects C and whose volume goes to zero as $t \to \infty$. This happens in such a way that the product of the volume of C and C_0 must lie below a finite upper limit. The infinitely many particles which become massless as $t \to \infty$ are the excitations of an effective string obtained by wrapping a D3-brane along C_0. As we will see, the properties of this string depend crucially on the fact that the curve C_0 has vanishing self-intersection $C_0 \cdot C_0 = 0$. This implies that the elliptic fibration whose base allows for a limit (2.16) necessarily exhibits a $K3$-fibration.

While the properties of the string associated with C_0 are the subject of the subsequent sections, we will first analyze this degenerate limit of the base B_2 in more detail. To this end, let us formulate our above findings concerning the limit in the Kähler cone in more precise terms:

1. Our first key result is that in the limit (2.16), with the finite volume of B_2 normalised to 1, the Kähler form J of the base B_2 must take the universal form

$$J = tJ_0 + \sum_\nu s_\nu I_\nu \,, \quad \text{as} \ \ t \to \infty \,. \tag{2.17}$$

Here $J_0 \in H^{1,1}(B_2, \mathbb{Z})$ is a Kähler cone generator satisfying

$$J_0 \cdot J_0 = 0 \,, \qquad 2m := \int_C J_0 \geq 1 \,. \tag{2.18}$$

The remaining Kähler cone generators $I_\nu \in H^{1,1}(B_2, \mathbb{Z})$ have the property that

$$\sum_\nu n_\nu s_\nu \to \frac{1}{t} \qquad \text{as} \ \ t \to \infty \,, \tag{2.19}$$

where

$$n_\nu = J_0 \cdot I_\nu \tag{2.20}$$

and at least one $n_\nu \neq 0$. The non-negative expansion parameters s_ν stay finite as $t \to \infty$ and are chosen such that $\mathrm{vol}_J(B_2) = 1$.

To summarize, the Kähler form asymptotes to the direction of one Kähler cone generator, J_0, and the fact that $\int_C J_0 \geq 1$ guarantees that the volume of C goes to infinity as $t \to \infty$. This happens in a very special way such that the total volume of B_2 remains finite.

2. The second key result is that there exists a rational curve C_0 which intersects C and whose volume goes to zero in the limit (2.16) as

$$\mathrm{vol}_J(C_0) \to \frac{1}{t} \,. \tag{2.21}$$

It is of crucial importance that its self-intersection vanishes, $C_0 \cdot C_0 = 0$. This leads to the behaviour

$$\mathrm{vol}_J(C)\,\mathrm{vol}_J(C_0) \to 2m + \frac{s}{t} \tag{2.22}$$

for the product of the volumes of the two curves, in the limit where C becomes infinitely large while B_2 is kept finite. Here

$$s := \sum_\nu s_\nu \int_C I_\nu \tag{2.23}$$

is a non-negative real number which remains finite in the limit.

3. The curve C_0 is the only curve whose volume vanishes in the limit with the property that $C_0 \cdot C_0 = 0$. All other vanishing curves have negative self-intersection.

4. Whenever the base admits a limit of the form (2.15), the elliptic fibration over it admits the structure of a $K3$ fibration,

$$\begin{array}{ccc} \rho: & K3 & \to & Y_3 \\ & & & \downarrow \\ & & & C' \end{array} \tag{2.24}$$

over some curve, C'. The class of the K3-fiber is

$$\widehat{C}_0 := \pi^{-1}(C_0) \in H_4(Y_3),\qquad(2.25)$$

i.e. the restriction of the elliptic fibration (2.1) to C_0. In general, this $K3$ fibration is not compatible with the elliptic fibration (2.1).

In appendix B we derive these four results (Assertions 1–4). The busy reader not interested in the technical details is invited to skip the derivation on a first reading.

2.3 The global limit as a point at infinite distance

As explained in the previous section, we are interested in taking a limit in Kähler moduli space in which a self-intersection zero curve C_0 shrinks to zero volume. Before analyzing in more detail the physical consequences of this limit in the next section, note that on a surface B_2 a curve of non-negative self-intersection cannot be contractible in the sense of algebraic geometry. In fact, the limit in Kähler moduli space in which C_0 shrinks to zero volume is strikingly different from the limit where a *contractible* curve assumes vanishing volume. In the latter case, we can take the curve volume to zero without any other curve on B_2 acquiring infinite volume. The result of such a contraction is a canonical singularity on B_2, which, for F-theory bases, is of the local form [74] $\mathbb{C}^2/G$ with G a discrete subgroup of U(2). If we view the degeneration leading to a canonical singularity in complex structure moduli space, then these occur at finite distance with respect to the Weil-Petersson metric [75]. The degeneration of B_2 in the present context, however, is much more severe. In particular, C_0 cannot be considered in isolation because we have shown that there must exist another curve C with $C \cdot C_0 \neq 0$ with the property

$$\mathrm{vol}_J(C) \to \infty \qquad \text{as} \quad \mathrm{vol}_J(C_0) \to 0.\qquad(2.26)$$

In fact, as we show now, the point $t \to \infty$ in (2.17) lies at infinite distance in the Kähler moduli space.[2]

To see this, let us consider a basis ω_α, $\alpha = 0,\ldots,n_T$ of $H^{1,1}(B_2)$. We recall first that the intersection form $\Omega_{\alpha\beta} = \int_{B_2} \omega_\alpha \wedge \omega_\beta$ on the compact Kähler surface has signature $\mathrm{SO}(1,n_T)$ and can be assumed to take the form

$$\Omega_{\alpha\beta} = \mathrm{diag}(+1,-1,\ldots,-1),\qquad(2.27)$$

upon an appropriate diagonalization. In physics terms, $n_T = h^{1,1}(B_2) - 1$ is the number of tensor multiplets in the 6d $N = (1,0)$ F-theory vacuum. Any Kähler form J normalised such that $\frac{1}{2}\int_{B_2} J^2 = 1$ can therefore be expanded in the diagonal basis as

$$J = \sqrt{2}j^\alpha \omega_\alpha.\qquad(2.28)$$

Thus, we may parameterise the coordinates on the hypersurface $\mathcal{M}_T$ of constant six-dimensional Planck mass, namely,

$$j \cdot j = j^\alpha \Omega_{\alpha\beta} j^\beta = (j^0)^2 - \sum_{i=1}^{n_T}(j^i)^2 = 1,\qquad(2.29)$$

---

[2]The following computation was worked out in collaboration with Diego Regalado.

as

$$j^0 = \cosh x \qquad j^i = \sinh x \, u^i(\phi^1, \ldots, \phi^{n_T - 1}) \,. \tag{2.30}$$

Here x takes any real value and the functions $u^i(\phi^1, \ldots, \phi^{n_T-1})$ are chosen such that $\sum_i (u^i)^2 = 1$. Hence the coordinates ϕ^A ($A = 1, \ldots, n_T - 1$) parameterise the unit $(n_T - 1)$-sphere $\mathbb{S}$.

Now, the metric on the $(n_T + 1)$-dimensional Kähler moduli space spanned by j^α is known to take the form

$$g_{\alpha\beta} = 2\Omega_{\alpha\rho}\Omega_{\beta\kappa}j^\rho j^\kappa - \Omega_{\alpha\beta} \,. \tag{2.31}$$

Given (2.30) we may compute the induced metric on the hypersurface $\mathcal{M}_T$ of unit volume surfaces, which reads

$$ds^2 = dx^2 + \sinh^2 x \, d\Omega_{\mathbb{S}} \,, \tag{2.32}$$

where $d\Omega_{\mathbb{S}} = h_{AB}(\phi) \, d\phi^A d\phi^B$ denotes the standard metric on the unit sphere $\mathbb{S}$. The distance between any two points $P, Q \in \mathcal{M}_T$ is given by

$$L(P, Q) = \int_{x_P}^{x_Q} dx \sqrt{1 + \sinh^2 x \, h_{AB}(\phi(x)) \frac{d\phi^A}{dx} \frac{d\phi^B}{dx}} \tag{2.33}$$

where we used x itself to parameterise the path between the two points. The important point is that for any path connecting P, Q, including the geodesic path, we have that

$$L(P, Q) \geq |x_P - x_Q| \tag{2.34}$$

since $\sinh^2 x \, h_{AB}(\phi(x)) \frac{d\phi^A}{dx} \frac{d\phi^B}{dx} \geq 0$ for any path $\phi^A(x)$.

It is interesting to compare the parameterisation of the Kähler form (2.28) with the one in (2.17). Since the behaviour of the moduli parameterising the finite volume sphere $\mathbb{S}$ is irrelevant for our purposes, it suffices to take $h^{1,1}(B_2) = 2$ for simplicity. In this case (2.28) reads

$$J = \sqrt{2}(\cosh x \, \omega_0 + \sinh x \, \omega_1) \,. \tag{2.35}$$

On the other hand, the ansatz (2.17) for $h^{1,1}(B_2) = 2$ can be written explicitly as

$$J = \frac{1}{\sqrt{\frac{n_1}{2} + \frac{n_{11}}{8\tilde{t}^2}}} \left(\tilde{t} J_0 + \frac{1}{2\tilde{t}} I_1 \right) \,, \qquad I_1 \cdot I_1 =: n_{11} \,, \quad J_0 \cdot I_1 =: n_1 \,, \tag{2.36}$$

where J_0 and I_1 are the two generators of the Kähler cone. Then the parameter t in (2.17) controlling the asymptotic volume of C is given by

$$t = \left(\frac{n_1}{2} + \frac{n_{11}}{8\tilde{t}^2} \right)^{-\frac{1}{2}} \tilde{t} \tag{2.37}$$

and asymptotically proportional to $\tilde{t}$. We can now make an ansatz for J_0 and I_1 in terms of ω_0 and ω_1 and fix the coefficients by comparing the intersection forms. For large t, we find that

$$t \simeq \frac{1}{2} e^x \,. \tag{2.38}$$

In particular, we are interested in measuring the geodesic distance d between two points with $t(x_P)$ finite and $t(x_Q) = \infty$. In view of (2.34) and (2.38), this means that $x_Q = \infty$, and therefore the distance between an arbitrary point in $\mathcal{M}_T$ and the boundary $x \to \infty$ is infinite. In general, due to the term proportional to $\sinh^2 x$ inside the square root in (2.33), even the geodesic distance d acquires a "correction" term from the lower bound (2.34). However, such a correction is finite and we can still estimate d to be the same as x up to subleading terms for large t, so that

$$t \simeq \frac{1}{2} e^d \, . \tag{2.39}$$

In conclusion this confirms the expectation that the limit in which a gauge symmetry becomes global in F-theory indeed lies at infinite distance in Kähler moduli space.

2.4 Asymptotically tensionless heterotic strings

The appearance of a genus-zero curve C_0 in B_2 with zero or almost zero volume has important physical consequences: a D3-brane wrapped on C_0 gives rise to a string propagating in 6 dimensions, which becomes tensionless as $\text{vol}_J(C_0) \to 0$. Evidently this tensionless string is quite different from the familiar non-critical strings that have been intensively investigated after the initial works of [48, 76] in the context of $N = (1,0)$ superconformal field theories (SCFTs) (see [45] for a recent review).

The special properties of the string associated with C_0 stem from the fact that $C_0 \cdot C_0 = 0$, and, consequently, that the restriction of the fibration to C_0 describes a $K3$ surface, $\widehat{C}_0$. The appearance of this $K3$ surface has already been stressed at the end of section 2.2. By contrast, the familiar non-critical strings which appear in the context of $N = (1,0)$ SCFTs are due to D3-branes wrapping vanishing curves Γ of *negative* self-intersection. Correspondingly the surface $\widehat{\Gamma} = \pi^{-1}(\Gamma)$ is never a K3 surface because[3]

$$-2 = 2g(\Gamma) - 2 = \Gamma \cdot (\Gamma - \bar{K}) \quad \Rightarrow \quad \Gamma \cdot \bar{K} \leq 1 \quad \text{if} \quad \Gamma \cdot \Gamma < 0 \, . \tag{2.40}$$

This means that the surface cannot contain 24 I_1 fibers which would have been required for an elliptic K3. This is because these are located at the intersection points of Γ with the discriminant divisor Σ in class $12\bar{K}$, and by the above $\Gamma \cdot \Sigma \leq 12$.

The 2d supersymmetric field theory on the world-volume of the effective string can be deduced [41, 42] with the help of a certain topological duality twist [78] along C_0, and turns out to have $(0,4)$ world-sheet supersymmetry. Given a general curve C_β in the base B_2, the complex structure of the elliptic fiber of Y_3 is identified with the complexified gauge coupling of the $N = 4$ SYM theory on the D3-brane along C_β, which therefore varies holomorphically over C_β. Using the topological duality twist [41, 42, 78] the effective world-sheet theory can be obtained by dimensional reduction of the underlying $N = 4$ SYM. The spectrum of massless fields as determined by the twist is shown in table 1. More specifically, the space $\mathbb{R}^4 \subset \mathbb{R}^{1,5}$ normal to the string world-sheet comes with a transverse rotation group

$$\mathrm{SO}(4)_T = \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_- \, , \tag{2.41}$$

---

[3]The first equality follows because a contractible curve on an F-theory base is always rational [77].

Fermions		Bosons		(0,4)	Multiplicity
$(\mathbf{2},\mathbf{1})_1$	ψ_+	$(\mathbf{1},\mathbf{1})_0$, $(\mathbf{1},\mathbf{1})_0$	$\bar{a}, \bar{\sigma}$	Hyper	$g - 1 + \bar{K} \cdot C_\beta$
$(\mathbf{2},\mathbf{1})_1$	$\tilde{\psi}_+$	$(\mathbf{1},\mathbf{1})_0$, $(\mathbf{1},\mathbf{1})_0$	a, σ		
$(\mathbf{1},\mathbf{2})_1$	μ_+	$(\mathbf{2},\mathbf{2})_0$	φ	Twisted	1
$(\mathbf{1},\mathbf{2})_1$	$\tilde{\mu}_+$			Hyper	
$(\mathbf{1},\mathbf{2})_{-1}$	$\tilde{\rho}_-$			Fermi	$g(C_\beta)$
$(\mathbf{1},\mathbf{2})_{-1}$	ρ_-				
$(\mathbf{1},\mathbf{1})_{-1}$	λ_-			half-Fermi	$8\bar{K} \cdot C_\beta$

Table 1. Spectrum of massless 2d $N = (0,4)$ multiplets of the effective world-sheet theory of the string that arises from a D3-brane that wraps $C_\beta \subset B_2$ (not contained in the discriminant). The representations are under $\mathrm{SU}(2)_+ \times \mathrm{SU}(2)_- \times \mathrm{SO}(1,1)$. The table is a modification of table 3 in [42].

and the spectrum organizes into $(0,4)$ supermultiplets distinguished by their $\mathrm{SU}(2)_+ \times \mathrm{SU}(2)_-$ quantum numbers, as well as by their spin with respect to the $\mathrm{SO}(1,1)$ Lorentz algebra along the string world-sheet. The fields shown in the first three rows of table 1 arise from dimensional reduction of the underlying $N = 4$ supermultiplet, while the half-Fermi multiplets in the last line localize at the intersection of C_β with the discriminant divisor Σ. While the naive number of such half-Fermi multiplets would be $12\bar{K} \cdot C_\beta$ given that Σ is in the class of $12\bar{K}$, not all of these modes are independent due to $\mathrm{SL}(2,\mathbb{Z})$ monodromies. The correct counting is $8\bar{K} \cdot C_\beta$ and can be deduced [41, 42] e.g. from the requirement of gravitational anomaly cancellation or by duality with MSW strings [79]. These half-Fermi multiplets carry net charge with respect to the gauge group of the 6d $N = (1,0)$ field theory. As a result, also the excitations of the effective string will be charged.

Applied to the rational curve C_0 with $C_0 \cdot C_0 = 0$ and $C_0 \cdot \bar{K} = 2$, we recover 16 left-moving chiral fermions (λ_-) along with $4 + 4$ left-moving scalars. The latter describe the center-of-mass motion in the four extended normal directions (φ) as well as the internal degrees of freedom of the gauge field ($a, \bar{a}$) and the normal bundle modes within B_2 ($\sigma, \bar{\sigma}$). The right-moving sector consists of the right-moving counterparts of these scalars along with their their fermionic superpartners.

This spectrum identifies the effective string with the heterotic string. This can be equivalently understood by going to the M-theory picture and recalling that the M5-brane, when wrapped on a $K3$ surface, is dual to the heterotic string [46, 80]. More precisely, in our situation, where the $K3$ is embedded in an elliptic threefold, we have less supersymmetries than in the usually discussed situation, where the heterotic string is compactified on T^4. Rather we will obtain a chiral heterotic string with $(1,0)$ space-time supersymmetry in $d = 6$, which means that it is, morally speaking, compactified on some $K3$ (generically with gauge bundles on top, potentially dressed by pointlike instantons, and not necessarily weakly coupled). We will denote this generalized $K3$ compactification geometry by $\mathcal{K}$. Note that there is no reason why $\mathcal{K}$ should coincide with the vanishing $K3$ surface, $\widehat{C}_0$, in our F-theory setup.

Hence the strings of our present interest are orthogonal to the familiar non-critical strings that appear in 6d SCFTs. This is of course as expected, because the latter arise in the limit of decoupling gravity at finite distance in the moduli space, while the reason for the appearance of a zero-volume curve C_0 in our context was the requirement of preserving gravity, and keeping the Planck mass fixed. Thus it makes sense that we obtain a critical string theory (albeit compactified), rather than a non-critical one.

Since this is an important point, let us investigate the relation to the heterotic string in more detail. Before discussing the general case, assume for a moment that the base B_2 is given by one of the Hirzebruch surfaces, $\mathbb{F}_a$. While we will review the pertinent properties of these surfaces in more detail in section 4.1, what is important for us here is that they have the structure of a $\mathbb{P}^1$-fibration

$$
\begin{array}{rcl}
p: & \mathbb{P}^1_f & \to & \mathbb{F}_a \\
& & & \downarrow \\
& & & \mathbb{P}^1_b
\end{array}
\tag{2.42}
$$

As we will see in section 4.1, the limit (2.17) is compatible with this structure if and only if we identify

$$
\mathbb{P}^1_f = C_0 \,.
\tag{2.43}
$$

This means that the elliptic $K3$ surface, $\widehat{C}_0 = \pi^{-1}(C_0)$, which we advertised on general grounds at the end of section 2.2, turns out to be in itself fibered over the base, $C' = \mathbb{P}^1_b$, of the Hirzebruch surface. Thus, in this case, the elliptic fibration of Y_3 is compatible with the $K3$ fibration. F-theory on this $K3$ fibration Y_3 is then dual to the heterotic string on a $K3$ surface $\mathcal{K}$ which is an elliptic fibration by itself:

$$
\begin{array}{rcl}
r: & T^2 & \to & \mathcal{K} \\
& & & \downarrow \\
& & & \mathbb{P}^1_b
\end{array}
\tag{2.44}
$$

To avoid confusion, let us recapitulate and spell out again that we generally deal with two $K3$ surfaces, which are given by $\widehat{C}_0$ on the F-theory side and $\mathcal{K}$ on the heterotic side, respectively. That is, the nearly tensionless heterotic string that is obtained by wrapping a D3-brane on the shrinking base C_0 of $\widehat{C}_0$ coincides with the heterotic string compactified on $\mathcal{K}$. In the situation where the base B_2 of Y_3 is a Hirzebruch surface, $\widehat{C}_0$ is an elliptic $K3$ with base $C_0 = \mathbb{P}^1_f$, which by itself is fibered over $\mathbb{P}^1_b$. Moreover on the heterotic side, $\mathcal{K}$ is an elliptic fibration over $\mathbb{P}^1_b$.

Having straightened out any possible confusion, we now continue with an analysis of the various couplings. The coupling of the heterotic string (measured in the heterotic string frame) is determined by the duality as [37]

$$
(g_s^h)^2 = \frac{\mathrm{vol}_J(\mathbb{P}^1_f)}{\mathrm{vol}_J(\mathbb{P}^1_b)} \,.
\tag{2.45}
$$

In our limit (2.17), $\mathrm{vol}_J(\mathbb{P}^1_f) = \mathrm{vol}_J(C_0) \to 0$ while $\mathrm{vol}_J(\mathbb{P}^1_b) \to \infty$. Therefore the string under consideration indeed asymptotically describes the weakly coupled heterotic

string. The fact that the heterotic string is weakly coupled is a priori true only in the specific limit (2.17), and away from this limit quantitative statements are harder to make. Fortunately, this is sufficient for our purposes as we are interested just in the asymptotic behaviour.

We can be more precise about the dual heterotic theory in the weak coupling limit, where we continue to work for now in the string frame when discussing the heterotic side of the duality, and in the Einstein frame when referring to the Type IIB side. First note that the tension of the string associated with C_0 in the Type IIB Einstein frame takes the form

$$T = \frac{2\pi}{\ell_s^4}\text{vol}(C_0)\,, \tag{2.46}$$

where we have reinstated the Type IIB string length ℓ_s (outside of this section, we will generally set $\ell_s \equiv 1$.). The precise normalization is computed in appendix A, which also states our conventions. Equating this with the heterotic string tension determines the heterotic string length ℓ_h (measured in heterotic string frame) as

$$\frac{2\pi}{\ell_h^2} = \frac{2\pi}{\ell_s^4}\text{vol}_J(C_0)\,. \tag{2.47}$$

Furthermore, the unwrapped D3-brane corresponds to a heterotic 5-brane on the fiber T^2 of (2.44), and comparing the tensions leads to

$$\frac{2\pi}{\ell_h^6(g_s^h)^2}\text{vol}(T^2) = \frac{2\pi}{\ell_s^4}\,. \tag{2.48}$$

This gives an alternative expression for the heterotic coupling,

$$(g_s^h)^2 = \frac{\text{vol}(T^2)\text{vol}(\mathbb{P}_f^1)}{\ell_h^4}\,. \tag{2.49}$$

Combining (2.49) and (2.47) we find for the Planck scale of the heterotic theory

$$(M_{\text{Pl}}^h)^4 = \frac{4\pi}{\ell_s^8}\text{vol}(\mathbb{P}_f^1)\,\text{vol}(\mathbb{P}_b^1)\,. \tag{2.50}$$

If we apply the limit (2.17) to the special case of a base space with $h^{1,1}(B_2) = 2$, the product of C_0 and the remaining curve class indeed remains constant, see (2.22), and therefore the Planck scale of the heterotic theory remains fixed. On the other hand, the heterotic gauge coupling is related to the heterotic Planck scale as

$$\frac{1}{(g_{\text{YM}}^h)^2} = (M_{\text{Pl}}^h)^4\frac{\ell_h^2}{16\pi^2}\,. \tag{2.51}$$

In summary, in the limit (2.17), the heterotic string becomes tensionless, $\ell_h \to \infty$, and weakly coupled, $g_s^h \to 0$. Gravity remains dynamical as M_{Pl}^h stays constant, while the gauge coupling g_{YM}^h scales to zero.

We recall that the above discussion applies to geometries where Y_3 is elliptically fibered over a Hirzebruch surface. Let us now discuss the situation for general base spaces, B_2,

that are compatible with the limit (2.17). What we have established is the existence of a $K3$ fibration of the form (2.24), which, however, is in general not compatible with the elliptic fibration of Y_3. Nevertheless, we will argue that in the tensionless limit we will again obtain the critical heterotic string.

To see this, we use a chain of dualities as follows. First, we use F/M-theory duality to relate F-theory on the elliptic fibration Y_3 times a circle S^1 to M-theory on the same 3-fold, Y_3. This duality makes use of the elliptic fibration (2.1) in the usual manner. The D3-brane wrapping C_0 but not wrapping the circle S^1 dualises to an M5-brane wrapped on $\widehat{C}_0$. We are therefore interested in the effective theory on the string world-sheet associated with this M5-brane. M-theory on the 3-fold Y_3 is dual to strongly coupled Type IIA theory on Y_3, with the M5-brane dualising to an NS5-brane on $\widehat{C}_0$.

Finally, we make use of the $K3$ fibration (2.24): type IIA string theory compactified on the $K3$ fibration (2.24) is dual to a 4d $\mathcal{N} = 2$ compactification of heterotic string theory. This is a fibered version of the 6d IIA - heterotic duality, which equates the (strongly coupled) Type IIA string on K3 with the weakly coupled heterotic string on T^4 [81]. Famously, under this duality the NS5-brane wrapped along K3 turns into the heterotic string [46, 80]. Adiabatically fibering the 6d duality over a rational curve C' leads to the duality with the heterotic string in four dimensions [82]. If the $K3$ fibration (2.24) were compatible with the elliptic fibration of the $K3$ fiber, then the dual heterotic string theory would be compactified on $T^2 \times \mathcal{K}$. Here $\mathcal{K}$ is the $K3$ surface (2.44) that appears also in the duality between F-theory on Y_3 and the heterotic string in six dimensions. When in addition the base of the $K3$ fibration of Y_3 becomes large, the dual heterotic string is weakly coupled, as discussed above.

More generally, however, the K3-surface $\mathcal{K}$ of the dual heterotic compactification space $\mathcal{K} \times T^2$ is not elliptically fibered *globally*. Type IIA - heterotic duality under such more general conditions has been revisited in the recent work [47]. More specifically, degenerations of the $K3$ fiber in the fibration (2.24) of Y_3 can lead to heterotic 5-brane defects in the dual theory. Even in the limit of large base C', such localised defects on the heterotic side can spoil a fully perturbative CFT description. Nonetheless, as long as the base C' is large, it is reasonable to think of the resulting theory as being 'weakly coupled away from the defects', while these provide important further degrees of freedom that render the full theory consistent.

Despite these complications, we will find evidence in the course of this paper that the string that arises from wrapping a D3 brane on C_0, shares essential properties of the heterotic string, in particular its elliptic genus. As we will see, this includes a generalization which describes not just single wrapped, but also multiply wrapped strings, which makes it evident that we really deal with heterotic strings that arise from wrapping D3 branes on C_0.

2.5   U(1)-charge spectrum of the heterotic string

To summarise the discussion so far, we have seen that in the asymptotic limit in Kähler moduli space, the string from a D3-brane wrapped on a shrinking curve, C_0, becomes tensionless and can be identified with a critical heterotic string on some $K3$ surface, $\mathcal{K}$.

The excitation spectrum of this string takes the usual form

$$M_n^2 = \alpha \, 2\pi \, T \, (n - 1) \,, \qquad T = 2\pi \, \text{vol}_J(C_0) \,, \tag{2.52}$$

where we have set $\ell_s = 1$ in the expression for the string tension, and n refers to the (left-moving) excitation level. This will be discussed in more detail in section 3.2. Furthermore for the critical heterotic string the overall normalization is

$$\alpha = 4 \,. \tag{2.53}$$

Note that, strictly speaking, we can argue for the specific relation (2.52) only as we approach the tensionless limit, which coincides with the weak coupling limit (modulo complications for general B_2, as discussed above). In this limit we are essentially changing the duality frame from F-theory/Type IIA to the heterotic one. In particular, we will find later that the massless sector of the heterotic string correctly reproduces the massless F-theory spectrum, at least at the level of the elliptic genus that we will be computing. This fact holds true both if the base is a Hirzebruch surface and in more general situations.

Before we discuss the role of the (nearly tensionless) heterotic string in the context of the Weak Gravity Conjecture in the next section, let us first analyze the charges of the string excitations in some more detail. Indeed the excitations necessarily carry charge under the gauge symmetries that become global in the tensionless string limit. The reason is that, as we have shown, the D3-brane wrapped around C_0 intersects the stack of 7-branes on C (see Point 2 made in section 2.2). The intersection locus between C_0 and C hosts massless 3-7 strings between the D3-brane on C_0 and the 7-brane on C. The resulting fermionic zero-modes have already been discussed in section 2.4 in terms of the chiral (half-)Fermi supermultiplets charged under the 7-brane gauge group. From the space-time point of view, these charges translate into global "flavor" charges of the string excitations.

In order to be more quantitative, we will focus on a single U(1) gauge symmetry. The detailed technical arguments will be developed in section 3, while for now we present the following key results:

1. For each value of U(1) charge $\mathfrak{q} \in \mathbb{Z}$ there exists at least one state in the excitation spectrum of a single string along C_0. All of these states become massless in the limit $\text{vol}_J(C_0) \to 0$, which coincides with the limit $g_{\text{YM}} \to 0$. Oftentimes, we will only discuss non-negative charges $\mathfrak{q} \in \mathbb{Z}_{\geq 0}$ given that the states in the theory come in pairs of U(1) charges $\mathfrak{q}$ and $-\mathfrak{q}$.

2. At each excitation level n, the spectrum of these string excitations contains states of maximal charge

$$\mathfrak{q}_{\text{max}}^2(n) \geq \beta(n) \, n \,, \tag{2.54}$$

for some numerical prefactor β that is fixed by the specifics of the abelian gauge group and its realisation in F-theory. For large n, this prefactor can be taken as a
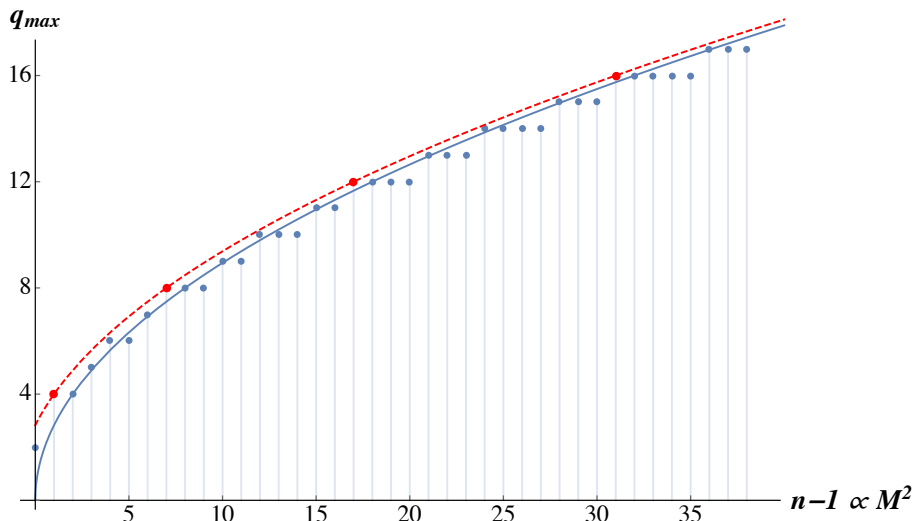
Figure 1. Maximal charge $\mathfrak{q}_{\max}(n)$ per excitation level n for a 6d F-theory compactification on Y_3 with base $B_2 = \mathbb{F}_1$. The model refers to the values $(x, y) = (4, 4)$ in the notation of section 4.1, and has charge index $m = \frac{1}{2}C \cdot C_0 = 2$. The solid blue curve is given by $\mathfrak{q}(n) = \sqrt{4m(n-1)}$, which corresponds to the modified, scalar weak gravity bound derived in ref. [83] for the relevant extremal black holes. Observe that the charges for some excitations lie just barely above this curve, as a consequence of the plateau pattern in conjunction with the offset of the vacuum energy by -1. The maximally superextremal states, marked in red, lie on the dashed curve given by $\mathfrak{q}_{\max}(n) = \sqrt{4mn}$ and populate a charge sublattice with spacing given by $\Delta \mathfrak{q} = 2m = 4$. Together with the additional superextremal states lying between the red and the blue curves they populate the full charge lattice. This is a feature of this particular example.

positive constant with an upper bound given by $2C \cdot C_0$, in the following sense: for each value of $\epsilon \in (0, 1)$, we can find some $N(\epsilon)$ such that

$$\beta(n) \geq (1 - \epsilon)\, 4m =: \beta_\epsilon \qquad \forall\, n \geq N(\epsilon)\,, \tag{2.55}$$

with

$$m = \frac{1}{2}C \cdot C_0 \tag{2.56}$$

as introduced in (2.18). This implies, in particular, that

$$\mathfrak{q}_{\max}^2(n) \geq \beta_\epsilon\, (n-1) \qquad \forall\, n \geq N(\epsilon)\,. \tag{2.57}$$

As we will explain in section 3.4 and appendix D, these properties of the excitation spectrum can be deduced on general grounds without reference to a concrete model. We can reduce the bound $N(\epsilon)$ at the cost of increasing ϵ and therefore decreasing β_ϵ. In concrete examples we will give an even lower value for β_ϵ such that the bound holds for all values of n.

3. Conversely, there exists a *sublattice* of charges of index $2m$ such that for each $\mathfrak{q}_k$ in the sublattice, i.e. for each

$$\mathfrak{q}_k = 2m\, k\,, \qquad k \in \mathbb{Z}\,, \tag{2.58}$$

there exists a state at excitation level $n(k) = \frac{\mathfrak{q}_k^2}{4m} = m\,k^2$. Put differently, for each charge in the sublattice (2.58), there exists a state with excitation level n such that

$$\mathfrak{q}_k^2 = 4m\,n(k)\,. \tag{2.59}$$

A derivation will be given in section 3.4. The importance of these states is that they are superextremal with respect to a (dilatonic) charged black hole and hence an extremal such black hole can decay into them, see section 2.6 for a detailed discussion. In general, there may be further superextremal states (possibly even corresponding to a smaller index sublattice), but already the states associated with the sublattice (2.58) suffice for the Sublattice Weak Gravity Conjecture to hold.

This general behaviour is illustrated in figure 1, which refers to a particular example that will be discussed in more detail in section 4.1. Depicted is the maximal U(1) charge per excitation level, $\mathfrak{q}_{\max}(n)$, for a D3-brane wrapping a curve C_0 in the base $B_2 = \mathbb{F}_1$ with $C_0{}^2 = 0$ and $m = \frac{1}{2}C_0 \cdot C = 2$. Note that at given n, each value of the charges with $|\mathfrak{q}| \leq \mathfrak{q}_{\max}(n)$ is populated by string states. We can see that the maximally superextremal excitations indeed obey eq. (2.59) and form the sublattice defined by (2.58). In this particular example, the full set of superextremal states in fact happens to populate the *full* charge lattice.

In section 3 of this paper we will substantiate the claims (2.54), (2.55) as well as (2.58), (2.59) by computing the elliptic genus for the nearly tensionless heterotic strings we are encountering. Our main focus will be on the massive states that are charged under a U(1) gauge symmetry that becomes global in the limit $t \to \infty$. A key role will be played by the charge refinement of the elliptic genus in terms of the height pairing associated with the rational section underlying U(1).

2.6 Relation to quantum gravity conjectures

As $\mathrm{vol}_J(C_0) \to 0$, the tension of the heterotic string discussed in the previous section tends to zero, and so do the masses of its excitations. A priori we need to distinguish between the appearance of the asymptotically tensionless (BPS protected) heterotic *string* as such and the (non-BPS) *particle* type excitations of the string. The latter result in a tower of nearly massless, charged (particle) states. In this section we discuss the implications of the tower of particle states in the context of four interrelated Quantum Gravity conjectures, reserving some speculations about the role of the string for section 5.

Before entering the discussion let us note that the results of this section can be read in two ways: from the outset we view them as statements about the 6d F-theory compactification and its relation to the Quantum Gravity Conjectures. Equivalently, however, we can interpret them directly in the duality frame of the heterotic string introduced in section 2.4, and as such they make sense for 6d heterotic string compactifications *per se*, without any reference to an F-theory dual.

1) Censorship against global symmetries in Quantum Gravity. The appearance of infinitely many massless particle-like states in the tensionless limit is responsible for the

breakdown of the 6d effective field theory. Such a breakdown is expected and, in fact, required by quantum gravity arguments: in the limit $t \to \infty$, a global symmetry emerges while at the same time gravity remains dynamical as the 6d Planck mass remains constant; this would be in stark contrast with general reasoning (see [4, 5] and references therein) that continuous global symmetries are incompatible with quantum gravity. In this sense, the appearance of a tensionless string and its associated infinite tower of massless modes acts as a censor that prevents the theory from an inconsistency with general quantum gravity arguments: the field theory breaks down before reaching a catastrophic global symmetry limit [4, 5].

2) The sublattice weak gravity conjecture. The particle excitations of the nearly tensionless string play an important role also in the context of the so-called Sublattice Weak Gravity Conjecture (SWGC) [17, 39]. In the sequel, we will show that, under certain assumptions, the string excitation spectrum contains charged particle-like states whose charge-to-mass ratio, near the tensionless limit, is in accordance with the SWGC bounds.

Applied to a single 1-form U(1) gauge symmetry in d-dimensional spacetime, the SWGC posits that there must exist some finite index sublattice of the charge lattice such that for each value of charges $\mathfrak{q}$ in that sublattice, there exists an elementary particle-like state of mass M with the property that its charge-to-mass ratio exceeds that of a charged (non-BPS) extremal black hole. This gives rise to a bound of the form

$$\mathfrak{q}^2 g_{\mathrm{YM}}^2 \overset{!}{\geq} \mu \frac{M^2}{M_{\mathrm{Pl}}^4} \,. \tag{2.60}$$

The precise value of the numerical constant μ depends on the type of extremal black hole with respect to which the particles must be super-extremal.

Certainly in six dimensions many kinds of black objects are known to exist, in particular black strings[4] (see, for instance, [84–87] and references therein for a sample of the vast possibilities in higher dimensions). Given the enormous richness of possible black objects in six dimensions, we adopt the working assumption that the relevant objects with respect to which super-extremal states are to exist are certain extremal Reissner-Nordström (RN) black holes; in this we follow ref. [17]. If we could completely ignore the influence of massless scalar field fluctuations, the corresponding black objects to consider would be *non-dilatonic* extremal RN black holes [88] in d-dimensional Einstein-Maxwell theory. With respect to these, the constant μ in (2.60) takes the value [17]

$$\mu|_{d=6,\mathrm{non-dil.}} = \frac{d-3}{d-2}|_{d=6} = \frac{3}{4} \,. \tag{2.61}$$

However, in a supergravity context, the effect of massless scalar field fluctuations cannot be neglected [17, 25]. In the forthcoming work [83] we will derive in detail that including the massless scalar field fluctuations in the tensor multiplets increases the numerical value of the bound μ in (2.60) to

$$\mu|_{d=6,\mathrm{dil.}} = \frac{d-3}{d-2}|_{d=6} + \frac{1}{4} = 1 \,. \tag{2.62}$$

---

[4]Unlike black holes, these can be BPS in six dimensions with $N = (1,0)$ supersymmetry; however, BPS objects are not of interest here in the context of the SWGC.

We proceed by evaluating whether the charge spectrum (2.54) obeys the requirement (2.60) with this value $\mu = \mu|_{d=6,\text{dil.}} = 1$. Such a comparison is meaningful a priori only in the tensionless limit $t \to \infty$, where we have perturbative control of the heterotic string spectrum and trust in particular the mass relation (2.52). With this in mind, first deduce from (2.52) and (2.54) that

$$\mathfrak{q}_{\max}^2(n) \geq \beta(n) \left(\frac{1}{4\pi^2\alpha} \frac{M_n^2}{\text{vol}(C_0)} + 1 \right) \tag{2.63}$$

near the tensionless limit. Next we multiply both sides with g_{YM}^2, given in (2.11), and furthermore express the result in terms of the 6d Planck mass (2.2), using that we have fixed $\text{vol}_J(B_2) = 1$. As a result we obtain

$$\mathfrak{q}_{\max}^2(n)\, g_{\text{YM}}^2 \geq \beta(n) \left(\frac{2}{\alpha} \frac{1}{\text{vol}(C)\,\text{vol}(C_0)} \frac{M_n^2}{M_{\text{Pl}}^4} + g_{\text{YM}}^2 \right) . \tag{2.64}$$

In the limit where $\text{vol}_J(C) \sim t \to \infty$, which coincides with the tensionless limit in which we are working, we have established the behaviour (2.22) for the product $\text{vol}_J(C)\,\text{vol}_J(C_0)$. Hence in this limit we can estimate

$$\mathfrak{q}_{\max}^2(n)\, g_{\text{YM}}^2 \geq \beta(n) \left(\frac{1}{\alpha\, m} \frac{M_n^2}{M_{\text{Pl}}^4} + g_{\text{YM}}^2 \right) \qquad \text{as} \quad t \to \infty. \tag{2.65}$$

At this stage we recall that there exists a sublattice of charges $\mathfrak{q}_k^2$, see (2.58), with the property (2.59). These charges hence saturate the inequality (2.65) with $\beta(n)$ replaced by $4m$. Using $\alpha = 4$ in (2.54), as is appropriate for the perturbative heterotic string, we find for this charge sublattice the relation

$$\mathfrak{q}_k^2\, g_{\text{YM}}^2 = \frac{M_n^2}{M_{\text{Pl}}^4} + 4m g_{\text{YM}}^2 > \frac{M_n^2}{M_{\text{Pl}}^4} \qquad \text{as} \quad t \to \infty, \tag{2.66}$$

where $g_{\text{YM}}^2 \to 0$ as $t \to \infty$. This beautifully matches the Weak Gravity bound (2.60) for the value $\mu = 1$ as in (2.62). Specifically, after dividing by g_{YM}^2 the two sides of this inequality correspond to the two curves shown in figure 1, in the context of an explicit example. We see that that the plateau-like structure of the maximal charges, in combination with the offset of the vacuum energy by -1, is crucially responsible for the existence of superextremal string exitations that lie just minimally above the Sublattice Weak Gravity bound. This is close in spirit to, and generalizes, the original analysis of the Weak Gravity Conjecture for heterotic strings compactified on T^6 [5].

We find it quite remarkable that the purely number theoretic relation (2.59) implies that the corresponding states in this sublattice just narrowly satisfy the (Scalar) Weak Gravity Conjecture bound, which after all has been derived in a completely independent manner that also takes the effects of scalar fluctuations [17, 25, 83] into account.

Before claiming victory, however, we hasten to point out a few caveats. The inequality (2.66) has been established only asymptotically for $t \to \infty$. Specifically, eq. (2.66) makes use of the behaviour (2.22), which applies only for $t \to \infty$ in Kähler moduli space. Moreover, only in this limit can we rely on the perturbative mass relation (2.52) of the

non-BPS string excitations, for geometries where the base space is rationally fibered. For more general base spaces, we have argued that in the tensionless limit the heterotic string can be trusted too. Away from the limit, however, it is expected that the precise value of the mass per excitation level is subject to renormalisation, in particular since the states are not BPS protected. A related point is that the Weak Gravity relation (2.60) rests on the formula for the mass of an extremal Reissner-Nordström black hole in 6d. A priori one could expect it to receive quantum corrections due to absence of the BPS property. It would be interesting to understand these effects and their potential interplay. In particular it is tempting to conjecture that the various corrections away from the limit $t \to \infty$ conspire such that the string states continue to satisfy the Sublattice Weak Gravity Conjecture, but this is pure speculation at this point.

3) The Completeness Conjecture. The Sublattice Weak Gravity Conjecture is a variant of the Completeness Conjecture [4, 40], according to which each point of the charge lattice should be populated by a state. As stressed in section 2.5, for the excitations of a string from a single D3-brane wrapping C_0, we find that indeed each charge $\mathfrak{q} \in \mathbb{Z}$ is realized for some mass in the spectrum.

While this fact is pretty self-evident in terms of the weakly coupled heterotic dual string, it can be contrasted with the subsector of charged matter states in F-theory which are due to 7-7 strings. For these states the SWGC and in fact even the completeness hypothesis appears to be badly violated, at least at the level of stable one-particle states. To see this, consider again an abelian U(1) gauge symmetry and suppose that the massless string excitations contain states of certain charge. Then the tower of massive 7-7 string states above the massless ground states does not contain any charges higher than that. From a perturbative Type IIB perspective, the U(1) charges are a consequence of the Chan-Paton factors of open strings stretched between 7-branes, and these do not change at higher string excitation levels. One might wonder if this situation is remedied in F-theory due to the appearance of multi-pronged strings, but this is not the case: under F/M-theory duality the spectrum of (p, q) string states maps to M2-branes wrapping exclusively curves in the fiber in the dual M-theory. Computation of the BPS invariants for these curves shows that no charges appear beyond the ones present already at the massless level in the F-theory spectrum. By contrast, M2-branes wrapping linear combinations of charged fibral curves, the full fiber and curves in the base do give to rise to arbitrarily highly charged states. In fact, these are precisely encoded in the Gromov-Witten invariants which will be computed in section 4. But the associated states in M-theory are the 5d BPS states obtained from wrapped 6d strings [48]. So again the string is needed to render the spectrum complete.

Let us now come back to the fate of the heterotic string excitations away from the tensionless, perturbative limit $t \to \infty$. It is natural to speculate that these states continue to be realized as physical states even though we cannot trust the weak coupling expression (2.52) for their masses. In fact, we can turn tables around and use the Completeness Conjecture to argue for the existence of these states, as otherwise the states of high charge would appear to be missing in the F-theory spectrum.

The situation seems reminiscent of the duality between Type I string theory and the $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string in 10 dimensions: the D1-string of Type I theory becomes

light in the limit of large Type I string coupling. In this limit its physical excitation spectrum coincides with the spectrum of the weakly coupled dual heterotic string. Unlike the fundamental Type I string, this includes excitations in spin representations of $\mathrm{Spin}(32)/\mathbb{Z}_2$. As the Type I string coupling becomes smaller, the description of the D1-string becomes non-perturbative in heterotic language, but it is reasonable to expect that the spinorial representations stay in the spectrum of the Type I theory, albeit at high masses.

4) Swampland conjectures and emergence of gauge symmetries. As shown in section 2.3, the limit $t \to \infty$ occurs at infinite distance in Kähler moduli space. According to the general Swampland Conjecture of [6] and its refinement in [21], as one approaches a point at infinite distance in moduli space, a tower of infinitely many states becomes massless, and the mass scale of the tower of states which become massless must be at least exponentially suppressed by the distance in moduli space. This fits perfectly with the behaviour in the concrete situation studied in this paper. Indeed, from (2.21) and (2.39) we notice that as we take the limit $t \to \infty$, the volume of the curve C_0 goes to zero parametrically as

$$\mathrm{vol}_J(C_0) \sim \frac{1}{t} \sim e^{-d}, \tag{2.67}$$

where d is the (geodesic) distance in moduli space as we approach the point $t \to \infty$. At the same time, from (2.52) we see that the masses of the particles that arise from excitations of the D3-brane wrapped on C_0 are suppressed by $e^{-d/2}$. Hence we confirm the expectations of [6, 21] that at infinite distance in moduli space a tower of infinitely many states becomes exponentially light.

However, as we have discussed, the tower of massless states is also charged under the U(1) gauge symmetry that becomes a global symmetry at infinite distance in moduli space. A very detailed analysis of a similar phenomenon in the context of 4d $\mathcal{N} = 2$ compactifications of Type IIB string theory on a Calabi-Yau 3-fold has been presented in [8]; the relevant points lie at infinite distance in complex structure moduli space, and a tower of infinitely many BPS states has been identified that become massless in the vicinity of these points. The points at infinite distance considered in [8] indeed coincide with the limit where the couplings of the RR gauge fields vanish. The appearance of infinitely many charged massless particles is in perfect agreement with field theoretic expectations: such massless states can be thought of as inducing a running of the gauge coupling in such a way as to enforce its vanishing in the extreme limit where all of them become massless [7, 8, 21, 31, 89]. Whereas the appearance of a finite number of charged massless states leads to a divergence of $1/g_{\mathrm{YM}}^2$ which is logarithmic in the mass, as in the prototypical example of the conifold transition [90, 91], an infinite number of charged massless states gives rise to a polynomial divergence of $1/g_{\mathrm{YM}}^2$ with the particle mass [8].

In the six-dimensional F-theory context that we are considering, the gauge coupling indeed vanishes polynomially in the Kähler moduli as

$$\frac{1}{g_{\mathrm{YM}}^2} = \frac{1}{2\pi}\mathrm{vol}_J(C) \sim t \qquad \text{as} \quad t \to \infty. \tag{2.68}$$

By a very crude estimate, we can relate this behaviour to the appearance of the tower of massless charged string excitations in the spirit of the above discussion, as follows.

We are considering the effective theory at an IR scale Λ_{IR}, which is identified with the scale of the charged particle with lowest mass. Above this scale, a tower of massive charged states arises. The field theory description breaks down at the UV cutoff Λ_{UV}, where gravitational effects are non-negligible. If we denote by N the number of massive states below Λ_{UV}, then in a theory in d spacetime dimensions, the UV cutoff is to be identified [7, 8] with the so-called species scale Λ_{UV} determined by [92–94] $\Lambda_{\text{UV}}^{D-2} = \frac{M_{\text{Pl}}^{D-2}}{N}$, which for $d = 6$ gives

$$\Lambda_{\text{UV}}^4 = \frac{M_{\text{Pl}}^4}{N} \,. \tag{2.69}$$

The fact that, in the presence of a tower of light states, the correct cutoff scale is indeed the species scale was stressed in [7, 8] and is an important ingredient in the following analysis. In our context, we identify N with the excitation level of the highest mass state which lies below the UV cutoff. This is, a priori, a significant oversimplification because we know that the excitation numbers already from the single string sector at a given excitation level n are certainly far from uniform and, in fact, considerably larger than one. Nonetheless, as far as the parametric behaviour is concerned, this simplification leads to the desired result. Indeed, with the above interpretation, and using that the mass M_n at level n scales with $\sqrt{n}$, see (2.52), we can alternatively express Λ_{UV} as

$$\Lambda_{\text{UV}} = \sqrt{(4\pi^2 \alpha)} \sqrt{N \, \text{vol}_J(C_0)} \,, \tag{2.70}$$

where we neglect the constant offset in (2.52) for large N. Combining both relations and ignoring unimportant numerical prefactors, we arrive at the relation

$$N^3 \sim \frac{M_{\text{Pl}}^4}{\text{vol}_J^2(C_0)} \tag{2.71}$$

for the maximal excitation level of states below Λ_{UV}.

The 1-loop running of the inverse gauge coupling from the UV to the IR cutoff is - up to potential subleading contributions -

$$\left. \frac{1}{g_{\text{YM}}^2} \right|_{\text{IR}} = \left. \frac{1}{g_{\text{YM}}^2} \right|_{\text{UV}} - x \sum_{n=1}^{N} \mathfrak{q}_n^2 \, M_n^2 \,, \tag{2.72}$$

where the numerical prefactor x depends on the specific type of state considered. To obtain the dominant contribution to the running one can hence very crudely estimate, with $\mathfrak{q}_n^2 \sim n$ from (2.54) and $M_n^2 \sim n \, \text{vol}_J(C_0)$ from (2.52), that

$$\sum_{n=1}^{N} \mathfrak{q}_n^2 \, M_n^2 \sim \sum_{n=1}^{N} n^2 \text{vol}_J(C_0) \sim N^3 \, \text{vol}_J(C_0) \sim \frac{1}{\text{vol}_J(C_0)} \sim t \,. \tag{2.73}$$

The underlying approximation is that at each excitation level n, the dominant contribution to the running comes from the states of the highest charge $\mathfrak{q}_n$. This indeed reproduces the vanishing of the gauge coupling as the infinite distance point $t \to \infty$ is attained.

3 U(1)-charge spectrum from elliptic genus

In this section we address the counting of charged states of the (nearly) tensionless heterotic string, by evaluating its elliptic genus. The main result of our analysis is an underpinning of our claim (2.54) for the maximal charge per excitation level. In particular, we will see that the counting of string states charged under an abelian group U(1) is determined geometrically via the intersection product between the curve C_0 wrapped by the D3-brane and the height pairing associated with the section S. This result may be of interest also to other applications.

We begin by reviewing the overall strategy how to determine the charge spectrum, or at least a characteristic portion thereof, of a nearly tensionless heterotic string that arises from a given elliptically fibered threefold, Y_3. Following the pioneering work of refs. [48, 76] (as reviewed in [45]), the procedure consists of several logical steps which are connected as follows:

$$\text{topological string on } Y_3 \longrightarrow \text{elliptic genus of } \mathcal{K} \longrightarrow \text{charge spectrum} \qquad (3.1)$$

In the next sections we will discuss each step in detail, while roughly following the arrows in reverse order. In section 3.1, we give some overview and comments on the charge spectrum. Then in section 3.2 we will proceed with a technical review of the elliptic genus in terms of weak Jacobi forms. We will also review the relation between the elliptic genus and the topological string partition function, which will later be used in actual computations for concrete examples. In section 3.3 we will derive the abovementioned result about the relation between the "fugacity index m" of the elliptic parameter related to the U(1) gauge symmetry in the 6d theory, and the height pairing of its underlying section. Finally in section 3.4 we will summarize the preceding technical sections, and the casual reader is invited to directly jump there.

3.1 Perturbative and non-perturbative elliptic genera

Our aim is to determine the charge spectrum (or rather a characteristic part of it) of the nearly tensionless heterotic string in six dimensions with $N = (1, 0)$ space-time supersymmetry, with respect to a nearly global U(1) symmetry whose specific embedding is determined by the given elliptic threefold, Y_3.

A familiar method is to use the elliptic genus associated with the 6-dimensional heterotic string theory. Computations of elliptic genera of the various non-critical strings have been a long-standing theme over the years, following the works of refs. [48, 76]. There one starts with the compactification of such strings on S^1 to 5 dimensions, making use of the fact that the wrapped strings can be straightforwardly counted as BPS particles in one dimension lower. Indeed, ref. [48] considers the non-critical E-string, obtained in F-theory by wrapping a D3-brane on a rational curve Γ with $\Gamma \cdot \Gamma = -1$. The authors then analyze the level-matching condition for the 6d non-critical E-string wrapped on an S^1 with winding number κ and Kaluza-Klein level n. The total excitation number N of the string turns out to be related to the winding number and KK level as

$$\kappa \, n = N \,. \qquad (3.2)$$

In particular for a single wrapped string, $\kappa = 1$, one can identify the KK momentum n with the excitation number N. In this way one can write down a counting function for the excited string states, which defines, in the limit of large S^1, what is meant by the elliptic genus of the 6d non-critical string. Concretely, for the non-critical E-string one obtains [48, 76] the expression

$$Z_{E_8}(\tau) \;=\; \frac{\Theta_{E_8}(\tau)}{\eta^{12}(\tau)}\,, \tag{3.3}$$

where $\Theta_{E_8}(\tau)$ is the partition function of the root lattice of E_8.

As is well-known, a crucial property of the elliptic genus is that it is a meromorphic modular form (for early works, see [60, 61, 64, 95, 96]), and this often allows us to determine it with very little physical input; for instance, for the E-string it is given in terms of an Eisenstein series, $\Theta_{E_8}(\tau) = E_4(\tau)$. When adding background fields, the elliptic genus still maintains good modular properties and turns into a meromorphic, in general Weyl-invariant Jacobi form [66, 97, 98].

We now turn to the nearly tensionless heterotic strings under consideration. The simplest situation occurs when the heterotic string is perturbative and weakly coupled. As we have discussed in section 2.4, it arises by wrapping a D3-brane along the fiber C_0 of a Hirzebruch surface in the tensionless limit (2.17). Being weakly coupled, the string can be formulated in terms of a 2d superconformal world-sheet theory, which guarantees a well-defined partition function. From modular invariance and the mere fact that we have chiral $N = (1,0)$ space-time supersymmetry in $d = 6$, it immediately follows [61] that, in the absence of background fields, the elliptic genus of this perturbative heterotic string must be given by

$$Z_{K3}(\tau) \;\equiv\; \mathrm{Tr}_R\left[(-1)^F F^2 q^{H_L} \bar{q}^{H_R}\right] \;=\; 2\frac{E_4(\tau)E_6(\tau)}{\eta^{24}(\tau)} \;=\; \frac{2}{q} - 480 - 282888q + \dots . \tag{3.4}$$

Here, as usual $q = e^{2\pi i \tau}$, and F denotes the right-moving fermion number.[5] This expression incorporates besides the conformal field theoretical, $\mathcal{N} = (0,4)$ world-sheet supersymmetric elliptic genus pertaining to any $K3$ surface [63, 97, 98], also the extra left-moving degrees of freedom of the perturbative heterotic string. In the following we will always implicitly assume this when we refer to the elliptic genus of $K3$, and in particular this applies to the elliptic genus related to the specific, not necessarily perturbative compactification geometry $\mathcal{K}$ that was introduced in section 2.4.

We will be interested in a refinement induced by the U(1) symmetry that is determined by the elliptic fibration Y_3. In terms of $\mathcal{K}$ this refinement translates into certain bundles on top such that the only unbroken gauge symmetry is just this U(1). For reasons of modularity, the elliptic genus must then take the more general form [64, 98]

$$Z_{\mathcal{K}}(\tau, z) = \frac{\Phi_{10,m}(\tau, z)}{\eta^{24}(\tau)}\,, \tag{3.5}$$

---

[5]The factor of F^2 is required in order to obtain a non-vanishing result, where the trace is taken over the entire Hilbert space. Equivalently, one can factor out the spin content of a half-hypermultiplet and only trace over the remaining quantum numbers of the states in the Hilbert space, but without insertion of F^2.

where $\Phi_{10,m}(\tau, z)$ is a weak Jacobi form of weight 10. The concept of weak Jacobi forms and their behaviour under modular transformations is reviewed in appendix C. The U(1) 'fugacity index' m depends on the specific U(1) embedding and will be determined in section 3.3 (z denotes the field strength of the U(1) background field, or its 'fugacity'). As is familiar in this context, and will be re-addressed later in section 3.4, the ring of such modular forms is finitely generated and this is why one can easily determine $Z_{\mathcal{K}}(\tau, z)$ for concrete examples.

Note that the elliptic genus (3.5) is not a partition function for physical states, at least not directly. This is because the left-moving tower of states lacks level-matching; most notably, the tachyon is not physical. Rather, it needs to be properly combined with right-moving excitations. This means that these states cannot be BPS protected (and indeed there are no BPS particles for 6d $(1, 0)$ supersymmetry in the first place).

We can put a slightly different perspective on this trivial fact by employing space-time supersymmetry, which can be implemented by spectral flow of the elliptic genus [64, 99, 100] from the R to the NS sector. This transports the tower of left-moving states paired with the right-moving Ramond ground states to a partition function summed over all fermionic periodicities.[6] Since the left sector will not be touched by this operation, the total sum will still be holomorphic, now as a consequence of space-time supersymmetry rather than of world-sheet supersymmetry.

The partition function can then be disentangled into bosonic and fermionic space-time sectors, and one can infer the existence of physical, level-matched states coming from the super-Virasoro modules over the bosonic and fermionic ground states (corresponding to massless representations of the $\mathcal{N} = (0, 4)$ world-sheet supersymmetry [64]). Certainly this does not give, by far, the full partition function, rather it gives a piece of it that is holomorphically factorized. This subsector is protected, and in particular moduli independent, by an interplay between modularity and world-sheet/space-time supersymmetry; see figure 2 for an illustration. While this is indeed only a very small subsector of the theory, it suffices to demonstrate the existence of physical states with certain charge-to-mass ratios, and this is all we need for discussing weak gravity conjectures as in section 2.6.

Note that these arguments apply primarily when we have a well-defined world-sheet formulation in terms of conformal field theory, i.e. to perturbative heterotic strings in the limit of vanishing coupling. As mentioned in section 2.4, these arise when we consider threefolds Y_3 that are elliptic fibrations over Hirzebruch bases, $\mathbb{F}_a$, in the limit $\mathrm{vol}_J(C_0) \to 0$. However, tensionless heterotic strings can also arise from shrinking zero-intersection curves C_0 on non-Hirzebruch bases. These will in general contain more than one massless tensor field and will not be easily describable in terms of conformal field theory.

For such strongly coupled heterotic strings the physical particle spectrum is less clear, since the massive excitations are not BPS protected and a world-sheet formulation is also not at hand. For such cases, one may adopt as working assumption that it suffices to consider the elliptic genus at face value, even though it may not directly count physical states.

---

[6]One can see this more directly by making use of certain generalized Riemann theta function identities that map the parity-odd sector to a sum over all sectors [101, 102].

Figure 2. Shown are left- and right-moving spectra relevant for the elliptic genus of a perturbative heterotic string with $(1, 0)$ supersymmetry in $d = 6$. In the right-moving periodic Ramond sector, the massive towers cancel due to world-sheet supersymmetry, but they can also be mapped into towers that cancel between bosons and fermions due to space-time supersymmetry. In this way we see that the left-moving, U(1)-charged excitations can be level-matched to excitations in the vacuum modules of the right-moving sector, to form physical supermultiplets with fixed masses.

As we will show below, an elliptic genus can indeed be defined also for strongly coupled heterotic strings via dualities, analogously as for the non-critical E-strings. For such heterotic strings the elliptic genus is still of the form (3.5), where however $\Phi_{10,m}(\tau, z)$ is in general only quasi-modular (which means that may involve the quasi-modular Eisenstein series, $E_2(\tau)$). The deviation from the familiar conformal field theoretic expression (3.4), for which $\Phi_{10,m}(\tau, 0) = E_4(\tau)E_6(\tau)$, is due to the modified anomaly associated with the extra massless tensor fields. The mild non-modularity reflects that we do not have a standard world-sheet description in this case. We will continue to refer to this more general object as $Z_{\mathcal{K}}(\tau, z)$, even though it may not necessarily coincide with the familiar, conformal field theoretical elliptic genus associated with a $K3$ surface. In the following we explain how to compute $Z_{\mathcal{K}}(\tau, z)$ also in these cases.

3.2 From elliptic genus to topological string partition function

We now review and discuss some technical aspects of the elliptic genus that are relevant to us. The discussion will be structured according to the following four different aspects:

1. For a string arising from a D3-brane wrapping a curve C_β on the base of an F-theory model, it is convenient to define a refined notion of an elliptic genus, $Z_{C_\beta}(\tau, \lambda_s, z)$,

as in (3.7) below, by introducing an extra fugacity parameter λ_s with respect to the group of rotations transverse to the string. We will sometimes refer to $Z_{C_\beta}(\tau, \lambda_s, z)$ as the 'spin-refined elliptic genus'. The elliptic genus we are ultimately interested in, namely $Z_{\mathcal{K}}(\tau, z)$ in (3.5), is contained in this more general notion of the elliptic genus as we will explain.

2. The spin-refined elliptic genus $Z_{C_\beta}(\tau, \lambda_s, z)$ enjoys good modular properties and can hence be expanded in terms of suitable weak Jacobi forms.

3. Duality with M-theory relates $Z_{C_\beta}(\tau, \lambda_s, z)$ to the topological string partition function on the elliptic 3-fold Y_3 as in (3.21) and (3.25) below.

4. The topological string genus-zero free energy determines the elliptic genus $Z_{\mathcal{K}}(\tau, z)$ of the heterotic string, as shown in (3.34). For perturbative heterotic strings, this relation reproduces the conformal field theoretical elliptic genus, and we will take it as a definition for the non-perturbative generalisation with extra massless tensor fields.

Most of the material corresponding to the first three points is well-known in the literature (see for example [41, 49, 50, 52–59] and the review [45] for further references), and we will review it below to set the notation and to make the article self-contained. The informed reader (or those not interested primarily in the technical details) may skip directly to point 4.

1) The spin-refined elliptic genus of wrapped strings. We consider strings that arise from a D3-brane wrapping some curve, $C_\beta \in H_2(B_2, \mathbb{Z})$, in the base B_2 of an elliptically fibered Calabi-Yau, Y_3. We assume that the curve is at general position, i.e. that it is not a component of the discriminant. As pointed out in section 2.4, the restriction of the elliptic fibration to C_β describes an in general non-trivially fibered elliptic surface, $\widehat{C}_\beta \in H_4(Y_3, \mathbb{Z})$. For the nearly tensionless heterotic string that we consider, this surface turns out to be a $K3$ surface $\widehat{C}_0$. This is not to be confused with $\mathcal{K}$, which represents the $K3$ surface on which the dual heterotic string is compactified, whose elliptic genus we are up to determine.

Following [48, 76] we consider a compactification of the 6d string on S^1, while also making the time direction periodic. Thus we can think of space-time formally to be given by

$$\mathbb{R}^4 \times T^2. \tag{3.6}$$

An elliptic genus-like partition function, $Z_{C_\beta}(\tau, \lambda_s, z)$, can then be defined [48, 76] as a trace over the spectrum of 5d BPS momentum and winding states of the wrapped string, with periodic boundary conditions:

$$Z_{C_\beta}(\tau, \lambda_s, z) = \mathrm{Tr}_R \left[(-1)^F F^2 q^{H_L} \bar{q}^{H_R} u^{2J_-} \xi^J \right] . \tag{3.7}$$

Here $u = e^{2\pi i \lambda_s}$, where the parameter λ_s implements a refinement by the universally present global $\mathrm{SU}(2)_-$ symmetry with generator J_-, which is part of the transverse Lorentz

group (2.41). Thus, the insertion of u^{2J_-} in the trace weighs the BPS states depending on their $SU(2)_-$ spin. Moreover

$$\xi^J = e^{2\pi i z J} \tag{3.8}$$

induces a grading with respect to the almost global $U(1)$ symmetry, with generator J. It is here that the dependence of the BPS spectrum on the gauge symmetry of the specific F-theory background enters. Moreover, we have defined

$$q = e^{2\pi i \tau} , \tag{3.9}$$

where τ refers to the torus T^2 and does, a priori, *not* refer as usual to the string worldsheet. Rather, $Z_{C_\beta}(\tau, \lambda_s, z)$ counts the index of winding and Kaluza-Klein 5d BPS states arising from the wrapped string on S^1.

The spin-refined elliptic genus enjoys an expansion of the form

$$Z_{C_\beta}(\tau, \lambda_s, z) = q^{E_0} \left(f^{(0)}(\lambda_s, z) + \sum_{n=1}^{\infty} q^n f^{(n)}(\lambda_s, z) \right) , \tag{3.10}$$

where n corresponds to the Kaluza-Klein level of the string. The shift in vacuum energy is due to the Casimir energy of the $\mathcal{N} = (0, 4)$ worldsheet theory [55, 58],

$$E_0 = -\frac{1}{2} C_\beta \cdot \bar{K} , \tag{3.11}$$

where $\bar{K}$ is the anti-canonical bundle of B_2.

The important point [48, 76] is that the functions $f^{(n)}(\lambda_s, z)$ for $n \geq 1$ appearing in (3.10) have an equivalent interpretation directly in terms of the string propagating in $\mathbb{R}^{1,5}$, namely as counting left-moving string excitations at level $N = n$. This makes contact with the conformal field theoretical elliptic genus that we mentioned in the previous section, for which q refers to the modular parameter of a stringy world-sheet.

2) Modular properties of the elliptic genus. As is well-known, key to determining the elliptic genus is to make use of the specific behaviour under modular transformations of τ and the elliptic parameters λ_s and z. That is, it must transform as a Jacobi form of given weight w and index m as defined in eqs. (C.2) and (C.3) of the appendix. Modular invariance implies that the total weight w of $Z_{C_\beta}(\tau, \lambda_s, z)$ must vanish, $w = 0$. The indices of $Z_{C_\beta}(\tau, \lambda_s, z)$ with respect to the elliptic parameters λ_s and z are encoded in the modular anomaly equation[7] [49, 52, 53, 55–57, 103]

$$\left(\frac{\partial}{\partial E_2(\tau)} + \frac{(2\pi)^2}{12} f_{C_\beta}(\lambda_s, z) \right) Z_{C_\beta}(\tau, \lambda_s, z) = 0 . \tag{3.12}$$

Indeed, as reviewed in appendix C, the index of a weak Jacobi form with respect to the elliptic parameters can be read off from the quadratic terms in $f_{C_\beta}(\lambda_s, z)$, see eq. (C.24). A general strategy to determine the function $f_{C_\beta}(\lambda_s, z)$ is via the t' Hooft anomaly polynomial of the $(0, 4)$ theory on the string [55, 56, 59]. In section 3.3 we will show how to determine

---

[7]The factor of $(2\pi)^2$ reflects our conventions that λ_s and z are quasi-periodic as in (C.3) or (C.12).

the index m related to the U(1) symmetry in terms of the height pairing of a rational section on the elliptic fibration Y_3.

An analysis of the pole structure of the elliptic genus suggests that it can be written in terms of a ratio of weak Jacobi forms, with an explicit expression for the denominator. Specifically, for the special case of a single string associated to a curve C_β corresponding to one of the generators of the Mori cone, the elliptic genus takes the form [41, 53]

$$Z_{C_\beta}(\tau, \lambda_s, z) = \left(\frac{1}{\eta^2(\tau)}\right)^{6C_\beta \cdot \bar{K}} \frac{\Phi_{W,L,m}(\tau, \lambda_s, z)}{\varphi_{-2,1}(\tau, \lambda_s)} . \tag{3.13}$$

Here $\varphi_{-2,1}(\tau, \lambda_s)$ is the unique weak Jacobi form of weight -2 and index 1 with respect to λ_s, see eq. (C.14). The numerator is then a Jacobi form whose weight W is determined by the requirement that $Z_{C_\beta}(\tau, \lambda_s, z)$ has total weight zero, where we recall that $\eta^2(\tau)$ transforms like a modular form of weight 1. The total index m_- of $Z_{C_\beta}(\tau, \lambda_s, z)$ with respect to the elliptic parameter λ_s is determined by the modular anomaly equation (3.12) to be

$$m_- = \frac{1}{2} C_\beta \cdot (C_\beta + K). \tag{3.14}$$

Altogether this leads to

$$W = 6C_\beta \cdot \bar{K} - 2, \qquad L = \frac{1}{2} C_\beta \cdot (C_\beta + K) + 1 = g(C_\beta), \tag{3.15}$$

where $g(C_\beta)$ is the genus of C_β, and K the canonical bundle of B_2. The index m of Φ with respect to the U(1) fugacity agrees with that of $Z_{C_\beta}(\tau, \lambda_s, z)$, because the denominator does not involve z. The reason is that the only modes charged under the global symmetries are the half-Fermi multiplets in the last line of table 1, which are purely fermionic and whose contribution therefore appears only in the numerator of the elliptic genus.

More generally, suppose that the curve C_β is a linear combination

$$C_\beta = \sum_{i=1}^{k} \beta^i C_i \tag{3.16}$$

of some Mori cone generators, C_i. Then the pole structure is slightly more complicated and the total elliptic genus takes the form (see e.g. [41])

$$Z_{C_\beta}(\tau, \lambda_s, z) = \left(\frac{1}{\eta^2(\tau)}\right)^{6C_\beta \cdot \bar{K}} \frac{\Phi_{W,L,m}(\tau, \lambda_s, z)}{\prod_{i=1}^{k} \prod_{s_i=1}^{\beta^i} \varphi_{-2,1}(\tau, s_i \lambda_s)} . \tag{3.17}$$

The weight W and index L of the numerator now change accordingly, such that $Z_{C_\beta}(\tau, \lambda_s, z)$ is still of weight zero and of SU(2)$_-$ fugacity index as given in (3.14).

Once modular indices and weight are given, the elliptic genus $Z_{C_\beta}(\tau, \lambda_s, z)$ is fixed up to a finite number of parameters. This well-known feature is a consequence of the fact that the ring of weak Jacobi forms is finitely generated [104]. So all one needs to do is to fix these parameters by matching the general ansatz for $Z_{C_\beta}(\tau, \lambda_s, z)$, with given modular indices and weight, to some extra physical input.

3) Relation to the topological string on Y_3. The elliptic genus $Z_{C_\beta}(\tau, \lambda_s, z)$, as given in (3.7), is closely related to the topological string partition function Z_{top} on the elliptically fibered Calabi-Yau Y_3, via a chain of dualities [50, 52]. This was already used in the original work [48] in the context of non-critical E-strings. More generally, the topological string partition function on elliptic Calabi-Yau 3-folds has been studied in detail in [49, 53, 103].

Specifically, it is known by F/M-theory duality that the 6d effective action of F-theory on Y_3 reduced on S^1 coincides with the 5d effective action of M-theory on Y_3. A critical or non-critical string from a D3-brane on C_β can either wrap the S^1 or not. If it does, it produces a BPS particle in 5d. In the dual M-theory description, the 5d BPS particle arises from an M2-brane wrapping some curve on Y_3. If we consider for simplicity a generic Weierstrass model Y_3, then the string with wrapping number $\kappa = 1$ along S^1 and Kaluza-Klein momentum level n maps to a 5d BPS particle that arises from the M2-brane wrapping the curve

$$C_\beta + n C_{\mathcal{E}} \,. \tag{3.18}$$

Here $C_{\mathcal{E}}$ is the class of the elliptic fiber while C_β is the curve in the base. Indeed, the Kaluza-Klein U(1) symmetry of the S^1 reduction is generated by the divisor [71]

$$\tilde{S}_0 = S_0 + \frac{1}{2} \pi^{-1} \bar{K} \,, \tag{3.19}$$

where S_0 is the zero-section of the elliptic fibration Y_3 and $\bar{K}$ the anti-canonical divisor of the base. Hence an M2-brane along $C_\beta + n C_{\mathcal{E}}$ has KK charge

$$\tilde{S}_0 \cdot_{Y_3} (C_\beta + n C_{\mathcal{E}}) = -\frac{1}{2} \bar{K} \cdot_{B_2} C_\beta + n = E_0 + n \,, \tag{3.20}$$

where we used the relation $S_0 \cdot S_0 = -S_0 \cdot \pi^{-1}(\bar{K})$ for the zero-section of an elliptic fibration.[8] Note that the shift by $-\frac{1}{2} \bar{K} \cdot C_\beta$ beautifully coincides with the shift of the energy of the wrapped string in (3.10), which is due to the Casimir energy E_0.

To describe the precise relation between the elliptic genus and Z_{top}, we introduce the following notation. A basis of $H_2(Y_3, \mathbb{Z})$ can be split into a basis $\{C_i^{\text{b}}\}$ of base curve classes and a basis $\{C_{\mathcal{E}}, C_a^{\text{f}}\}$ of fibral curves, where in the latter we have singled out $C_{\mathcal{E}}$ as the class of the generic elliptic fiber. To these basis elements we assign the Kähler parameters $\mathbf{t} = \{t_i\}$, τ and $\mathbf{z} = \{z_a\}$, respectively. Given a base curve $C_\beta \in H_2(B_2, \mathbb{Z})$ with expansion $C_\beta = \beta^i C_i^{\text{b}}$, its complexified Kähler volume can then be expanded as $t_\beta = \beta^i t_i$, and likewise for a fibral curve $n C_{\mathcal{E}} + r^a C_a^{\text{f}}$, the volume is given as $n\tau + r^a z_a$. Hereafter, we will restrict ourselves to the geometries with exactly one independent non-zero section, i.e., to those effective theories with a single U(1), and the subscripts in C_a^{f} and z_a, etc. will thus be removed. Furthermore, we will assume w.l.o.g. that an appropriate basis for the fibral curves $\{C_{\mathcal{E}}, C^{\text{f}}\}$ has been chosen such that the M2-brane wrapped on C^{f} has U(1) charge $\mathfrak{q} = 1$ and KK charge 0.

---

[8]More precisely, for a curve $C_\beta \in H_2(B_2)$ in the base, the corresponding base curve $\mathfrak{C}_\beta \in H_2(Y_3)$ we consider in Y_3 is given as $\mathfrak{C}_\beta := S_0 \cdot_{Y_3} \pi^{-1}(C_\beta)$. By abuse of notation, we often denote this also by C_β. Then, $S_0 \cdot_{Y_3} \mathfrak{C}_\beta = S_0 \cdot_{Y_3} S_0 \cdot_{Y_3} \pi^{-1}(C_\beta) = -S_0 \cdot_{Y_3} \pi^{-1}(\bar{K}) \cdot_{Y_3} \pi^{-1}(C_\beta) = -\bar{K} \cdot_{B_2} C_\beta$. Furthermore $S_0 \cdot_{Y_3} C_{\mathcal{E}} = 1$ because the fiber intersects the zero-section once.

In this notation, the topological string partition function on Y_3 takes the form

$$Z_{\text{top}} = \exp\left(\mathcal{F}(\lambda_s, \tau, \mathbf{t}, z)\right) = \exp\left(\sum_{g \geq 0} \mathcal{F}^{(g)}(\tau, \mathbf{t}, z)\lambda_s^{2g-2}\right). \tag{3.21}$$

Here $\mathcal{F}(\lambda_s, \tau, \mathbf{t}, z)$ denotes the topological string free energy, which enjoys a perturbation series in contributions of genus g maps, where the expansion parameter λ_s plays the role of the string coupling. Each genus g free energy, $\mathcal{F}^{(g)}(\tau, \mathbf{t}, z)$, can in turn be expanded in terms of the exponentiated Kähler parameters of Y_3,

$$q = e^{2\pi i \tau}, \qquad Q_\beta = e^{2\pi i t_\beta}, \qquad \xi = e^{2\pi i z}, \tag{3.22}$$

as

$$\mathcal{F}^{(g)}(\tau, \mathbf{t}, z) = \sum_{C_\beta \in H_2(B_2, \mathbb{Z})} \mathcal{F}^{(g)}_{C_\beta}(\tau, z) Q_\beta. \tag{3.23}$$

Here, the object

$$\mathcal{F}^{(g)}_{C_\beta}(\tau, z) = \sum_{n, \gamma} N^{(g)}_{C_\beta}(n, r) q^n \xi^r \tag{3.24}$$

gives an expansion in terms of the genus g Gromov-Witten invariants, which are a priori rational numbers. However, for a single wrapped curve classe these agree with the integral Gopakumar-Vafa BPS invariants. In the present context, single wrapping refers to the class C_β not being a multiple of some other curve class.

Now we are ready to write the relation between the topological partition function on Y_3, Z_{top}, and the elliptic genus, $Z_{C_\beta}(\tau, \lambda_s, z)$, of the 6d, $(1, 0)$ supersymmetric string theory obtained by compactification of F-theory on Y_3 [50, 52]:

$$Z_{\text{top}} = Z_0(\tau, \lambda_s, z)\left(1 + \sum_{C_\beta} Z_{C_\beta}(\tau, \lambda_s, z)\widehat{Q}_\beta\right). \tag{3.25}$$

The overall factor $Z_0(\tau, \lambda_s, z)$ is the contribution to the 5d index from 5d BPS states due to the 6d supergravity modes and their Kaluza-Klein tower. The expansion parameter in (3.23) is shifted as [53, 56]

$$\widehat{Q}_\beta = \sigma(\tau, z) Q_\beta. \tag{3.26}$$

This shift has been motivated in [53] by requiring that $\mathcal{F}^{(g)}_{C_\beta}(\tau, z)$ enjoys good modular properties. Specifically, if the base curve C_β is not contained in the discriminant, the shift is given as $\sigma(\tau, z) = e^{\pi i (C_\beta \cdot \bar{K})\tau}$, that is,

$$\widehat{Q}_\beta = q^{\frac{1}{2}C_\beta \cdot \bar{K}} Q_\beta = e^{2\pi i (t_\beta + \frac{1}{2}(C_\beta \cdot \bar{K})\tau)}. \tag{3.27}$$

Observe, importantly, that by virtue of the duality, τ can be viewed as either the modular parameter of the elliptic genus, or as the Kähler parameter of the elliptic fiber of Y_3. Similarly, λ_s can be interpreted as either the elliptic parameter with respect to SU(2)$_-$, or as the topological string coupling; moreover z as either the elliptic parameter with respect to the global U(1) symmetry of the string, or as the fibral Kähler parameter of Y_3.

We can elucidate the relation between the free energy of the topological string and the elliptic genus by taking the logarithm of (3.21) and expanding

$$\sum_{g \geq 0} \mathcal{F}^{(g)}(\tau, \mathbf{t}, z) \lambda_s^{2g-2} = \log Z_0(\tau, \lambda_s, z) + \log\left(1 + \sum_{\beta} Z_{C_\beta}(\tau, \lambda_s, z)\,\widehat{Q}_\beta\right) \qquad (3.28)$$

$$= \log Z_0(\tau, \lambda_s, z) + \sum_{\beta} Z_{C_\beta}(\tau, \lambda_s, z)\,\widehat{Q}_\beta$$

$$- \frac{1}{2} \sum_{C_{\beta'}, C_{\beta''}} Z_{C_{\beta'}} Z_{C_{\beta''}} \widehat{Q}_{\beta'} \widehat{Q}_{\beta''} + \dots.$$

A considerable simplification occurs if C_β corresponds to one of the generators of the Mori cone and therefore cannot be written as the sum of two effective curves $C_{\beta'}$ and $C_{\beta''}$. In this case, the elliptic genus obeys the simpler form (3.13). Furthermore, in view of (3.23) we can identify

$$\sum_{g \geq 0} \mathcal{F}_{C_\beta}^{(g)}(\tau, z)\,\lambda_s^{2g-2} = q^{\frac{1}{2} C_\beta \cdot \bar{K}}\, Z_{C_\beta}(\tau, \lambda_s, z) = \left(\frac{q^{\frac{1}{2} C_\beta \cdot \bar{K}}}{\eta(\tau)^{12 C_\beta \cdot \bar{K}}}\right) \frac{\Phi_{W,L,m}(\tau, \lambda_s, z)}{\varphi_{-2,1}(\tau, \lambda_s)}. \qquad (3.29)$$

The important factor of $q^{\frac{1}{2} C_\beta \cdot \bar{K}}$ is due to the shift in $\widehat{Q}_\beta$. In view of (3.10) and (3.11) it precisely cancels the overall factor q^{E_0} in $Z_{C_\beta}(\tau, \lambda_s, z)$, as is necessary for the match with the topological string prepotential to work. Due to this extra factor, while $Z_{C_\beta}(\tau, \lambda_s, z)$ is a weak Jacobi form of weight zero, the l.h.s. and r.h.s. of (3.29) do not have good modular properties.

One may wonder what this implies for the modular behaviour of the $\mathcal{F}^{(g)}(\tau, z)$, viewed as expansion coefficients of the free energy with regard to the elliptic parameter λ_s. For this, note that a weak Jacobi form of weight w and index m has the expansion

$$\varphi_{w,m}(\tau, z) = \sum_{j=0}^{\infty} \phi_j(\tau) z^j, \qquad (3.30)$$

where $\phi_j(\tau)$ are in general only *quasi-modular*[9] forms of weight $j + w$. As explained at the end of appendix C, this means that in general the $\phi_j(\tau)$ must lie in the ring generated by the modular Eisenstein functions $E_4(\tau)$ and $E_6(\tau)$, as well as the quasi-modular Eisenstein function $E_2(\tau)$; the quasi-modular transformation behaviour of $E_2(\tau)$ can be found in (C.10). In the same spirit, we may expand the numerator on the r.h.s. of eq. (3.29),

$$\Phi_{W,L,m}(\tau, \lambda_s, z) = \Phi_{W,m}(\tau, z) + \mathcal{O}(\lambda_s), \qquad (3.31)$$

as well as use the expansion (C.19) of the denominator,

$$\varphi_{-2,1}(\tau, \lambda_s) = -\lambda_s^2 \left(1 + \mathcal{O}(\lambda_s^2)\right). \qquad (3.32)$$

---

[9]See, e.g., page 16 of [57] for a review of this fact. The same conclusion has been obtained in [49, 53, 103] without making use of the duality, solely based on properties of the topological string partition function for elliptic fibrations.

Comparing the terms of order λ_s^{-2} of both sides of (3.29) finally identifies the genus-zero prepotential with the elliptic genus as

$$\mathcal{F}_{C_\beta}^{(0)}(\tau, z) = -\frac{q^{\frac{1}{2}C_\beta \cdot \bar{K}}}{\eta(\tau)^{12C_\beta \cdot \bar{K}}}\Phi_{W,m}(\tau, z)\,. \tag{3.33}$$

Note that the expansion (3.31) has higher order terms only if the index L (as given in (3.15)) is non-zero. In this case, the function $\Phi_{W,m}(\tau, z)$ will be generically only a quasi-modular form, which follows from the remarks above. This fits with the fact that the genus-zero prepotential satisfies an in general non-trivial modular anomaly equation [49, 53, 103]. On the other hand, if C_β is a genus-zero curve corresponding to a Mori cone generator, then $L = 0$ and hence $\Phi_{W,m}(\tau, z)$ gives the complete numerator of eq. (3.29), because $\Phi_{W,L=0,m}(\tau, \lambda_s, z)$ is independent of λ_s. In this case $q^{-\frac{1}{2}C_\beta \cdot \bar{K}}\mathcal{F}_{C_\beta}^{(0)}(\tau, z)$ is by itself a good modular Jacobi form and fully determines the elliptic genus. Recall, however, that to arrive at this result, we have had to make the assumption that C_β corresponds to a Mori cone generator as otherwise (3.13) would not hold. In this case, the relation between $Z_{C_\beta}(\tau, \lambda_s, z)$ and the genus-zero prepotential is more complicated, as will be explained below; what is still correct, even in this more general situation, is the ansatz (3.33) for $\mathcal{F}_{C_\beta}^{(0)}(\tau, z)$, with $\Phi_{W,m}(\tau, z)$ now being only quasi-modular.

4) Relation to $\mathcal{N} = (0, 4)$ elliptic genus $Z_\mathcal{K}$. After these preparations we are ready to apply the formalism reviewed above to our object of interest, namely the elliptic genus $Z_\mathcal{K}(\tau, z)$ of the heterotic sting on the $K3$ geometry $\mathcal{K}$. There are two cases to distinguish. Recall from section 2.4 that if the self-intersection-zero rational curve C_0 is the fiber of a Hirzebruch surface $\mathbb{F}_a$, there exists a perturbative heterotic dual, and in the tensionless limit under consideration the heterotic string becomes weakly coupled. As will be explained in more detail in section 4.1, the fiber of a Hirzebruch surface is indeed a generator of the Mori cone and therefore the spin-refined elliptic genus $Z_{C_0}(\tau, \lambda_s, z)$ takes the simple form (3.13). Furthermore, the elliptic genus without the spin refinement is the leading term in a λ_s expansion of $Z_{C_0}(\tau, \lambda_s, z)$ around $\lambda_s = 0$. According to the discussion around (3.33) this coincides up to a sign and a factor of q with the genus-zero prepotential of the topological string on Y_3. That is, we can identify:

$$Z_\mathcal{K}(\tau, z) = -q^{-1}\mathcal{F}_{C_0}^{(0)}(\tau, z) = -q^{-1}\sum N_{C_0}^{(0)}(n, r)\, q^n \xi^r\,. \tag{3.34}$$

This fits perfectly with the fact that, up to the factor of q, the genus-zero prepotential is by itself a modular form of precisely the correct weight to furnish the elliptic genus (3.5) for the perturbative heterotic string.[10]

More generally, suppose that C_0 lies in a non-Hirzebruch surface. In this case, it will generically not be one of the Mori cone generators. Assume for definiteness that $C_0 = C_1 + C_2$ for two Mori cone generators C_1 and C_2. The general relation (3.28) now allows us to write

$$Z_{C_0}(\tau, \lambda_s, z) = \mathcal{F}_{C_0}(\tau, \lambda_s, z) + \frac{1}{2}\mathcal{F}_{C_1}(\tau, \lambda_s, z)\mathcal{F}_{C_2}(\tau, \lambda_s, z)\,, \tag{3.35}$$

---

[10]Note that for this be true, the GW invariants $N_{C_0}^{(0)}(n, r)$ must be integers, which is the case for the expansion (3.23) where we consider only terms linear in Q_0.

where $\mathcal{F}_{C_i}(\tau, \lambda_s, z) = \sum_{g \geq 0} \lambda_s^{2g-2} \mathcal{F}_{C_i}^{(g)}(\tau, \lambda_s, z)$ is the all-genus prepotential that multiplies Q_{C_i} in the expansion of the topological string free energy. From (3.17) we see that the denominator of Z_{C_0} contains the product $\varphi_{-2,1}^2(\tau, \lambda_s)$. The leading term in the λ_s expansion (3.32) hence goes as $1/\lambda_s^4$. This term must equal the $1/\lambda_s^4$ contribution from $\frac{1}{2}\mathcal{F}_{C_1}\mathcal{F}_{C_2}$. The next term is of order $1/\lambda_s^2$ and receives contributions from both the single string prepotential $\mathcal{F}_{C_0}$ and the 2-string prepotential $\mathcal{F}_{C_1}\mathcal{F}_{C_2}$. The '1-string irreducible' contribution is only due to the leading piece in $\mathcal{F}_{C_0}$, which again coincides with the genus-zero prepotential. Hence it is natural to identify this with the non-spin-refined elliptic genus describing the dual, non-perturbative heterotic string. This reasoning leads to the same expression (3.34). The genus-zero free energy has to have the correct modular properties for this to be possible: up to the same factor of q, it must be quasi-modular in accordance with the modular anomaly equation [49, 53, 103], which under the present assumptions takes the form

$$\frac{\partial \mathcal{F}_{C_0}^{(0)}}{\partial E_2} = \frac{1}{24} \sum_{C_1 + C_2 = C_0} (C_1 \cdot C_2) \, \mathcal{F}_{C_1}^{(0)} \mathcal{F}_{C_2}^{(0)} \,. \tag{3.36}$$

This generically leads to a dependence on the quasi-modular Eisenstein series $E_2(\tau)$ for the non-spin-refined elliptic genus advertised in section 3.2. We will exemplify this behaviour for an F-theory base dP_2 in section 4.2. In particular we will verify that the genus-zero prepotential associated with C_0 correctly encodes the massless spectrum as expected from the geometry.

5) Multi-wrappings. As a side remark, it is also instructive to consider the situation where the D3-brane is wrapped not just once, but k times on C_0. This can be investigated by computing via mirror symmetry the terms with higher powers of Q_0 of the prepotential

$$\mathcal{F}^{(0)}(Q_0, q, \xi) = \mathcal{F}_{\text{class}}^{(0)} + \sum n_{k \cdot C_0}^{(0)}(n, r) \operatorname{Li}_3(Q_0{}^k q^n \xi^r) \,. \tag{3.37}$$

Here $n_{k \cdot C_0}^{(0)}(n, r)$ denotes the (integral) Gopakumar-Vafa invariants. In fact, multi-wrapped heterotic strings are expected [41, 48, 76] not to form bound states, so that the elliptic genus for multiple heterotic strings should be given [51, 105] in terms of the elliptic genus of the k-fold symmetric product of $\mathcal{K}$. Taking multi-coverings into account, this translates into

$$\mathcal{F}_{k \cdot C_0}^{(0)}(q, \xi) = -q^k Z_{Sym^k \mathcal{K}}(q, \xi) - q^k \mathcal{F}^{(0)}(Q_0, q, \xi)\big|_{Q_0^k}, \quad k > 1, \tag{3.38}$$

where for simplicity we assumed k to be prime. Above,

$$Z_{Sym^k \mathcal{K}}(q, \xi) = T_k^{(-2)}(Z_{\mathcal{K}}(\tau, z)) \,, \tag{3.39}$$

where the Hecke transform of a Jacobi form φ of modular weight w is defined by [104]

$$T_k^{(w)} \varphi(\tau, z) = k^{w-1} \sum_{\substack{ad=k \\ b \bmod d}} d^{-w} \varphi\left(\frac{a\tau + b}{d}, a\, z\right) \,. \tag{3.40}$$

We have checked for the examples we will consider later that (3.38) is indeed satisfied, to some low orders of expansions in Q_0, q, ξ. This implies for the relevant Gopakumar-Vafa BPS invariants that $n_{k \cdot C_0}^{(0)} = 0$ for $k > 1$, which in turn means that the heterotic

strings do not form bound states, as expected. While this fact has been known since long for perturbative heterotic strings [48, 76], it may appear slightly less trivial for non-perturbative heterotic strings. This is why we will re-address this point for the example in section 4.2.

3.3 Global Mordell-Weil U(1) symmetries

It is well-known [37, 70–72] that U(1) gauge symmetries arise in F-theory when the elliptic 3-fold Y_3 exhibits a non-trivial Mordell-Weil group of rational sections. The U(1) gauge symmetries translate into global symmetries of the tensionless string. Although we will consider geometries with just one independent non-zero section, and hence models with a single U(1) factor, the generalization to multiple U(1) factors is immediate. We now explain how the index m of the elliptic genus $Z_{C_\beta}(\tau, \lambda_s, z)$,(3.13) or (3.17), with respect to the background field z is determined by the geometry of the Mordell-Weil group of the elliptic fibration.[11]

As already pointed out, this index is given by the terms of $f_{C_\beta}(\lambda_s, z)$ in (3.12) which are quadratic in z as a consequence of the general relation (C.24). The function $f_{C_\beta}(\lambda_s, z)$ in turn can be identified with the 't Hooft anomaly potential of the world-sheet theory of the string [55, 56, 59, 107].

The anomaly polynomial of a single $\mathcal{N} = (0, 4)$ world-sheet supersymmetric string has been already computed in [42, 54, 108–110], however for situations in which the the F-theory gauge group only contains non-abelian factors. In our notation it takes the form

$$I_4 = -\frac{1}{4}(\bar{K} \cdot C_\beta) \, p_1(T) - \frac{1}{2} \sum_I \mathrm{Tr} F_I^2 \, ([\Sigma_I] \cdot C_\beta) + \frac{1}{2} \, \mathrm{tr} \, F_{\mathcal{R}}^2$$
$$+ \frac{1}{2} \, \mathrm{tr} \, F_+^2 \left(\frac{1}{2} C_\beta \cdot C_\beta + \frac{1}{2} \bar{K} \cdot C_\beta \right) + \frac{1}{2} \, \mathrm{tr} \, F_-^2 \left(-\frac{1}{2} C_\beta \cdot C_\beta + \frac{1}{2} \bar{K} \cdot C_\beta \right) . \tag{3.41}$$

Here F_- and F_+ refer to the field strengths associated with the global symmetries $\mathrm{SU}(2)_-$ and $\mathrm{SU}(2)_+$ in (2.41), and $F_{\mathcal{R}}$ denotes the SU(2) R-symmetry of the 6d $N = (1, 0)$ theory that is inherited by the string.[12] Finally F_I is the field strength associated with a non-abelian gauge algebra $\mathfrak{g}_I$ of a 7-brane wrapping the divisor component Σ_I. The trace is normalised with respect to the trace in the fundamental representation via $\mathrm{Tr} F_I^2 = \frac{1}{\lambda_I} \mathrm{tr} F_I^2$, where λ_I is the Dynkin index (in particular, $\lambda_I = 1$ for $\mathfrak{su}(N)$).

To read off the index with respect to the global symmetry $\mathrm{SU}(2)_-$, one identifies I_4 with $f_{C_\beta}(\lambda_s, z)$ upon replacing

$$\frac{1}{2} \, \mathrm{tr} \, F_-^2 \to -\lambda_s^2, \qquad p_1(T) \to 0, \qquad \frac{1}{2} \, \mathrm{tr} \, F_I^2 \to 0, \qquad \frac{1}{2} \, \mathrm{tr} \, F_+^2 \to 0 . \tag{3.42}$$

This reproduces the result (3.14) for the value of the index with respect to λ_s. Similarly one can obtain the index with respect to non-abelian global symmetries of the string from the term $-\frac{1}{2} \sum_I \mathrm{Tr} F_I^2 \, ([\Sigma_I] \cdot C_\beta)$ in the anomaly polynomial.

---

[11]Already in [48, 106] BPS numbers for examples of elliptic fibrations with extra rational sections have been computed and a refinement with respect to the associated U(1) charges has been observed. Our results hold very generally for any type of rational section generating an abelian gauge symmetry in F-theory.

[12]In table 1, the fermions of the twisted hypermultiplet transform as a **2** of this symmetry, while all other fermions are singlets. This explains why the 't Hooft anomaly coefficient is a universal constant.

We now come to the case of interest for us where we have a single U(1) gauge symmetry. The 't Hooft anomaly polynomial of the stringy world-sheet contains an additional term

$$I_4|_{\text{U(1)}} = -\frac{1}{2}F^2\left(b \cdot C_\beta\right), \tag{3.43}$$

where $b = -\pi_*(\sigma(S) \cdot \sigma(S))$ is the height pairing of the section S as introduced in section 2.1. The precise form of this term including its normalization has been derived from first principles in [111]. The normalization is in conventions where the lowest charge with respect to U(1) is given by $\mathfrak{q}_{\min} = 1$. Note that, on the other hand, the index m of a weak Jacobi form is defined in a normalization where the weight lattice is $\mathbb{Z}/\sqrt{2}$, or equivalently, that the smallest co-weight has norm 2, see appendix C. Since the charges are determined by intersection numbers of the Shioda map $\sigma(S)$ with suitable fibral curves of Y_3, we must rescale the section by $\frac{1}{\sqrt{2}}$ and hence the height pairing b by $\frac{1}{2}$. Taking this into account, and making the replacement

$$\frac{1}{2}F^2 \rightarrow -z^2, \tag{3.44}$$

in analogy to (3.42), we conclude that the index with respect to the U(1) fugacity z is

$$m = -\frac{1}{2}\pi_*(\sigma(S) \cdot \sigma(S)) \cdot C_\beta = \frac{1}{2}b \cdot C_\beta \,. \tag{3.45}$$

Note that this quantity is guaranteed to be integer because b is an even divisor as can be seen from (2.13).

3.4 Synopsis: determining the maximal charge per excitation level

We now combine the arguments of the foregoing sections and outline how to practically determine (a subset of) the U(1) charge spectrum of the nearly tensionless heterotic string we consider. Recall that the string arises from a wrapped D3-brane on a shrinking curve C_0 in the base of an elliptic 3-fold, Y_3. As a first step one makes a general ansatz for the quasi-modular form $\Phi_{W,m}(\tau,z)$ in eq. (3.33) of weight $W = 6C_0 \cdot \bar{K} - 2 = 10$ and U(1) fugacity index $m = \frac{1}{2}C_0 \cdot b$. The ansatz consists of a linear combination of all monomials in the generators of the ring of quasi-modular Jacobi forms (see appendix C), i.e.,

$$\Phi_{10,m}(\tau,z) = \sum c_{a_0,a_1,a_2,a_3,a_4} E_2^{a_0}(\tau)\, E_4^{a_1}(\tau)\, E_6^{a_2}(\tau)\, \varphi_{0,1}^{a_3}(\tau,z)\, \varphi_{-2,1}^{a_4}(\tau,z) \tag{3.46}$$

with

$$2a_0 + 4a_1 + 6a_2 - 2a_4 = 6C_0 \cdot \bar{K} - 2 = 10, \qquad a_3 + a_4 = \frac{1}{2}C_0 \cdot b = m\,. \tag{3.47}$$

As furthermore explained in appendix C, the only source of quasi-modularity in $\Phi_{10,m}^{(0)}(\tau,z)$ are possible factors of $E_2(\tau)$. We have discussed in section 3.2 that if C_0 is a Mori cone generator and of genus zero, there cannot be an explicit dependence on $E_2(\tau)$, as in this case the prepotential is modular, not quasi-modular. This immediately implies that $\Phi_{10,m}(\tau,0) = E_4(\tau)E_6(\tau)$, which is a classic result for perturbative heterotic strings on $K3$ [61]. This is in particular the case where Y_3 is a fibration over a Hirzebruch surface, $\mathbb{F}_a$.

For non-perturbative heterotic strings, however, $\Phi_{10,m}(\tau, 0)$ may not reduce to $E_4(\tau)E_6(\tau)$ due to the possible appearance of extra terms involving $E_2(\tau)$; this reflects the fact that there can be a variable number of extra massless tensor fields which contribute to the gauge and gravitational anomalies in six dimensions.

In any concrete model, the ansatz (3.46) leaves us with a finite number of free parameters, which can be fixed by comparison with the genus-zero prepotential via eq. (3.33). Specifically, for the geometry we consider we have

$$\mathcal{F}_{C_0}^{(0)}(\tau, z) \equiv \sum N_{C_0}^{(0)}(n, r)\, q^n\, \xi^r \tag{3.48}$$
$$= -\frac{q}{\eta(\tau)^{24}}\, \Phi_{10,m}(\tau, z) \;\equiv\; -q\, Z_{\mathcal{K}}(\tau, z)\,.$$

It suffices to consider a small number of Gromow-Witten invariants $N_{C_0}^{(0)}(n, r)$ of Y_3, which can be straightforwardly computed by using mirror symmetry. Having obtained in this way an exact, analytic expression for the U(1)-graded elliptic genus $Z_{\mathcal{K}}(\tau, z)$, one can then go on and analyze the charge-versus-mass spectrum in order to investigate various weak gravity conjectures. This will be done explicitly for a number of examples in the following section 4 and in appendix E.

However, even without analyzing concrete models, one can deduce important general properties of the U(1) charges of the string excitations that are encoded in the elliptic genus. These properties follow from the modular properties of Jacobi forms. One way to make use of these is to note that the generators $\varphi_{0,1}(\tau, z)$ and $\varphi_{-2,1}(\tau, z)$ that appear in the ansatz (3.46), can be rewritten in terms of the Eisenstein-Jacobi series $E_{w,l}(\tau, z)$ via the identities (C.15). This property of the Eisenstein-Jacobi series allows us to establish a lower bound for the maximal charge of the states at any given mass level n, as discussed in detail in appendix D. The result of this analysis is the general behaviour that we mentioned already in section 2.5: namely, in a model with U(1) index $m = \frac{1}{2}C_0 \cdot b$, the maximal charge per excitation level n is bounded as follows:

$$\mathfrak{q}_{\mathrm{max}}^2(n) \geq (1 - \epsilon)4\, m\, n =: \beta_\epsilon\, n \qquad \forall\, n \geq N(\epsilon)\,. \tag{3.49}$$

This part of the analysis is completely general and does not rely on any explicit evaluation of the elliptic genus for a given model. On the other hand, in order to find the actual value of $N(\epsilon)$ for given ϵ, we must explicitly analyze the elliptic genus by following the steps outlined above. In particular, this will lead to lower, universal bounds for $\mathfrak{q}_{\mathrm{max}}^2(n)$ valid for all values of n and not just for n above a certain limit, $N(\epsilon)$.

In a similar vein one can make use of the fact that any weak Jacobi form, and thus in particular the elliptic genus, can be expanded as [104][13]

$$\varphi_{w,m}(\tau, z) \;=\; \sum_{\ell \in \mathbb{Z} \bmod 2m} h_\ell(\tau) \Theta_{m,\ell}(\tau, z)\,. \tag{3.50}$$

---

[13]Correspondingly one may describe the U(1) sector of the theory in terms of a free periodic boson H that is associated with a Kac-Moody current of level m of the form $J = i\sqrt{m}\partial H$. Its winding states are then counted by the theta functions below. This is analogous to the familiar U(1) current that is part of an $N = 2$ superconformal algebra on the world-sheet.

Here

$$\Theta_{m,\ell}(\tau, z) \;=\; \sum_{k \in \mathbb{Z}} q^{(\ell+2mk)^2/4m} \xi^{\ell+2mk} \tag{3.51}$$

is the standard theta function of index m and characteristic ℓ, and the coefficient functions $h_\ell(\tau)$ are certain vector-valued modular forms of weight $w - 1/2$. A particular distinguished subset of states is defined by $\ell = 0$, for which we can read off

$$n(k) = m\, k^2 \,, \qquad \mathfrak{q}_k = 2\, m\, k \,, \qquad k \in \mathbb{Z} \,. \tag{3.52}$$

These are extremal in the sense that they saturate the bound (3.49)

$${\mathfrak{q}_k}^2 \;=\; 4\, m\, n(k) \,, \tag{3.53}$$

which amounts to a vanishing "discriminant", $D \equiv 4mn - \mathfrak{q}^2 = 0$. Note that they form a sublattice with spacing $\Delta \mathfrak{q} = 2m$. Obviously, all these charges strictly satisfy the weak gravity bound (2.60), which translates to

$$\mathfrak{q}(n)^2 \;\geq\; 4\, m\, (n-1) \,. \tag{3.54}$$

The other states with $\ell \neq 0$ decompose into conjugacy classes of charge lattices shifted by ℓ and will generically not saturate the bound (3.53). Which precise subset of these states satisfies the weak gravity bound (3.54) depends on the model and may be determined by a case-by-case analysis. Recall, however, that given that the $\ell = 0$ states already satisfy the gravity bound, the $\ell \neq 0$ states are irrelevant for consistency of the theory with the Sublattice Weak Gravity Conjecture.

4 Examples of F-theory models with U(1) symmetry

In this section we explore various explicit 6-dimensional F-theory compactifications with a single U(1) gauge group. Our main purpose is to explicitly compute the elliptic genus of heterotic strings that arise from wrapped D3-branes on shrinking rational curves, C_0, of self-intersection zero on the base B_2 of an elliptic 3-fold Y_3. The results of this section serve as concrete examples for the advertised general properties of the massive, U(1) charged string excitations, in the limit of vanishing tension and global U(1) symmetry.

We will start in section 4.1 by analyzing two explicit models on the Hirzebruch surface $B_2 = \mathbb{F}_1$. In this case the string associated with the vanishing curve C_0 is the perturbative, critical heterotic string. We will see that indeed, starting from the geometry and applying mirror symmetry, the correct U(1) refinement of the elliptic genus is obtained for the perturbative heterotic string theory on the $K3$-surface $\mathcal{K}$, defined in (2.44). Exactly the same method can systematically be applied to models for more general toric bases B_2, e.g., any other Hirzebruch surfaces $\mathbb{F}_a$ and the toric del Pezzo surfaces, dP_r ($r \leq 3$). For more Hirzebruch examples, the results for $\mathbb{F}_2$ are collected in appendix E.

For a further illustration, a fibration over $B_2 = dP_2$ will be analyzed in section 4.2. On this non-Hirzebruch base the string associated with C_0 does not become a perturbative

heterotic string, and correspondingly we will observe that the elliptic genus is different from the familiar elliptic genus of the perturbative heterotic string.

In all these examples the type of U(1) gauge symmetry is engineered by realising the fiber of the elliptic 3-folds Y_3 as a most general hypersurface of degree 4 within the space $\text{Bl}_1\mathbb{P}^2_{112}$. Such an elliptic fibration has a Mordell-Weil group of rational sections of rank one [72]. It is cut out by the hypersurface

$$\hat{P} = sw^2 + b_0 s^2 u^2 w + b_1 suvw + b_2 v^2 w + c_0 s^3 u^4 + c_1 s^2 u^3 v + c_2 su^2 v^2 + c_3 uv^3 \,, \quad (4.1)$$

where $[u : v : w : s]$ are the homogeneous coordinates of the toric fibral ambient space $\text{Bl}_1\mathbb{P}^1_{112}$. The b_i and c_i are sections of certain bundles on B_2, the associated cohomology classes of which take the form

$$
\begin{aligned}
&[b_0] = \beta \,, &&[b_1] = \bar{K} \,, &&[b_2] = 2\bar{K} - \beta \,, &&[c_0] = 2\beta \,, \\
&[c_1] = \beta + \bar{K} \,, &&[c_2] = 2\bar{K} \,, &&[c_3] = 3\bar{K} - \beta \,, &&[c_4] = 4\bar{K} - 2\beta \,.
\end{aligned}
\quad (4.2)
$$

Apart from the concrete choice of base space B_2, the fibration is therefore specified by a class $\beta \in H^2(B_2, \mathbb{Z})$. It must satisfy the condition

$$0 \leq \beta \leq 2\bar{K} \quad (4.3)$$

in order for all b_i and c_i to be realisable as holomorphic polynomials on B_2. Here, an inequality between two divisor classes indicates that the difference of the divisors is effective. The class β also defines the transformation behaviour of the fiber coordinates, which are sections of the bundles

$$
\begin{aligned}
&u \in H^0(Y_3, \mathcal{L}_u)\,, \quad v \in H^0(Y_3, \mathcal{L}_u \otimes \mathcal{L}_s \otimes \mathcal{O}(\beta - \bar{K}))\,, \quad w \in H^0(Y_3, \mathcal{L}_u^2 \otimes \mathcal{L}_s \otimes \mathcal{O}(\beta))\,, \\
&s \in H^0(Y_3, \mathcal{L}_s)\,.
\end{aligned}
\quad (4.4)
$$

Let us denote by

$$S_0 = \{u = 0\}\,, \qquad S = \{s = 0\} \quad (4.5)$$

the divisor classes associated with the zero-section and an extra independent rational section, respectively. The extra section S is responsible for the U(1) gauge symmetry whose global limit we investigate. Since $\pi_*(S_0 \cdot S) = b_2$, the Shioda homomorphism (2.9) takes the form[14]

$$\sigma(S) = S - S_0 - \pi^{-1}(\bar{K} + b_2)\,, \quad (4.6)$$

where the last term is computed via (2.10) as

$$\mathfrak{D} = \pi^{-1}(\pi_*(S - S_0) \cdot S_0) = \pi^{-1}(b_2 + \bar{K})\,. \quad (4.7)$$

This divisor can be seen to generate the U(1) gauge symmetry in the 6d F-theory effective action. Of special importance for us is the height-pairing

$$b = -\pi_*(\sigma(S) \cdot \sigma(S)) = 2\bar{K} + 2[b_2] = 6\bar{K} - 2\beta\,. \quad (4.8)$$

---

[14]By slight abuse of notation, we denote by b_2 also the divisor associated with $b_2 = 0$ on B_2.

As reviewed in section 2.1, we can define a holomorphic curve C with class $[C] = b$ such that its volume determines the gauge coupling of U(1) as in (2.11).

The fibration Y_3 over B_2 contains two types of curve classes: first, as recalled in section 3.2, the classes in $C_\beta \in H_2(B_2)$ can be used to define the base curve classes $S_0 \cdot \pi^{-1}(C_\beta) \in H_2(Y_3, \mathbb{Z})$, both of which will be denoted by C_β by abuse of notation. The fibral curve classes, on the other hand, are given by the full fiber class $C_\mathcal{E}$ and one more fibral curve class C^{f} with the respective properties

$$S_0 \cdot C_\mathcal{E} = 1\,, \qquad S \cdot C_\mathcal{E} = 1 \qquad \Longrightarrow \qquad \sigma(S) \cdot C_\mathcal{E} = 0 \tag{4.9}$$

$$S_0 \cdot C^{\mathrm{f}} = 0\,, \qquad S \cdot C^{\mathrm{f}} = 1 \qquad \Longrightarrow \qquad \sigma(S) \cdot C^{\mathrm{f}} = 1\,. \tag{4.10}$$

There are two types of massless, charged hypermultiplets in the 6d F-theory compactification, which are localised in the fiber over the following two loci on B_2 [72],

$$
\begin{aligned}
C_I &= \{b_2 = 0\} \cap \{c_3 = 0\}: & \mathfrak{q}_I &= 2 \\
C_{II} &= V(f_1, f_2) \setminus C_I: & \mathfrak{q}_{II} &= 1\,,
\end{aligned}
\tag{4.11}
$$

where $V(f_1, f_2)$ denotes the vanishing ideal of the functions

$$
\begin{aligned}
f_1 &= -c_1 b_2^4 + b_1 b_2^3 c_2 + b_0 b_2^3 c_3 - b_1^2 b_2^2 c_3 - 2 b_2^2 c_2 c_3 + 3 b_1 b_2 c_3^2 - 2 c_3^3 \\
f_2 &= -c_3^2 b_0^2 + b_1 b_2 b_0^2 c_3 - b_1 b_2^2 b_0 c_1 + b_2^2 c_1^2 + b_1^2 b_2^2 c_0 - 4 b_1 b_2 c_0 c_3 + 4 c_0 c_3^2\,.
\end{aligned}
\tag{4.12}
$$

Both C_I and C_{II} represent a set of isolated points on B_2. Over each of these points, the elliptic fiber factorises into two components. One of the two fibral components over the locus C_I is a rational curve in class $2C^{\mathrm{f}}$, and one of the components over C_{II} is the rational curve in class C^{f}. M2-branes wrapping these isolated rational curves give rise to states of charge $\mathfrak{q}_I = 2$ and $\mathfrak{q}_{II} = 1$, respectively. These become massless in the F-theory limit and furnish the advertised charged hypermultiplets. Note that for the special value $\beta = 2\bar{K}$, the class $[b_2] = 0$ and hence the charge-two state locus C_I is absent.

4.1 Hirzebruch surface base $B_2 = \mathbb{F}_1$ with extra section

Let us first recall some topological properties of the Hirzebruch surfaces $\mathbb{F}_a$. A Hirzebruch surface can be viewed as a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$ of the form (2.42), i.e. as the total space of the projectivisation of the bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-a)$ over a base curve of genus zero. The space $H^2(\mathbb{F}_a, \mathbb{Z})$ of its divisor classes is spanned by the class h of the base $\mathbb{P}^1_b$ and the class f of the fiber $\mathbb{P}^1_f$ with intersection numbers

$$h \cdot h = -a\,, \quad f \cdot f = 0\,, \quad h \cdot f = 1\,, \tag{4.13}$$

and the anti-canonical divisor has the class

$$\bar{K} = 2h + (2 + a)f\,. \tag{4.14}$$

The Mori cone $\mathbf{M}$ and (the closure of) the Kähler cone $\mathbf{K}$, both viewed as a cone in $H^2(\mathbb{F}_a, \mathbb{R})$, are then described, respectively, as

$$\mathbf{M}(\mathbb{F}_a) = \mathrm{Span} \langle f, h \rangle\,, \tag{4.15}$$

$$\overline{\mathbf{K}(\mathbb{F}_a)} = \mathrm{Span} \langle f, h + af \rangle\,, \tag{4.16}$$

	ν_{z_1}	ν_{z_2}	ν_{z_3}	ν_{z_4}	ν_u	ν_v	ν_w	ν_s
$U(1)_h$	0	1	0	1	0	$x-2$	x	0
$U(1)_f$	1	a	1	0	0	$y-(2+a)$	y	0
$U(1)_U$	0	0	0	0	1	1	2	0
$U(1)_S$	0	0	0	0	0	1	1	1

Table 2. GLSM charges of the toric coordinates of the $\mathrm{Bl}_1\mathbb{P}^2_{112}$ fibration over $\mathbb{F}_a$.

where $\mathrm{Span}\langle\cdot\rangle$ denotes the set of non-negative linear combinations of the indicated divisor classes.

Let us begin by specifying the toric data for the $\mathrm{Bl}_1\mathbb{P}^2_{112}[4]$ fibrations with base $B_2 = \mathbb{F}_a$ in terms of the abelian GLSM charges. The divisor class $\beta \in H^2(\mathbb{F}_a, \mathbb{Z})$ introduced above is parametrized as

$$\beta(x,y) = x\,h + y\,f\,. \tag{4.17}$$

The constraint (4.3) implies that $\beta(x,y)$ has to take values within the finite range

$$0 \leq x \leq 4\,, \qquad 0 \leq y \leq 4+2a\,. \tag{4.18}$$

If we introduce toric coordinates ν_{z_i} for the base space and ν_u, ν_v, ν_w, ν_s for the fiber, then (4.4) translates into the GLSM charges shown in table 2.

The inverse gauge coupling of the gauge group $U(1)$ is controlled by the volume of the curve

$$C = b = 6\bar{K} - 2\beta = (12 - 2x)h + (12 + 6a - 2y)f\,. \tag{4.19}$$

We are interested in the limit (2.17) in Kähler moduli space, where the volume of C goes to infinity. For a base with $h^{1,1}(B_2) = 2$ this limit is conveniently parametrized as in (2.36). We therefore need to identify a boundary ray J_0 of $\mathbf{K}(\mathbb{F}_a)$ such that $\int_{\mathbb{F}_a} J_0 \wedge J_0 = 0$ and $\int_C J_0 \neq 0$. Since the two boundary rays f and $h + af$ have self-intersection numbers 0 and a, respectively, the only possibility to satisfy the first requirement is to take $J_0 = f$ (when $a = 0$ both f and h can be taken as J_0, but in this case the fibration is trivial so that f and h may be treated on an equal footing). This choice also satisfies the second constraint, that is,

$$\int_C J_0 = 12 - 2x \geq 4 \tag{4.20}$$

within the range (4.18) so that $\int_C J_0 \neq 0$. Altogether we are therefore interested in the limit

$$J = \frac{1}{\sqrt{\frac{1}{2} + \frac{a}{8\tilde{t}^2}}} \left(\tilde{t}\,f + \frac{1}{2\tilde{t}}\,(h + af) \right)\,, \tag{4.21}$$

for $t := \tilde{t}(\frac{1}{2} + \frac{a}{8\tilde{t}^2})^{-1/2} \to \infty$. In agreement with the general discussion of section 2.2, the class $J_0 = f$ is the class of a holomorphic curve

$$C_0 = f\,. \tag{4.22}$$

This is, in fact, the only curve class (modulo integer multiples) whose volume with respect to J_0 vanishes as $t \to \infty$.

With this preparation, the elliptic genus Z_{C_0} of the nearly tensionless heterotic string associated with C_0 can be determined by following the procedure that was outlined in section 3.4:

- For a Hirzebruch surface $\mathbb{F}_a$ an important simplification occurs, as we recall from section 3.2: since C_0 is a Mori cone generator, the full elliptic genus Z_{C_0} is given by (3.13). Since $L = g(C_0) = 0$, the numerator is a weak Jacobi form of weight $W = 6C_0 \cdot \bar{K} - 2 = 10$ and U(1) fugacity index

$$m = \frac{1}{2}C_0 \cdot b = 6 - \mathrm{x}, \qquad (4.23)$$

so in total we have:

$$Z_{C_0}(\tau, \lambda_s, z) = \frac{1}{\eta^{24}(\tau)} \frac{\Phi_{10,6-\mathrm{x}}(\tau, z)}{\varphi_{-2,1}(\tau, \lambda_s)}. \qquad (4.24)$$

- Since C_0 is a Mori cone generator and thus the numerator does not depend on λ_s, the genus-zero topological string partition function $\mathcal{F}_{C_0}$ is given by

$$\mathcal{F}_{C_0}^{(0)}(\tau, z) = -\frac{q}{\eta^{24}(\tau)} \Phi_{10,6-\mathrm{x}}(\tau, z) \qquad (4.25)$$

$$= -q Z_{\mathcal{K}}(\tau, z), \qquad (4.26)$$

where $Z_{\mathcal{K}}(\tau, z)$ is the U(1)-refined heterotic elliptic genus associated to the elliptically fibered $K3$ surface $\mathcal{K}$ defined in eq. (2.44). The minus sign in (4.25) originates from the minus sign of the leading term in the expansion $\varphi_{-2,1}(\tau, \lambda_s) = -\lambda_s^2 (1 + \mathcal{O}(\lambda_s^2))$, which is needed for matching the r.h.s. of (4.24) to the genus-zero free energy.

- As per (3.46), we can now make an ansatz for $\Phi_{10,6-\mathrm{x}}(\tau, z)$ as a sum of products of $E_4(\tau)$, $E_6(\tau)$, $\varphi_{0,1}(\tau, z)$, $\varphi_{-2,1}(\tau, z)$ of total weight 10 and fugacity index $6 - \mathrm{x}$. To fix the finite number of coefficients in this ansatz, we compare it with a finite number of genus-zero Gromov-Witten invariants of curve classes of the form

$$\Gamma_{C_0}(n, r) := C_0 + nC_{\mathcal{E}} + rC^{\mathrm{f}}, \quad n = 0, 1 \ldots, \qquad (4.27)$$

for a small number of choices of n. The fibral classes $C_{\mathcal{E}}$ and C^{f} have the properties (4.9) and (4.10). The expansion (3.24) of the generating function

$$\mathcal{F}_{C_0}^{(0)}(\tau, z) = \sum_{n \geq 0} \sum_{|r| \leq \mathfrak{q}_{\max}(n)} N_{C_0}^{(0)}(n, r) q^n \xi^r \qquad \text{with } q = e^{2\pi i \tau}, \ \xi = e^{2\pi i z} \qquad (4.28)$$

can then be obtained to the chosen finite order, where $\mathfrak{q}_{\max}(n)$ denotes the maximal power of ξ appearing in the expansion at a given level n in q. By computing a sufficient number of low-degree Gromov-Witten invariants $N_{C_0}^{(0)}(n, r)$, one can proceed to fix the coefficients in the ansatz for $\Phi_{10,6-\mathrm{x}}$. This provides an analytic expression for $\mathcal{F}_{C_0}^{(0)}$ and thus for $Z_{\mathcal{K}}(\tau, z)$, as well as for its spin-refined cousin, $Z_{C_0}(\tau, \lambda_s, z)$.

In the remainder of this subsection we will consider $a = 1$, that is $B_2 = \mathbb{F}_1$. Although one may in principle analyze all possible choices of β within the range (4.18), we will focus here on the two models with $(x, y) = (4, 4)$ and $(4, 6)$, respectively. Some further models over $\mathbb{F}_2$ can be found in appendix E.

Model 1: Hirzebruch base $\mathbb{F}_1$ with extra section, for $(x, y) = (4, 4)$. Some results for the Hirzebruch base $\mathbb{F}_1$ with $(x, y) = (4, 4)$ have already been presented in section 2.5 to exemplify the general behaviour of the charge spectrum of the nearly tensionless heterotic string. Using the toric data as summarized in appendix E.1, we can compute via mirror symmetry a collection of low order genus-zero Gromov-Witten invariants. These assemble into the expansion (4.28) as follows:

$$
\begin{aligned}
\mathcal{F}_{C_0}^{(0)}(\tau, z) &\equiv \sum N_{C_0}^{(0)}(n, r)\, q^n \xi^r \\
&= -2 + \left(216 + 128\xi^{\pm 1} + 4\xi^{\pm 2}\right) q \\
&\quad + \left(121964 + 70528\xi^{\pm 1} + 9808\xi^{\pm 2} + 128\xi^{\pm 3} - 2\xi^{\pm 4}\right) q^2 \\
&\quad + \mathcal{O}(q^3)\,.
\end{aligned}
\tag{4.29}
$$

Here and in the sequel we use the abbreviation

$$
\xi^{\pm n} := \xi^n + \xi^{-n}\,.
\tag{4.30}
$$

These Gromov-Witten invariants are more than enough to determine the unknown coefficients in the ansatz (3.46), and this yields:

$$
\begin{aligned}
\mathcal{F}_{C_0}^{(0)}(\tau, z) &= -q\, Z_{\mathcal{K}}(\tau, z) \\
&= \frac{q}{\eta^{24}} \left(\frac{1}{54} E_4^3 \varphi_{-2,1}\varphi_{0,1} + \frac{1}{108} E_6^2 \varphi_{-2,1}\varphi_{0,1} - \frac{1}{72} E_4^2 E_6 \varphi_{-2,1}^2 - \frac{1}{72} E_4 E_6 \varphi_{0,1}^2 \right).
\end{aligned}
\tag{4.31}
$$

As a non-trivial consistency check, we have confirmed that the Gromov-Witten invariants $N_{C_0}^{(0)}(2, r)$ at order $n = 2$ in q are correctly reproduced. Moreover, in the un-refined limit $z = 0$, for which $\varphi_{-2,1} = 0$ and $\varphi_{0,1} = 12$, the well-known expression (3.4) for the conformal field theoretic elliptic genus $Z_{K3}(\tau)$ is recovered.

Based on the analytic expression for $\mathcal{F}_{C_0}^{(0)}(\tau, z)$, the maximal charge per excitation level, $\mathfrak{q}_{\max}(n)$, can easily be extracted at a given order n in q. See figure 1. We confirm the general behaviour as announced already in section 2.5, namely that there exist superextremal states whose charges lie on a sublattice of the charge lattice and satisfy the Sublattice Weak Gravity bound.

Let us now interpret the expansion coefficients in (4.29) in terms of a nearly tensionless, weakly coupled heterotic string. The left-moving excitation number $n = 0$ is associated with the tachyonic mode of the heterotic string, which is not physical because of lack of level-matching. The sector at $n = 1$ is therefore the first physical sector and corresponds to the U(1)-refined index of the massless modes. This sector must reproduce the massless sector of the heterotic string, which in turn has to agree with the massless sector of the F-theory

model itself. More specifically, at the massless level, the U(1)-refined elliptic genus must reproduce the chiral index which counts the difference of anti-chiral and chiral fermions,

$$(n_- - n_+)_{\mathfrak{q}} = 2(n_V + 3 + n_T - n_H)_{\mathfrak{q}}. \tag{4.32}$$

Here n_V, n_T and n_H denote the numbers of vector multiplets, tensor multiplets, and hypermultiplets, respectively. The universal contribution of $+3$ comes from the gravitino. The subscript indicates that we must distinguish the states according to the U(1) charges $\mathfrak{q}$. Of course, the gravitino and the tensor multiplets are always uncharged under the U(1) symmetry we consider.

The coefficients of the term q, $q\xi$ and $q\xi^2$ in (4.29) therefore represent, up to the overall sign flip and the shift of q-power in (4.26), the piece in the heterotic elliptic genus counting such massless states with U(1) charge 0, 1 and 2, respectively. Indeed, these numbers agree precisely with our expectations based on F-theory/heterotic duality. Consider first the uncharged sector. In the dual F-theory, the relevant combination of uncharged massless fields adds up to

$$2\,(n_V + 3 + n_T - n_H)_0 = 2\,(h^{1,1}(Y_3) - h^{2,1}(Y_3)) = \chi(Y_3), \tag{4.33}$$

where $\chi(Y_3)$ is the topological Euler characteristic of Y_3. This follows from the general relations between the Hodge numbers of the elliptic fibration and the massless uncharged spectrum of F-theory on Y_3 [112],

$$h^{1,1}(Y_3) = 1 + \mathrm{rk}(G) + h^{1,1}(B_2) = 1 + n_{V_0} + n_T + 1, \qquad h^{2,1}(Y_3) = n_{H_0} - 1. \tag{4.34}$$

The first equation uses that the uncharged vectors correspond to the Cartan generators of the gauge group G, whose number agrees with the rank of the gauge group, and the second equation follows from the fact that the uncharged moduli are given by the complex structure moduli plus one universal modulus. Explicit computation via PALP using the toric data listed in appendix E.1 gives the Hodge numbers

$$h^{1,1}(Y_3) = 4 \qquad h^{2,1}(Y_3) = 112, \qquad \chi(Y_3) = -216. \tag{4.35}$$

In particular, the Euler number agrees with (4.29) if we take into account the overall sign. Furthermore, the expansion coefficients in front $q\,\xi$ and $q\,\xi^2$ equal the number of points in the set C_{II} and C_I introduced in (4.11), which count the number of half-hypermultiplets of charge 1 and 2, respectively. Taking into account also the half-hypermultiplets of charge -1 and -2, encoded in the coefficients of $q\,\xi^{-1}$ and $q\,\xi^{-2}$, gives the number of charged hypermultiplets. Since the gauge group is abelian, these are the only charged states. This fact has the following geometrical meaning: by mirror symmetry, the expansion coefficients represent the BPS numbers for M2-branes wrapping the curves

$$q\,\xi: \ \Gamma_{C_0}(1,1) := C_0 + C_{\mathcal{E}} + C^{\mathrm{f}} \tag{4.36}$$

$$q\,\xi^2: \Gamma_{C_0}(1,2) := C_0 + C_{\mathcal{E}} + 2C^{\mathrm{f}} \tag{4.37}$$

in the dual 5d M-theory compactification. As mentioned around (4.11), the classes C^{f} and $2C^{\mathrm{f}}$ are associated with isolated rational curves localised in the fiber over the base loci C_{II}

and C_I. The number of isolated curves in these two classes hence equals the number of points in the sets C_{II} and C_I. These numbers coincide exactly with the BPS numbers for the curves $C_{\mathcal{E}} + 2C^{\mathrm{f}}$ and $C_{\mathcal{E}} + C^{\mathrm{f}}$ combined with the base curve C_0, i.e. the Gromov-Witten invariants for the two classes $\Gamma_{C_0}(1,2)$ and $\Gamma_{C_0}(1,1)$, respectively.

In summary, we confirm that the U(1)-refined elliptic genus at level $n = 1$ precisely counts the index (4.32) of massless multiplets in F-theory, which agrees, by duality, with the corresponding index in the massless sector of the perturbative heterotic string on the $K3$ surface $\mathcal{K}$. Extrapolating this to higher excitation levels we obtain information about the tower of charged excitations, at least in the tensionless, perturbative limit.

Model 2: Hirzebruch base $\mathbb{F}_1$ with extra section, for $(\mathrm{x}, \mathrm{y}) = (4, 6)$. This choice of parameters is special in that the class $[b_2]$ in the defining equation (4.1) is now trivial. From the discussion after (4.11) this means that the 6d F-theory model has only one type of massless charged hypermultiplets, of charge $\mathfrak{q}_{II} = 1$, localised over the point set C_{II}. We will see that this has interesting consequences also for the excitation numbers of the tensionless heterotic string. Using the toric data given in in appendix E.2, the generating function (4.28) for low-lying Gromov-Witten invariants can be explicit computed as follows:

$$
\begin{aligned}
\mathcal{F}_{C_0}^{(0)}(\tau, z) &\equiv \sum N_{C_0}^{(0)}(n, r)\, q^n \xi^r \\
&= -2 + \left(288 + 96\xi^{\pm 1}\right) q \\
&\quad + \left(123756 + 69280\xi^{\pm 1} + 10192\xi^{\pm 2} + 96\xi^{\pm 3} - 2\xi^{\pm 4}\right) q^2 + \mathcal{O}(q^3),
\end{aligned}
\tag{4.38}
$$

where we recall the notation (4.30). As expected, the expansion coefficient for $q\,\xi^{\pm 2}$ vanishes, because for $(\mathrm{x}, \mathrm{y}) = (4, 6)$ there are no rational curves in the class $2C^{\mathrm{f}}$. The absence of these excitations hints at a non-trivial cancellation, which can indeed occur only for special linear combinations of the Jacobi forms in the ansatz which we will now determine.

Since the U(1) fugacity index for the model with $(\mathrm{x}, \mathrm{y}) = (4, 6)$ continues to be given by $m = 6 - \mathrm{x} = 2$, we make the same ansatz for the elliptic genus in terms of weak Jacobi forms as before. Comparison with (4.38) at levels $n = 0, 1$ in q fixes the ansatz uniquely as follows:

$$
\begin{aligned}
\mathcal{F}_{C_0}^{(0)}(\tau, z) &= -q\, Z_{\mathcal{K}}(\tau, z) \\
&= \frac{q}{\eta(\tau)^{24}} \left(-\frac{1}{72} E_4(\tau)^2 E_6(\tau) \varphi_{-2,1}(\tau, z)^2 + \frac{7}{432} E_4(\tau)^3 \varphi_{-2,1}(\tau, z) \varphi_{0,1}(\tau, z) \right. \\
&\quad \left. + \frac{5}{432} E_6^2(\tau) \varphi_{-2,1}(\tau, z) \varphi_{0,1}(\tau, z) - \frac{1}{72} E_4(\tau) E_6(\tau) \varphi_{0,1}^2(\tau, z) \right).
\end{aligned}
\tag{4.39}
$$

The Gromov-Witten invariants $N_{C_0}^{(0)}(2, r)$ at level $n = 2$ in q can be confirmed to be correctly reproduced. As before, the elliptic genus $Z_{K3}(\tau)$ in (3.4) is recovered in the limit $z = 0$.

The maximal charge $\mathfrak{q}_{\max}(n)$ can easily be determined at a given level n in q. See figure 3 for a plot.. Evidently it displays the same rough features as for the first example, in particular a sub-lattice of charges formed by the isolated points peeking above the solid line associated with extremal black holes.
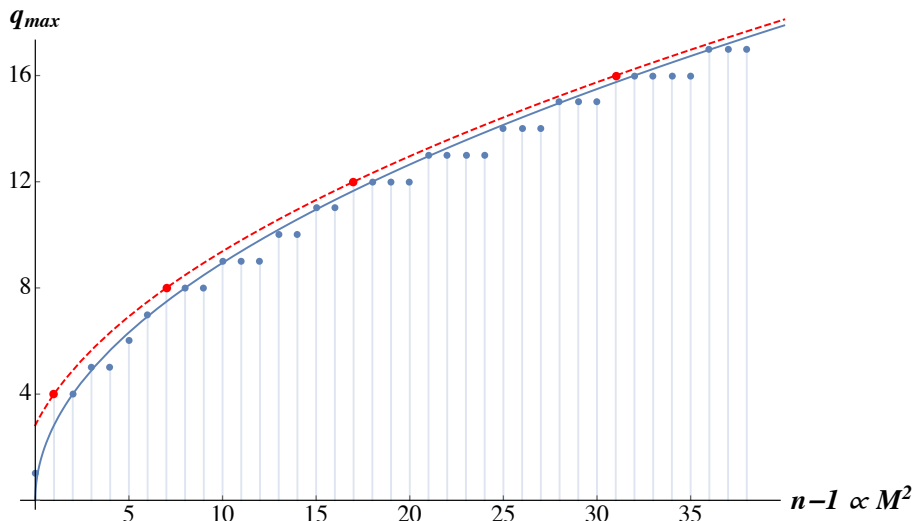
Figure 3. Maximal charge $\mathfrak{q}_{\max}(n)$ for a $\mathrm{Bl}_1\mathbb{P}^2_{112}[4]$ fibration over $\mathbb{F}_1$ with $(\mathrm{x},\mathrm{y}) = (4, 6)$. Again we oberve that maximally superextremal excitations exist whose charges form a sublattice of index $2m = 4$. However, in contrast to the previous example, the set of all superextremal states does not populate the full charge lattice.

Furthermore, the elliptic genus as encoded in the Gromov-Witten invariants at order q correctly reproduces the massless spectrum of the nearly tensionless heterotic string as defined via the geometry in F-theory. For the uncharged states this follows from

$$h^{1,1}(Y_3) = 4 \qquad h^{2,1}(Y_3) = 148\,, \qquad \chi(Y_3) = -288\,, \tag{4.40}$$

while the expansion coefficients of $q\xi$ and $q\xi^{-1}$ correctly count the number of half-hypermultiplets of charges $\mathfrak{q} = +1$ and $\mathfrak{q} = -1$, respectively. Hence the number of charged massless hypermultiplets with $|\mathfrak{q}| = 1$ is 96.

4.2 Del Pezzo surface base $B_2 = dP_2$ with extra section

We will now discuss an example of a U(1) fibration over the base dP_2. Since this base is not rationally fibered, the structure of the nearly tensionless heterotic string is more subtle, as explained in detail in section 2.4. In particular the string won't be perturbative due to the appearance of small instantons which lead to an extra massless tensor field.

Let us first recall some topological properties of the del Pezzo surfaces dP_r. They are constructed by blowing up a $\mathbb{P}^2$ at r generic points ($r = 0, \cdots, 8$). The space $H^2(dP_r, \mathbb{Z})$ of its divisor classes is spanned by the hyperplane class l and the blow-up classes e_i ($i = 1, \ldots, r$). Their non-zero intersection numbers are given as

$$l \cdot l = 1\,, \quad e_i \cdot e_i = -1 \ (i = 1, \cdots, r)\,, \tag{4.41}$$

and the anti-canonical divisor is

$$\bar{K} = 3l - \sum_{i=1}^{r} e_i\,. \tag{4.42}$$

	ν_{z_1}	ν_{z_2}	ν_{z_3}	ν_{z_4}	ν_{z_5}	ν_u	ν_v	ν_w	ν_s
$U(1)_l$	1	1	1	0	0	0	$x - 3$	x	0
$U(1)_{l+e_1}$	0	1	0	1	0	0	$x - y_1 - 2$	$x - y_1$	0
$U(1)_{l+e_2}$	1	0	0	0	1	0	$x - y_2 - 2$	$x - y_2$	0
$U(1)_U$	0	0	0	0	0	1	1	2	0
$U(1)_S$	0	0	0	0	0	0	1	1	1

Table 3. GLSM data for $\mathrm{Bl}_1\mathbb{P}^2_{112}$ fibration over dP_2.

The generators for the Mori cone $\mathbf{M}$ and (the closure of) the Kähler cone $\mathbf{K}$ look rather different for different values of r. Suffice it here to state them for the case of $r = 2$,

$$\mathbf{M}(dP_2) = \mathrm{Span}\langle l - e_1 - e_2, e_1, e_2 \rangle \,, \tag{4.43}$$

$$\overline{\mathbf{K}(dP_2)} = \mathrm{Span}\langle l, l - e_1, l - e_2 \rangle \,. \tag{4.44}$$

Let us now move on to the toric data for $\mathrm{Bl}_1\mathbb{P}^2_{112}[4]$ fibrations over dP_2 in terms of the abelian GLSM charges. The class β appearing in (4.2) is parameterized as

$$\beta(x, y_1, y_2) = xl - y_1 e_1 - y_2 e_2 \tag{4.45}$$

and enters the divisor classes of the toric coordinates as displayed in table 3. Both β and $2\bar{K} - \beta$ have to be effective, which implies that $\beta(x, y_1, y_2)$ must take values in the following finite range,

$$0 \leq x \leq 6, \qquad x - 4 \leq y_i \leq x, \quad \text{for } i = 1, 2 \,. \tag{4.46}$$

We are interested in taking the limit in Kähler cone where the volume of the height pairing

$$C = b = 6\bar{K} - 2\beta = (18 - 2x)l - (6 - 2y_1)e_1 - (6 - 2y_2)e_2 \tag{4.47}$$

associated with the $U(1)$ gauge theory becomes infinite. To realize the limiting base Kähler form (2.17), we need to take a boundary ray J_0 of $\mathbf{K}(dP_2)$ such that $\int_{dP_2} J_0 \wedge J_0 = 0$ and $J_0 \cdot C \neq 0$. Given the most general ansatz for J_0 in terms of the Kähler cone generators,

$$J_0 = al + \sum_{i=1,2} a_i(l - e_i) \,, \quad a, a_i \geq 0 \,, \tag{4.48}$$

we have

$$\int_{dP_2} J_0 \wedge J_0 = (a + a_1 + a_2)^2 - a_1^2 - a_2^2 \tag{4.49}$$

$$= a^2 + 2a(a_1 + a_2) + 2a_1 a_2 \,, \tag{4.50}$$

which only vanishes if $a = 0$ and one of the two a_i vanishes. Without loss of generality we may thus take

$$J_0 = l - e_1 \,. \tag{4.51}$$

Since $(l - e_1) \cdot b = 24 - 2\mathrm{x} - 2\mathrm{y}_1$, this ray has non-zero intersection with the height-pairing unless $\mathrm{x} = \mathrm{y}_1 = 6$, in which case it is not possible to take a limit with $\mathrm{vol}(C) \to \infty$ while keeping $\mathrm{vol}(B_2) = 1$.

With the choice (4.48) fixed as (4.51), let us now find the curves of volume zero. These correspond to those curves that do not intersect J_0. Given an effective curve of the general form

$$C_0 = c(l - e_1 - e_2) + \sum_{i=1}^{2} c_i e_i \,, \quad c, c_i \geq 0 \,, \tag{4.52}$$

its volume, with respect to J_0, follows as

$$C_0 \cdot (l - e_1) = c_1 \,. \tag{4.53}$$

The vanishing volume curves in the limit in Kähler cone are therefore in the class $C_0 = c(l - e_1 - e_2) + c_2 e_2$, with self-intersection number $C_0 \cdot C_0 = -(c - c_2)^2$. In agreement with the general theory of section 2.2, there exists precisely one self-intersection zero curve class C_0 (up to multiple wrapping), corresponding to $c = c_2$, and the lowest wrapping number is associated with the curve

$$C_0 \equiv J_0 = l - e_1 \,. \tag{4.54}$$

It is, however, interesting to keep in mind that there may be additional massless charged states from D3 branes wrapping the other possible curves of negative self-intersection obtained by choosing $c \neq c_2$ in the ansatz for C_0. This realizes the third point in section 2.2. The U(1) fugacity index m is parameterised as

$$m = \frac{1}{2} C_0 \cdot b = 6 - \mathrm{x} + \mathrm{y}_1 \,. \tag{4.55}$$

In the sequel we focus on the particular example of a fibration corresponding to the parameters $(\mathrm{x}, \mathrm{y}_1, \mathrm{y}_2) = (6, 2, 2)$, such that the U(1) fugacity index is $m = 2$.

By a mirror symmetry computation similar to the one of the previous section, the expansion of the generating function for the U(1)-refined genus-zero Gromov-Witten invariants up to order $n = 3$ in q reads

$$\begin{aligned}
\mathcal{F}_{C_0}^{(0)} = {} & -2 + \left(252 + 84\xi^{\pm 1} \right) q \\
& + \left(116580 + 65164\xi^{\pm 1} + 9448\xi^{\pm 2} + 84\xi^{\pm 3} - 2\xi^{\pm 4} \right) q^2 \\
& + \left(6238536 + 3986964\xi^{\pm 1} + 965232\xi^{\pm 2} + 65164\xi^{\pm 3} + 252\xi^{\pm 4} \right) q^3 \\
& + \mathcal{O}(q^4) \,.
\end{aligned} \tag{4.56}$$

Unlike for the models on the Hirzebruch surfaces, the genus-zero prepotential is expected to be not a modular function by itself, but only a quasi-modular function. This is in line with the general discussion of the modular properties of the topological string prepotential in section 3. The failure of $\mathcal{F}_{C_0}^{(0)}$ to be modular is measured by the modular anomaly equation [49, 53, 103],

$$\frac{\partial \mathcal{F}_{C_0}^{(0)}}{\partial E_2} = \frac{1}{24} \sum_{C_1 + C_2 = C_0} (C_1 \cdot C_2) \, \mathcal{F}_{C_1}^{(0)} \mathcal{F}_{C_2}^{(0)} \,, \tag{4.57}$$

where the non-trivial E_2-dependence appears due to possible splits of C_0 into two effective pieces. Given the Mori cone (4.43), it is straightforward to see that $C_0 = l - e_1$ can only split into the sum of $C_1 = l - e_1 - e_2$ and $C_2 = e_2$. Interestingly, both C_1 and C_2 have self-intersection number -1 and hence correspond to an E-string, of which $\mathcal{F}^{(0)}$ is known not to depend on E_2. Alternatively, since both C_1 and C_2 are Mori cone generators, they cannot split into two non-trivial effective pieces. Therefore, $\mathcal{F}^{(0)}_{C_{i=1,2}}$ cannot have a non-trivial modular anomaly. This means that the r.h.s. of (4.57) does not involve E_2 and hence that $\mathcal{F}^{(0)}_{C_0}$ can depend on E_2 at worst linearly. This considerably simplifies the ansatz (3.46) for $\mathcal{F}^{(0)}_{C_0}$.

Upon requiring that the Gromov-Witten invariants in (4.56) match the ansatz (3.46), the coefficients are uniquely fixed and the following generating function is found:

$$
\begin{aligned}
\mathcal{F}^{(0)}_{C_0}(\tau, z) &= -q\, Z_{\mathcal{K}}(\tau, z) \\
&= \frac{q}{\eta^{24}} \left(-\frac{23}{1728} E_4^2 E_6 \varphi_{-2,1}^2 + \frac{1}{64} E_4^3 \varphi_{-2,1}\varphi_{0,1} + \frac{19}{1728} E_6^2 \varphi_{-2,1}\varphi_{0,1} - \frac{23}{1728} E_4 E_6 \varphi_{0,1}^2 \right. \\
&\quad \left. + E_2 \left(-\frac{1}{1728} E_6^2 \varphi_{-2,1}^2 + \frac{1}{864} E_4 E_6 \varphi_{-2,1}\varphi_{0,1} - \frac{1}{1728} E_4^2 \varphi_{0,1}^2 \right) \right).
\end{aligned}
\tag{4.58}
$$

As expected, when switching off the U(1) background field, $z = 0$, the familiar, conformal field theoretic elliptic genus $Z_{K3}(\tau)$ in (3.4) is not reproduced, but rather we arrive at[15]

$$
Z_{\mathcal{K}}(\tau, 0) \ = \ Z_{K3}(\tau) + \frac{1}{\eta^{24}} \left(\frac{1}{12} E_2 E_4{}^2 - \frac{1}{12} E_4 E_6 \right) \ = \ \frac{2}{q} - 420 - 265968 q + \dots \tag{4.59}
$$

This makes it evident that, unlike for models on a Hirzebruch base, we do not obtain a perturbative heterotic dual. However, due to existence of a $K3$ fibration of Y_3, as argued in point 4 of section 2.2, we can still invoke Type IIA-heterotic duality, albeit with a more complicated heterotic side. In particular, the heterotic dual involves one more tensor multiplet, which is associated with a heterotic 5-brane defect which defies a fully perturbative description. As we see, this leaves its imprint in the spectrum of the heterotic theory. In fact, we observe that the genus-zero contribution to the elliptic genus correctly reproduces the chiral index of the F-theory model at the massless level: the expansion coefficient in eq. (4.56) at level $q\xi^0$ again agrees with the Euler characteristic of the F-theory 3-fold Y_3 because

$$
h^{1,1}(Y_3) = 5 \qquad h^{2,1}(Y_3) = 131\,, \qquad \chi(Y_3) = -252\,. \tag{4.60}
$$

Furthermore, the expansion coefficient of $q\,\xi$ and $q\,\xi^{-1}$ agrees with the number of $\mathfrak{q} = 1$ and $\mathfrak{q} = -1$ half-hypermultiplets, respectively. These are the only massless charged states of the F-theory model. This is because the fibration we are considering does not contain any charge-two loci C_I, like for the Hirzebruch model with $(\mathrm{x}, \mathrm{y}) = (4, 6)$ discussed before. Summing over all massless states gives the chiral index at the massless level,

$$
\sum_{\mathfrak{q}} (n_- - n_+)_{\mathfrak{q}} = 2(n_V + n_T + 3 - n_H) = -420\,, \tag{4.61}
$$

---

[15]Note that in this case, the full spin-refined elliptic genus $Z_{C_0}(\tau, \lambda_s, z)$ is not completely fixed by the genus-zero prepotential, but this is not of interest in the present context.
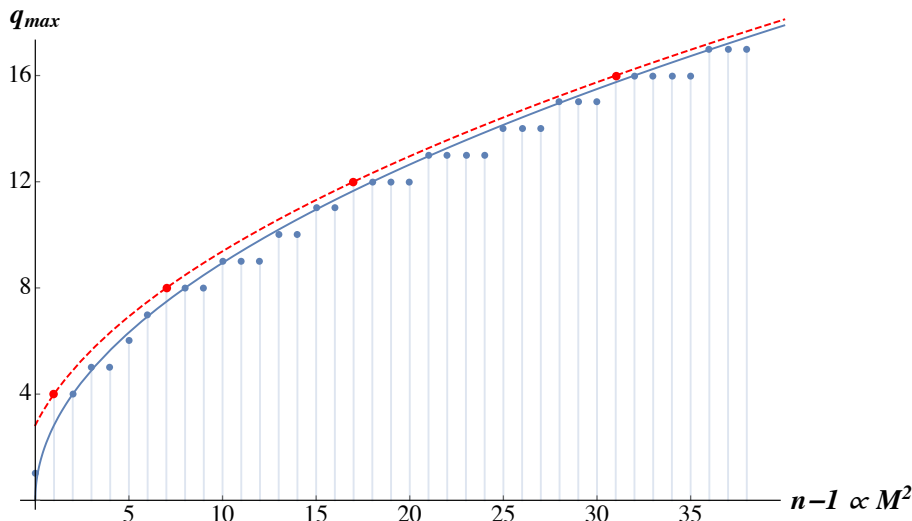
Figure 4. Maximal charge $\mathfrak{q}_{\max}(n)$ for a $\mathrm{Bl}_1\mathbb{P}^2_{112}[4]$ fibration over dP_2, with $(\mathrm{x}, \mathrm{y}_1, \mathrm{y}_2) = (6, 2, 2)$. The general features of this strongly coupled theory are completely in line with the findings of the previous examples, which correspond to weakly coupled strings.

which reproduces the constant term in eq. (4.59), so all is consistent. As stressed, this result differs from the value -480 which pertains to the perturbative heterotic string on K3 in eq. (3.4). Indeed, the model on dP_2 under consideration is related by a non-perturbative tensor transition [38, 43, 113] with the model with $(\mathrm{x}, \mathrm{y}) = (4, 6)$ on $\mathbb{F}_1 = dP_1$. Starting from dP_1 we obtain dP_2 by a single blowup, which increases the number of tensors by one. At the same time, the number of vector multiplets is $n_V = 1$ in both cases. The 6d gravitational anomaly relation

$$n_H - n_V + 29 n_T = 273 \tag{4.62}$$

requires that the total number of hypermultiplets in going from the dP_1 to the dP_2 model decreases by 29. In total this indeed leads to the correct shift of the index by

$$2(\Delta n_T - \Delta n_H) = 2(1 - (-29)) = 60 \,. \tag{4.63}$$

We view the fact that the elliptic genus (4.58) correctly reproduces the index of the massless states as evidence that it has been computed correctly, and thus that it also encodes the proper charge-to-mass spectrum. See figure 4 for a plot, which confirms again our expectations based on the Weak Gravity Conjecture. This is reassuring as we lack a completely perturbative description on the heterotic side, due to the tensor transition induced by the putative heterotic 5-brane.

In a similar vein, we have checked whether this non-perturbative heterotic string forms bound states when wrapped multiple times. Certainly the perturbative ones do not [48, 76], and we confirm the same behaviour also for the non-perturbative heterotic string discussed

in this section. Concretely, we compute via mirror symmetry the following sequence:

$$\mathcal{F}_{2C_0}^{(0)} = \left(252 + 84\xi^{\pm 1}\right) q^2 \tag{4.64}$$
$$+ \left(6238536 + 3986964\xi^{\pm 1} + 965232\xi^{\pm 2} + 65164\xi^{\pm 3} + 252\xi^{\pm 4}\right) q^3 + \mathcal{O}(q^4),$$
$$\mathcal{F}_{3C_0}^{(0)} = \left(252 + 84\xi^{\pm 1}\right) q^3 \tag{4.65}$$
$$+ \left(161704980 + 110586648\xi^{\pm 1} + 33904224\xi^{\pm 2}\right.$$
$$\left. + 3986964\xi^{\pm 3} + 116580\xi^{\pm 4} + 84\xi^{\pm 5}\right)q^4 + \mathcal{O}(q^5).$$

This pattern reproduces the r.h.s. of eq. (3.38), which means that all terms above arise from multi-covers of the basic invariants, $N_{C_0}^{(0)}$. In other words, the Gopakumar-Vafa BPS invariants associated with the higher wrappings vanish, $N_{k \cdot C_0}^{(0)} = 0$ for $k > 1$, and consequently there are no bound states.

5 Discussion and outlook

Our analysis has revealed an intricate interplay between various conjectured properties of quantum gravity and the geometric realisation of gauge symmetries in string compactifications. We have demonstrated that whenever a gauge symmetry — abelian or non-abelian — becomes asymptotically a global symmetry in 6d F-theory compactifications on some threefold Y_3, while gravity is being kept dynamical, there appears a nearly tensionless, critical heterotic string in the theory. At the level of algebraic geometry, this is because there must exist a rational curve C_0 of self-intersection zero in the base of the elliptic 3-fold Y_3 whose volume tends to zero in the limit. Wrapping a D3-brane on this curve gives rise to an effective string which becomes tensionless when C_0 shrinks to zero volume. This leads to a breakdown of the effective field theory near the point where the global symmetry is reached. Such behaviour is in agreement with by now familiar arguments on the properties of quantum gravity [3–5], and similar phenomena have been observed in various other corners of the landscape such as [7, 8, 30, 31].

Our analysis directly addresses the question of having a weakly coupled gauge symmetry in the 'open-string' sector. Somewhat ironically maybe, the crucial ingredient responsible for consistency with quantum gravity is nonetheless a closed string in disguise: the string which becomes tensionless is dual to the critical heterotic string compactified to six dimensions. We have shown that the tensionless limit can be taken only if the elliptic 3-fold Y_3 admits a $K3$ fibration, whose fiber class is the pullback of C_0 to Y_3. Even though in general this fibration is not compatible with the elliptic fibration of Y_3, Type IIA/heterotic duality still allows us to infer the appearance of a nearly tensionless heterotic dual.

Our approach to analyze the nearly tensionless string has been to compute its elliptic genus, making use of a number of beautiful results in the literature, pioneered in [48] and reviewed e.g. in [45]; especially concerning the relation between the elliptic genus and the topological string partition function on an elliptically fibered 3-fold Y_3 [50, 51]. At a technical level, we have extended the general techniques to situations in which the 3-fold has general extra rational sections, which translate into global U(1) symmetries of the string.

While the elliptic genus as such does not directly represent a partition function of physical excitations of the string in six dimensions, it describes a small, moduli-independent sub-sector of left-moving string excitations that can be paired with appropriate right-moving oscillators such as to provide a subset of the physical spectrum. The charges of these states are encoded entirely in the left-moving sector and can therefore be read off from the suitably refined elliptic genus as such. Our working assumption is that at least in the limit of very weak coupling, we can rely on the perturbative mass relation for the states at given excitation level. Fortunately, the heterotic weak coupling limit coincides with the limit where the gauge symmetry becomes global. This way we arrive at an intriguing relation between the charges and masses of the states that become light as the gauge group becomes global. We argue that, under certain assumptions, the charge-to-mass ratio in particular asymptotically satisfies the (Sub)Lattice Weak Gravity Conjecture bound [5, 17, 39].

Further away from the tensionless limit we cannot a priori trust the formula for the masses of the states. However, it is tempting to speculate that the states remain in the spectrum. This is plausible as they are apparently needed to satisfy the Completeness Hypothesis [40], according to which every site in the charge lattice should be populated by a physical state. The 'perturbative' open string spectrum in (Type IIB language) by itself does not contain arbitrarily high charges, not even at the massive higher excitation levels, because the charges are encoded entirely in the Chan-Paton factors, and this fact remains true taking into account also multi-pronged strings in F-theory.

One of our key results is the quantitative confirmation of the Sublattice Weak Gravity Conjecture for six-dimensional heterotic strings, at least for the tensionless limit. The relevant bound with which we need to compare the charge/mass ratio of the excitations corresponds to certain extremal Reissner-Nordström black holes, where we additionally take the effect of massless scalar field fluctuations [17, 25, 83] into account. Indeed we observe the existence of superextremal string excitations that just minimally lie above this bound and whose charges populate at least a sublattice of the full charge lattice.

As far as the heterotic string is concerned, our findings are completely in line with the original discussion in ref. [5]. There the heterotic string was compactified on T^6, while here we consider compactification on $K3$ with a single unbroken U(1) gauge symmetry. As we have argued in section 3.4, the modular properties of the elliptic genus guarantee that this U(1) gauge symmetry can formally be associated with a single boson compactified on an S^1. It is precisely the charge/mass properties of the winding states in this sector that ultimately forms the common ground between our work and that of ref. [5] (as well as [17, 39]).

As far as extremal black holes are concerned, we find it quite remarkable that the balance of gravitational and scalar forces [25] conspires, in a narrowly consistent manner, with the charge pattern of the heterotic string, which is encoded in the number theoretical properties of weak Jacobi forms.

Our focus in this work has been on the role of the particle excitations of the nearly tensionless string in the context of Quantum Gravity conjectures. At the same time, it is natural to wonder about the implications of the nearly tensionless BPS *string* as such.

First, to argue for a breakdown of the effective field theory in the limit where the gauge symmetry becomes global, it probably suffices to consider the tensionless string. This string is a priori charged with respect to the 2-form gauge symmetries associated with the tensor multiplets of the 6d $N = (1,0)$ supergravity; however, due to the 6d Green-Schwarz mechanism the shift symmetry of the tensors is gauged with respect to the 1-form U(1) symmetry which becomes global in the limit we are considering. This implies an effective charge also of the string itself. Microscopically this charge can be attributed to the appearance of charged fermionic zero-modes propagating on its world-sheet.

Concerning the Weak Gravity Conjecture, one might alternatively wish to consider not the non-BPS black holes charged under the abelian gauge symmetry, but rather black string solutions of the 6d $N = (1,0)$ supergravity. In this case one would have to check for the appearance of super-extremal strings with respect to the charge-to-tension ratio of the black strings. Given that the BPS strings which become tensionless in our limit carry net charge also under the 1-form U(1) symmetry via the Green-Schwarz mechanism, as noted above, it would be interesting to quantify this relation further.

As far as the Completeness Conjecture is concerned, it is rather obvious that the lattice of *string* charges is indeed realized explicitly by physical states due to the appearance of BPS saturated strings from wrapped D3-branes. In this sense the Completeness Conjecture with respect to the 1-form U(1) gauge symmetry for the particle-type excitations is far more non-trivial to check, along the lines spelled out in this paper.

On the formal side, one of the outcomes of this article is the determination of the quasi-modular elliptic genus of certain non-perturbative $K3$ compactifications of the heterotic string, which involve extra massless tensor multiplets. This is the situation in which the $K3$ fibration of the F-theory 3-fold is not compatible with the elliptic fibration. Such configurations are related by a tensor transition to F-theory on Hirzebruch surfaces [38], which has a more conventional perturbative heterotic dual. We have verified that the proposed elliptic genus of the dual non-perturbative heterotic string correctly captures the chiral index of F-theory massless modes, for the example of F-theory on a dP_2 base. It would be interesting to analyze this relation further.

Another direction to take would be to relate the 6d F-theory compactifications studied in this work to various other dual setups. Compactification on an extra torus takes us to a 4d $\mathcal{N} = 2$ compactification of the Type IIA string on the same Calabi-Yau Y_3. The 'open string U(1)' in the F-theory frame becomes a U(1) gauge symmetry in the Ramond-Ramond sector of the 4d theory. The wrapped string states become 4d BPS particles associated with wrapped D2 (and D4) branes in Type IIA. Mirror symmetry then relates this to Type IIB string theory with a U(1) gauge symmetry due to the Ramond-Ramond 4-form C_4 with charged BPS states from the sector of wrapped D3-branes. This is exactly the duality frame analyzed in [8]. It might be illuminating to work out the mirror map in detail and compare our results to the behaviour of this Type IIB theory in the vicinity of infinite distance points in complex structure moduli space.

Let us end with a few more speculations: the nearly tensionless critical 6d heterotic string that has been the star of the present analysis, appears to behave very differently from the non-critical 6d strings which are associated with 6d superconformal field theories

without gravity. These non-critical strings are due to D3-branes wrapping contractible curves of strictly negative self-intersection on the base. Shrinking such curves to zero volume, while keeping the volume of the base fixed, is possible at finite distance in moduli space. This is to be contrasted with the critical heterotic string, which is associated with a non-contractible curve of self-intersection zero: as we have seen, shrinking its volume to zero while keeping the base volume fixed is possible only if we take a drastic limit in which another curve approaches infinite volume, and this limit lies at infinite distance in moduli space.

From the perspective of Swampland Distance Conjecture [6], the fact that this limit is at infinite distance seems to be successfully explained by the appearance of a tower of infinitely many critical string excitations coupling to gravity which become light. This raises the obvious question why the tensionless point of the non-critical string lies at finite distance in moduli space. A priori, of course, the Swampland Distance Conjecture does not need to hold in the reverse, i.e. it does not postulate that *every* tower of infinitely many states which becomes massless at a point in moduli space requires this point to be at infinite distance.

Nonetheless, it is not immediately clear how to reconcile the existence of such a tower of charged states e.g. with the following fact: if the curve associated with a critical or non-critical string transversally intersects a 7-brane curve, the 7-brane gauge group becomes a flavour symmetry of the string, and its excitations, if present, would be charged with respect to this symmetry. The limit of shrinking the non-critical string curve is compatible with keeping the transverse 7-brane curve at finite volume. If the string gives rise to a tower of infinitely many massless charged states in the limit, the running of these states should affect the gauge coupling of the 7-brane theory [7, 8, 31]. As argued at the end of section 2.6 this is exactly the situation for the critical heterotic string; by contrast this is apparently not the case for the non-critical strings as the gauge coupling associated with the transverse 7-brane stays finite if we contract the curve associated with the string.

A bold resolution would be that the non-critical string does not give rise to infinitely many massless physical particle states in six dimensions as we approach the tensionless limit. To the best of our understanding this seems to be consistent with all we know about these mysterious strings: as the curve contracts, the 6d *BPS string* becomes tensionless, which is a BPS protected statement. While the wrapped string does give rise to 5d BPS particles, the 6d particle excitations are non-BPS, as stressed several times, and hence are a priori not protected against large corrections to their mass. In the case of the critical heterotic string we have invoked an explicit duality with a weakly coupled heterotic string to argue for the appearance of massless particle excitations in the tensionless limit, but for non-critical strings no such arguments are immediately available at first sight. If correct, this argument would suggest that (all but possibly a finite number of) the particle excitations of the non-critical string sit at non-zero mass even in the tensionless limit due to non-trivial interaction effects, while the BPS string as such becomes tensionless. In the SCFT limit this mass must of course be infinite as otherwise there would be a scale in the theory. A less radical resolution would be that the asymptotically massless excitations couple both to gravity and to the gauge theories of the other branes in a fundamentally

different manner even infinitesimally away from the tensionless point. It would be very desirable to understand this point better.

Acknowledgments

A Conventions for dimensional reduction

Here we collect our conventions for the effective action of 7-branes and D3-branes. Our starting point is the 10d string frame Einstein-Hilbert action together with the Dirac-Born-Infeld (DBI) action for the 7-branes and 3-branes, whose relevant terms are

$$
S = \frac{2\pi}{\ell_s^8} \int_{\mathbb{R}^{1,9}} d^{10}x \, e^{-2\phi} \sqrt{-\det g} \, R \, + \frac{2\pi}{\ell_s^8} \int_{\mathrm{D7}} d^8x \, e^{-\phi} \sqrt{-\det(g + 2\pi\alpha' F)}
$$
$$
+ \frac{2\pi}{\ell_s^4} \int_{\mathrm{D3}} d^4x \, e^{-\phi} \sqrt{-\det(g + \ldots)} + \ldots \, .
\tag{A.1}
$$

In the sequel we will set the string length to unity,

$$
\ell_s = 2\pi\sqrt{\alpha'} \equiv 1 \, .
\tag{A.2}
$$

We dimensionally reduce the Einstein-Hilbert action on the 2-complex dimensional space B_2, which serves as base of the elliptic threefold, Y_3. Moreover, we reduce the DBI actions on the curves C and C_0 that are wrapped by the 7-brane and the 3-brane, respectively. This produces an effective action whose relevant terms read

$$
S = 2\pi \, \mathrm{vol}(B_2) \int_{\mathbb{R}^{1,5}} d^6x \, \sqrt{-\det g} \, e^{-2\phi} \, R + \frac{\mathrm{vol}(C)}{2\pi} \int_{\mathbb{R}^{1,5}} d^6x \, \sqrt{-\det g} \, e^{-\phi} \frac{1}{4} F_{\mu\nu} F^{\mu\nu}
$$
$$
+ 2\pi \, \mathrm{vol}(C_0) \int_{\mathbb{R}^{1,1}} d^2x \, e^{-\phi} \sqrt{-\det g} + \ldots \, .
\tag{A.3}
$$

After rescaling the metric $g \to e^{\frac{\phi}{2}} g$ we obtain the Einstein-Hilbert action in the 6d Einstein frame. The dilaton dependence both of the 7-brane gauge kinetic term and of the D3-brane tension term is removed by this rescaling. We can therefore rewrite the relevant part of the effective action as

$$
S = \int d^6x \sqrt{-\det g} \left(\frac{M_{\mathrm{Pl}}^4}{2} R + \frac{1}{4g_{\mathrm{YM}}^2} F_{\mu\nu} F^{\mu\nu} \right) + T \int_{\mathbb{R}^{1,1}} \sqrt{-\det g} + \ldots
\tag{A.4}
$$

and identify

$$M_{\mathrm{Pl}}^4 = 4\pi\,\mathrm{vol}(B_2) \tag{A.5}$$

$$\frac{1}{g_{\mathrm{YM}}^2} = \frac{1}{2\pi}\mathrm{vol}(C) \tag{A.6}$$

$$T = 2\pi\,\mathrm{vol}(C_0)\,. \tag{A.7}$$

Here T denotes the tension of the string obtained by wrapping the D3-brane along C_0.

B   The geometric limit with $g_{\mathrm{YM}} \ll M_{\mathrm{Pl}}^{-1}$

In this appendix we derive the four key facts characterizing the limit (2.16) of the Kähler form J of a compact Kähler surface presented in section 2.2.

B.1   The degenerate limit

As stated already in the main text, the general limiting behaviour of the Kähler form J can be summarized in

Assertion 1. *For* (2.3) *and* (2.15) *to be satisfied simultaneously, the Kähler form J on B_2 must be chosen to lie infinitesimally close to a boundary ray of the Kähler cone. To be more specific, the Kähler form must asymptote to*

$$J = tJ_0 + \sum_\nu s_\nu I_\nu\,, \quad \text{as}\ \ t \to \infty\,. \tag{B.1}$$

Here $J_0 \in H^{1,1}(B_2,\mathbb{Z})$ is a Kähler cone generator satisfying

$$J_0 \cdot J_0 = 0\,, \qquad \int_C J_0 \geq 1\,. \tag{B.2}$$

The remaining Kähler cone generators $I_\nu \in H^{1,1}(B_2,\mathbb{Z})$ have the property that

$$\sum_\nu n_\nu s_\nu \leq \frac{1}{t}\,, \tag{B.3}$$

where

$$n_\nu = J_0 \cdot I_\nu \tag{B.4}$$

and at least one $n_\nu \neq 0$. In fact

$$\sum_\nu n_\nu s_\nu \to \frac{1}{t} \qquad as \ \ t \to \infty\,. \tag{B.5}$$

The non-negative expansion parameters s_ν stay finite as $t \to \infty$ and are chosen such that $\mathrm{vol}_J(B_2) = 1$.

The first relation in (B.2) implies in particular that J_0 must lie on the boundary $\partial \mathbf{K}(B_2)$ of the Kähler cone of B_2. The volume of C, which governs the inverse gauge coupling, obeys

$$\text{vol}_J(C) = t \int_C J_0 + \sum_\nu s_\nu \int_C I_\nu \tag{B.6}$$
$$= 2mt + s \,,$$

where

$$2m := \int_C J_0 \tag{B.7}$$

is a positive integer fixed by the topology and

$$s := \sum_\nu s_\nu \int_C I_\nu \tag{B.8}$$

is a non-negative real number that remains finite in our geometric limit.

To derive this, note first that the Kähler form J of any compact Kähler surface B_2 can be expanded as

$$J = \sum_a t_a J_a + \sum_\nu s_\nu I_\nu \,, \qquad t_a, s_\nu \geq 0 \,, \tag{B.9}$$

where J_a and I_ν are the Kähler cone generators in $H^{1,1}(B_2, \mathbb{Z})$. We have split the set of Kähler parameters into t_a, which go to infinity in the limit (2.16) we are considering, and s_ν, which remain finite in the limit. Being Kähler cone generators, the J_a and I_ν have the important property that

$$J_a \cdot J_b \geq 0 \,, \qquad J_a \cdot I_\nu \geq 0 \,, \qquad I_\nu \cdot I_\mu \geq 0 \,. \tag{B.10}$$

Here, for notational simplicity, the integration of forms over the base has been expressed in terms of intersection products. By assumption, at least one of the J_a has a non-zero integral over C so that $\int_C J \to \infty$ as $t_a \to \infty$, as intended.

Let us now start imposing various constraints on J. First in order to have

$$\text{vol}_J(B_2) = \frac{1}{2} J \cdot J = 1 \tag{B.11}$$

in the limit $t_a \to \infty$, the J_a must obey a trivial intersection form,

$$J_a \cdot J_b = 0 \,. \tag{B.12}$$

This follows from the fact that all t_a and s_ν are non-negative, together with (B.10), which makes cancellations between infinite terms impossible.

Now, (B.12) implies that there can be only one Kähler cone generator J_a which appears with a coefficient $t_a \to \infty$ in the ansatz (B.9), i.e. we can write

$$J = t J_0 + \sum_\nu s_\nu I_\nu \tag{B.13}$$

with $J_0 \in H^{1,1}(B_2, \mathbb{Z})$ a Kähler cone generator satisfying

$$J_0 \cdot J_0 = 0 , \qquad \int_C J_0 \geq 1 . \qquad (B.14)$$

In (B.13) the index ν runs over the remaining generators of the Kähler cone different from J_0.

The reason why only one Kähler cone generator can appear with a large Kähler parameter is that in order for any two elements J_a and J_b to satisfy

$$J_a \cdot J_a = 0 , \qquad J_b \cdot J_b = 0 , \qquad J_a \cdot J_b = 0 \qquad (B.15)$$

on a compact Kähler surface, we must have $J_b = \lambda J_a$ for some $\lambda \in \mathbb{R}$ as will be shown below. Hence there is only one choice of Kähler cone generators which can appear with a coefficient t_a which goes to infinity. To see this, recall that on any compact Kähler surface we can consider a basis $\{\omega_0, \omega_i\}$ of $H^{1,1}(B_2)$ with intersection form

$$\omega_0 \cdot \omega_0 = +1 , \qquad \omega_0 \cdot \omega_i = 0 , \qquad \omega_i \cdot \omega_j = -\delta_{ij} \qquad (B.16)$$

because the intersection form is known to have signature $(+1, -1, \ldots, -1)$. Let us expand

$$J_a = a \, \omega_0 + a_i \, \omega_i , \quad J_b = b \, \omega_0 + b_i \, \omega_i \qquad (B.17)$$

and introduce a vector notation $(\vec{a})_i = a_i$ and $(\vec{b})_i = b_i$. Then (B.15) is equivalent to requiring that

$$a^2 = \vec{a} \cdot \vec{a} , \qquad b^2 = \vec{b} \cdot \vec{b} , \qquad ab = \vec{a} \cdot \vec{b} . \qquad (B.18)$$

Hence

$$|ab|^2 = |\vec{a} \cdot \vec{b}|^2 \leq |\vec{a}^2||\vec{b}^2| = a^2 b^2 \qquad (B.19)$$

with equality only for $\vec{b} = \lambda \vec{a}$. The only possibility to satisfy (B.18) is then to take also $b = \lambda a$ and hence $J_a = \lambda J_b$, which proves the claim.

Having established the parametrization (B.13) of the Kähler form, we note that the argument after Assertion 2 below establishes that there exists a holomorphic curve C_0 with class $[C_0] = J_0$. Being a holomorphic curve, its volume must be strictly positive inside the Kähler cone of B_2, i.e. away from the asymptotic limit $t \to \infty$ for the Kähler form (B.13). Since $\int_{C_0} J_0 = J_0 \cdot J_0 = 0$, this requires that there must exist at least one I_ν with a non-zero intersection $n_\nu := J_0 \cdot I_\nu$. Then, the volume of B_2 is given as

$$\mathrm{vol}_J(B_2) = t \sum_\nu s_\nu (J_0 \cdot I_\nu) + \frac{1}{2} \sum_{\nu, \mu} s_\nu s_\mu (I_\nu \cdot I_\mu) \geq t \sum_\nu n_\nu s_\nu , \qquad (B.20)$$

where in the last step the non-negative terms that do not involve t have been dropped. Requiring that $\mathrm{vol}_J(B_2) = 1$ implies the bound

$$\sum_\nu n_\nu s_\nu \leq \frac{1}{t} \qquad (B.21)$$

on the parameters s_ν in the ansatz (B.13), and in fact $\sum_\nu n_\nu s_\nu \to \frac{1}{t}$ as $t \to \infty$.

It is important to note that $\sum_\nu n_\nu s_\nu$ has to asymptote precisely to $\frac{1}{t}$. In particular it cannot become *parametrically* smaller than $1/t$. For this to happen, we would need $s_{\nu_0} s_{\mu_0} (I_{\nu_0} \cdot I_{\mu_0})$ to be of order one (so that $\mathrm{vol}_J(B_2)$ can stay finite) for at least one pair of generators I_{ν_0} and I_{μ_0} with $\nu_0 \neq \mu_0$, or that at least one $s_{\nu_0}^2 (I_{\nu_0} \cdot I_{\nu_0})$ is of order one. Since all s_ν remain finite as $t \to \infty$, as proved above, this requires both s_{ν_0} and s_{μ_0} to be of order one, which in turn implies that $J_0 \cdot I_{\nu_0}$ and $J_0 \cdot I_{\mu_0}$ must both vanish in order for $\mathrm{vol}_J(B_2)$ to stay finite. The last requirement, however, is not possible for the same reason as given in the proof of Assertion 3 in section B.3.

B.2 Existence of the curve C_0

The second observation we make is summarized in

Assertion 2. *Modulo the technical assumption (B.33), as the Kähler form asymptotes to (B.1), the base B_2 necessarily contains a rational curve C_0 with*

$$C_0 \cdot C_0 = 0 , \qquad C_0 \cdot \bar{K} = 2 , \qquad C_0 \cdot C \neq 0 \tag{B.22}$$

whose volume vanishes in the limit,

$$\lim_{t \to \infty} \mathrm{vol}_J(C_0) = 0 \qquad as \qquad \lim_{t \to \infty} \mathrm{vol}_J(C) = \infty . \tag{B.23}$$

The curve class C_0 coincides with the class J_0 in (B.1),

$$C_0 = J_0 . \tag{B.24}$$

Indeed, if there exists an effective curve C_0 in this class, then by (B.2)

$$C_0 \cdot C_0 = 0 , \qquad C_0 \cdot C \neq 0 , \tag{B.25}$$

and it is clear that its volume vanishes for $t \to \infty$ as

$$\mathrm{vol}_J(C_0) = C_0 \cdot J = J_0 \cdot J = \sum_\nu n_\nu s_\nu \to \frac{1}{t} . \tag{B.26}$$

For later purposes note that this implies the following behaviour for the product of the two volumes:

$$\mathrm{vol}_J(C) \, \mathrm{vol}_J(C_0) \to 2m + \frac{s}{t} . \tag{B.27}$$

The question is then if an effective curve exists in the class J_0. By construction, the asymptotic Kähler form is in the closure of the Kähler cone, $J_0 \in \overline{\mathbf{K}(B_2)}$. For a surface B_2, the latter is contained in $\overline{NE(B_2)}$, the closure of the Mori cone $NE(B_2)$ of effective curves on B_2. According to Mori's cone theorem, when applied to B_2, there exist countably many rational curves C_i satisfying $0 < \bar{K} \cdot C_i \leq 3$ such that

$$\overline{NE(B_2)} = \overline{NE(B_2)}_{\bar{K} \leq 0} + \sum_i \mathbb{R}_{\geq 0} \, [C_i] . \tag{B.28}$$

The second term on the r.h.s. of (B.28) is manifestly in the Mori cone, i.e. describes effective curves, while the first term may not be effective (since it is a priori only in the closure of

the Mori cone). It is the subset of $\overline{NE(B_2)}$ with non-positive intersection number with the anti-canonical class $\bar{K}$ of B_2.

Now, recall that the anti-canonical divisor must be effective for there to exist a Calabi-Yau Weierstrass model on B_2, $\bar{K} \in NE(B_2)$, and the asymptotic Kähler form is by construction in the closure of the Kähler cone, $J_0 \in \overline{\mathbf{K}(B_2)}$. Therefore

$$J_0 \cdot \bar{K} \geq 0\,. \tag{B.29}$$

This leaves two possibilities: either $J_0 \cdot \bar{K} = 0$ and $J_0 \in \overline{NE(B_2)}_{\bar{K} \leq 0}$, or $J_0 \cdot \bar{K} > 0$ and $J_0 \in \sum_i \mathbb{R}_{\geq 0}\,[C_i]$. In the latter case, J_0 is an effective curve class itself, and the genus of the associated holomorphic curve C_0 can be computed by the Hirzebruch-Riemann-Roch theorem as

$$2g(C_0) - 2 = C_0 \cdot (C_0 + K) = J_0 \cdot (J_0 + K) = -J_0 \cdot \bar{K} < 0\,. \tag{B.30}$$

The only possibility is

$$g(C_0) = 0\,, \quad \text{i.e.} \quad C_0 \cdot \bar{K} = 2\,. \tag{B.31}$$

Together with (B.25), C_0 hence has all properties advertised in (B.22).

The question is now whether the other scenario, $J_0 \cdot \bar{K} = 0$, is possible. To address it, let us first assume that the curve C whose volume goes to infinity as $t \to \infty$ supports a non-abelian gauge algebra $\mathfrak{g}_I$, corresponding to the first case in (2.12). In view of (2.8) and (2.7), this means that the class of the discriminant of the fibration must satisfy the relation

$$12\bar{K} = [\Sigma] = C + [\Sigma']\,, \tag{B.32}$$

where both C and $[\Sigma']$ are effective curve classes. If $J_0 \cdot \bar{K} = 0$, while at the same time $J_0 \cdot C > 0$ (which is required by the assumption that $\text{vol}_J(C) \to \infty$ as $t \to \infty$), then necessarily $J_0 \cdot [\Sigma'] < 0$. But this is not possible because Σ' is an effective curve corresponding to the residual components of the discriminant locus.

It remains to exclude $J_0 \cdot \bar{K} = 0$ also for the second case in (2.12), where C corresponds to the height pairing $b = -\pi_*(\sigma(S) \cdot \sigma(S))$ of a rational section. Consider an F-theory model with a single abelian gauge group factor, and suppose that there exists $n > 0$ such that

$$n\,\bar{K} = b + \delta\,, \qquad \delta \quad \text{effective.} \tag{B.33}$$

Then the same argument as above shows that $J_0 \cdot \bar{K} = 0$ requires $J_0 \cdot \delta < 0$, in clash with the effectiveness of δ. The question is then if for every Y_3 with a rational section, such an n can be found. While we cannot present a general proof for arbitrary B_2, this property is realized in all models we know with abelian gauge symmetries in F-theory. Consider first the so-called Morrison-Park model [72], which is a rather general (though not the most general [114–116]) type of an elliptic fibration with a Mordell-Weil group of rank one. In this case, $b = 6\bar{K} - 2\beta$ with β an effective class such that $2\bar{K} - \beta$ is also effective. Hence (B.33) is true with $n = 6$ and $\delta = 2\beta$. In the presence of additional non-abelian gauge algebras, b changes by the subtraction of multiples of the divisor classes Σ_I associated with $\mathfrak{g}_I$ from b, so that δ becomes even more effective.

On the other hand, more general U(1) models are known to exist in which the height pairing does exceed the specific bound $b \leq 6\bar{K}$. This comes with the inclusion of higher charges beyond $\mathfrak{q} = 1$ and $\mathfrak{q} = 2$. As shown in [115], all of these models are related to a non-generic Weierstrass model of the form [72], which is however not Calabi-Yau because the class of f and g is $4\bar{K} + 4D$ and $6\bar{K} + 6D$, respectively, with D a priori an effective class. The elliptic Calabi-Yau 3-fold describing the U(1) is related to this model by a non-minimal transformation which changes the class of f and g to $4\bar{K}$ and $6\bar{K}$, as required for an elliptic Calabi-Yau. The height pairing of the section is in this case

$$b = 6\bar{K} + 4D \, . \tag{B.34}$$

However, extra constraints on D may arise by requiring that the Weierstrass model has a smooth birational resolution. Explicit examples of this type were constructed in [114, 116] with charged matter with $\mathfrak{q} = 1, 2, 3$, and in all these types of fibrations (B.33) is satisfied for some higher value of n.

For favorable examples of base spaces, on the other hand, (B.33) is readily proved to hold for any conceivable rational section and associated height pairing: indeed, for any base B_2 where the effective divisor $\bar{K}$ lies strictly inside the Mori cone (as opposed to merely on its boundary), given any effective divisor b we can find an integer $n > 0$ such that $n\bar{K} - b$ is effective. This is simply because by assumption $n\bar{K}$ has a nontrivial positive component along each generator of the Mori cone, which can be made arbitrarily large by scaling up n. This is in particular the case for all Hirzebruch surfaces $\mathbb{F}_a$ and del Pezzo surfaces dP_r ($r \leq 8$).

In the sequel we are making the technical assumption that (B.33) indeed holds. In this case, we have proved rigorously that in the limit where an abelian gauge symmetry becomes global, a rational curve C_0 with $C_0^2 = 0$ attains zero volume. Note that for non-abelian gauge symmetries, we have proved the analogous statement even without any further assumptions. Importantly, the curve C_0 has non-trivial intersection with the curve C supporting the non-abelian or abelian gauge symmetry.

B.3 Uniqueness of C_0 and K3-fibration

If $h^{1,1}(B_2) \geq 3$ one might wonder if there are more curves on B_2 whose volume tend to zero in the limit $t \to \infty$. The answer is that C_0 is the only curve class of self-intersection zero with this property, and every additional curve class with asymptotically vanishing volume must have negative self-intersection. This is summarized in

Assertion 3. *Every curve class $\tilde{C}$ with asymptotically vanishing volume in the limit (2.17) must satisfy $\tilde{C} \cdot \tilde{C} \leq 0$, and such curve satisfies $\tilde{C} \cdot \tilde{C} = 0$ if and only if $\tilde{C} = C_0$ up to a muliplicative factor.*

This is a simple consequence of the well-known lightcone-like structure of the intersection form on the compact Kähler surface B_2. On B_2 we can always find a basis

$$\{\omega_0, \omega_i\} \, , \quad i = 1, \ldots n_T \tag{B.35}$$

of $H^{1,1}(B_2)$ with diagonal intersection form

$$\omega_0 \cdot \omega_0 = 1 \qquad \omega_i \cdot \omega_j = -\delta_{ij} \qquad \omega_0 \cdot \omega_i = 0 \,. \tag{B.36}$$

We can use the freedom to define the basis in order to set w.l.o.g. $J_0 = j(\omega_0 + \omega_1)$. The condition that $\mathrm{vol}_J(\tilde{C}) \to 0$ in the limit $t \to \infty$ for (2.17) requires $J_0 \cdot \tilde{C} = 0$. Expanding $\tilde{C} = a_0\omega_0 + \sum_{i=1}^{n_T} a_i\omega_i$, this implies $a_0 = a_1$ so that $\tilde{C} = a_0(\omega_0 + \omega_1) + \sum_{i=2}^{n_T} a_i\omega_i$. We therefore have

$$\tilde{C} \cdot \tilde{C} = -\sum_{i=2}^{n_T} a_i^2 \le 0 \,, \tag{B.37}$$

where the equality, $\tilde{C} \cdot \tilde{C} = 0$, holds if and only if $\tilde{C} = a_0(\omega_0 + \omega_1) = \frac{a_0}{j} J_0$. Hence the class of $\tilde{C}$ is proportional to that of C_0, as claimed.

The curve C_0 has the following important property: if we restrict the elliptic fibration $\pi : Y_3 \to B_2$ to C_0, we obtain an elliptically fibered surface

$$\widehat{C}_0 = \pi^{-1}(C_0) \in H_4(Y_3) \,, \tag{B.38}$$

which is an elliptic $K3$ surface. To see this, note that the curve C_0 intersects the discriminant divisor Σ of the elliptic fibration Y_3 in isolated points whose numbers is computed, via (B.31), to be

$$C_0 \cdot \Sigma = 12\, C_0 \cdot \bar{K} = 24 \,. \tag{B.39}$$

The elliptically fibered surface $\widehat{C}_0$ therefore has *generically* 24 I_1-fibers at the location of the 24 intersection points of C_0 with the discriminant. This uniquely identifies $\widehat{C}_0$ as a K3 surface. This surface plays an important role because of our final

Assertion 4. *If the base B_2 is compatible with a limit of the form (2.17), the Calabi-Yau 3-fold Y_3 admits not only an elliptic fibration (2.1) over B_2, but in addition a K3 fibration*

$$\rho : \quad \begin{array}{ccc} K3 & \to & Y_3 \\ & & \downarrow \\ & & C' \end{array} \tag{B.40}$$

over a curve C', where the class of the K3-fiber is $\widehat{C}_0 \in H_4(Y_3)$. In general, this K3 fibration is not compatible with the elliptic fibration (2.1).

For the proof we recall the criterion of Ooguiso [117] in order for a given Calabi-Yau 3-fold to be $K3$ fibered. The criterion states that this is the case if Y_3 possesses a nef (numerically effective) divisor D with the property that $D \cdot D = 0$ and $c_2(Y_3) \cdot D > 0$. The class of the nef divisor D then serves as the class of the generic $K3$ fiber of Y_3 (up to a multiplicative factor). In the present case we would like to identify the class of D with the class of the divisor $\widehat{C}_0$.

First, note that manifestly $\widehat{C}_0 \cdot \widehat{C}_0 = 0$: since $\widehat{C}_0$ is the pullback under π of C_0 to Y_3, the only possible non-zero intersection is with a section S, but $\widehat{C}_0 \cdot \widehat{C}_0 \cdot S = C_0 \cdot_{B_2} C_0 = 0$. To check if $\widehat{C}_0$ is nef, recall that a divisor is nef if and only if it lies in the dual cone to $\overline{NE(Y_3)}$, the closure of the Mori cone of effective curves on Y_3. This dual cone is precisely

$\overline{\mathbf{K}(Y_3)}$, the closure of the Kähler cone. In the present case we are interested in the divisor given by the pullback of $C_0 = J_0$, with J_0 one of the generators of the Kähler cone of the base. Its pullback to Y_3 must then lie in the Kähler cone of Y_3, and hence $\widehat{C}_0$ is nef.

It remains to check for strict positivity of the product

$$\widehat{C}_0 \cdot_{Y_3} c_2(Y_3) = C_0 \cdot_{B_2} \pi_*(c_2(Y_3)) \overset{?}{>} 0 \,. \tag{B.41}$$

Let us first consider a smooth Weierstrass model over a base B_2, assuming that it exists. By standard computation,

$$c_2(Y_3) = 11c_1^2(B_2) + c_2(B_2) + 12S_0 \cdot c_1(B_2) \,, \tag{B.42}$$

where S_0 is the zero-section. Hence we confirm

$$\widehat{C}_0 \cdot_{Y_3} c_2(Y_3) = 12\widehat{C}_0 \cdot S_0 \cdot c_1(B_2) = 12\, C_0 \cdot_{B_2} \bar{K} > 0 \tag{B.43}$$

and therefore Y_3 is $K3$ fibered. This makes use of the fact $\widehat{C}_0$ is the pullback of a base divisor, whence the first two terms in (B.42) do not contribute, and the last step relies on the important property (B.22) of C_0. In presence of an extra abelian or non-abelian gauge symmetry, we are not to consider a smooth Weierstrass model Y_3, but a singular Weierstrass model and its resolution $\hat{Y}_3$. In this case, Theorem 2.1 in [118] applies to all crepant smooth resolutions of a singular Weierstrass model, which ensures that

$$\widehat{C}_0 \cdot_{Y_3} c_2(\hat{Y}_3) = 12\, C_0 \cdot_{B_2} \bar{K} > 0 \tag{B.44}$$

still holds. The only possible complication can occur if no such crepant resolution exists, i.e. if the partially resolved space $\hat{Y}_3$ is left with $\mathbb{Q}$-factorial terminal singularities in the elliptic fiber over codimension-two points on B_2 [119, 120]. Since these are localised at isolated points on the base B_2, however, they cannot affect the result for the intersection of the divisor C_0 with $c_2(\hat{Y}_3)$.

C The ring of weak Jacobi forms

In this appendix we collect a few well-known and useful properties of weak Jacobi forms. These have been introduced in [104] and are reviewed e.g. in [121]. The relation between Jacobi forms and the topological string prepotential has been noticed as early as in ref. [98], which also includes a review of many of their properties.

Weyl invariant Jacobi forms. Given a Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$, a Weyl invariant Jacobi form $\varphi_{w,\mathbf{m}}(\tau, \mathbf{z})$ with weight $w \in \mathbb{N}$ and index $\mathbf{m} \in \mathbb{N}$ is a holomorphic function

$$\begin{aligned} \varphi_{w,\mathbf{m}} : \mathbb{H} \times \mathfrak{h}_{\mathbb{C}} &\to \mathbb{C} \\ (\tau, \mathbf{z}) &\mapsto \varphi_{w,\mathbf{m}}(\tau, \mathbf{z}) \end{aligned} \tag{C.1}$$

which satisfies the following four properties [122]:

- It transforms under an $SL(2, \mathbb{Z})$ transformation of the modular parameter τ as

$$\varphi_{w,\mathbf{m}} \left(\frac{a\tau + b}{c\tau + d}, \frac{\mathbf{z}}{c\tau + d} \right) = (c\tau + d)^w e^{2\pi i \frac{\mathbf{m} c}{c\tau + d} \frac{(\mathbf{z},\mathbf{z})}{2}} \varphi_{w,\mathbf{m}}(\tau, \mathbf{z}) \,. \tag{C.2}$$

Here $(\mathbf{z}, \mathbf{z})$ is the invariant bilinear form on $\mathfrak{h}_{\mathbb{C}}$, which is normalised in such a way that the norm square of the smallest co-root is 2.

- It is quasi-periodic in shifts of the elliptic parameter by elements of the λ, μ in the co-weight lattice of $\mathfrak{g}$,

$$\varphi_{w,\mathbf{m}} \left(\tau, \mathbf{z} + \lambda\tau + \mu \right) = e^{-2\pi i \mathbf{m}(\frac{(\mathbf{z},\mathbf{z})}{2}\tau + 2\frac{(\lambda,\mathbf{z})}{2})} \varphi_{w,\mathbf{m}}(\tau, \mathbf{z}) \,. \tag{C.3}$$

- It is invariant under the action of the Weyl group $W_{\mathfrak{g}}$ of $\mathfrak{g}$,

$$\varphi_{w,\mathbf{m}}(\tau, s\mathbf{z}) = \varphi_{w,\mathbf{m}}(\tau, \mathbf{z}) \,, \qquad s \in W_{\mathfrak{g}} \,. \tag{C.4}$$

- The form $\varphi_{w,\mathbf{m}}(\tau, \mathbf{z})$ can be expanded as

$$\varphi_{w,\mathbf{m}}(\tau, \mathbf{z}) = \sum c(n, r) q^n y^r \,, \qquad q = e^{2\pi i\tau}, \ y = e^{2\pi i\mathbf{z}} \,, \tag{C.5}$$

where the quasi-periodicity (C.3) implies that

$$c(n, r) = C(4n\mathbf{m} - r^2, r) \tag{C.6}$$

for some function C.

The ring of Weyl invariant Jacobi forms associated with Lie algebra $\mathfrak{g}$ is usually denoted by

$$J_{*,*}(\mathfrak{g}) = \oplus_{l,m} J_{l,m}(\mathfrak{g}) \,. \tag{C.7}$$

For even weight it is a freely generated (for $\mathfrak{g} \neq \mathfrak{e}_8$) polynomial ring over the ring of modular forms. A modular form of weight w is a holomorphic function on $\mathbb{H}$ transforming under an $SL(2, \mathbb{Z})$ transformation as

$$f \left(\frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^w f(\tau) \,. \tag{C.8}$$

An important role for us is played by the Eisenstein series $E_{2k}(\tau)$ for $k \geq 1$, which is defined as

$$E_{2k}(\tau) = 1 + \frac{2}{\zeta(1 - 2k)} \sum_{n=1}^{\infty} \frac{n^{2k-1} q^n}{1 - q^n} \,. \tag{C.9}$$

For $k \geq 2$, $E_{2k}(\tau)$ is a modular form of weight $w = 2k$, and $E_4(\tau)$ and $E_6(\tau)$ generate the space of modular forms. On the other hand, $E_2(\tau)$ is only a quasi-modular modular form of weight 2 because it suffers a modular anomaly

$$E_2 \left(\frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^2 E_2(\tau) + \frac{6c}{\pi i} (c\tau + d) \,. \tag{C.10}$$

Jacobi forms. For the special case $\mathfrak{g} = \mathfrak{su}(2)$, the elliptic parameter of a Weyl invariant Jacobi form takes values in $\mathfrak{h}_\mathbb{C} \simeq \mathbb{C}$. Taking into account the normalization of the invariant bilinear form a Weyl invariant Jacobi form for $\mathfrak{g} = \mathfrak{su}(2)$ hence reduces to a weak Jacobi form. Indeed, by definition a Jacobi form of weight w and index m is a holomorphic function from $\mathbb{H} \times \mathbb{C}$ to $\mathbb{C}$ with the analogous transformation behaviour

$$\varphi_{w,m}\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^w e^{2\pi i \frac{m\,c}{c\tau + d} z^2} \varphi_{w,m}(\tau, z)\,, \tag{C.11}$$

$$\varphi_{w,m}(\tau, z + \lambda\tau + \mu) = e^{-2\pi i m(z^2\tau + 2\lambda z)}\varphi_{w,m}(\tau, z) \quad \lambda, \mu \in \mathbb{Z}\,. \tag{C.12}$$

A Jacobi form $\varphi_{w,m}(\tau, z)$ is called

- holomorphic Jacobi form if $c(n, r) = 0$ unless $4mn \geq r^2$,

- Jacobi cusp form if $c(n, r) = 0$ unless $4mn > r^2$,

- weak Jacobi form if $c(n, r) = 0$ unless $n \geq 0$.

One furthermore defines a weakly (or nearly) holormorphic Jacobi form with similar properties except that $c(n, r) = 0$ unless $n \geq n_0$ for negative integer n_0. More information can also be found for instance in [98, 121].

The ring $J_{2k,*}(\mathfrak{su}(2))$ is freely generated by the Weyl invariant Jacobi forms $\varphi_{0,1}(\tau, z)$ and $\varphi_{-2,1}(\tau, z)$ with coefficients being polynomials in $E_4(\tau)$ and $E_6(\tau)$ [104] (see also e.g. the appendix A.2 of [57] or section 4.3 of [121]). Here $\varphi_{0,1}(\tau, z)$ and $\varphi_{-2,1}(\tau, z)$ can be defined in terms of the Dedekind function $\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 - q^n)$ and the Jacobi theta functions as

$$\varphi_{0,1}(\tau, z) = 4\left(\frac{\vartheta_2(\tau, z)^2}{\vartheta_2(\tau, 0)^2} + \frac{\vartheta_3(\tau, z)^2}{\vartheta_3(\tau, 0)^2} + \frac{\vartheta_4(\tau, z)^2}{\vartheta_4(\tau, 0)^2}\right) \tag{C.13}$$

$$\varphi_{-2,1}(\tau, z) = \frac{\vartheta_1(\tau, z)^2}{\eta^6(\tau)}\,. \tag{C.14}$$

For given weight and elliptic index, only a finite number of coefficients need to be fixed in the expansion in terms of this generating system.

An alternative presentation is

$$\varphi_{-2,1}(\tau, z) = \frac{\varphi_{10,1}(\tau, z)}{\Delta}\,, \qquad \varphi_{0,1}(\tau, z) = \frac{\varphi_{12,1}(\tau, z)}{\Delta} \tag{C.15}$$

where the discriminant function, given in terms of the the Dedekind eta-function,

$$\Delta(\tau) = \eta^{24}(\tau)\,, \tag{C.16}$$

is a modular function of weight 12. The specific Jacobi functions $\varphi_{10,1}(\tau, z)$ and $\varphi_{12,1}(\tau, z)$ can be expressed in terms of the Eisenstein-Jacobi series $E_{4,1}(\tau, z)$ and $E_{6,1}(\tau, z)$ as [104]

$$\varphi_{10,1}(\tau, z) = \frac{1}{144}\left(E_6(\tau)E_{4,1}(\tau, z) - E_4(\tau)E_{6,1}(\tau, z)\right) \tag{C.17}$$

$$\varphi_{12,1}(\tau, z) = \frac{1}{144}\left(E_4^2(\tau)E_{4,1}(\tau, z) - E_6(\tau)E_{6,1}(\tau, z)\right)\,. \tag{C.18}$$

We will discuss the Eisenstein-Jacobi series in more detail in appendix D.

Finally, note that $\varphi_{0,1}(\tau, z)$ and $\varphi_{-2,1}(\tau, z)$ have a polynomial expansion in terms of z whose coefficient functions are polynomials in $E_2(\tau)$, $E_4(\tau)$ and $E_6(\tau)$ [57]:

$$\varphi_{-2,1}(\tau, z) = -z^2 + \frac{E_2(\tau)}{12}z^4 + \frac{E_4(\tau) - 5E_2(\tau)^2}{1440}z^6 + \dots \tag{C.19}$$

$$\varphi_{0,1}(\tau, z) = 12 - E_2(\tau)z^2 + \frac{E_2(\tau)^2 + E_4(\tau)}{24}z^4 + \dots. \tag{C.20}$$

Modular anomalies and quasi-modularity. A weak Jacobi form of weight w and index m can in general be expanded as a power series in the elliptic parameter [104]

$$\varphi_{w,m}(\tau, z) = \sum_{j=0}^{\infty} \phi_j(\tau) z^j \,, \tag{C.21}$$

where the coefficient functions $\phi_j(\tau)$ are *quasi-modular* functions of weight $j + w$. As pointed out e.g. in [57], as a consequence of the quasi-modularity (C.10) of $E_2(\tau)$ the product

$$\Phi_{w,0}(\tau, z) = e^{\frac{\pi^2}{3}mz^2 E_2(\tau)} \varphi_{w,m}(\tau, z) \tag{C.22}$$

behaves like a modular function, i.e. like a Jacobi form of weight w and index 0. Since $\Phi_{w,0}(\tau, z)$ is modular in τ, the deviation from modularity in the quasi-modular expansion coefficients $\phi_j(\tau)$ of $\varphi_{w,m}(\tau, z)$ is therefore cancelled by the $E_2(\tau)$ dependent prefactor in $\Phi_{w,0}(\tau, z)$. It follows that the only source of quasi-modularity in $\phi_j(\tau)$ can be due to monomials in $E_2(\tau)$, which are cancelled term by term in $\Phi_{w,0}(\tau, z)$.

Another conclusion is that $\varphi_{w,m}(\tau, z)$ satisfies the differential equation [57]

$$\begin{aligned}
0 &= \frac{\partial}{\partial E_2} \Phi_{w,0}(\tau, z) \\
&= \frac{\partial}{\partial E_2} e^{\frac{\pi^2}{3}mz^2 E_2(\tau)} \varphi_{w,m}(\tau, z) = e^{\frac{\pi^2}{3}mz^2 E_2(\tau)} \left(\frac{\partial}{\partial E_2} + \frac{(2\pi)^2}{12}mz^2 \right) \varphi_{w,m}(\tau, z) \,. \tag{C.23}
\end{aligned}$$

This differential equation allows us to read off the elliptic index of a weak Jacobi form. Conversely, a power series in z with quasi-modular expansion functions satisfying periodicity under $z \to z + 1$ and the equation

$$\left(\frac{\partial}{\partial E_2} + \frac{(2\pi)^2}{12}mz^2 \right) \varphi_{w,m}(\tau, z) = 0 \tag{C.24}$$

is a weak Jacobi form of index m.

D Maximal charges via Eisenstein-Jacobi forms

In this appendix we prove the key fact that the maximal charge per excitation level n as encoded in the elliptic genus obeys a bound of the form (2.54). This makes use of various properties of the Eisenstein-Jacobi forms that we will collect below. The relevance of the Eisenstein-Jacobi forms is that, as described in section 3.4, the elliptic genus can always be expressed via an ansatz of the form (3.46). The charge dependence of the states is encoded

in the weak Jacobi forms $\varphi_{-2,1}(\tau,z)$ and $\varphi_{0,1}(\tau,z)$ introduced in appendix C, which in turn have an expression in terms of the Eisenstein-Jacobi series as given in (C.15).

The Eisenstein-Jacobi forms $E_{4,m}(\tau,z)$ and $E_{6,m}(\tau,z)$ are holomorphic Jacobi forms of index m. The coefficients $c(n,r)$ in the series

$$E_{w,m}(\tau,z) = \sum_{n,r} c(n,r)\, q^n\, \xi^r\,, \qquad q = e^{2\pi i\tau},\, \xi = e^{2\pi iz}, \tag{D.1}$$

have the property that [104]

$$
\begin{aligned}
c(n,r) &\neq 0 && \text{for} \quad r^2 < 4mn \\
c(n,r) &= \begin{cases} 1 & \text{if } r \equiv 0 \ (\text{mod } 2m) \\ 0 & \text{otherwise} \end{cases} && \text{for} \quad r^2 = 4mn
\end{aligned}
\tag{D.2}
$$

An analytic expression for the $c(n,r)$ can be found in [104] in terms of Cohen's function. Since the Eisenstein-Jacobi forms are invariant under $\xi \leftrightarrow \xi^{-1}$, we will mostly focus on non-negative powers of ξ.

Eq. (D.2) implies that the maximal ξ-power $r_{\max}(n)$ which appears in the expansion multiplying a term q^n is

$$
\begin{aligned}
r_{\max}(n) &= \lfloor \sqrt{4\,m\,n} \rfloor && \text{if } 4mn \text{ is } not \text{ a perfect square} \\
r_{\max}(n) &= \begin{cases} \sqrt{4\,m\,n} & \text{if } n = m\,l^2 \text{ for } l \in \mathbb{N}_0 \\ \sqrt{4\,m\,n} - 1 & \text{otherwise} \end{cases} && \text{if } 4mn \text{ is a perfect square}
\end{aligned}
\tag{D.3}
$$

In other words, the maximal ξ-power is

$$r_{\max}(n) = \lfloor \sqrt{4\,m\,n} \rfloor - a(n)\,, \tag{D.4}$$

where the shift function a is generically zero. To be precise, $a(n)$ is given as

$$a(n) = \begin{cases} 1 & \text{if } 4mn \text{ is a perfect square and } \frac{n}{m} \text{ is not a square of an integer} \\ 0 & \text{otherwise} \end{cases} \tag{D.5}$$

Note first that $r_{\max}(n)$ is a non-decreasing function of n even in the presence of a non-generic shift by -1. Furthermore, one can immediately observe that $r_{\max}(n)$ continues to have a plateau-like structure as n increases. An intuition for the latter property is gained from the fact that $r_{\max}(n)$ is generically given as $\lfloor \sqrt{4\,m\,n} \rfloor$. Since the values of $\sqrt{4\,m\,n}$ at $n + \Delta n$ and n differ by

$$\sqrt{4\,m\,(n + \Delta n)} - \sqrt{4\,m\,n} = \sqrt{4\,m\,n}\left(\sqrt{1 + \frac{\Delta n}{n}} - 1\right) = \sqrt{m\frac{\Delta n}{n}}\left(1 + \mathcal{O}\left(\frac{\Delta n}{n}\right)\right),\tag{D.6}$$

which can be made arbitrarily small for a small $\frac{\Delta n}{n}$, it follows that $r_{\max}(n)$ stays constant in the interval $[n, n + \Delta n]$ if $\frac{\Delta n}{n}$ is small enough.

Now, given the expressions (D.4) and (D.5), we have the following lower bound for $r_{\max}(n)$,

$$r_{\max}(n) = \lfloor \sqrt{4\,m\,n} \rfloor - a(n) \geq \sqrt{4\,m\,n} - 1\,. \tag{D.7}$$

We will then use the symbol "$\gtrsim$" to denote an "approximate" asymptotic inequality,

$$r_{\max}(n) \gtrsim \sqrt{4\,m\,n}\,, \tag{D.8}$$

in the sense that for any small $\hat{\epsilon} > 0$ there exists a finite $n_0 = n_0(\hat{\epsilon})$ such that[16]

$$r_{\max}(n) > \sqrt{(4m - \hat{\epsilon})n} \qquad \text{if } n > n_0(\hat{\epsilon})\,. \tag{D.9}$$

Let us now consider the product of two Eisenstein-Jacobi series $E_{w_1,m_1} E_{w_2,m_2}$, which is itself a Jacobi form of weight $w_1 + w_2$ and index $m_1 + m_2$ with expansion

$$E_{w_1,m_1} E_{w_2,m_2} = \sum_{n_1,r_1} c_1(n_1,r_1)\, q^{n_1} \xi^{r_1} \sum_{n_2,r_2} c_2(n_2,r_2)\, q^{n_2} \xi^{r_2} = \sum_{n,r} c(n,r) q^n \xi^r\,. \tag{D.10}$$

For the series expansions of E_{w_1,m_1} and E_{w_2,m_2}, we will denote by $r_{\max}^{(1)}(n)$ and $r_{\max}^{(2)}(n)$, respectively, the maximal ξ-powers which appear in the expansions multiplying a term q^n. Then, for a given n, there is a non-trivial contribution to $c(n,r)$ from the product of the following two terms,

$$q^n \xi^{r_{\max}^{(1)}(n)}\,, \qquad q^0 \xi^0\,, \tag{D.11}$$

in the two expansions, respectively. This implies that the maximal ξ-power for the series expansion of the product obeys

$$r_{\max}(n) \geq r_{\max}^{(1)}(n) \geq \sqrt{4\,m_1\,n} - 1\,. \tag{D.12}$$

Similarly, we also have

$$r_{\max}(n) \geq r_{\max}^{(2)}(n) \geq \sqrt{4\,m_2\,n} - 1\,, \tag{D.13}$$

and hence, eventually, the following inequality holds,

$$r_{\max}(n) \geq \max(r_{\max}^{(1)}(n), r_{\max}^{(2)}(n)) \geq \sqrt{4\,(\max\{m_1,m_2\})\,n} - 1\,. \tag{D.14}$$

In formulating these bounds, we are assuming that no complete cancellations can occur amongst the maximal ξ-power terms at given n, arising from multiplying two monomials $q^{n_1}\xi^{r_1}$ and $q^{n_2}\xi^{r_2}$ with $n_1 + n_2 = n$, for the case where more than one maximal term exists. Such cancellations could in principle occur for specific values of n, spoiling the bound (D.14) for the $r_{\max}(n)$ of the product. However, explicit analysis of the behaviour for various examples suggests that this coincidence does not happen in generic situations. On the other hand, we are aware of at least one instance where such a cancellation does happen for a specific model at a specific value of n, namely the model described by $(\mathrm{x}, \mathrm{y}) = (4, 6)$ in section 4.1, at $n = 1$, as explained in more detail in the main text.

In general, given a generic linear combination of various products of Eisenstein-Jacobi and Eisenstein series of the form

$$\sum_I c_I \prod_i E_{w_{I,i}, m_{I,i}} \prod_{i'} E_{w'_{I,i'}}\,, \tag{D.15}$$

---

[16]The small parameter $\hat{\epsilon}$ is related to the small parameter ϵ used in (2.55) and the following discussion in the main text via $\hat{\epsilon} = 4m\epsilon$.

we must have

$$r_{\max}(n) \geq \sqrt{4\left(\max_I\{\max_i\{m_{I,i}\}\}\right)n} - 1\,, \tag{D.16}$$

where $m_{I,i}$ denote the indices of the $E_{w_{I,i},m_{I,i}}$ appearing in each product summand. In particular, the following very conservative bound holds,

$$r_{\max}(n) \geq \sqrt{4\,n} - 1\,, \tag{D.17}$$

where we have used the fact that the indices of the Eisenstein-Jacobi series obey $m_{I,i} \geq 1$. Again, all of this is true as long as there is no complete cancellation amongst individual contributions to the term $q^n \xi^r$.

Now, it is interesting to observe that the bound (D.16) can be hugely improved in the asymptotic regime as

$$r_{\max}(n) \gtrsim \sqrt{4\,m\,n}\,, \tag{D.18}$$

where $m = \sum_i m_{I,i}$ is the index of any product of Eisenstein-Jacobi series. Once again, the symbol "$\gtrsim$" denotes an approximate asymptotic inequality in the sense defined in eqs. (D.8) and (D.9), where the asymptotic bound (D.18) is now claimed for a generic linear combination (D.15). To see how this improved bound arises, let us first note that the shift of -1 may be ignored in the asymptotic regime. Then, we have the obvious bound

$$r_{\max}(n) \gtrsim \max\Big\{ \sum_i \sqrt{4\,m_i\,n_i} \,\Big|\, \sum_i n_i = n,\, 0 \leq n_i \leq n,\, n_i \in \mathbb{N}_0 \Big\}\,, \tag{D.19}$$

provided that there are no cancellations of the aforementioned kind. Note that we are denoting $m_{I,i}$ by m_i for simplicity. To estimate the r.h.s. of (D.19) we first consider the function

$$f(\{\rho_i\}) = \sum_i \sqrt{m_i \rho_i}\,, \tag{D.20}$$

and find that the maximum of f in the region $0 \leq \rho_i \leq 1$ under the constraint $\sum_i \rho_i = 1$ is

$$f_{\max} = \sqrt{\sum_i m_i} = \sqrt{m}\,. \tag{D.21}$$

One can see this by demanding that all the partial derivatives of f with respect to $\rho_{i>1}$ vanish upon taking $\rho_1 = 1 - \sum_{i>1} \rho_i$, that is,

$$0 = \frac{\partial f}{\partial \rho_i} = -\frac{\sqrt{m_1}}{2\sqrt{\rho_1}} + \frac{\sqrt{m_i}}{2\sqrt{\rho_i}}\,, \quad i > 1\,. \tag{D.22}$$

This means that an extremum, which turns out to be the maximum, occurs at $\rho_i = \frac{m_i}{m}$ with the extremal value

$$f_{\max} = \sum_i \sqrt{\frac{m_i^2}{m}} = \frac{\sum_i m_i}{\sqrt{m}} = \sqrt{m}\,, \tag{D.23}$$

as claimed. Since $\sum_i \sqrt{4\, m_i\, n_i} = \sqrt{4n} \sum_i \sqrt{m_i \frac{n_i}{n}} = \sqrt{4n} f(\{\frac{n_i}{n}\})$, we may in turn express (D.19) as

$$r_{\max}(n) \gtrsim \max\left\{ \sqrt{4n} f(\{\rho_i\}) \,\Bigg|\, \sum_i \rho_i = 1,\, 0 \le \rho_i \le 1,\, \rho_i n \in \mathbb{N}_0 \right\} \gtrsim \sqrt{4n} f_{\max} = \sqrt{4\, m\, n}\,. \tag{D.24}$$

In the first step eq. (D.19) has been rewritten and in the second step we have neglected the fact that $\rho_i n \in \mathbb{N}_0$, thereby obtaining an approximate asymptotic inequality. The latter inequality actually saturates for n's that are an integer multiple of m but in general, it only works as an asymptotic bound.

Note finally that when we multiply a (product or linear combination of products of) Eisenstein-Jacobi series by a modular function $f(\tau) = a_n q^n$, the conservative bounds (D.16) and (D.17), as well as the improved, asymptotic bound (D.24), continue to hold as long as $a_0 \ne 0$. If the expansion of the function $f(\tau)$ starts with a value of $-n_0 < 0$, then the bounds get even more improved. For example, for the conservative bound (D.16) we have the following improvement,

$$r_{\max}(n) \gtrsim \sqrt{4\,(\max_I\{\max_i\{m_{I,i}\}\})(n + n_0)} - 1 \ge \sqrt{4\,(\max_I\{\max_i\{m_{I,i}\}\})n} - 1\,, \tag{D.25}$$

because of a shift of all levels by $n_0 > 0$, and similarly, the other bounds also get improved. All of this again assumes that no cancellations happen at a given level, n.

E   Analysis of U(1) models with elliptic bases $\mathbb{F}_1$ and $\mathbb{F}_2$

In this appendix we collect some mathematical details that are needed for the computation of the Gromow-Witten invariants via mirror symmetry, which in turn will determine the U(1)-refined elliptic genera for the bases $B_2 = \mathbb{F}_a$ with $a = 1, 2$. The actual computations proceed along the steps described in section 4.1.

E.1   Hirzebruch base $\mathbb{F}_1$ with extra section, for $(x, y) = (4, 4)$

We begin by providing the geometric details underlying the results (4.29) for the computation of the lowest-lying Gromov-Witten invariants of the U(1) fibration with $(x, y) = (4, 4)$ over the base space $\mathbb{F}_1$.

To compute the Gromov-Witten invariants $N_{C_0}^{(0)}(n, r)$ for some low values of n, we employ the machinery of mirror symmetry for Calabi-Yau three-folds in the framework of toric geometry [123, 124]. The required input are the Mori cone generators and the triple intersection numbers, both of which can easily be obtained via PALP [125, 126]. Based on the toric data in table 2 one finds three possible triangulations, one of which is compatible with a flat elliptic fibration.[17] The Mori cone and the intersection numbers associated with

---

[17]A necessary condition for this is that there must exist a 2×4 block of zeros in the 4×8 Mori cone matrix since the two fibral curves should not intersect with the four toric divisors that are pulled back from the four base toric divisors. This criterion singles out the phase presented here.

this phase are as follows:

$$l^{(1)} = (\ 0,\ 0,\ -1,\ \ 0,\ \ 0,\ 0,\ 1,\ \ 1)$$
$$l^{(2)} = (\ 1,\ 0,\ \ 0,\ \ 1,\ \ 0,\ 0,\ 0,\ -2)$$
$$l^{(3)} = (\ 0,\ 1,\ \ 0,\ -1,\ -1,\ 1,\ 0,\ \ 0)$$
$$l^{(4)} = (\ 0,\ 0,\ \ 2,\ \ 0,\ \ 1,\ 0,\ 0,\ -1)$$

(E.1)

$$\mathcal{I} = -3J_1^2 J_3 + 3J_1 J_2 J_3 + 3J_1 J_3^2 + 14J_2 J_3^2 + 80J_3^3 - J_1^2 J_4$$
$$+ J_1 J_2 J_4 - J_1 J_4^2 - 2J_2 J_4^2 + 8J_4^3.$$

(E.2)

Here, the Mori cone generators $l^{(i)}$ are described in terms of their intersection numbers with the 8 toric divisors $d_i = \{\nu_i = 0\}$, with the classes of ν_i given in table 2. The toric divisors are in turn expressible via the 4 basis elements J_i of $H^{1,1}(X, \mathbb{Z})$ as

$$d_1 = J_1 + J_2\,, \qquad d_2 = J_2\,, \quad d_3 = -4J_1 - 4J_2 + J_3 - 2J_4\,, \quad d_4 = J_1\,,$$
$$d_5 = -2J_1 - 3J_2 + J_3 - J_4\,, \quad d_6 = J_2\,, \quad d_7 = J_3\,, \qquad\qquad d_8 = J_4\,.$$

(E.3)

This data is sufficient to obtain [123, 124] the low-degree Gromov-Witten invariants in the basis of curves $l^{(i)}$. In order to extract the relevant invariants $N_{C_0}^{(0)}(n, r)$ from them, we need to express $\Gamma_{C_0}(n, r) = C_0 + nC_\mathcal{E} + rC^{\mathrm{f}}$ in terms of $l^{(i)}$. The correct identification is

$$C_0 = l^{(2)}\,, \quad C_\mathcal{E} = 3l^{(1)} + 2l^{(4)}\,, \quad C^{\mathrm{f}} = l^{(1)} + l^{(4)}\,,$$

(E.4)

leading to

$$\Gamma_{C_0}(n, r) = (3n + r)l^{(1)} + l^{(2)} + (2n + r)l^{(4)}\,.$$

(E.5)

Using standard methods of mirror symmetry, the lowest expansion terms of the genus-zero prepotential (4.28) can then be computed with the result as given in (4.29).

E.2 Hirzebruch base $\mathbb{F}_1$ with extra section, for $(x, y) = (4, 6)$

The elliptic fibration with $(x, y) = (4, 6)$ over $\mathbb{F}_1$ admits two inequivalent triangulations, one of which is compatible with the existence of a flat fibration. The Mori cone and the intersection numbers of this topological phase are

$$l^{(1)} = (\ 0,\ -1,\ \ 0,\ 0,\ 0,\ 0,\ 1,\ \ 1)$$
$$l^{(2)} = (\ 1,\ \ 0,\ -1,\ 1,\ 0,\ 0,\ 0,\ -1)$$
$$l^{(3)} = (\ 0,\ \ 0,\ \ 1,\ 0,\ 1,\ 0,\ 0,\ -2)$$
$$l^{(4)} = (\ 0,\ \ 2,\ \ 0,\ 0,\ 0,\ 1,\ 0,\ -1)$$

(E.6)

$$\mathcal{I} = 2J_1^2 J_2 + 6J_1 J_2^2 + 16J_2^3 + 3J_1^2 J_3 + 12J_1 J_2 J_3 + 32J_2^2 J_3$$
$$+ 21J_1 J_3^2 + 64J_2 J_3^2 + 120J_3^3 + J_1^2 J_4 - 3J_1 J_4^2 + 8J_4^3.$$

(E.7)

The Mori cone generators $l^{(i)}$ are described in terms of their intersection numbers with the 8 toric divisors $d_i = \{\nu_i = 0\}$

$$d_1 = -2J_1 - J_2 + J_3 - J_4\,, \quad d_2 = 2J_2 - J_3\,,$$
$$d_3 = 3J_1 + J_2 - J_3 + J_4\,, \quad d_4 = -2J_1 - J_2 + J_3 - J_4\,,$$
$$d_5 = J_1\,, \qquad\qquad\qquad\qquad d_6 = J_2\,,$$
$$d_7 = J_3\,, \qquad\qquad\qquad\qquad d_8 = J_4\,.$$

(E.8)
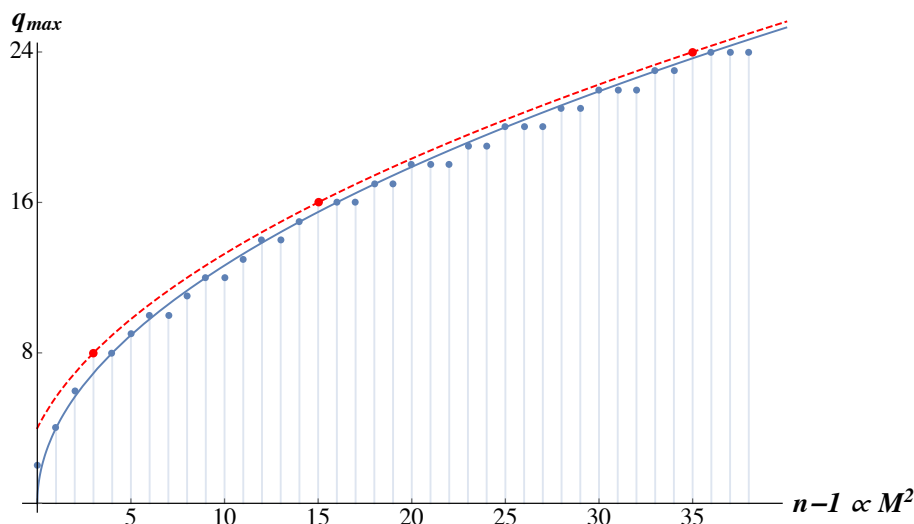
Figure 5. Maximal charge $\mathfrak{q}_{\max}(n)$ per excitation level for $\mathrm{Bl}_1\mathbb{P}^2_{112}[4]$ fibration over $\mathbb{F}_2$ with $(\mathrm{x},\mathrm{y}) = (2,4)$. The maximally superextremal states form a sublattice of spacing $2m = 8$.

The correct identification with the base curve C_0 as well as the fibral curves (4.9) and (4.10) is

$$C_0 = l^{(3)}, \quad C_{\mathcal{E}} = 3l^{(1)} + 2l^{(4)}, \quad C^{\mathrm{f}} = l^{(1)} + l^{(4)}, \tag{E.9}$$

leading to

$$\Gamma_{C_0}(n,r) = (3n+r)l^{(1)} + l^{(3)} + (2n+r)l^{(4)}. \tag{E.10}$$

This data in turn determines the lowest order Gromov-Witten invariants as given in (4.38).

E.3 Hirzebruch base $\mathbb{F}_2$ with extra section, for $(\mathrm{x},\mathrm{y}) = (2,4)$

Following the analogous procedure as in sections E.1 and E.2, we obtain the following expansion of the genus-zero generation function for an U(1) elliptic fibration with $(\mathrm{x},\mathrm{y}) = (2,4)$ over $\mathbb{F}_2$:

$$\begin{aligned}
\mathcal{F}^{(0)}_{C_0} &= -2 + \left(192 + 128\xi^{\pm 1} + 16\xi^{\pm 2}\right)q \\
&\quad + \left(87360 + 65664\xi^{\pm 1} + 26880\xi^{\pm 2} + 4992\xi^{\pm 3} + 228\xi^{\pm 4}\right)q^2 + \mathcal{O}(q^3).
\end{aligned} \tag{E.11}$$

The fugacity index of the generating function is computed as

$$m = 6 - \mathrm{x} = 4, \tag{E.12}$$

so that the general ansatz for $\mathcal{F}^{(0)}_{C_0}$ is

$$\mathcal{F}^{(0)}_{C_0} = -\frac{q}{\eta^{24}}\Phi_{10,4}, \tag{E.13}$$

where $\Phi_{10,4}$ is a weak Jacobi form of weight 10 and index 4. It is thus expanded as a general linear combination of suitable monomials $E_4^{a_1}E_6^{a_2}\varphi_{0,1}^{a_3}\varphi_{-2,1}^{a_4}$ with $4a_1 + 6a_2 - 2a_4 = 10$ and $a_3 + a_4 = 4$ for $a_i \geq 0$. Upon requiring that the GW invariants in (E.11) be obtained for

order $n = 0, 1$ in q, the coefficients in the linear combination ansatz are fixed and the full generating function is found as follows:

$$
\begin{aligned}
\mathcal{F}_{C_0}^{(0)} &= -q\, Z_{\mathcal{K}}(\tau, z) \tag{E.14} \\
&= \frac{q}{\eta^{24}} \bigg(-\frac{1}{31104} E_4^3 E_6 \varphi_{-2,1}^4 - \frac{1}{15552} E_6^3 \varphi_{-2,1}^4 + \frac{1}{10368} E_4^4 \varphi_{-2,1}^3 \varphi_{0,1} \\
&\quad + \frac{1}{3456} E_4 E_6^2 \varphi_{-2,1}^3 \varphi_{0,1} - \frac{1}{1728} E_4^2 E_6 \varphi_{-2,1}^2 \varphi_{0,1}^2 + \frac{7}{31104} E_4^3 \varphi_{-2,1} \varphi_{0,1}^3 \\
&\quad + \frac{5}{31104} E_6^2 \varphi_{-2,1} \varphi_{0,1}^3 - \frac{1}{10368} E_4 E_6 \varphi_{0,1}^4 \bigg) .
\end{aligned}
$$

The maximal charges per excitation level are plotted in figure 5, which again confirms the expectations based on the Sublattice Weak Gravity Conjecture.

E.4 Hirzebruch base $\mathbb{F}_2$ with extra section, for $(\mathrm{x}, \mathrm{y}) = (4, 8)$

The fibration with $(\mathrm{x}, \mathrm{y}) = (4, 8)$ over $\mathbb{F}_2$ gives rise to a non-generic model in which $[b_2] = 0$ and hence cancellations at the level of states with $n = 1$, $\mathfrak{q} = 2$ are expected, in line with the general discussion of section 4.1. Indeed, the expansion of the generation function up to order $n_0 = 2$ in q is found as follows:

$$
\begin{aligned}
\mathcal{F}_{C_0}^{(0)} &= -2 + \left(288 + 96 \xi^{\pm 1} \right) q \tag{E.15} \\
&\quad + \left(123756 + 69280 \xi^{\pm 1} + 10192 \xi^{\pm 2} + 96 \xi^{\pm 3} - 2 \xi^{\pm 4} \right) q^2 + \mathcal{O}(q^3) ,
\end{aligned}
$$

exhibiting, as expected, no states at $n = 1$, $\mathfrak{q} = 2$. This is exactly the same expression as for the model with $(\mathrm{x}, \mathrm{y}) = (4, 6)$ on $\mathbb{F}_1$ presented in section 4.1, so the same conclusions apply.

References

[1] C. Vafa, *The string landscape and the swampland*, hep-th/0509212 [INSPIRE].

[2] T.D. Brennan, F. Carta and C. Vafa, *The string landscape, the swampland and the missing corner*, PoS(TASI2017)015 [arXiv:1711.00864] [INSPIRE].

[3] T. Banks and L.J. Dixon, *Constraints on string vacua with space-time supersymmetry*, *Nucl. Phys.* **B 307** (1988) 93 [INSPIRE].

[4] T. Banks and N. Seiberg, *Symmetries and strings in field theory and gravity*, *Phys. Rev.* **D 83** (2011) 084019 [arXiv:1011.5120] [INSPIRE].

[5] N. Arkani-Hamed, L. Motl, A. Nicolis and C. Vafa, *The string landscape, black holes and gravity as the weakest force*, *JHEP* **06** (2007) 060 [hep-th/0601001] [INSPIRE].

[6] H. Ooguri and C. Vafa, *On the geometry of the string landscape and the swampland*, *Nucl. Phys.* **B 766** (2007) 21 [hep-th/0605264] [INSPIRE].

[7] B. Heidenreich, M. Reece and T. Rudelius, *The weak gravity conjecture and emergence from an ultraviolet cutoff*, *Eur. Phys. J.* **C 78** (2018) 337 [arXiv:1712.01868] [INSPIRE].

[8] T.W. Grimm, E. Palti and I. Valenzuela, *Infinite distances in field space and massless towers of states*, *JHEP* **08** (2018) 143 [arXiv:1802.08264] [INSPIRE].

[9] J. Brown, W. Cottrell, G. Shiu and P. Soler, *Fencing in the swampland: quantum gravity constraints on large field inflation*, *JHEP* **10** (2015) 023 [arXiv:1503.04783] [INSPIRE].

[10] J. Brown, W. Cottrell, G. Shiu and P. Soler, *On axionic field ranges, loopholes and the weak gravity conjecture*, *JHEP* **04** (2016) 017 [arXiv:1504.00659] [INSPIRE].

[11] T.C. Bachlechner, C. Long and L. McAllister, *Planckian axions and the weak gravity conjecture*, *JHEP* **01** (2016) 091 [arXiv:1503.07853] [INSPIRE].

[12] A. Hebecker, P. Mangat, F. Rompineve and L.T. Witkowski, *Winding out of the swamp: evading the weak gravity conjecture with F-term winding inflation?*, *Phys. Lett.* **B 748** (2015) 455 [arXiv:1503.07912] [INSPIRE].

[13] A. Hebecker, F. Rompineve and A. Westphal, *Axion monodromy and the weak gravity conjecture*, *JHEP* **04** (2016) 157 [arXiv:1512.03768] [INSPIRE].

[14] L.E. Ibáñez, M. Montero, A. Uranga and I. Valenzuela, *Relaxion monodromy and the weak gravity conjecture*, *JHEP* **04** (2016) 020 [arXiv:1512.00025] [INSPIRE].

[15] T. Rudelius, *Constraints on axion inflation from the weak gravity conjecture*, *JCAP* **09** (2015) 020 [arXiv:1503.00795] [INSPIRE].

[16] B. Heidenreich, M. Reece and T. Rudelius, *Weak gravity strongly constrains large-field axion inflation*, *JHEP* **12** (2015) 108 [arXiv:1506.03447] [INSPIRE].

[17] B. Heidenreich, M. Reece and T. Rudelius, *Sharpening the weak gravity conjecture with dimensional reduction*, *JHEP* **02** (2016) 140 [arXiv:1509.06374] [INSPIRE].

[18] F. Baume and E. Palti, *Backreacted axion field ranges in string theory*, *JHEP* **08** (2016) 043 [arXiv:1602.06517] [INSPIRE].

[19] M. Montero, G. Shiu and P. Soler, *The weak gravity conjecture in three dimensions*, *JHEP* **10** (2016) 159 [arXiv:1606.08438] [INSPIRE].

[20] B. Heidenreich, M. Reece and T. Rudelius, *Axion experiments to algebraic geometry: testing quantum gravity via the weak gravity conjecture*, *Int. J. Mod. Phys.* **D 25** (2016) 1643005 [arXiv:1605.05311] [INSPIRE].

[21] D. Klaewer and E. Palti, *Super-Planckian spatial field variations and quantum gravity*, *JHEP* **01** (2017) 088 [arXiv:1610.00010] [INSPIRE].

[22] G. Shiu, P. Soler and W. Cottrell, *Weak gravity conjecture and extremal black hole*, arXiv:1611.06270 [INSPIRE].

[23] A. Hebecker and P. Soler, *The weak gravity conjecture and the axionic black hole paradox*, *JHEP* **09** (2017) 036 [arXiv:1702.06130] [INSPIRE].

[24] Y. Hamada and G. Shiu, *Weak gravity conjecture, multiple point principle and the standard model landscape*, *JHEP* **11** (2017) 043 [arXiv:1707.06326] [INSPIRE].

[25] E. Palti, *The weak gravity conjecture and scalar fields*, *JHEP* **08** (2017) 034 [arXiv:1705.04328] [INSPIRE].

[26] D. Lüst and E. Palti, *Scalar fields, hierarchical UV/IR mixing and the weak gravity conjecture*, *JHEP* **02** (2018) 040 [arXiv:1709.01790] [INSPIRE].

[27] I. Valenzuela, *Backreaction in axion monodromy, 4-forms and the swampland*, PoS(CORFU2016)112 [arXiv:1708.07456] [INSPIRE].

[28] R. Blumenhagen, I. Valenzuela and F. Wolf, *The swampland conjecture and F-term axion monodromy inflation*, *JHEP* **07** (2017) 145 [arXiv:1703.05776] [ɪɴSPIRE].

[29] L.E. Ibáñez and M. Montero, *A note on the WGC, effective field theory and clockwork within string theory*, *JHEP* **02** (2018) 057 [arXiv:1709.02392] [ɪɴSPIRE].

[30] S. Andriolo, D. Junghans, T. Noumi and G. Shiu, *A tower weak gravity conjecture from infrared consistency*, *Fortsch. Phys.* **66** (2018) 1800020 [arXiv:1802.04287] [ɪɴSPIRE].

[31] B. Heidenreich, M. Reece and T. Rudelius, *Emergence of weak coupling at large distance in quantum gravity*, *Phys. Rev. Lett.* **121** (2018) 051601 [arXiv:1802.08698] [ɪɴSPIRE].

[32] G. Aldazabal and L.E. Ibáñez, *A note on 4D heterotic string vacua, FI-terms and the Swampland*, *Phys. Lett.* **B 782** (2018) 375 [arXiv:1804.07322] [ɪɴSPIRE].

[33] E. Gonzalo and L.E. Ibáñez, *The fundamental need for a SM Higgs and the weak gravity conjecture*, *Phys. Lett.* **B 786** (2018) 272 [arXiv:1806.09647] [ɪɴSPIRE].

[34] R. Blumenhagen, D. Klaewer, L. Schlechter and F. Wolf, *The Refined Swampland Distance Conjecture in Calabi-Yau Moduli Spaces*, *JHEP* **06** (2018) 052 [arXiv:1803.04989] [ɪɴSPIRE].

[35] R. Blumenhagen, *Large field inflation/quintessence and the refined swampland distance conjecture*, PoS(CORFU2017)175 [arXiv:1804.10504] [ɪɴSPIRE].

[36] C. Vafa, *Evidence for F-theory*, *Nucl. Phys.* **B 469** (1996) 403 [hep-th/9602022] [ɪɴSPIRE].

[37] D.R. Morrison and C. Vafa, *Compactifications of F-theory on Calabi-Yau threefolds. 1*, *Nucl. Phys.* **B 473** (1996) 74 [hep-th/9602114] [ɪɴSPIRE].

[38] D.R. Morrison and C. Vafa, *Compactifications of F-theory on Calabi-Yau threefolds. 2*, *Nucl. Phys.* **B 476** (1996) 437 [hep-th/9603161] [ɪɴSPIRE].

[39] B. Heidenreich, M. Reece and T. Rudelius, *Evidence for a sublattice weak gravity conjecture*, *JHEP* **08** (2017) 025 [arXiv:1606.08437] [ɪɴSPIRE].

[40] J. Polchinski, *Monopoles, duality and string theory*, *Int. J. Mod. Phys.* **A 19S1** (2004) 145 [hep-th/0304042] [ɪɴSPIRE].

[41] B. Haghighat, S. Murthy, C. Vafa and S. Vandoren, *F-theory, spinning black holes and multi-string branches*, *JHEP* **01** (2016) 009 [arXiv:1509.00455] [ɪɴSPIRE].

[42] C. Lawrie, S. Schäfer-Nameki and T. Weigand, *Chiral 2d theories from N = 4 SYM with varying coupling*, *JHEP* **04** (2017) 111 [arXiv:1612.05640] [ɪɴSPIRE].

[43] N. Seiberg and E. Witten, *Comments on string dynamics in six-dimensions*, *Nucl. Phys.* **B 471** (1996) 121 [hep-th/9603003] [ɪɴSPIRE].

[44] E. Witten, *Phase transitions in M-theory and F-theory*, *Nucl. Phys.* **B 471** (1996) 195 [hep-th/9603150] [ɪɴSPIRE].

[45] J.J. Heckman and T. Rudelius, *Top Down Approach to 6D SCFTs*, arXiv:1805.06467 [ɪɴSPIRE].

[46] J.A. Harvey and A. Strominger, *The heterotic string is a soliton*, *Nucl. Phys.* **B 449** (1995) 535 [*Erratum ibid.* **B 458** (1996) 456] [hep-th/9504047] [ɪɴSPIRE].

[47] A.P. Braun and T. Watari, *Heterotic-type IIA duality and degenerations of K3 surfaces*, *JHEP* **08** (2016) 034 [arXiv:1604.06437] [ɪɴSPIRE].

[48] A. Klemm, P. Mayr and C. Vafa, *BPS states of exceptional noncritical strings*, *Nucl. Phys. Proc. Suppl.* **58** (1997) 177 [hep-th/9607139] [ɪɴSPIRE].

[49] A. Klemm, J. Manschot and T. Wotschke, *Quantum geometry of elliptic Calabi-Yau manifolds*, arXiv:1205.1795 [INSPIRE].

[50] B. Haghighat et al., *M-Strings*, *Commun. Math. Phys.* **334** (2015) 779 [arXiv:1305.6322] [INSPIRE].

[51] B. Haghighat, G. Lockhart and C. Vafa, *Fusing E-strings to heterotic strings: $E + E \to H$*, *Phys. Rev.* **D 90** (2014) 126012 [arXiv:1406.0850] [INSPIRE].

[52] B. Haghighat, A. Klemm, G. Lockhart and C. Vafa, *Strings of minimal* 6d *SCFTs*, *Fortsch. Phys.* **63** (2015) 294 [arXiv:1412.3152] [INSPIRE].

[53] M.X. Huang, S. Katz and A. Klemm, *Topological string on elliptic CY 3-folds and the ring of Jacobi forms*, *JHEP* **10** (2015) 125 [arXiv:1501.04891] [INSPIRE].

[54] H.-C. Kim, S. Kim and J. Park, 6d *strings from new chiral gauge theories*, arXiv:1608.03919 [INSPIRE].

[55] M. Del Zotto and G. Lockhart, *On exceptional instanton strings*, *JHEP* **09** (2017) 081 [arXiv:1609.00310] [INSPIRE].

[56] M. Del Zotto et al., *Topological strings on singular elliptic Calabi-Yau 3-folds and minimal* 6d *SCFTs*, *JHEP* **03** (2018) 156 [arXiv:1712.07017] [INSPIRE].

[57] J. Gu, M.-x. Huang, A.-K. Kashani-Poor and A. Klemm, *Refined BPS invariants of* 6d *SCFTs from anomalies and modularity*, *JHEP* **05** (2017) 130 [arXiv:1701.00764] [INSPIRE].

[58] J. Kim, K. Lee and J. Park, *On elliptic genera of* 6d *string theories*, *JHEP* **10** (2018) 100 [arXiv:1801.01631] [INSPIRE].

[59] M. Del Zotto and G. Lockhart, *Universal features of BPS strings in six-dimensional SCFTs*, *JHEP* **08** (2018) 173 [arXiv:1804.09694] [INSPIRE].

[60] A.N. Schellekens and N.P. Warner, *Anomalies and modular invariance in string theory*, *Phys. Lett.* **B 177** (1986) 317 [INSPIRE].

[61] A.N. Schellekens and N.P. Warner, *Anomalies, characters and strings*, *Nucl. Phys.* **B 287** (1987) 317 [INSPIRE].

[62] W. Lerche, B.E.W. Nilsson, A.N. Schellekens and N.P. Warner, *Anomaly cancelling terms from the elliptic genus*, *Nucl. Phys.* **B 299** (1988) 91 [INSPIRE].

[63] T. Eguchi, H. Ooguri, A. Taormina and S.-K. Yang, *Superconformal algebras and string compactification on manifolds with* SU(N) *holonomy*, *Nucl. Phys.* **B 315** (1989) 193 [INSPIRE].

[64] T. Kawai, Y. Yamada and S.-K. Yang, *Elliptic genera and $N = 2$ superconformal field theory*, *Nucl. Phys.* **B 414** (1994) 191 [hep-th/9306096] [INSPIRE].

[65] T. Kawai, *String duality and modular forms*, *Phys. Lett.* **B 397** (1997) 51 [hep-th/9607078] [INSPIRE].

[66] V. Gritsenko, *Elliptic genus of Calabi-Yau manifolds and Jacobi and Siegel modular forms*, math/9906190 [INSPIRE].

[67] F. Denef, *Les Houches lectures on constructing string vacua*, *Les Houches* **87** (2008) 483 [arXiv:0803.1194] [INSPIRE].

[68] W. Taylor, *TASI lectures on supergravity and string vacua in various dimensions*, arXiv:1104.2051 [INSPIRE].

[69] T. Weigand, *TASI lectures on F-theory*, arXiv:1806.01854 [INSPIRE].

[70] T.W. Grimm and T. Weigand, *On abelian gauge symmetries and proton decay in global F-theory GUTs*, *Phys. Rev.* **D 82** (2010) 086009 [arXiv:1006.0226] [INSPIRE].

[71] D.S. Park, *Anomaly equations and intersection theory*, *JHEP* **01** (2012) 093 [arXiv:1111.2351] [INSPIRE].

[72] D.R. Morrison and D.S. Park, *F-theory and the Mordell-Weil group of elliptically-fibered Calabi-Yau threefolds*, *JHEP* **10** (2012) 128 [arXiv:1208.2695] [INSPIRE].

[73] S.-J. Lee, D. Regalado and T. Weigand, *6d SCFTs and* U(1) *flavour symmetries*, arXiv:1803.07998 [INSPIRE].

[74] J.J. Heckman, D.R. Morrison and C. Vafa, *On the classification of* 6D *SCFTs and Generalized ADE Orbifolds*, *JHEP* **05** (2014) 028 [*Erratum ibid.* **06** (2015) 017] [arXiv:1312.5746] [INSPIRE].

[75] C.L. Wang, *On the incompleteness of the Weil-Petersson metric along degenerations of Calabi-Yau manifolds*, *Math. Res. Lett.* **4** (1997) 157.

[76] J.A. Minahan, D. Nemeschansky, C. Vafa and N.P. Warner, *E strings and* $N = 4$ *topological Yang-Mills theories*, *Nucl. Phys.* **B 527** (1998) 581 [hep-th/9802168] [INSPIRE].

[77] D.R. Morrison and W. Taylor, *Classifying bases for* 6D *F-theory models*, *Central Eur. J. Phys.* **10** (2012) 1072 [arXiv:1201.1943] [INSPIRE].

[78] L. Martucci, *Topological duality twist and brane instantons in F-theory*, *JHEP* **06** (2014) 180 [arXiv:1403.2530] [INSPIRE].

[79] J.M. Maldacena, A. Strominger and E. Witten, *Black hole entropy in M-theory*, *JHEP* **12** (1997) 002 [hep-th/9711053] [INSPIRE].

[80] S.A. Cherkis and J.H. Schwarz, *Wrapping the M-theory five-brane on* K3, *Phys. Lett.* **B 403** (1997) 225 [hep-th/9703062] [INSPIRE].

[81] E. Witten, *String theory dynamics in various dimensions*, *Nucl. Phys.* **B 443** (1995) 85 [hep-th/9503124] [INSPIRE].

[82] S. Kachru and C. Vafa, *Exact results for* $N = 2$ *compactifications of heterotic strings*, *Nucl. Phys.* **B 450** (1995) 69 [hep-th/9505105] [INSPIRE].

[83] S.-J. Lee, W. Lerche and T. Weigand, *A stringy test of the scalar weak gravity conjecture*, arXiv:1810.05169 [INSPIRE].

[84] M. Cvetič and D. Youm, *Near BPS saturated rotating electrically charged black holes as string states*, *Nucl. Phys.* **B 477** (1996) 449 [hep-th/9605051] [INSPIRE].

[85] M. Cvetič and C.M. Hull, *Black holes and U duality*, *Nucl. Phys.* **B 480** (1996) 296 [hep-th/9606193] [INSPIRE].

[86] R. Emparan and H.S. Reall, *Black holes in higher dimensions*, *Living Rev. Rel.* **11** (2008) 6 [arXiv:0801.3471] [INSPIRE].

[87] R. Emparan, T. Harmark, V. Niarchos and N.A. Obers, *New horizons for black holes and branes*, *JHEP* **04** (2010) 046 [arXiv:0912.2352] [INSPIRE].

[88] M.J. Duff, H. Lü and C.N. Pope, *The black branes of M-theory*, *Phys. Lett.* **B 382** (1996) 73 [hep-th/9604052] [INSPIRE].

[89] D. Harlow, *Wormholes, emergent gauge fields and the weak gravity conjecture*, *JHEP* **01** (2016) 122 [arXiv:1510.07911] [INSPIRE].

[90] A. Strominger, *Massless black holes and conifolds in string theory*, *Nucl. Phys.* **B 451** (1995) 96 [hep-th/9504090] [INSPIRE].

[91] C. Vafa, *A stringy test of the fate of the conifold*, *Nucl. Phys.* **B 447** (1995) 252 [hep-th/9505023] [INSPIRE].

[92] S. Dimopoulos, S. Kachru, J. McGreevy and J.G. Wacker, *N-flation*, *JCAP* **08** (2008) 003 [hep-th/0507205] [INSPIRE].

[93] G. Dvali, *Black holes and large N species solution to the hierarchy problem*, *Fortsch. Phys.* **58** (2010) 528 [arXiv:0706.2050] [INSPIRE].

[94] G. Dvali and M. Redi, *Black hole bound on the number of species and quantum gravity at LHC*, *Phys. Rev.* **D 77** (2008) 045027 [arXiv:0710.4344] [INSPIRE].

[95] E. Witten, *Elliptic genera and quantum field theory*, *Commun. Math. Phys.* **109** (1987) 525 [INSPIRE].

[96] O. Alvarez, T.P. Killingback, M.L. Mangano and P. Windey, *String theory and loop space index theorems*, *Commun. Math. Phys.* **111** (1987) 1 [INSPIRE].

[97] T. Kawai, *K3 surfaces, Igusa cusp form and string theory*, in the proceedings of *Topological field theory, primitive forms and related topics. Proceedings, 38th Taniguchi Symposium*, December 9–13, Kyoto, Japan, and *RIMS Symposium*, December 16–19, Kyoto, Japan (1996), hep-th/9710016 [INSPIRE].

[98] T. Kawai, *String duality and enumeration of curves by Jacobi forms*, in the proceedings of the *Integrable systems and algebraic geometry. Taniguchi Symposium*, June 30–July 4, Kobe, Japan (1997), hep-th/9804014 [INSPIRE].

[99] W. Lerche and N.P. Warner, *Index theorems in $N = 2$ superconformal theories*, *Phys. Lett.* **B 205** (1988) 471 [INSPIRE].

[100] J.A. Harvey and G.W. Moore, *Algebras, BPS states and strings*, *Nucl. Phys.* **B 463** (1996) 315 [hep-th/9510182] [INSPIRE].

[101] W. Lerche, *Elliptic index and superstring effective actions*, *Nucl. Phys.* **B 308** (1988) 102 [INSPIRE].

[102] W. Lerche, A.N. Schellekens and N.P. Warner, *Ghost triality and superstring partition functions*, *Phys. Lett.* **B 214** (1988) 41 [INSPIRE].

[103] M. Alim and E. Scheidegger, *Topological strings on elliptic fibrations*, *Commun. Num. Theor. Phys.* **08** (2014) 729 [arXiv:1205.1784] [INSPIRE].

[104] M. Eichler and D. Zagier, *The theory of Jacobi forms*, Birkhäuser, Germany (1995).

[105] R. Dijkgraaf, G.W. Moore, E.P. Verlinde and H.L. Verlinde, *Elliptic genera of symmetric products and second quantized strings*, *Commun. Math. Phys.* **185** (1997) 197 [hep-th/9608096] [INSPIRE].

[106] M.-X. Huang, A. Klemm and M. Poretschkin, *Refined stable pair invariants for E-, M- and $[p, q]$-strings*, *JHEP* **11** (2013) 112 [arXiv:1308.0619] [INSPIRE].

[107] F. Benini, R. Eager, K. Hori and Y. Tachikawa, *Elliptic genera of 2d $\mathcal{N} = 2$ gauge theories*, *Commun. Math. Phys.* **333** (2015) 1241 [arXiv:1308.4896] [INSPIRE].

[108] D.S. Berman and J.A. Harvey, *The self-dual string and anomalies in the M5-brane*, *JHEP* **11** (2004) 015 [hep-th/0408198] [INSPIRE].

[109] K. Ohmori, H. Shimizu, Y. Tachikawa and K. Yonekura, *Anomaly polynomial of general 6D SCFTs*, *PTEP* **2014** (2014) 103B07 [arXiv:1408.5572] [INSPIRE].

[110] H. Shimizu and Y. Tachikawa, *Anomaly of strings of 6D $\mathcal{N} = (1,0)$ theories*, *JHEP* **11** (2016) 165 [arXiv:1608.05894] [INSPIRE].

[111] T. Weigand and F. Xu, *The Green-Schwarz mechanism and geometric anomaly relations in 2d* $(0,2)$ *F-theory vacua*, *JHEP* **04** (2018) 107 [arXiv:1712.04456] [INSPIRE].

[112] R. Wazir, *Arithmetic on elliptic threefolds*, *Comp. Math.* **140** (2001) 567 [math/0112259].

[113] O.J. Ganor and A. Hanany, *Small E_8 instantons and tensionless noncritical strings*, *Nucl. Phys.* **B 474** (1996) 122 [hep-th/9602120] [INSPIRE].

[114] D. Klevers and W. Taylor, *Three-index symmetric matter representations of* SU(2) *in F-theory from non-Tate form Weierstrass models*, *JHEP* **06** (2016) 171 [arXiv:1604.01030] [INSPIRE].

[115] D.R. Morrison and D.S. Park, *Tall sections from non-minimal transformations*, *JHEP* **10** (2016) 033 [arXiv:1606.07444] [INSPIRE].

[116] N. Raghuram, *Abelian F-theory models with charge-3 and charge-4 matter*, *JHEP* **05** (2018) 050 [arXiv:1711.03210] [INSPIRE].

[117] K. Oguiso, *On algebraic fiber spaces structures on a Calabi-Yau 3-fold*, *Int. Jour. Math.* **4** (1993) 439.

[118] M. Esole and M.J. Kang, *Characteristic numbers of crepant resolutions of Weierstrass models*, arXiv:1807.08755 [INSPIRE].

[119] P. Arras, A. Grassi and T. Weigand, *Terminal singularities, Milnor numbers and matter in F-theory*, *J. Geom. Phys.* **123** (2018) 71 [arXiv:1612.05646] [INSPIRE].

[120] A. Grassi and T. Weigand, *On topological invariants of algebraic threefolds with (Q-factorial) singularities*, arXiv:1804.02424 [INSPIRE].

[121] A. Dabholkar, S. Murthy and D. Zagier, *Quantum black holes, wall crossing and mock modular forms*, arXiv:1208.4074 [INSPIRE].

[122] K. Wirthmüller, *Root systems and Jacobi forms*, *Comp. Math.* **82** (1992) 293.

[123] S. Hosono, A. Klemm, S. Theisen and S.-T. Yau, *Mirror symmetry, mirror map and applications to Calabi-Yau hypersurfaces*, *Commun. Math. Phys.* **167** (1995) 301 [hep-th/9308122] [INSPIRE].

[124] S. Hosono, A. Klemm, S. Theisen and S.-T. Yau, *Mirror symmetry, mirror map and applications to complete intersection Calabi-Yau spaces*, *Nucl. Phys.* **B 433** (1995) 501 [hep-th/9406055] [INSPIRE].

[125] M. Kreuzer and H. Skarke, *PALP: a Package for analyzing lattice polytopes with applications to toric geometry*, *Comput. Phys. Commun.* **157** (2004) 87 [math/0204356] [INSPIRE].

[126] A.P. Braun et al., *PALP — A user manual*, in *Strings, gauge fields, and the geometry behind: the legacy of Maximilian Kreuzer*, A. Rebhan et al. eds., World Scientific, Singapore, (2012), arXiv:1205.4147.