Spatial Geometry of the Electric Field Representation of Non-Abelian Gauge Theories*

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A unitary transformation $\Psi[E] = \exp(i\Omega[E]/g)F[E]$ is used to simplify the Gauss law constraint of non-abelian gauge theories in the electric field representation. This leads to an unexpected geometrization because $\omega_i^a \equiv -\delta\Omega[E]/\delta E^{ai}$ transforms as a (composite) connection. The geometric information in ω_i^a is transferred to a gauge invariant spatial connection Γ^i_{jk} and torsion by a suitable choice of basis vectors for the adjoint representation which are constructed from the electric field E^{ai}. A metric is also constructed from E^{ai}. For gauge group $SU(2)$, the spatial geometry is the standard Riemannian geometry of a 3-manifold, and for $SU(3)$ it is a metric preserving geometry with both conventional and unconventional torsion. The transformed Hamiltonian is local. For a broad class of physical states, it can be expressed entirely in terms of spatial geometric, gauge invariant variables.

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I. INTRODUCTION

The canonical commutation relations and Gauss law constraint of Hamiltonian gauge theories in temporal gauge are invariant under spatial diffeomorphisms of the canonical variables $A_i^a(x)$ and $E^{ai}(x)$. This local $GL(3)$ symmetry is broken in the Hamiltonian in a simple way because of the appearance of the Cartesian metric δ_{ij} of flat space, and the energy density transforms as a $GL(3)$ tensor density. In this paper we discuss a formulation of non-abelian gauge theories in which the Gauss law constraint is easily implemented and the Hamiltonian is expressed in terms of variables which are gauge invariant or covariant and also geometric, i.e. they are $GL(3)$ tensors, connections or curvatures. The resulting theory has an elegant mathematical structure but it is far from clear that the spatial geometry will be helpful for dynamical calculations or offer any advantages over such well-developed approaches as lattice gauge theories.

We choose to work in the electric representation of gauge theories in which states $\Psi[E^{ai}]$ are functionals of the electric field. In common with an earlier non-geometric approach to the $SU(2)$ theory [1] the key element of our work is a unitary transformation $\Psi[E] = \exp(i\Omega[E]/g)F[E]$ of the theory which simplifies the form of the Gauss law constraint. The phase $\Omega[E]$ is a local $GL(3)$ invariant functional of the electric field, whose variation under infinitesimal gauge transformations is $\delta\Omega[E] = \int d^3x \quad \theta^a \partial_i E^{ai}$. These gauge and $GL(3)$ properties of $\Omega[E]$ imply that the quantity $\omega_i^a[E] = -\delta\Omega[E]/\delta E^{ai}$ transforms as a Lie algebra valued connection on the initial value surface

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$I\!R^3$. Thus a composite gauge connection $\omega_i^a[E]$ appears and plays a central role in our formulation although the fundamental variable E^{ai} transforms homogeneously under gauge transformations. The Hamiltonian is local but, as in earlier work [1,2] it involves functional derivatives $\delta/\delta E^{ai}$ up to fourth order.

For gauge group $SU(2)$, $\omega_i^a[E]$ is simply the standard Riemannian spin connection on a three-manifold with frame 1-form $e_i^a(x)$ related to the electric field by $E^{ai} = \varepsilon^{abc}\varepsilon^{ijk}e_j^b e_k^c/2$. One can argue that under fairly general assumptions one can restrict to wave functionals $F[G_{ij}]$ where $G_{ij} = e_i^a e_j^a$ is a composite metric (we actually use the tensor variable $\varphi^{ij} = \det G\, G^{ij}$ for reasons explained in Sec. 3). Such states satisfy the Gauss law constraint, and the Hamiltonian acting on them can be rewritten in terms of the Christoffel connection Γ_{ij}^k and curvature $R^k{}_{\ell ij}$. Thus a Riemannian spatial geometry underlies $SU(2)$ gauge theory. It is actually known [3,4] from work on Ashtekhar variables in gravity that the spin connection on a 3-manifold is the variational derivative of the local functional $\Omega[E]$. It is not lost upon us that the Ashtekhar approach makes gravity look a lot like gauge theory, while our approach makes gauge theory look a lot like gravity.

One could view the structure described above as the accidental consequence of the fact that the gauge group $SU(2)$ coïncides with the tangent space group of a three-manifold. However we are able to give a formula for the phase $\Omega[E]$ for a general gauge group G. The formula is not entirely explicit because it involves the inverse of a matrix of dimension $3\dim G \times 3\dim G$ which is a quadratic function of E^{ai}. But it is explicit enough to see that the general structure of the theory is similar to $SU(2)$ but that the associated spatial geometry, which we outline for $SU(3)$, is more complicated. It can be described as a metric-preserving geometry with an unconventional torsion.

One may also study the spatial geometry of a magnetic formulation of gauge theory. Indeed we drew our inspiration from a recent study [5] of the $SU(2)$ theory in which a curious Einstein space geometry with torsion appeared. The geometry is correct, but the application made to Hamiltonian dynamics in [5] failed because of the Wu-Yang ambiguity [6] which is generically continuous in three spatial dimensions [7]. A new magnetic formulation [8] avoids the problem and leads to a Hamiltonian which is second order in functional derivatives $\delta/\delta G_{ij}$ with respect to a composite metric variable, but is non-local.

We also wish to cite recent papers involving a geometrical approach to gauge theories in the Lagrangian formalism by Lunev [9] and others [10,11] in which a spatial metric has appeared in studies of gauge theories. Finally, there are recent extensive studies of Hamiltonian dynamics for gauge theory in light-cone gauge [12].

II. THE UNITARY TRANSFORMATION AND ITS CONSEQUENCES

The canonical variables of a non-abelian gauge theory are the vector potential $A_i^a(x)$ and electric field $E^{ai}(x)$ which satisfy the commutation relations

$$\left[A_i^a(x), E^{bj}(x')\right] = i\delta^{ab}\delta_i^j\,\delta^{(3)}(x-x'). \tag{2.1}$$

In temporal gauge, $A_0^a(x) = 0$, the generator of spatial gauge transformations with parameter $\theta^a(x)$ is

$$\begin{aligned} \mathcal{G}[\theta] &= \textstyle\int d^3x\ \theta^a(x)\ \mathcal{G}^a(x) \\ \mathcal{G}^a(x) &= \frac{1}{g}\ D_i E^{ai}(x) = \frac{1}{g}\left(\partial_i E^{ai}(x) + g\, f^{abc}A_i^b(x)E^{ci}(x)\right), \end{aligned} \tag{2.2}$$

and Eq. (2.1) implies the quantum gauge transformation rules

$$\begin{aligned} \delta A_i^a(x) &= -i\left[\mathcal{G}[\theta], A_i^a(x)\right] = \frac{1}{g}\left(\partial_i\theta^a(x) + g\, f^{abc}A_i^b(x)\theta^c(x)\right) \\ \delta E^{ai}(x) &= -i\left[\mathcal{G}[\theta], E^{ai}(x)\right] = f^{abc}E^{bi}(x)\theta^c(x). \end{aligned} \tag{2.3}$$

Using the magnetic field

$$B^{ai}(x) = \varepsilon^{ijk}\left(\partial_j A_k^a(x) + \frac{1}{2}\ g f^{abc}A_j^b(x)A_k^c(x)\right) \tag{2.4}$$

which transforms homogeneously, i.e. as the electric field in Eq. (2.3), the Hamiltonian can be written as

$$H = \frac{1}{2}\int d^3x\ \delta_{ij}\left(E^{ai}(x)E^{aj}(x) + B^{ai}(x)B^{aj}(x)\right)\ . \tag{2.5}$$

We now observe that Eqs. (2.1-2.4) are covariant under coordinate transformations $x^i \to y^\alpha$ on the domain $I\!R^3$ provided that

1. $A_i^a(x)$ transforms as a covariant vector

$$A'^a_\alpha(y) = \frac{\partial x^i}{\partial y^\alpha} A_i^a(x) \tag{2.6}$$

which is implied by the 1-form interpretation $A^a = A_i^a dx^i$ of the vector potential and

2. $E^{ai}(x)$ transforms as a contravariant vector density

$$E'^{a\alpha}(y) = \left|\frac{\partial x}{\partial y}\right| \frac{\partial y^\alpha}{\partial x^i} E^{ai}(x) \tag{2.7}$$

which is consistent with its implementation as a functional derivative $E^{ai}(x) = -i\delta/\delta A_i^a(x)$ in the familiar magnetic representation of (2.1).

Note that the gauge parameters $\theta^a(x)$ transform as $GL(3)$ scalars and that $\mathcal{G}^a(x)$ is a scalar density. No connection Γ^i_{jk} is required in Eq. (2.2) because E^{ai} is a density of weight one. The magnetic field is also a contravariant vector density of weight one.

The Hamiltonian fails to be $GL(3)$ invariant because the fixed cartesian metric appears, but one sees that the energy density transforms as the δ_{ij} trace of a contravariant symmetric tensor density of weight two. The Hamiltonian is gauge invariant, viz., $[\mathcal{G}[\theta], H] = 0$, and the dynamical problem of gauge theories can be formally stated as the problem of diagonalizing H on the physical subspace of gauge invariant states Ψ which satisfy the Gauss law constraint

$$\mathcal{G}^a(x)\Psi = \frac{1}{g}\left(\partial_i E^{ai}(x) + g\, f^{abc} A_i^b(x) E^{ci}(x)\right)\Psi = 0\ . \tag{2.8}$$

Our goal here is to formulate this dynamical problem in a way which maintains the $GL(3)$ properties of the theory.

We work in electric field representation with state functionals $\Psi[E]$. Then $E^{ai}(x)$ is realized by simple multiplication and $A_i^a(x) = i\delta/\delta E^{ai}(x)$ by functional differentiation. It would be easy to implement the Gauss law constraint if the gauge generator contained only the second term

$$\bar{\mathcal{G}}^a(x) = -if^{abc}E^{bi}(x)\frac{\delta}{\delta E^{ci}(x)} \tag{2.9}$$

because this operator simply generates local group rotations without spatial transport. Note that both $\mathcal{G}^a(x)$ and $\bar{\mathcal{G}}^a(x)$ satisfy the group algebra in the local form

$$\left[\mathcal{G}^a(x), \mathcal{G}^b(x')\right] = if^{abc}\delta^{(3)}(x-x')\,\mathcal{G}^c(x). \tag{2.10}$$

In the spirit of [1], we shall make a unitary transformation on the states and operators of the theory in order to simplify the gauge generators. We write

$$\begin{aligned}\Psi[E] &= \exp(i\Omega[E]/g)\ F[E]\\ \mathcal{O}(x) &= \exp(i\Omega[E]/g)\ \bar{\mathcal{O}}(x)\ \exp(-i\Omega[E]/g)\end{aligned} \tag{2.11}$$

and require that

$$\mathcal{G}^a(x)\ \exp(i\Omega[E]/g)\ F[E] = \exp(i\Omega[E]/g)\ \bar{\mathcal{G}}^a(x)\ F[E]. \tag{2.12}$$

The phase $\Omega[E]$ thus satisfies

$$\begin{aligned}\exp(-i\Omega[E]/g)\,\mathcal{G}^a(x)\ \exp(i\Omega[E]/g) &= \mathcal{G}^a(x) + \frac{i}{g}\left[\mathcal{G}^a(x), \Omega[E]\,\right]\\ &= \bar{\mathcal{G}}^a(x)\end{aligned} \tag{2.13}$$

This is equivalent to the requirement that the gauge variation of $\Omega[E]$ be

$$\delta\Omega[E] = -i\ \left[\mathcal{G}[\theta]\,, \Omega[E]\,\right] = \int d^3x\ \theta^a(x)\partial_i E^{ai}(x)\ . \tag{2.14}$$

We also require that the phase $\Omega[E]$ be $GL(3)$ invariant, so that the unitary transformation preserves the behavior of the theory under spatial diffeomorphisms. Note that for an abelian gauge group $U(1)$ any $\Omega[E]$ is gauge invariant, so that we cannot satisfy Eq. (2.14). Thus our treatment must be restricted to non-abelian groups.

We will now show that the form of the resulting theory is essentially determined by these two requirements on $\Omega[E]$. In subsequent sections we will give local formulae for the phase, i.e. of the form $\Omega[E] = \int d^3x\ f(E(x), \partial E(x))$ first for gauge group $SU(2)$ and then for general G.

So we now assume the existence of a $GL(3)$ invariant phase whose gauge variation is given by Eq. (2.14), and work out the structure of the unitary transformed theory. The transformed canonical variables are

$$\begin{aligned}
\bar{E}^{ai}(x) &= \exp(-i\Omega[E]/g)\ E^{ai}(x)\ \exp(i\Omega[E]/g) = E^{ai}(x) \\
\bar{A}^a_i(x) &= \exp(-i\Omega[E]/g)\ A^a_i(x)\ \exp(i\Omega[E]/g) \\
&= A^a_i(x) + \frac{i}{g}\,[A^a_i(x), \Omega[E]\,] \\
&= A^a_i(x) - \frac{1}{g}\frac{\delta}{\delta E^{ai}(x)}\Omega[E] \\
&\equiv i\frac{\delta}{\delta E^{ai}(x)} + \frac{1}{g}\ \omega^a_i(x)
\end{aligned} \tag{2.15}$$

The quantity $\omega^a_i(x)$ is the variational derivative of a $GL(3)$ invariant functional with respect to a vector density so $\omega^a_i(x)$ is a covariant vector under spatial diffeomorphisms. Its gauge variation is

$$\begin{aligned}
\delta\omega^a_i(x)/g &= i\left[\mathcal{G}[\theta], \tfrac{\delta\Omega[E]}{\delta E^{ai}(x)}\right]/g \\
&= [\mathcal{G}[\theta], [\Omega[E], A^a_i(x)]\,] \\
&= [\Omega[E], [\mathcal{G}[\theta], A^a_i(x)]\,] - [\,[\mathcal{G}[\theta], \Omega[E]\,]\,, A^a_i(x)] \\
&= -i\,[\Omega[E], D_i\theta^a(x)]\,/g - i\left[\textstyle\int d^3y\ \theta^b(y)\partial_j E^{bj}(y), A^a_i(x)\right] \\
&= -f^{abc}\frac{\delta}{\delta E^{bi}(x)}\Omega[E]\ \theta^c(x) + \frac{1}{g}\ \partial_i\theta^a(x) \\
&= \frac{1}{g}\ \left(\partial_i\theta^a(x) + f^{abc}\omega^b_i(x)\theta^c(x)\right) \equiv \frac{1}{g}\ \hat{D}_i\theta^a(x)
\end{aligned} \tag{2.16}$$

Thus $\omega^a_i[E]/g$ is a local composite function of E^{ai} which transforms as a gauge potential. One could almost derive this result by inspection of Eq. (2.15), since the gauge variation of $\bar{A}^a_i$ is

$$\delta\bar{A}^a_i = -i\left[\bar{\mathcal{G}}[\theta], \bar{A}^a_i\right] = \frac{1}{g}\left(\partial_i\theta^a + g\ f^{abc}\bar{A}^b_i\theta^c\right). \tag{2.17}$$

Since $i\delta/\delta E$ transforms homogeneously, the second term in Eq. (2.15), ω/g, must transform as a potential. However the longer derivation in Eq. (2.16) has the virtue of emphasizing that if the gauge variation of any functional $\Omega[E]$ is given by Eq. (2.14) then $\delta\Omega/\delta E^{ai}$ transforms as a gauge connection.

In the unitary transformed theory, $\hat{D}_i$ will denote a gauge covariant derivative formed with the composite connection ω^a_i. The magnetic field formed using Eq. (2.4) with A replaced by ω/g and removing a factor $1/g$ will be denoted by $\hat{B}^{ai} = \varepsilon^{ijk}\left(\partial_j\omega^a_k + \frac{1}{2}f^{abc}\omega^b_j\omega^c_k\right)$. It also follows from the trivial relation

$$\delta\Omega[E] = \int d^3x\ \frac{\delta\Omega[E]}{\delta E^{ai}}\delta E^{ai} \tag{2.18}$$

and use of Eq. (2.14) with $\delta E^{ai} = f^{abc}E^{bi}\theta^c$ that a "Bianchi identity" holds in the form

$$\hat{D}_i E^{ai} = 0. \tag{2.19}$$

The transformed Hamiltonian is

$$\bar{H} = \frac{1}{2}\int d^3x\ \delta_{ij}(\bar{E}^{ai}\bar{E}^{aj} + \bar{B}^{ai}\bar{B}^{aj}). \tag{2.20}$$

The electric term is quite simple. One can define the gauge invariant symmetric tensor variable

$$\varphi^{ij} = E^{ai}E^{aj} \tag{2.21}$$

and express the electric energy density as the multiplication operator $\frac{1}{2}\delta_{ij}\varphi^{ij}$.

The magnetic field $\bar{B}^{ai}(x)$ applied to a state $F[E]$ is

$$\begin{aligned}\bar{B}^{ai}F[E] &= \varepsilon^{ijk}\left(\partial_j \bar{A}^a_k + \tfrac{1}{2}g\, f^{abc}\bar{A}^b_j\bar{A}^c_k\right)F[E] \\ &= \left[\frac{1}{g}\,\hat{B}^{ai} + i\,\varepsilon^{ijk}\hat{D}_j\frac{\delta}{\delta E^{ak}} - \frac{g}{2}\,\varepsilon^{ijk}f^{abc}\frac{\delta}{\delta E^{bj}}\frac{\delta}{\delta E^{ck}} \right. \\ &\qquad \left. + i\,g\,\varepsilon^{ijk}f^{abc}\frac{\delta\omega^c_k}{\delta E^{bj}}\right]F[E]\end{aligned} \tag{2.22}$$

The beginning of a geometric structure is evident in the first two terms, namely the composite magnetic field and the ω-covariant derivative of $\delta F/\delta E$. The third term contains the second functional derivative $\delta^2/\delta E\delta E$ which is characteristic of the electric representation of non-abelian theories [1,2]. The Hamiltonian therefore contains terms up to fourth order in $\delta/\delta E$. The fourth term in Eq. (2.22) comes from the operator reordering $\left[\delta/\delta E^{bj}(x), \omega^c_k(x)\right]$ which was necessary to obtain the the $\hat{D}_j$ covariant derivative. As will be seen explicitly for the $SU(2)$ case, this ordering term involves the singular objects $\partial\delta(0)$ and $\delta(0)$ and is one troublesome feature of a nonlinear theory with functional derivatives. Similar terms also were present in [1]. Our derivation of the Hamiltonian has been rather formal and requires regularization. We shall argue in the appendix that this particular ordering term vanishes if covariant point splitting regularization is used, but one must study the additional ordering terms in the magnetic energy density which is quadratic in $\bar{B}$.

We will discuss the Hamiltonian further in later sections, after we elucidate its spatial geometric structure. We close this section with a remark concerning the uniqueness of $GL(3)$-invariant functionals ,which satisfy Eq. (2.14). One must not expect a unique solution for a given gauge group , but the difference $\Omega'[E] - \Omega[E]$ between any two functionals which satisfy the requirements must be both gauge and $GL(3)$-invariant. For example one could have

$$\Omega'[E] - \Omega[E] \propto \int d^3x (\det \varphi^{ij})^{1/4}. \tag{2.23}$$

III. THE $SU(2)$ THEORY

In this section we study the $SU(2)$ gauge theory in more detail. We first give explicit formulas for the phase $\Omega[E]$ and composite gauge connection ω^a_i and then develop the associated spatial geometry which turns out to be the standard Riemannian geometry of a 3-manifold.

The simplest phase candidate one can write using the electric field E^{ai} and its matrix inverse E^a_i, i.e. $E^{ai}E^b_i = \delta^{ab}$, turns out to be successful. This is

$$\Omega[E] = \frac{1}{2}\int d^3x\ \varepsilon^{abc}\ E^{ai}(x)E^{bj}(x)\partial_i E^c_j(x)\ . \tag{3.1}$$

It is $GL(3)$ invariant because the integrand has density weight $+1$ and terms arising from the ∂_i derivative of the coördinate change of E^c_j, which is a covariant vector density, cancel. Although we need only the infinitesimal gauge variation to confirm Eq. (2.14), it is no more difficult to study the finite gauge transformation $E^{ai} \to T^{ab}E^{bi}$ where T^{ab} is an $SO(3)$ matrix. We have

$$\begin{aligned}\Omega[TE] &= \frac{1}{2}\int d^3x\ \varepsilon^{abc}\ T^{a\bar{a}}T^{b\bar{b}}\left\{T^{c\bar{c}}E^{\bar{a}i}E^{\bar{b}j}\partial_i E^{\bar{c}}_j + \partial_i T^{c\bar{b}}E^{\bar{a}i}\right\} \\ &= \Omega[E] - \frac{1}{2}\int d^3x\ \varepsilon^{abc}\ (T^{-1}\partial_i T)^{bc}E^{ai}\ .\end{aligned} \tag{3.2}$$

Group invariance of the structure constants was used to obtain the first term, and the invariant 1-forms $T^{-1}\partial T$ appear in the second term, whose infinitesimal limit is Eq. (2.14).

We already know that $\omega^a_i = -\delta\Omega/\delta E^{ai}$ is an $SO(3)$ gauge connection, so it should not be a great surprise that it turns out to be a familiar object. We define a new variable e^a_i by

$$E^{ai} = \frac{1}{2}\varepsilon^{ijk}\varepsilon^{abc}e^b_j e^c_k \ , \tag{3.3}$$

so that e^a_i has dimension $+1$, and is a gauge covariant, $GL(3)$ vector. These are exactly the properties of the frame 1-form (dreibein) on a 3-manifold with tangent space group $SO(3)$ and metric

$$G_{ij} = e^a_i e^a_j \ . \tag{3.4}$$

By straightforward computation one can show that

$$\begin{aligned}\omega^a_i &= -\frac{\delta\Omega}{\delta E^{ai}} = -\frac{1}{2}\,\varepsilon^{abc}\left\{e^{bj}\partial_i e^c_j - e^{bj}e^c_k\Gamma^k_{ij}\right\} \\ &= -\frac{1}{2}\,\varepsilon^{abc}\omega^{bc}_i \ . \end{aligned}\tag{3.5}$$

Here Γ^k_{ij} is the Christoffel symbol for the metric G_{ij}, and ω^{ab}_i is just the standard spin connection on a Riemannian 3-manifold. Thus the composite gauge potential ω^a_i of $SU(2)$ gauge theory is the well-known spin connection, and we now see that a conventional Riemannian spatial geometry underlies $SU(2)$ gauge theory.

A corollary of our discussion above is the fact that in three spatial dimensions the spin connection is the variational derivative of the local functional $\Omega[E]$ of Eq. (3.1). This was established in studies of the Ashtekar formalism for gravity in which the form of $\Omega[E]$ with Eq. (3.3) inserted was used, viz.,

$$\Omega[E] = \frac{1}{2}\int d^3x\ \varepsilon^{ijk}\ e^a_i(x)\partial_j e^a_k(x) \ , \tag{3.6}$$

showing that $\Omega[E]$ is the integral of a natural 3-form.

Actually we have been a little too hasty in the above. The definition Eq. (3.3) actually implies that $\det E^{ai} \geq 0$, whereas both signs of $\det E$ occur in gauge theory. So we should actually define

$$E^{ai} = \pm\frac{1}{2}\varepsilon^{ijk}\varepsilon^{abc}e^b_j e^c_k \ , \tag{3.7}$$

with $\pm$ according to whether $\det E > 0$ or < 0. For each sign above, there are two solutions for $e[E]$ which differ by a sign. We make the convention to choose the solution with $\det e^a_i > 0$, so that we take

$$e^a_i = \pm\sqrt{|\det E^{ai}|}\ E^a_i \tag{3.8}$$

as the solution to Eq. (3.7). One can show that Eqs. (3.4) and (3.5) remain valid (but Eq. (3.6) acquires a $\pm$ sign), so that ω^a_i is the same standard connection for both signs of $\det E$. Since ω^a_i is an even function of the frame, it can be reexpressed as an even function of E^{ai} and the sign in Eq. (3.8) cancels.

Note that

$$E^{ai} = \pm e^{ai}\det e \tag{3.9}$$

is a "densitized" inverse frame. One can show using Eqs. (3.5) and (3.7) that the total covariant derivative vanishes, i.e.,

$$\nabla_i E^{ak} \equiv \partial_i E^{ak} + \Gamma'^k_{ij}E^{aj} + \varepsilon^{abc}\omega^b_i E^c_k = 0 \ , \tag{3.10}$$

where Γ'^k_{ij} is a not-often-used but standard connection for the covariant differentiation of densities, namely

$$\begin{aligned}\Gamma'^k_{ij} &= -\frac{1}{2}\delta^k_j\partial_i \ln\det G_{mn} + \Gamma^k_{ij} \\ &= +\frac{1}{4}\delta^k_{[i}\partial_{j]}\ln\det\varphi^{mn} + \frac{1}{2}\varphi^{k\ell}\left[\partial_i\varphi_{j\ell} + \partial_j\varphi_{i\ell} - \partial_\ell\varphi_{ij}\right] \ , \end{aligned}\tag{3.11}$$

where, for reasons stated below, we have used the relation $\varphi^{ij} = \det G\, G^{ij}$ between the tensor density φ^{ij} introduced in the previous section and the inverse metric G^{ij}. One can solve Eq. (3.10) for ω^a_i and obtain a form equivalent to Eq. (3.5). The fact that $\nabla_i E^{ak} = 0$ solidifies the geometric interpretation of the electric field.

It is easy to see [5] that the curvature tensors of Γ' and Γ coïncide, since the density term cancels:

$$R^\ell_{kij}(\Gamma') \;=\; \partial_{[i}\Gamma'^\ell_{j]k} + \Gamma'^\ell_{m[i}\Gamma'^m_{j]k} \;=\; R^\ell_{kij}(\Gamma) \tag{3.12}$$

One can also show that the composite magnetic field, defined above Eq. (2.18) is related to the standard curvature by

$$\begin{aligned} \hat{B}^{ai} &= -\frac{1}{2}\,\varepsilon^{ijk}\varepsilon^{abc}R^{bc}_{jk}(\omega) \\ &= -\frac{1}{2}\,\varepsilon^{ijk}\hat{\varepsilon}_{mnq}E^{aq}R^{mn}{}_{jk}(\Gamma) \\ &= 2E^{aq}(R^i_q - \frac{1}{2}\delta^i_q R)\;. \end{aligned} \tag{3.13}$$

The standard curvature of the spin connection in the first line is converted to space indices using the frame, and the representation of the curvature of a 3-manifold in terms of its Ricci and scalar contractions

$$R_{ijk\ell} = G_{ik}R_{j\ell} - G_{i\ell}R_{jk} - G_{jk}R_{i\ell} + G_{j\ell}R_{ik} - \frac{R}{2}(G_{ik}G_{j\ell} - G_{i\ell}G_{jk}) \tag{3.14}$$

is used in the final step. Note that $\hat{\varepsilon}_{mnq}$ has components $\pm 1, 0$, and transforms as a tensor density of weight -1.

Let us now consider whether E^{ai} or e^a_i, obtained through Eq. (3.8), is the better variable for the dynamics of $SU(2)$ gauge theory in this approach. Certainly e^a_i is more geometric and has lower dimension, but provisionally we prefer the electric field E^{ai} because the parity transformation $E^{ai}(x) \to -E^{ai}(-x)$ is very awkward to implement on e^a_i. So we shall use E^{ai}, φ^{ij} and Γ'^k_{ij} for the rest of the paper. It is not difficult to convert to e^{ai}, G_{ij} and Γ^k_{ij} if that proves to be desirable.

Finally we come to the question of implementing the Gauss law constraint, $\bar{\mathcal{G}}^a F[E] = 0$, within this approach to $SU(2)$ gauge theory. We shall describe several classes of gauge invariant states, but we are not certain that they comprise the "general solution" of the constraint.

Following similar discussions [5,13] for the magnetic representation, we note that E^{ai} contains 9 components. Since there are 3 gauge group "angles", we would expect that it takes 6 functions to describe the gauge invariant content of an electric field configuration. The symmetric tensor φ^{ij} has 6 independent components. Although $\det E$ is another local gauge invariant, one has $\det\varphi = (\det E)^2$, and only the sign of $\det E$ is independent of φ^{ij}. So the most general functional of the local invariants takes the form

$$F[E^{ai}] \;=\; F_+[\varphi^{ij}] + \int d^3x\;(\det E(x))\;F_-[\varphi^{ij}, x)\;. \tag{3.15}$$

For states which are invariant under spatial translations, $\varphi^{ij}(x) \to \varphi^{ij}(x+a)$, the two terms in Eq. (3.15) have opposite parity and can be considered separately. For simplicity we work only with the even parity term below, and refer to it as $F[\varphi]$.

Let us consider the "electric" Chern-Simons functional of the composite spin connection

$$CS[\omega] \;=\; \frac{1}{16\pi^2}\int d^3x\;\varepsilon^{ijk}\left[\omega^a_i\partial_j\omega^a_k + \frac{1}{3}\,\varepsilon^{abc}\omega^a_i\omega^b_j\omega^c_k\right]\;, \tag{3.16}$$

normalized to give $\hat{B}^{ai} = 8\pi^2\delta(CS)/\delta\omega^a_i$. With $T\omega$ denoting the finite gauge transformation of ω under $E^{ai} \to T^{ab}E^{bi}$, we have

$$CS[T\omega] \;=\; CS[\omega] - \frac{1}{96\pi^2}\int d^3x\;\varepsilon^{ijk}\left[T^{da}\partial_i T^{db}T^{eb}\partial_j T^{ec}T^{fc}\partial_k T^{fa}\right]\;. \tag{3.17}$$

The last term is the integer-valued winding number, so $CS[\omega]$ is certainly infinitesimally gauge-invariant, and satisfies $[\mathcal{G}^a(x), CS[\omega]\,] = 0$. But

$$CS[T_k\omega] \;=\; CS[\omega] + k \tag{3.18}$$

for a gauge transformation T_k with winding number k. All of the above is standard [14]. One then sees that states of the form

$$F[E,\theta] \;\equiv\; e^{i\theta CS[\omega]}\;F[\varphi] \tag{3.19}$$

transform as

$$F[T_k E,\theta] \;=\; e^{ik\theta}\; F[E,\theta]. \tag{3.20}$$

Thus, as in the magnetic representation [14], the Chern-Simons functional, here a composite functional of E^{ai}, can be used to relate states with nontrivial response to large gauge transformations to invariant states, here $F[\varphi]$.

We also want to discuss briefly a third class of states which obey the Gauss law constraint, namely functionals constructed from "electric" Wilson loops:

$$\begin{aligned} W[\omega,C] \;&=\; Tr\left[P \exp\; i\; \oint dx^i \omega_i\right] \\ \omega_i \;&=\; \frac{1}{2}\tau_a\omega_i^a \end{aligned} \tag{3.21}$$

where τ_1,τ_2,τ_3 are Pauli matrices and C is a closed curve in $I\!R^3$. Certainly $[\mathcal{G}^a(x), W[\omega,C]\,]=0$, and state functionals formed from $W[\omega,C]$ satisfy the gauge constraint, but it is not clear to us whether such states are an independent class of physical states or whether they can be expressed in the form $F[\varphi]$. Another general question concerns the relation between the electric Chern-Simons and Wilson loop functionals and their magnetic analogues. They do not appear to be simply related by the functional Fourier transform [1] between magnetic and electric representations of the theory.

The discussion above has ended in a less definite way than we would like, and we now return to a question on which definite calculations can be presented. Namely we wish to discuss the form of the Hamiltonian $\bar{H}$ of Eq. (2.20) acting on states $F[\varphi]$. We need to express $\bar{B}^{ai}(x)F[\varphi]$ of Eq. (2.22) in terms of φ using the chain rule

$$\begin{aligned} \frac{\delta}{\delta E^{ak}}F[\varphi] \;&=\; \frac{\delta\varphi^{pq}}{\delta E^{ak}}\frac{\delta F}{\delta\varphi^{pq}} \\ &=\; 2E^{ap}\frac{\delta F}{\delta\varphi^{pq}} \end{aligned} \tag{3.22}$$

and also Eqs. (3.10-3.13). It is not difficult to obtain

$$\begin{aligned} \bar{B}^{ai}F[\varphi] \;=\; 2\Big\{ &\frac{1}{g}\,E^{ap}(R_p^i - \frac{1}{2}\delta_i^p R) + i\varepsilon^{ijk}E^{ap}\nabla_j\frac{\delta}{\delta\varphi^{kp}} \\ &-g\varepsilon^{ijk}\varepsilon^{pqr}\,E_r^a \det E\;\frac{\delta}{\delta\varphi^{jq}}\frac{\delta}{\delta\varphi^{kr}}\Big\}\;F[\varphi] \end{aligned} \tag{3.23}$$

where

$$\nabla_j\frac{\delta F}{\delta\varphi^{kp}} \;=\; \partial_j\frac{\delta F}{\delta\varphi^{kp}} - \Gamma'^{q}_{jp}\frac{\delta F}{\delta\varphi^{kp}} - \Gamma^{q}_{jk}\frac{\delta F}{\delta\varphi^{pq}} \tag{3.24}$$

is exactly the standard spatial covariant derivative of a tensor density of weight -1, which is what $\delta F/\delta\varphi$ is. The Γ^q_{jk} connection term cancels in Eq. (3.23) due to symmetry. We have dropped the $\delta(0)$ ordering term of Eq. (2.22) in Eq. (3.23), because of the provisional conclusion of the Appendix, that this term vanishes after regularization.

We now consider the magnetic energy density

$$\mathcal{E}_M\; F[\varphi] \;=\; \frac{1}{2}\delta_{\bar{i}i}\bar{B}^{a\bar{i}}\bar{B}^{ai}F[\varphi]\;. \tag{3.25}$$

Even without a detailed computation, one sees that the gauge indices cancel, e.g. $E^{ai}E^{aj}\;=\;\varphi^{ij}$, so that the full Hamiltonian can be rewritten entirely in terms of the spatial geometric variables φ, Γ' and R. Whether useful or not, this is a remarkable transformation of the original gauge theory. See the final section for further discussion of this Hamiltonian.

One can also transform the functional measure used to compute matrix elements of $\bar{H}$ in states $F[\varphi]$. This can be done *via* the manipulation, at each point x,

$$\begin{aligned} \int\prod_{a,k}dE^{ak}\; f[\varphi] \;&=\; \int\prod_{m\le n}d\varphi^{mn}\; f[\varphi]\int\prod_{a,k}dE^{ak}\prod_{i\le j}\delta(\varphi^{ij}-E^{ai}E^{aj}) \\ &=\; 2\pi^2\int\prod_{m\le n}d\varphi^{mn}\;\frac{1}{\sqrt{\det\varphi}}\;f[\varphi]\;, \end{aligned} \tag{3.26}$$

where the second line is obtained after directly performing the integral over $\delta(\varphi^{ij} - E^{ai}E^{aj})$ by expanding in components.

The phase $\Omega[E]$ of Eq. (3.1) involves the matrix inverse of the electric field, so our transformation is singular when $\det E^{ai} = 0$. The composite connection ω_i^a as well as Γ'^k_{ij} are also singular here. One can see upon closer inspection of the magnetic energy density Eq. (3.23) that the singular terms always involve spatial derivatives $\partial_i \varphi^{jk}$. As in [1,13] we believe that these singularities are the functional analogue of the angular momentum barrier for central forces in quantum mechanics. Any finite energy wave functional must "know how to behave itself" as such singular field configurations are approached, otherwise it would not have finite energy.

IV. GENERAL GAUGE GROUPS

The extension of the present methodology to gauge groups larger than $SU(2)$ is important for two reasons. First the realistic color group of the strong interactions is $SU(3)$. Second, we must show that the geometrization found for $SU(2)$ is not an accidental consequence of the fact that $SO(3)$ ($\approx SU(2)$) is the tangent space group of a 3-dimensional Riemannian space.

Technically, it was easy to construct the phase $\Omega[E]$ for $SU(2)$ because the electric field $E^{ai}(x)$ is a 3×3 matrix with a matrix inverse $E_i^a(x)$ which respects gauge and $GL(3)$ covariance. For larger groups, $E^{ai}(x)$ is a rectangular matrix, and there is no inverse. The major problem in constructing the phase $\Omega[E]$ for other semi-simple groups is to find an appropriate substitute for the inverse. In this section we present such a construction.

To begin with, we attempt to generalize Eq. (3.1) where, however, since we do not have an $E_i^a(x)$ available, we write instead

$$\Omega[E] \;\equiv\; \frac{1}{2} \int d^3x \, \frac{f^{abc} E^{ai}(x) E^{bj}(x)}{(\det \varphi)^{1/4}} \, \partial_i R_j^c(x) \;\equiv\; \frac{1}{2} \int d^3x \, \varepsilon^{ijk} L_i^a(x) \partial_j R_k^a(x) \; , \tag{4.1}$$

with the variable $R_i^a(x)$ to be determined so that $\Omega[E]$ is $GL(3)$ invariant with gauge variation Eq. (2.14). The quantity $L_i^a(x)$ above is simply shorthand for

$$L_i^a(x) \;=\; \frac{1}{2} \, \hat{\varepsilon}_{ijk} \, \frac{f^{abc} E^{bj}(x) E^{ck}(x)}{(\det \varphi)^{1/4}} \; . \tag{4.2}$$

We have divided by $(\det \varphi)^{1/4}$ in order to make $L_i^a(x)$ a covariant vector rather than a density, and we see from the last equality in Eq. (4.1) that $\Omega[E]$ is the integral of a 3-form, and therefore $GL(3)$ invariant, if $R_i^a(x)$ is also a covariant vector. Note that it was not necessary to insert the determinantal factor for $SU(2)$ because $R_i^a(x)$ in that case is the matrix inverse of $E^{ai}(x)$, and this was sufficient for $GL(3)$ invariance. $R_i^a(x)$ is now fixed as a function of $E^{ai}(x)$ by our first requirement on $\Omega[E]$, namely, that it satisfy Eq. (2.14). We now examine that requirement. The gauge variation of $\Omega[E]$ in Eq. (4.1) is easily computed if we assume that $R_i^a(x)$ transforms in the adjoint representation:

$$\delta\Omega[E] \;=\; -\frac{1}{2} \int d^3x \, \varepsilon^{ijk} \, f^{abc} \, L_i^a(x) R_j^b(x) \partial_k \theta^c(x) \; . \tag{4.3}$$

The requirement that this is of the form of Eq. (2.14) gives the following condition on $R_i^a(x)$:

$$\frac{1}{2} \varepsilon^{ijk} \, f^{abc} \, L_i^a(x) R_j^b(x) \;\equiv\; M^{ck,bj}(x) R_j^b(x) \;=\; E^{ck}(x) \; . \tag{4.4}$$

This is a linear system and there is a unique solution for $R_i^a(x)$ provided that the determinant of the $3 \dim G \times 3 \dim G$ direct product matrix M is non-vanishing. It is also easy to show from the structure $R = M^{-1}E$ that $R_i^a(x)$ has the required gauge and $GL(3)$ properties assumed above. An analytic calculation of M^{-1} would be necessary to have a truly explicit construction of the phase $\Omega[E]$. This is a difficult task, and we shall be content here with the fact that we have reduced the problem to this point.

We end this section with a possible alternative procedure to determine the phase $\Omega[E]$. Again, faced with the same initial problem of not having an "inverse" electric field E_i^a, we try another generalization of the $SU(2)$ phase, by writing an *ansatz* identical in form to Eq. (3.6):

$$\Omega[E] \;=\; \frac{1}{2} \int d^3x \, \varepsilon^{ijk} \, e_i^a(x) \partial_j e_k^a(x) \; , \tag{4.5}$$

with the difference that now the variables $e_i^a(x)$ form a $3 \times \dim G$ matrix, as yet undefined. The requirement that this phase has the correct gauge transformation Eq. (2.14), then determines e_i^a implicitly in a similar way as for R_i^a above. This requirement reads:

$$E^{ai} = \frac{1}{2}\varepsilon^{ijk} f^{abc} e_j^b e_k^c \ . \tag{4.6}$$

One must then solve this set of $3 \dim G$ quadratic equations to obtain $e[E]$. We have not been able to do this (despite considerable effort for the group $SU(3)$), but we find that it is an intriguing algebra problem with a group-theoretic flavor. It is formally identical to the problem of finding, for a general group G, the gauge potential A_i^a (e_i^a here) given a constant magnetic field B^{ai} (here E^{ai}). The solution to this would yield a phase $\Omega[E]$ which would automatically have the proper gauge and $GL(3)$ transformation properties, and could possibly lead to a simpler formulation of the theory than the one based on Eq. (4.1).

V. $SU(3)$ GAUGE THEORY

We now explore the $SU(3)$ theory in order to ascertain the spatial geometry associated with a larger gauge group. The first step is to use the group theory and the physics to define a basis of eight vectors for the adjoint representation of the group. The basis is then used to define the connection, torsion, and curvature of the geometry. Then we identify the class of gauge invariant states analogous to $F[\varphi^{ij}]$ of Sec. 3, and show that the Hamiltonian acting on these states can be expressed in terms of gauge invariant and geometric quantities. The attitude we shall take is that all geometric information is contained in the $SU(3)$ gauge connection ω_i^a calculated from $\Omega[E]$ in Eq. (4.1). The basis of eight vectors is a generalized frame used to transfer this information to geometric variables with spatial indices only. This attitude is consistent with the situation for $SU(2)$, but little thought was required there because the geometry was completely standard.

The first three 8-vectors of the basis are simply the three spatial components E^{ai} of the electric field. These are linearly independent for generic field configurations in which the rectangular matrix has rank 3. Using the d-symbols of $SU(3)$ we construct six additional 8-vectors

$$E^{ajk} \ \equiv \ d^{abc} E^{bj} E^{ck} \ . \tag{5.1}$$

First, we orthogonalize these with respect to the first three by defining

$$\hat{E}^{ajk} \ \equiv \ E^{ajk} - E^{am}\varphi_{mn}\varphi^{njk} \ , \tag{5.2}$$

where φ_{mn} is the matrix inverse of φ^{ij} and

$$\varphi^{ijk} \ \equiv \ d^{abc} E^{ai} E^{bj} E^{ck} \ . \tag{5.3}$$

The six $\hat{E}^{ajk}$ span an orthogonal subspace to that of E^{ai}. Within that subspace, the trace $\hat{E}^a = \hat{E}^{amn}\varphi_{mn}$ is generically linearly related to the 5 traceless combinations

$$\hat{E}^{a\{ij\}} \ \equiv \ \hat{E}^{aij} - \frac{1}{3}\varphi^{ij}\hat{E}^a \ , \tag{5.4}$$

and these are generically linearly independent. So as a basis of 8 vectors we take the set

$$\{ \ E^{ai}, \hat{E}^{a\{jk\}} \ \} \ . \tag{5.5}$$

when the mutual orthogonality is useful, and otherwise the set

$$\{ \ E^{ai} \, , E^{a\{ij\}} = E^{aij} - \frac{1}{3}\varphi^{ij}\varphi_{mn}E^{amn} \ \}. \tag{5.6}$$

We shall not characterize precisely the non-generic configurations in which the five $E^{a\{jk\}}$ fail to be linearly independent. Presumably this occurs when the span of any two of the three vectors E^{ai} determines an $SU(2)$ subalgebra of $SU(3)$.

Connections for the $SU(3)$ geometry are defined by the pair of equations

$$\hat{D}_i E^{ak} \equiv -\hat{\Gamma}'^{k}_{ij} E^{aj} - T^{ak}_i \tag{5.7}$$
$$E^{aj} T^{ak}_i \equiv 0 \ , \tag{5.8}$$

which is equivalent to the fact that $\hat{D}_i E^{ak}$ can be expanded uniquely in the basis of Eq. (5.5). Note that Eqs. (5.7-5.8) comprise $72+27$ equations for $27+72$ components of $\hat{\Gamma}'$ and T. One can see that $\hat{\Gamma}'^{k}_{ij}$ transforms as a connection (for densities of weight one), while T^{ak}_i is a gauge adjoint $GL(3)$ tensor density.

We now contract Eq. (5.7) with $E^{a\ell}$ and symmetrize in $k\ell$, which leads to

$$\partial_i \varphi^{k\ell} + \hat{\Gamma}'^{k}_{ij} \varphi^{j\ell} + \hat{\Gamma}'^{\ell}_{ij} \varphi^{kj} = 0 \ . \tag{5.9}$$

This is simply the metric compatible relation between $\hat{\Gamma}'$ and the "densitized metric" φ_{ij}. It then follows from simple algebra that $\hat{\Gamma}'$ takes the form of a (densitized) connection with torsion, namely

$$\hat{\Gamma}'^{k}_{ij} = \Gamma'^{k}_{ij} - K^{k}_{ij} \ , \tag{5.10}$$

where Γ'^{k}_{ij} is just the Riemannian Γ' of Eq. (3.11) and K is the contortion tensor, which satisfies the antisymmetry property

$$K_{ijk} = -K_{ikj} \tag{5.11}$$
$$K_{ijk} \equiv K_{ij}{}^{\ell} G_{\ell k} \ . \tag{5.12}$$

Because of orthogonality to E^{ai}, T^{ak}_i is determined entirely by the spatial tensor density

$$K_i{}^{\{mn\}k} \equiv E^{a\{mn\}} T^{ak}_i = -E^{a\{mn\}} \hat{D}_i E^{ak} \ . \tag{5.13}$$

Using the components $E^a_i, E^a_{\{mn\}}$ of the 8×8 matrix inverse of the basis Eq. (5.5) we see that

$$T^{ak}_i = E^a_{\{mn\}} K_i{}^{\{mn\}k} \ . \tag{5.14}$$

We regard T^{ak}_i or $K_i{}^{\{mn\}k}$ as a new type of torsion. The Bianchi identity Eq. (2.19) implies that all torsions are traceless:

$$K_{ij}{}^{i} = T^{ai}_i = K^{\{mn\}i}_i = 0 \ . \tag{5.15}$$

The torsions are local functions of E and ∂E which can be found from the definition Eq. (5.7-5.8), once we have the explicit form of ω^a_i. In turn this requires the construction of the matrix $R^a_i[E]$ which enters the phase $\Omega[E]$ of Eq. (4.1). Note that Eq. (5.7) can be expressed in terms of the total covariant derivative ∇_i of Eq. (3.10), but now we have

$$\nabla_i E^{ak} = -T^{ak}_i \ , \tag{5.16}$$

so the frame is no longer covariantly constant, but φ_{ij} and also G_{ij} are, since Eq. (5.9) is equivalent to

$$\nabla_i \varphi^{jk} = 0 \ . \tag{5.17}$$

The next step is to study the curvature by taking a further gauge derivative of Eq. (5.7) and antisymmetrizing to obtain

$$[\hat{D}_i, \hat{D}_j] E^{ak} = -\left\{ R^k{}_{\ell ij} E^{a\ell} + \hat{D}_{[i} T^{ak}_{j]} + \Gamma'^{k}_{[i\ell} T^{a\ell}_{j]} \right\}$$
$$= f^{abc} \hat{\varepsilon}_{ijm} \hat{B}^{bm} E^{ck} \ . \tag{5.18}$$

The sum of the last two terms in the first line is $GL(3)$ covariant, and we have used the gauge Ricci identity to obtain the last line.

We now wish to obtain the $SU(3)$ generalization of Eq. (3.13) and express the composite magnetic field $\hat{B}$ in terms of the curvature and torsion. This is awkward because E^{ck} itself does not have an inverse, but the full frame Eq. (5.5) can be brought to use as follows. The gauge covariant derivative of Eq. (5.1) can be evaluated as

$$\hat{D}_j E^{ak\ell} = -\Gamma'^k_{jm} E^{am\ell} - \Gamma'^\ell_{jm} E^{akm} + 2d^{abc} E^{bk} T^{c\ell}_j \tag{5.19}$$

and one also finds

$$\begin{aligned}
[\hat{D}_i, \hat{D}_j] E^{ak\ell} &= -\left\{ R^k{}_{mij} E^{am\ell} + R^\ell{}_{mij} E^{akm} + 2d^{abc} E^{bk} \left(\hat{D}_{[i} T^{c\ell}_{j]} + \Gamma'^\ell_{[im} T^{cm}_{j]} \right) \right\} \\
&= f^{abc} \hat{\varepsilon}_{ijm} \hat{B}^{bm} E^{ck\ell} \ .
\end{aligned} \tag{5.20}$$

With $\{...\}$ denoting symmetrization and removal of the trace, we obtain

$$f^{abc} \hat{\varepsilon}_{ijm} \hat{B}^{bm} E^{c\{k\ell\}} = -\left\{ R^{\{k}{}_{mij} E^{a\ell\}m} + d^{abc} E^{b\{k} \left(\hat{D}_{[i} T^{c\ell\}}_{j]} + \Gamma'^{\ell\}}_{[im} T^{cm}_{j]} \right) \right\} \ . \tag{5.21}$$

Using now the components of the inverse matrix $E^a_k, E^a_{\{k\ell\}}$ it is now simple to obtain $\hat{B}$ from Eqs. (5.18, 5.20):

$$\begin{aligned}
\hat{B}^{ai} &= -\frac{1}{6} f^{abc} \varepsilon^{ijk} \left\{ E^b_m \left[R^m{}_{njk} E^{cn} + \hat{D}_{[j} T^{cm}_{k]} + \Gamma'^m_{[jn} T^{cn}_{k]} \right] \right. \\
&\left. + E^b_{\{mn\}} \left[R^m_{\ell ij} E^{cn\ell} + d^{cde} E^{bm} \left(\hat{D}_{[j} T^{cn}_{k]} + \Gamma'^n_{[j\ell} T^{c\ell}_{k]} \right) \right] \right\} \ .
\end{aligned} \tag{5.22}$$

This is the desired expression for the composite magnetic field. One can go further and substitute the representation of Eq. (3.14) for $R^m{}_{njk}$, which holds with torsion [5], and one can use Eq. (5.14) to express $DT + \Gamma' T$ in terms of the total spatial covariant derivative of $K_i{}^{\{mn\}k}$ and $\nabla E^a_{\{mn\}}$. We shall not write the final resulting formula. Note that the matrix M of Eq. (4.4) is singular for electric fields which vanish except in an $SU(2)$ subalgebra of $SU(3)$, and Eq. (5.22) is also singular in this case.

The next stage of the discussion concerns gauge invariant states and local variables [13] for $SU(3)$. We start with the observation that the gauge invariant content of an $SU(3)$ electric field configuration can be described by $24 - 8 = 16$ variables. The symmetric tensor densities φ^{ij} and φ^{ijk} contain precisely $6 + 10 = 16$ independent components. There are other local invariants, such as the "efterminant" and "extended metric"

$$\begin{aligned}
eft\ E &\equiv \frac{1}{6} f^{abc} \hat{\varepsilon}_{ijk} E^{ai} E^{bj} E^{ck} \\
\varphi^{jk;\ell m} &\equiv E^{ajk} E^{a\ell m} \ .
\end{aligned} \tag{5.23}$$

We assume that the set $\varphi^{ij}, \varphi^{ijk}$ is an "essentially complete set" of local invariants. This means that $(eftE)^2$ and presumably $\varphi^{jk;\ell m}$ can be expressed as functions of φ^{ij} and φ^{ijk}. It also means that a representation analogous to Eq. (3.15) should hold with $\det E$ replaced by $eftE$. We have not proven these things, and we will discuss the situation further below after we examine the Hamiltonian to see what quantities need to be expressed in terms of φ^{ij} and φ^{ijk}.

It is clear that state functionals of the form $F[\varphi^{ij}, \varphi^{ijk}]$ are the $SU(3)$ generalizations of the states $F[\varphi^{ij}]$ considered for $SU(2)$ and we now study the form of the Hamiltonian on such states. We need the chain rule

$$\begin{aligned}
\frac{\delta}{\delta E^{ak}} F &= \frac{\delta \varphi^{pq}}{\delta E^{ak}} \frac{\delta F}{\delta \varphi^{pq}} + \frac{\delta \varphi^{pqr}}{\delta E^{ak}} \frac{\delta F}{\delta \varphi^{pqr}} \\
&= 2\, E^{ap} \frac{\delta F}{\delta \varphi^{pq}} + 3\, d^{abc} E^{bp} E^{cq} \frac{\delta F}{\delta \varphi^{pqk}} \ .
\end{aligned} \tag{5.24}$$

After some algebra one finds that the second term in $\bar{B}^{ai} F$ of Eq. (2.22) can be expressed in terms of $SU(3)$ connections and torsions as

$$\begin{aligned}
i\varepsilon^{ijk} \hat{D}_j \frac{\delta F}{\delta E^{ak}} &= i\varepsilon^{ijk} \left\{ 2E^{ap} \nabla_j \frac{\delta F}{\delta \varphi^{pk}} - T^{ap}_j \frac{\delta F}{\delta \varphi^{pq}} \right. \\
&\left. + 3\, E^{apq} \nabla_j \frac{\delta F}{\delta \varphi^{pqk}} - 6\, d^{abc} E^{bp} E^c_{mn} K_j^{\{mn\}q} \frac{\delta F}{\delta \varphi^{pqk}} \right\} \ ,
\end{aligned} \tag{5.25}$$

where $\nabla_j \frac{\delta F}{\delta \varphi^{pk}}$ has been defined in Eq. (3.24), and

$$\varepsilon^{ijk}\nabla_j \frac{\delta F}{\delta\varphi^{pqk}} = \varepsilon^{ijk}\left[\partial_j \frac{\delta F}{\delta\varphi^{pqk}} - \Gamma'^r_{jp}\frac{\delta F}{\delta\varphi^{rqk}} - \Gamma'^r_{jq}\frac{\delta F}{\delta\varphi^{prk}}\right] . \tag{5.26}$$

Similarly, the third term in Eq. (2.22) can be written as

$$-\frac{g}{2}\varepsilon^{ijk} f^{abc} \frac{\delta^2 F}{\delta E^{bj}\,\delta E^{ck}} = -\frac{g}{2}\varepsilon^{ijk} f^{abc}\left\{4E^{bp}E^{cq}\frac{\delta^2 F}{\delta\varphi^{pj}\,\delta\varphi^{qk}}\right.$$

$$\left. +12E^{bp}E^{crs}\frac{\delta^2 F}{\delta\varphi^{pj}\,\delta\varphi^{rsk}} + 9E^{bpq}E^{crs}\frac{\delta^2 F}{\delta\varphi^{pqj}\,\delta\varphi^{rsk}}\right\} . \tag{5.27}$$

No $\delta(0)$ ordering terms arise in Eq. (5.27), and we assume that the fourth term in Eq. (2.22) vanishes after regularization as discussed in the Appendix for $SU(2)$.

Consider now the magnetic energy density Eq. (3.25) with each factor $\bar{B}^{ai}$ expressed as the sum of Eqs. (5.22-5.25-5.27). It is clear that all gauge indices are contracted out in local invariant variables such as $\varphi^{pq;rs}$, $f^{abc}E^{apq}E^{br}E^{cs}$ and several others. If the hypothesis that the tensors $\varphi^{ij}, \varphi^{ijk}$ are an essentially complete set is correct, then all invariants which occur in Eq. (3.25) can be expressed in terms of φ^{ij} and φ^{ijk}. Similarly we expect that the torsions $K_{ij}{}^{k}$ and $K_i^{\{jk\}\ell}$ can be expressed in terms of the basic variables and their first spatial derivatives (in torsion-free covariant combinations). Symbolic manipulation programs can be useful to help find the required expressions which are necessary to express the $SU(3)$ gauge theory in complete geometric form.

This discussion has shown that our geometric ideas can be extended to the gauge group $SU(3)$, and that there is an interesting spatial geometry associated with this realistic color group. The theory is not yet in entirely explicit form. For this one must obtain the matrix R^a_j and the inverse frame components E^a_i and $E^a_{\{jk\}}$ as functions of E^{ai}, and one must solve the problem of independent $SU(3)$ invariants discussed in the previous paragraph. These "mechanical" problems are not necessarily easy, and we believe that the effort to solve them is justified only if the spatial geometry is shown to be useful for the dynamics in the $SU(2)$ theory of Sec. 3, which is far simpler.

VI. DISCUSSION

We have shown that it is possible to reexpress the geometry of non-abelian gauge theories in terms of a 3-dimensional spatial geometry. The first and most important step was the unitary transformation $\Psi[E] = \exp(i\Omega[E]/g)F[E]$ which allowed us to impose the Gauss law constraint on $F[E]$ and to exploit the fact that $\omega^a_i = -\delta\Omega/\delta E^{ai}$ transforms as a composite gauge connection.

For gauge group $SU(2)$, ω^a_i is just the standard spin connection of a Riemannian 3-manifold. We were naturally led to define metric- and connection-like variables φ^{ij} and Γ'^i_{jk} which are equivalent to the ordinary Riemannian metric and Christoffel connection. The $SU(2)$ theory essentially geometrizes itself, and a conventional Riemannian geometry underlies the theory.

For larger gauge groups, and for $SU(3)$ in particular, the same approach leads to a metric-preserving geometry with torsion of both standard and novel type. The construction of Secs. 4 and 5 was not quite explicit because certain "mechanical problems" of analytic matrix inversion and relations among group invariants remain to be solved. Apart from these problems, it is also possible that another choice of phase $\Omega[E]$ or basis E^{ai}, $E^{a\{ij\}}$ could lead to a simpler formulation.

Our initial motivation, beginning in [5], was to express the Hamiltonian in gauge invariant variables in order to develop a new approach to the non-perturbative dynamics of gauge theories. What has been achieved so far is just a formal structure, of some elegance we believe, but there are many difficulties to be overcome before it can be applied to real physics. The non-linear transformation to variables $\varphi^{ij} = E^{ai}E^{aj}$ may exacerbate the problem of Lorentz covariance in the Hamiltonian formalism. A suitable cutoff procedure must be found and one must cope with a Hamiltonian which is up to fourth order in functional derivatives. The fundamental unitary transformation is non-perturbative, so the composite magnetic field $\hat{B}^{ai}$ appears in (2.22) with coefficient $1/g$, and there are singular terms up to order $1/g^2$ in the Hamiltonian, as in [1,2]. These terms make it problematic to perform short distance calculations to test whether the transformed theory has the expected short distance behavior. But since these singular terms are the result of the exact treatment of the non-abelian gauge invariance, they may represent a significant non-perturbative aspect of the theory. Finally, the notion [1] that the behavior of physical wave functions at the singular points of the unitary transformation used is controlled by the energy barrier terms in H requires exploration. All of these problems appear to be substantial but we hope that the geometric structure of the formal theory provides the impetus to solve them.

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APPENDIX A

In this appendix we show explicitly for gauge group $SU(2)$ that the singular term in $\bar{B}^{ai}$, when properly regularized, vanishes. The singular term comes from the ill-defined quantity

$$\varepsilon^{ijk}\varepsilon^{abc}\frac{\delta\omega_k^c(x)}{\delta E^{bj}(x)} \tag{A1}$$

It is easy to see that the rest of $\bar{B}^{ai}$ has the gauge and tensorial properties of a magnetic field. Formally the singular term also does. So we have to look for a regularization that preserves these properties. The most obvious candidate would be to point-split, i.e., work with

$$\varepsilon^{ijk}\varepsilon^{abc}\frac{\delta\omega_k^c(y)}{\delta E^{bj}(x)} \tag{A2}$$

and take the limit $x = y$. However the quantity

$$\frac{\delta\omega_k^c(y)}{\delta E^{bj}(x)} = -\frac{\delta^2\Omega}{\delta E^{bj}(x)\delta E^{ck}(y)} \tag{A3}$$

does not transform as a geometric object at point x but as a "bi-geometric" object at points x and y (a gauge and contravariant spatial vector at x and y). This is clear from its definition but can also be checked on the explicit form of the second variation of Ω involving $\delta(x-y)$ and its first derivative. So the contraction of (A3) with $\varepsilon^{ijk}\varepsilon^{abc}$, which is covariant with respect to gauge and $GL(3)$ transformations at a single point, is not geometric. This is significant because (A3) is singular as $y \leftarrow x$.

A remedy for this is to introduce a linear operator $M_k^{k'cc'}(x,y)$ such that if $T_{k'}^{c'}$ is a gauge and contravariant spatial vector at y then $M_k^{k'cc'}(x,y)T_{k'}^{c'} = \tilde{T}_k^c$ has the same geometric properties at x. Then

$$M_k^{k'cc'}(x,y)\frac{\delta^2\Omega}{\delta E^{bj}(x)\delta E^{c'k'}(y)} \equiv \left[\frac{\delta^2\Omega}{\delta E^{bj}(x)\delta E^{ck}(y)}\right]^{(cov)} \tag{A4}$$

will be a geometric object at x.

In general a smooth choice of M is possible only locally. One must choose gauge and affine connections, and use these to parallel-transport $T_{k'}^{c'}$ along a path from y to x. So there are many ambiguities in the definition of M. But as stressed above,

$$\frac{\delta^2\Omega}{\delta E^{bj}(x)\delta E^{ck}(y)} \tag{A5}$$

is a local distribution of order 1, so that all what is needed is $M_k^{k'cc'}(x,x)$ and $\left(\partial^{(y)}M_k^{k'cc'}\right)(x,x)$, and this only involves the gauge and affine connections at point x.

To compute the second variation of Ω, the simplest way is to Taylor expand $\Omega[E+E'+E'']$ to first order in E' and E''. The result is

$$\int\int d^3x\ d^3y\ \frac{\delta^2\Omega}{\delta E^{bj}(x)\delta E^{ck}(y)} E'^{bj}(x)E''^{ck}(y) =$$

$$-\frac{1}{2}\int d^3z\ \varepsilon^{def}\ \left[E'^{d\ell}(\partial_\ell E''^{em})E^f_m - E^{d\ell}(\partial_\ell E''^{em})E^g_m E'^{gn}E^f_n - E'^{d\ell}(\partial_\ell E^{em})E^g_m E''^{gn}E^f_n \right.$$
$$\left. +E^{d\ell}(\partial_\ell E^{em})E^h_m E'^{hn}E^g_n E''^{gp}E^f_p + 4 \text{ terms with } E' \leftrightarrow E''\right] \ . \tag{A6}$$

Now, because the left-hand side is a geometric object, the right-hand side does not change if one replaces everywhere ordinary partial derivatives by total (gauge and affine) covariant derivatives acting on densities, making every term geometric.

As we have seen in Sec. 3, the electric formulation of the $SU(2)$ theory has brought natural (gauge and affine) connections to the fore, and it is more than natural to use these to define M and to rewrite (A6). In the covariant form

$$\int\int d^3x\ d^3y\ \frac{\delta^2\Omega}{\delta E^{bj}(x)\delta E^{ck}(y)} E'^{bj}(x)E''^{ck}(y) =$$

$$-\frac{1}{2}\int d^3z\ \varepsilon^{def}\ \left[E'^{d\ell}(\partial_\ell E''^{em} + \Gamma'^m_{\ell p}E''^{ep} + \varepsilon^{eha}\omega^a_\ell E''^{hm})E^f_m \right.$$
$$\left. -E^{d\ell}(\partial_\ell E''^{em} + \Gamma'^m_{\ell p}E''^{ep} + \varepsilon^{eha}\omega^a_\ell E''^{hm})E^g_m E'^{gn}E^f_n + 2 \text{ terms with } E' \leftrightarrow E''\right] \ . \tag{A7}$$

The second derivative of Ω is obtained by substituting $\delta^{ab}\delta^i_j\delta(x-z)$ for $E'^{ai}(z)$ (resp. $\delta^{ac}\delta^i_k\delta(y-z)$ for $E''^{ai}(z)$) in the right-hand side (A6). Note that these objects have the right geometric properties. We find

$$\frac{\delta^2\Omega}{\delta E^{bj}(x)\delta E^{c'k'}(y)} =$$
$$-\frac{1}{2}\varepsilon^{bef}\left\{(\partial^{(x)}_j\delta^{ec'}\delta^m_{k'} + \Gamma'^m_{jk'}(x)\delta^{ec'} + \varepsilon^{ec'a}\omega^a_j(x)\delta^m_{k'})\delta(x-y)\right\}E^f_m(x)$$
$$+\frac{1}{2}\varepsilon^{def}\left\{(\partial^{(x)}_\ell\delta^{ec'}\delta^m_{k'} + \Gamma'^m_{\ell k'}(x)\delta^{ec'} + \varepsilon^{ec'a}\omega^a_\ell(x)\delta^m_{k'})\delta(x-y)\right\}E^{d\ell}(x)E^b_m(x)E^f_j(x)$$
$$+ 2 \text{ terms with } (b\leftrightarrow c')(j\leftrightarrow k')(x\leftrightarrow y) \tag{A8}$$

The result is a distribution of order 1, and when we parallel-transport it, we can expand

$$M^{k'cc'}_k(x,y) = M^{k'cc'}_k(x,x) + (y^\ell - x^\ell)\left(\partial^{(y)}_\ell M^{k'cc'}_k\right)(x,x) + \cdots$$
$$= \delta^{k'}_k\delta^{cc'} - (y^\ell - x^\ell)(\Gamma^{k'}_{\ell k}(x)\delta^{cc'} + \omega^{cc'}_\ell(x)\delta^{k'}_k) + \cdots \tag{A9}$$

where the missing terms annihilate $\delta(x-y)$ and its first derivative, and consequently do not contribute. One should also expand the electric field in (A8) as

$$E^{ai}(y) = E^{ai}(x) + (y^\ell - x^\ell)(\partial_\ell E^{ai})(x) + \cdots$$
$$= E^{ai}(x) + (y^\ell - x^\ell)(-\Gamma'^i_{\ell m}E^{am} + \omega^{ad}_\ell E^{di}) + \cdots \tag{A10}$$

(and the corresponding equation for E^a_i), so that the evaluation point is always x. Then all that remains is a lengthy but straightforward computation. All the terms involving ω cancel either because of the antisymmetry of the structure constants of $SU(2)$ or because of the Jacobi identity. The terms involving Γ correspond to those involving Γ' with opposite signs, so that the final result is

$$\left[\frac{\delta^2\Omega}{\delta E^{bj}(x)\delta E^{ck}(y)}\right]^{(cov)} =$$
$$-\frac{1}{2}\varepsilon^{def}e^f_m(x)\left\{\delta^{bd}\delta^{ce}\delta^\ell_j\delta^m_k - \delta^{cd}\delta^{be}\delta^\ell_k\delta^m_j + (\delta^{bd}\delta^m_k e^c_j(x) - \delta^{cd}\delta^m_j e^b_k(x))e^{e\ell}(x)\right\}\partial^{(x)}_\ell\frac{\delta(x-y)}{\sqrt{G}} \tag{A11}$$

This is manifestly a tensorial object, and it is antisymmetric under the simultaneous exchange $(b \leftrightarrow c)(j \leftrightarrow k)$. Hence the contraction with $\varepsilon^{ijk}\varepsilon^{abc}$ vanishes identically, and the regulated version of

$$\varepsilon^{ijk}\varepsilon^{abc}\frac{\delta\omega_k^c(x)}{\delta E^{bj}(x)} \tag{A12}$$

vanishes as announced above.

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