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All partial breakings in $\mathcal{N} = 2$ supergravity with a single hypermultiplet

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Ignatios Antoniadis,[a,b] **Jean-Pierre Derendinger,**[b,a] **P. Marios Petropoulos**[c] **and Konstantinos Siampos**[d,b]

[a] *Laboratoire de Physique Théorique et Hautes Energies — LPTHE, Sorbonne Université, CNRS, 4 place Jussieu, 75005 Paris, France*

[b] *Albert Einstein Center for Fundamental Physics, Institute for Theoretical Physics, University of Bern, Sidlerstrasse 5, 3012 Bern, Switzerland*

[c] *Centre de Physique Théorique, Ecole Polytechnique, CNRS UMR 7644, 91128 Palaiseau Cedex, France*

[d] *Theoretical Physics Department, CERN, 1211 Geneva 23, Switzerland*

E-mail: antoniad@lpthe.jussieu.fr, derendinger@itp.unibe.ch, marios.petropoulos@cpht.polytechnique.fr, konstantinos.siampos@cern.ch

Abstract: We consider partial supersymmetry breaking in $\mathcal{N} = 2$ supergravity coupled to a single vector and a single hypermultiplet. This breaking pattern is in principle possible if the quaternion-Kähler space of the hypermultiplet admits (at least) one pair of commuting isometries. For this class of manifolds, explicit metrics exist and we analyse a generic electro-magnetic (dyonic) gauging of the isometries. An example of partial breaking in Minkowski spacetime has been found long ago by Ferrara, Girardello and Porrati, using the gauging of two translation isometries on $\mathrm{SO}(4,1)/\mathrm{SO}(4)$. We demonstrate that no other example of partial breaking of $\mathcal{N} = 2$ supergravity in Minkowski spacetime exists. We also examine partial-breaking vacua in anti-de Sitter spacetime that are much less constrained and exist generically even for electric gaugings. On $\mathrm{SO}(4,1)/\mathrm{SO}(4)$, we construct the partially-broken solution and its global limit which is the Antoniadis-Partouche-Taylor model.

Keywords: Extended Supersymmetry, Supergravity Models, Supersymmetry Breaking

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Contents

1 Introduction

Field theories invariant under global or local $\mathcal{N} = 2$ supersymmetry allow very large classes of vector, hyper or tensor multiplet interactions characterized by specific sigma-model geometries. The existence of realizations in which zero or one supersymmetry remains unbroken at the ground state of the theory is then a relatively vast and complicated subject which cannot be addressed in theories with more supersymmetries in which the class of allowed matter and gauge couplings is fatally restrictive.

Consider for instance the simplest global $\mathcal{N} = 2$ Maxwell theory, defined by an arbitrary prepotential $F(z)$. Since its scalar fields cannot break the $\mathrm{SU}(2)_R$ symmetry, a spontaneous breaking to $\mathcal{N} = 1$ is clearly impossible.[1] Antoniadis, Partouche and Taylor [1] (APT) have however invented many years ago a realization with partial breaking in

[1]This statement holds with an arbitrary number of Maxwell multiplets.

which the $SU(2)_R$ symmetry is violated by electric and magnetic Fayet-Iliopoulos (FI) terms inducing a nonlinear deformation of the second supersymmetry variation of one gaugino, defining it as the (single) goldstino. The ingredients of the model are then a *non-canonical* holomorphic prepotential and the FI constants. More recently [2], a similar mechanism has been shown to exist for a single hypermultiplet on a specific class of hyper-Kähler manifolds with a (translational) isometry, using its off-shell single-tensor dual formulation [3].

Local $\mathcal{N} = 2$ supersymmetry is more involved in several aspects. Firstly, the supergravity multiplet includes the graviphoton and electric-magnetic duality in the local super-Maxwell theory is extended and powerful [4–6].[2] Partial breaking requires the generation of a massive gravitino $\mathcal{N} = 1$ multiplet, with two spin one fields in a $6_\mathrm{B} + 6_\mathrm{F}$ (bosonic plus fermionic degrees of freedom) on-shell content. Fully spontaneous partial breaking requires then at least one physical Maxwell multiplet (for the second massive spin one state) and one hypermultiplet for the $SU(2)_R$ breaking. The minimal case of one hypermultiplet on the $SO(4,1)/SO(4)$ quaternion-Kähler manifold coupled to a single Maxwell multiplet has been studied in detail. It was shown that a partial breaking of $\mathcal{N} = 2$ supersymmetry can be realised for a generic prepotential, so that the APT model is obtained in an appropriate rigid globally supersymmetric limit [7]. A necessary ingredient[3] is the gauging of $\mathcal{N} = 2$ supergravity along magnetic directions of vector fields, or alternatively a standard electric gauging in a non-prepotential field basis [10].[4]

A more general analysis was also performed [13, 14] in a class of quaternionic manifolds of dimension $4(n+1)$ that are obtained by the so-called C-map from a special Kähler manifold of dimension $2n$, corresponding to the effective supergravity of the perturbative type II superstring compactified on a Calabi-Yau threefold [15]. The special Kähler manifold is associated to the scalars of vector multiplets of the mirror theory, while the extra scalar components are the $2n$ Ramond-Ramond fields and the universal hypermultiplet of the string dilaton parametrising for $n = 0$ an $SU(2,1)/SU(2) \times U(1)$ space broken to a quaternionic manifold with four isometries upon inclusion of the perturbative (one-loop) corrections [16–18]. For $n \neq 0$, it was shown that partial breaking can always be realised in either Minkowski or anti-de Sitter (AdS) vacuum by an appropriate choice of the embedding tensor that defines the directions of the gauging [19–21], which should have again some non-vanishing magnetic component. Finally, for the case of the single universal hypermultiplet ($n = 0$), no Minkowski $\mathcal{N} = 1$ vacuum was found.

In this work, we perform a general analysis of the $\mathcal{N} = 2$ partial breaking in supergravity theories containing a single hypermultiplet with two commuting isometries, gauged by the graviphoton and an additional vector multiplet. We work in a prepotential frame and use the embedding-tensor formalism [19, 20], for dyonic gaugings of the graviphoton and of the vector multiplet along two commuting isometries of the hypermultiplet manifold. Our goal is to provide a generic treatment for $\mathcal{N} = 1$ Minkowski vacua for arbitrary quaternion-Kähler manifolds, special-Kähler metrics and dyonic gaugings. In addition, we would like to

[2]The APT model does not have charged states and is invariant under electric-magnetic duality, upon a simultaneous transformation of the FI electric and magnetic constants.

[3]To avoid the obstruction described in refs. [8, 9].

[4]For earlier work, see [11, 12].

obtain the APT model [1] as an off-shell gravity-decoupling limit. A general quaternionic manifold of dimension four with two commuting isometries can be parametrised by the Calderbank-Pedersen (CP) metric [22], where we find a no-go result for $\mathcal{N}=1$ Minkowski vacua for a general special Kähler manifold of the vector multiplet, which seems to be in contradiction with the results obtained for the hyperbolic space $\mathrm{SO}(4,1)/\mathrm{SO}(4)$. We prove that this contradiction is only apparent because the latter space cannot be written in a CP form, with its torus symmetry identified within the three-dimensional abelian sub-algebra of $\mathrm{SO}(4,1)$, as a single exception.

The outline of this paper is as follows. In section 2, we present a brief review of the matter-coupled $\mathcal{N}=2$ supergravity. We first present the ungauged case exhibiting the electromagnetic duality transformations in the symplectic formalism (section 2.1). In passing, we show that a non-prepotential frame can exclusively arise from a magnetic duality transformation of the theory defined by the superconformal prepotential $F=-iX^0X^1$. We then summarize the gauging of isometries for the hypermultiplet manifold using the embedding-tensor formalism (section 2.2); in particular, we exhibit the relation of the scalar potential to the fermion shifts that provide a convenient way to look for partial supersymmetry breaking $\mathcal{N}=1$ vacua. In section 3, we make a systematic analysis in the case of one hypermultiplet with two isometries. We present the CP metric (section 3.1) and compute the fermion shifts upon gauging its isometries (section 3.2) proving a no-go theorem for partial breaking in Minkowski space (section 3.3). We also show that partial breaking in AdS is generically possible and we give an explicit example using a standard electric gauging of two shift isometries in the case of the universal dilaton hypermultiplet in type II superstrings compactified on a Calabi-Yau threefold (section 3.4). We then identify an obstruction for bringing the hyperbolic space in CP coordinates that allows partial breaking in Minkowski space (section 3.5). In section 4, we return to the general analysis of the $\mathrm{SO}(4,1)/\mathrm{SO}(4)$ which is actually the only quaternionic manifold that does not admit a CP metric when the two commuting isometries are shifts in the Poincaré coordinates (section 4.1). We construct explicitly the partial breaking Minkowski vacuum and study its off-shell gravity-decoupling limit (section 4.2), as well as non-supersymmetric Minkowski vacua (section 4.3). Section 5 contains some concluding remarks. Finally, we include four appendices. Appendix A contains useful formulae for the gauging of quaternionic manifolds with isometries, appendix B elaborates the hyperbolic space in CP coordinates, appendix C discusses coordinate transformations used to derive the CP metric and appendix D proves a result on $\mathcal{N}=0$ vacua of the $\mathrm{SO}(4,1)/\mathrm{SO}(4)$ model stated in section 4.3.

2 Matter-coupled $\mathcal{N}=2$ supergravities

2.1 The kinetic terms

The $\mathcal{N}=2$ target space $\mathcal{M}$ describing the scalar-field kinetic terms of a single hypermultiplet and n_{V} vector multiplets is factorized,

$$\mathcal{M}=\mathcal{M}_{\mathrm{H}}\times\mathcal{M}_{\mathrm{V}}\,, \tag{2.1}$$

JHEP08(2018)045

and both metrics only depend of the scalar fields of their respective multiplets, a property which by supersymmetry extends to all kinetic terms. The hypermultiplet scalar dynamics is encoded in the four-dimensional quaternion-Kähler metric $\mathcal{M}_{\rm H}$ with coordinates $q^u = (q^1, q^2, q^3, q^4)$,[5]

$$\mathcal{L}_{\rm hyper} = -\frac{e}{2\kappa^2}\, g^{\mu\nu}\, h_{uv}\, \partial_\mu q^u \partial_\nu q^v. \tag{2.2}$$

A generic quaternion-Kähler manifold for $n_{\rm H}$ hypermultiplets is Einstein with holonomy $\mathrm{Sp}(2n_{\rm H}) \times \mathrm{SU}(2)$,[6] and dimension $4n_{\rm H}$. For $n_{\rm H} = 1$, since $\mathrm{Sp}(2) \times \mathrm{SU}(2) \sim \mathrm{SO}(4)$, a particular characterization is needed: the quaternion-Kähler metric is Einstein with an (anti-) selfdual Weyl curvature tensor. As we will see in the next sections, four-dimensional metrics with these properties have been studied quite extensively when they admit one or several continuous isometries, which is the case of interest here.

The $\mathcal{N} = 2$ Maxwell sector is conveniently constructed in the superconformal formulation: it is then defined in terms of a prepotential $F(X^I)$ of $n_{\rm V} + 1$ complex scalar fields, with Weyl weight one. The index $I = 0, \ldots, n_{\rm V}$ includes a compensating multiplet. Its component fields include the propagating graviphoton, while its two gauginos and complex scalar are used to gauge-fix superconformal symmetries and solve field equations of auxiliary fields in the Weyl multiplet.[7] It is a common but unnecessary choice to set $I = 0$ as the compensator direction. Superconformal invariance requires that $F(X^I)$ has Weyl weight two:

$$F\left(X^I\right) = \left(X^0\right)^2 F\left(\frac{X^I}{X^0}\right) = \left(X^0\right)^2 F(1, z^a) = -i\,\left(X^0\right)^2 f(z^a), \qquad a = 1, \ldots, n_{\rm V}, \tag{2.3}$$

and $f(z^a)$ is an arbitrary function of the zero-weight scalar fields $z^a = X^a/X^0$ in the $n_{\rm V}$ physical Maxwell multiplets: the Poincaré theory is formulated in terms of the scalars z^a.

There is however a subtlety: electric-magnetic duality acts in the Maxwell sector as $\mathrm{Sp}(2(n_{\rm V}+1), \mathbb{R})$ linear transformations of the vector of sections

$$V = \begin{pmatrix} X^I \\ F_I \end{pmatrix}. \tag{2.4}$$

Choosing the section vector V (as a function of a given set of scalar fields) defines a *symplectic frame*: electric-magnetic duality would imply that the (ungauged, abelian) theory can be equivalently formulated in each symplectic frame obtained by the action of $\mathrm{Sp}(2(n_{\rm V}+1), \mathbb{R})$ on V.[8] In a *prepotential symplectic frame*, there exists $F(X)$ such that the sections are

$$X^I = (X^0, X^0 z^a), \quad F_I = \frac{\partial}{\partial X^I} F(X), \quad F_0 = -iX^0[2f(z) - z^a f_a], \quad F_a = -iX^0 f_a, \tag{2.5}$$

where $f_a = \frac{\partial}{\partial z^a} f(z)$. In a prepotential frame, the symplectic-invariant product $i(X^I \overline{F}_I - F_I \overline{X}^I)$ reads

$$- X^0 \overline{X}^0 [2(f + \overline{f}) - (z^a - \overline{z}^a)(f_a - \overline{f}_a)] = -X^0 \overline{X}^0\, \mathcal{Y} \tag{2.6}$$

[5] We use hypermultiplet scalars and metric with dimension mass^0.

[6] Or $G \times \mathrm{SU}(2)$, $G \subset \mathrm{Sp}(2n_{\rm H})$.

[7] These appear linearly in the lagrangian and then impose constraints.

[8] The symplectic orbit of V.

and $\mathcal{Y}$ will appear in the Kähler potential of the Poincaré fields z^a. Note that there is an ambiguity: this quantity vanishes if

$$\widehat{F}(X^I) = \alpha_{IJ} X^I X^J, \qquad \widehat{f}(z^a) = i[\alpha_{00} + 2\alpha_{0a} z^a + \alpha_{ab} z^a z^b] \tag{2.7}$$

with real coefficients α_{IJ} and two prepotentials differing by $\widehat{F}$ describe the same theory.

One may wonder if all frames in the symplectic orbit of a prepotential frame admit a prepotential, or if there exists orbits which relate prepotential and *non-prepotential* frames. The question of the existence of a prepotential frame has been discussed in general in ref. [23],[9] but the simple case $n_V = 1$, which is of interest here is very simple to solve explicitly.

Consider a symplectic transformation relating sections V and $\widetilde{V}$, assuming that V defines a prepotential frame with prepotential $F(X^0, X^1)$ and Poincaré scalar $z = X^0/X^1$. Assuming that we identify the compensators in both frames, $X^0 = \widetilde{X}^0$, $\mathrm{Sp}(2,2,\mathbb{R})$ duality reduces to $Sl(2,\mathbb{R})$ transformations

$$\widetilde{X}^1 = m_1 X^1 + m_2 F_1, \qquad \widetilde{F}_0 = F_0, \qquad \widetilde{F}_1 = m_3 X^1 + m_4 F_1, \qquad (m_1 m_4 - m_2 m_3 = 1), \tag{2.8}$$

which are electric-magnetic if $m_2 \neq 0$ or $m_3 \neq 0$. We wish to find a Poincaré scalar $\widetilde{z} = \widetilde{X}^1/\widetilde{X}^0$ and a prepotential $\widetilde{F}(\widetilde{X}^I) = -i(\widetilde{X}^0)^2 g(\widetilde{z})$, which identify sections $\widetilde{V}$ as a prepotential frame:

$$\begin{cases} \widetilde{X}_1 = \widetilde{X}^0 \widetilde{z} = X^0(m_1 z - i m_2 f_z), \\ \widetilde{F}_1 = -i\widetilde{X}^0 g_{\widetilde{z}} = X^0(m_3 z - i m_4 f_z), \\ \widetilde{F}_0 = -i\widetilde{X}^0[2g(\widetilde{z}) - \widetilde{z} g_{\widetilde{z}}] = -iX^0[2f(z) - z f_z], \end{cases} \Longrightarrow \begin{cases} \widetilde{z} = m_1 z - i m_2 f_z, \\ z = m_4 \widetilde{z} + i m_2 g_{\widetilde{z}}, \\ 2g(\widetilde{z}) - \widetilde{z} g_{\widetilde{z}} = 2f(z) - z f_z. \end{cases} \tag{2.9}$$

The three equations relating f and z with g and $\widetilde{z}$ are generated by the Legendre transformation

$$m_2 g(\widetilde{z}) - \frac{i}{2} m_4 \widetilde{z}^2 = -iz\widetilde{z} + m_2 f(z) + \frac{i}{2} m_1 z^2, \tag{2.10}$$

which exchanges z and $\widetilde{z}$. Clearly, the terms induced by m_1 or m_4 are irrelevant: they modify $f(z)$ or $g(\widetilde{z})$ by quadratic terms with imaginary coefficient which do not contribute to the theory. The only relevant case is then $m_2 = -m_3^{-1}$. The Legendre transformation implies

$$m_2^2\, g_{\widetilde{z}\widetilde{z}}\, f_{zz} = 1, \tag{2.11}$$

and it is singular only if $f(z)$ (or $g(\widetilde{z})$) is linear. Hence, the symplectic frame with sections $\widetilde{V}$ is a prepotential frame with Poincaré field $\widetilde{z}$ and prepotential $\widetilde{F} = -i(\widetilde{X}^0)^2 g\left(\frac{\widetilde{X}^1}{\widetilde{X}^0}\right)$ with a single exception,

$$F(X^I) = -i\alpha X^0 X^1, \qquad f(z) = \alpha z \qquad (\alpha \text{ real}), \tag{2.12}$$

for which (with $m_1 = m_4 = 0$)

$$\widetilde{X}^0 = X^0, \qquad \widetilde{X}^1 = -i\alpha m_2 X^0, \qquad \widetilde{F}_0 = -i\alpha X^1 = -i\alpha X^0 z, \qquad \widetilde{F}_1 = -m_2^{-1} X^1, \tag{2.13}$$

[9] Summarized in [24], section 21.2.2, page 474. See also ref. [25].

and

$$i(X^I\overline{F}_I - F_I\overline{X}^I) = -\alpha X_0\overline{X}_0(z+\overline{z}) \tag{2.14}$$

leading to Kähler potential $\mathcal{K} = -\ln(z+\overline{z})$. This simple discussion agrees with the general argument given in refs. [23] and [24].[10] The conclusion is that in the $n_{\rm V} = 1$ case, all symplectic orbits connect exclusively prepotential frames, with the single exception of the orbit of $F(X) = -iX^0X^1$ which includes non-prepotential frames.

The first example of partial $\mathcal{N} = 2$ breaking in supergravity [10] was found using precisely the sections (2.13). An electric gauging of two translation isometries of the hypermultiplet manifold $\mathrm{SO}(4,1)/\mathrm{SO}(4) \sim \mathrm{Sp}(2,2)/\mathrm{SU}(2)\times\mathrm{SU}(2)$ in this non-prepotential frame leads to a two-coupling theory with zero potential and $\mathcal{N} = 0$ for generic values of the couplings, $\mathcal{N} = 1$ when a linear relation is verified by the couplings, and $\mathcal{N} = 2$ for zero couplings.

Since the prepotential (2.12) is in the symplectic orbit of the non-prepotential frame (2.13) and since all other orbits include prepotential frames only, we are always allowed to work in a prepotential frame with sections (2.5) and to gauge isometries in this frame: since gauging fixes the electric-magnetic duality symmetry, the theory will then depend on the prepotential, the gauge couplings and the choice of hypermultiplet manifold.

The kinetic terms of the helicity $0, \pm\frac{1}{2}$ fields[11] in Poincaré Maxwell multiplets have a Kähler metric with Kähler potential

$$\mathcal{K} = -\ln\left[2(f+\overline{f}) - (z^a - \overline{z}^a)(f_a - \overline{f}_a)\right]. \tag{2.15}$$

For instance, for scalar fields (in a prepotential frame), the superconformal lagrangian includes

$$e^{-1}\mathcal{L}_{\rm kin.} = -g^{\mu\nu}N_{IJ}\left(D_\mu X^I\right)\left(D_\nu\overline{X}^J\right), \tag{2.16}$$

with

$$N_{IJ} = -iF_{IJ}+i\overline{F}_{IJ} = \frac{\partial^2 N}{\partial X^I\partial\overline{X}^J}, \qquad N = -iX^I(F_{IJ}-\overline{F}_{IJ})\overline{X}^J = i\left(X^I\overline{F}_I - \overline{X}^IF_I\right), \tag{2.17}$$

and with a covariant derivative $D_\mu X^I = (\partial_\mu - iA_\mu)X^I$ involving the gauge field of the superconformal $\mathrm{U}(1)_R$ symmetry. Eliminating this auxiliary vector field delivers[12]

$$e^{-1}\mathcal{L}_{\rm kin.} = -N\left[\frac{1}{4}(\partial_\mu\ln N)(\partial^\mu\ln N) + \frac{\partial^2\ln N}{\partial X^I\partial\overline{X}^J}\left(\partial_\mu X^I\right)\left(\partial^\mu\overline{X}^J\right)\right], \tag{2.18}$$

using the homogeneity of the prepotential. The Poincaré theory can then be obtained in field coordinates $X^I = X^0(1,z^a)$, $X^0 = \kappa^{-1}y(z,\overline{z})$ and sections $V = yU = y(Z^I(z),F_I(z))$ once the dilatation and $\mathrm{U}(1)_R$ gauge-fixing conditions

$$N = -\kappa^{-2} \quad\longrightarrow\quad (y\overline{y})^{-1} = 2(f+\overline{f}) - (z^a-\overline{z}^a)(f_a-\overline{f}_a) = \mathcal{Y}, \qquad y = \overline{y} \tag{2.19}$$

[10] But disagrees with statements in ref. [7] for instance.

[11] Propagating or auxiliary.

[12] Omitting fermions.

JHEP08(2018)045

have been applied. In terms of z^a then,

$$e^{-1}\,\mathcal{L}_{\text{kin.}} = -\frac{1}{\kappa^2}\,g_{a\bar{b}}\,(\partial_\mu z^a)(\partial^\mu \overline{z}^b), \qquad g_{a\bar{b}} = \frac{\partial^2\mathcal{K}}{\partial z^a \partial \overline{z}^b}, \qquad \mathcal{K} = -\ln\mathcal{Y}, \qquad y(z,\overline{z}) = \mathrm{e}^{\mathcal{K}/2}, \tag{2.20}$$

which leads to expression (2.15). The same metric appears in the kinetic terms of the Poincaré gauginos λ^{ia}. However the kinetic terms of gauge fields include further contributions due to the graviphoton:

$$e^{-1}\,\mathcal{L}_{\text{gauge}} = \frac{1}{4}\,\mathrm{Im}\,\mathcal{N}_{IJ}\,F^I_{\mu\nu}\,F^{\mu\nu J} - \frac{e}{8}\,\mathrm{Re}\,\mathcal{N}_{IJ}\,\varepsilon_{\mu\nu\rho\sigma}\,F^{\mu\nu I}\,F^{\rho\sigma J}\,, \tag{2.21}$$

with $n_{\mathrm{V}}+1$-dimensional metric

$$\mathcal{N}_{IJ} = \overline{F}_{IJ} + i\,\frac{N_{IK}X^K N_{JL}X^L}{N_{MN}X^M X^N}\,, \qquad\qquad I,J = 0,\ldots,n_{\mathrm{V}}. \tag{2.22}$$

Notice that $\mathrm{Im}\mathcal{N}_{IJ}$ is negative on physical fields.

In the following, we will explicitly consider the case $n_{\mathrm{V}} = 1$ only.

2.2 Fermion shifts, scalar potential, supersymmetry breaking

In $\mathcal{N} = 2$ supergravity, the scalar potential appears when isometries of the theory are gauged. With the graviphoton and the gauge field of a vector multiplet ($n_{\mathrm{V}} = 1$), we can gauge two commuting isometries, as required if partial supersymmetry breaking is envisaged [13]. This of course implies that two commuting isometries should exist and this defines a class of scalar manifolds for a single hypermultiplet for which explicit metrics are available. The problem of partial breaking can then be analytically studied in general.

The scalar potential in supergravity theories has a particular structure. The supersymmetry variation of all fermions ψ^A_I is of the form

$$\delta\,\psi^A_I \sim \mathcal{M}^A_{Ij}\,\epsilon^j + \cdots\,, \tag{2.23}$$

where the *fermion shift* $\mathcal{M}^A_{Ij}$ is a function of scalar fields (A runs over all supermultiplets, I over all fermions in multiplet A, j over all supersymmetries; in $\mathcal{N} = 2$ theories fermions are always in SU(2) doublets and $I = i$). If the supermultiplet admits an off-shell realization, as the $\mathcal{N} = 2$ Maxwell or single-tensor multiplets, the fermion shifts are in general auxiliary scalar fields. For instance, in a Maxwell $\mathcal{N} = 2$ multiplet with gauginos λ^i,

$$\delta\,\lambda^i \sim Y^{ij}\,\epsilon_j + \cdots\,, \qquad\qquad Y^{ij} = Y^{ji}, \tag{2.24}$$

and Y^{ij} is the SU(2) triplet of real (electric) auxiliary fields. For the gravitinos,

$$\delta\,\psi^i_\mu \sim \frac{1}{2}\,\kappa^2\,S^{ij}\,\gamma_\mu\epsilon_j + \cdots\,. \tag{2.25}$$

The scalar potential is then symbolically [26]

$$\mathcal{V} = e\sum \text{coeff.} \times \text{fermion shifts}^\dagger \times \text{metric} \times \text{fermion shifts}, \tag{2.26}$$

JHEP08(2018)045

where the sum is over all fermions and the coefficients are negative for gravitinos and positive for spin-$\frac{1}{2}$ fields and depend on the normalization chosen for the fermion fields. Hence, fermion shifts define the ground state of the theory and a nonzero value of a spin-$\frac{1}{2}$ fermion shift at the ground state indicates the presence of a goldstino, or several goldstinos, and then indicates spontaneous supersymmetry breaking. Analyzing the structure of the fermion shifts is fundamental when studying the breaking phases of a supersymmetric theory.

In order to obtain the fermion shifts, we need to specify the gauging applied in the theory. The gauge generators (associated with gauge field I) and the gauge variations can be defined by electric-magnetic symplectic vectors ${\Theta_I}^a \xi_a$: the embedding tensor ${\Theta_I}^a$ specifies a linear combination of the (commuting) isometries $\xi_a = \xi_a^u\,\partial_u$ of the quaternion-Kähler metric h_{uv}; it defines the coupling constants of the gauged theory. The index I defines ${\Theta_I}^a$ as a fixed symplectic vector associated with each isometry, but we will rather use

$$ {\Theta_I}^a = \Omega_{IJ} g^{Ja}, \tag{2.27} $$

and the coupling constants are the numbers g^{Ia}.

Consistency of the gauging is guaranteed by the locality constraint on the embedding tensor [19, 20]

$$ {\Theta_I}^a \Omega^{IJ} {\Theta_J}^b = 0\,, \qquad\qquad \Omega = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}. \tag{2.28} $$

The hypermultiplet scalar fields are coordinates q^u on a (four-dimensional) quaternion-Kähler space with metric h_{uv}. For each isometry vector $\delta_a q^u = \xi_a^u$, one can derive an SU(2) triplet of prepotentials (or moment maps) solving the differential equation

$$ P_a^x = -\frac{1}{2\kappa^2}\,(J^x)^u{}_v \nabla_u \xi_a^v\,, \qquad\qquad x = 1, 2, 3 \tag{2.29} $$

in terms of the triplet of complex structures J^x.[13] As usual, to describe the hypermultiplet fermions (hyperinos), we need a vielbein ${f^{iA}}_u$, which for $\mathcal{N}=2$ is defined by

$$ {f^{iA}}_u \Omega_{AB} {f^{jB}}_v = \frac{i}{2}(J_x)_{uv}(i\varepsilon\,\sigma^x)^{ij} + \frac{1}{2}\,h_{uv}\,\varepsilon^{ij} \quad\Longrightarrow\quad h_{uv} = {f^{iA}}_u\,\varepsilon_{ij}\Omega_{AB}{f^{jB}}_v \tag{2.30} $$

(i and A are respectively SU(2) and $\mathrm{Sp}(2n_{\mathrm{H}}) = \mathrm{Sp}(2)$ doublet indices and hyperinos carry index A). Then, for given quaternion-Kähler metric h_{uv}, complex structures J^x, isometries ξ_a^u and prepotentials

$$ P_a^{ij} = P_a^x (i\varepsilon\,\sigma^x)^{ij} = P_a^{ji}, \tag{2.31} $$

we obtain the following expressions for the fermion shifts:[14]

$$ \begin{aligned} &\text{Gravitinos:} && S^{ij} = \frac{1}{\kappa} e^{\mathcal{K}/2} P_a^{ij} U^I\,{\Theta_I}^a = S^{ji}\,, && \delta\,\psi_\mu^i = \frac{1}{2}\kappa^2 S^{ij}\gamma_\mu\epsilon_j + \cdots\,, \\ &\text{Gauginos:} && {W_\alpha}^{ij} = -\frac{1}{\kappa}\, e^{\mathcal{K}/2} P_a^{ij}\nabla_\alpha U^I\,{\Theta_I}^a = W_\alpha^{ji}\,, && \delta\,\lambda_i^\alpha = \kappa^2\, g^{\alpha\overline{\beta}}\overline{W}_{\overline{\beta}ij}\epsilon^j + \cdots\,, \\ &\text{Hyperinos:} && {N^i}_A = \frac{i}{\kappa}\, e^{\mathcal{K}/2} {f^{iB}}_u U^I\,{\Theta_I}^a \xi_a^u\,\Omega_{BA}, && \delta\,\zeta^A = {\overline{N}_i}^A\epsilon^i + \cdots\,, \end{aligned} \tag{2.32} $$

[13] Our SU(2) conventions are as in [24] — see also appendix A.

[14] These shifts hold for fermions with dimension mass$^{\frac{1}{2}}$. Similarly, the scalars and the metrics $g_{\alpha\overline{\beta}}$ and h_{uv} are dimensionless. We use Weyl spinors, ψ_μ^i, λ_i^α, ζ^A, ϵ^i are left-handed, $\psi_{\mu i}$, $\lambda^{\alpha i}$, ζ_A, ϵ_i are right-handed. The SU(2) indices are moved with $\lambda^i = \varepsilon^{ij}\lambda_j$ and $\lambda_i = \lambda^j\varepsilon_{ji}$.

JHEP08(2018)045

and of their conjugates ($P_{aij} = P_a^{ij\,*}$):

$$\begin{aligned}
&\text{Gravitinos:} && \overline{S}_{ij} = \frac{1}{\kappa} e^{\mathcal{K}/2} P_{aij} \overline{U}^I \, \Theta_I{}^a \,, \\
&\text{Gauginos:} && \overline{W}_{\bar\alpha ij} = -\frac{1}{\kappa} \, e^{\mathcal{K}/2} P_{aij} \overline{\nabla}_\alpha \overline{U}^I \, \Theta_I{}^a \,, \\
&\text{Hyperinos:} && \overline{N}_i{}^A = -\frac{i}{\kappa} \, e^{\mathcal{K}/2} f^{jA}{}_u \overline{U}^I \, \varepsilon_{ij} \Theta_I{}^a \xi_a^u \,.
\end{aligned} \tag{2.33}$$

The embedding tensor always appears in the combination $\Theta_I^a V^I = \kappa^{-1} e^{\mathcal{K}/2} \, \Theta_I^a U^I$. The notation ∇_α stands for Kähler-covariant derivatives. Since Kähler transformations act as

$$\mathcal{K} \to \mathcal{K} + \lambda(z) + \overline{\lambda}(\overline{z}), \qquad y \to e^{\lambda(z)} \, y, \qquad Z^I(z) \to e^{-\lambda(z)} \, Z^I(z), \tag{2.34}$$

the covariant derivatives are

$$\nabla_\alpha \, y = (\partial_\alpha - \mathcal{K}_\alpha) y = 0, \qquad \nabla_\alpha U^I = (\partial_\alpha + \mathcal{K}_\alpha) U^I, \qquad \mathcal{K}_\alpha = \frac{\partial}{\partial z^\alpha} \mathcal{K}. \tag{2.35}$$

Supersymmetry imposes the identity

$$\delta^i{}_j \mathcal{V} = \kappa^2 \Big(-3 \, S^{ik} \, \overline{S}_{jk} + W_\alpha{}^{ik} g^{\alpha\overline{\beta}} \, \overline{W}_{\overline{\beta} jk} \Big) + \frac{4}{\kappa^2} N_A^i \overline{N}_j{}^A, \tag{2.36}$$

and the gauging and fermion shifts lead then to the following $\mathcal{N} = 2$ scalar potential [24, 27, 28]:

$$e^{-1} \mathcal{V} = -\frac{1}{2} \, (\mathrm{Im}\mathcal{N})^{-1 IJ} \, \Theta_I{}^a \Theta_J{}^b P_a^x P_b^x + \overline{V}^I V^J \Theta_I{}^a \Theta_J{}^b \left(-4 \, \kappa^2 P_a^x P_b^x + \frac{2}{\kappa^2} \, h_{uv} \, \xi_a^u \xi_b^v \right), \tag{2.37}$$

where

$$-\frac{1}{2\kappa^2} \, (\mathrm{Im}\mathcal{N})^{-1 IJ} = \overline{V}^I V^J + g^{\alpha\bar\beta} \nabla_\alpha V^I \overline{\nabla}_{\bar\beta} \overline{V}^J \,, \qquad \nabla_\alpha V^I = (\partial_\alpha + \mathcal{K}_\alpha) \, V^I \tag{2.38}$$

in terms of the Kähler potential $\mathcal{K}$. For later use, we find useful to express the scalar potential (2.37) in terms of the anti-selfdual covariant derivatives k_{auv}^- defined in appendix A:

$$e^{-1} \mathcal{V} = -\frac{1}{2\kappa^4} \, (\mathrm{Im}\mathcal{N})^{-1 IJ} \, \Theta_I{}^a \Theta_J{}^b \, k_{auv}^- k_b^{-uv} + \frac{\overline{V}^I V^J \Theta_I{}^a \Theta_J{}^b}{\kappa^2} \left(-4 \, k_{auv}^- k_b^{-uv} + 2 \, h_{uv} \, \xi_a^u \xi_b^v \right), \tag{2.39}$$

using the identity

$$P_a^x P_b^x = \frac{1}{\kappa^4} \, k_{auv}^- \, k_b^{-uv} \,, \tag{2.40}$$

which can be proved using eqs. (A.3).

Supersymmetry breaking is then easily discussed. Firstly, at the ground state defined by the scalar potential, nonzero shifts of the spin-$\frac{1}{2}$ fermions indicate the presence of zero, one or two goldstinos, for a spontaneous breaking into $\mathcal{N} = 2$, 1 or 0 unbroken supersymmetry(ies). Secondly, if one or two supersymmetries remain unbroken, the value of the gravitino shift S^{ij} indicates the spacetime geometry of the ground state (AdS or

JHEP08(2018)045

Minkowski). Partial breaking $\mathcal{N}=2\to\mathcal{N}=1$ implies that there should be one (and only one) spinor ϵ_{1i} for which *three* conditions must be fulfilled:

$$\langle W_z{}^{ij}\rangle\epsilon_{1i}=0, \qquad \langle N^i{}_A\rangle\epsilon_{1i}=0, \qquad \langle S^{ij}\rangle\epsilon_{1i}=\frac{\mu}{\kappa^2}\,\epsilon_1^i, \tag{2.41}$$

and the scalar curvature of AdS spacetime is given by $\mathcal{R}=4\Lambda$, $\Lambda=-3|\mu|^2$. Furthermore, the second supersymmetry with spinor parameter ϵ_2 should verify either

$$\langle W_z{}^{ij}\rangle\epsilon_{2i}\neq 0 \qquad \text{or} \qquad \langle N^i{}_A\rangle\epsilon_{2i}\neq 0. \tag{2.42}$$

In the next sections, we analyze these conditions on a special Kähler geometry with arbitrary prepotential $F(X^I)$ and a generic quaternion-Kähler geometry for a single hypermultiplet.

3 The hypermultiplet with isometries and partial breaking

For one hypermultiplet, the four-dimensional quaternion-Kähler geometry is defined as an Einstein space with constant Ricci curvature proportional to κ^2 [29][15] and (anti-) selfdual Weyl curvature. With one or two isometries, metrics for generic quaternion-Kähler spaces have been thoroughly discussed. We will use two canonical forms: the Przanowski-Tod (PT) [30, 31] and the already quoted Calderbank-Pedersen (CP) [22]. Both are defined in terms of a solution of a differential equation, nonlinear (Toda) for the PT metric for spaces with one isometry, linear in the CP metric with two commuting isometries. Since we are interested in the latter case, we first consider the hypermultiplet metric in CP coordinates.

3.1 The CP metric

According to Calderbank and Pedersen [22], a four-dimensional quaternion-Kähler metric with two commuting isometries can be written in a set of coordinates $\rho>0$, η, ψ and φ for every solution $F(\rho,\eta)$ of the linear equation

$$\frac{\partial^2 F}{\partial\rho^2}+\frac{\partial^2 F}{\partial\eta^2}=\frac{3F}{4\rho^2}, \tag{3.1}$$

with isometries acting as shifts of ψ and φ. The line element $\mathrm{d}s^2=h_{uv}\,\mathrm{d}q^u\mathrm{d}q^v$ is

$$\mathrm{d}s^2=\frac{4\rho^2\left(F_\rho^2+F_\eta^2\right)-F^2}{4F^2}\,\mathrm{d}\ell^2+\frac{[(F-2\rho F_\rho)\alpha-2\rho F_\eta\beta]^2+[(F+2\rho F_\rho)\beta-2\rho F_\eta\alpha]^2}{F^2\left(4\rho^2\left(F_\rho^2+F_\eta^2\right)-F^2\right)}\,, \tag{3.2}$$

where

$$\alpha=\sqrt{\rho}\,\mathrm{d}\varphi\,, \qquad \beta=\frac{\mathrm{d}\psi+\eta\mathrm{d}\varphi}{\sqrt{\rho}}, \qquad \mathrm{d}\ell^2=\frac{\mathrm{d}\rho^2+\mathrm{d}\eta^2}{\rho^2}. \tag{3.3}$$

The metric determinant is

$$\frac{\left(4\rho^2\left(F_\rho^2+F_\eta^2\right)-F^2\right)^2}{16\,\rho^4F^8} \tag{3.4}$$

[15] For metric $g_{uv}=\kappa^{-2}h_{uv}$ as defined in eq. (2.2).

and positivity requires $4\rho^2\left(F_\rho^2+F_\eta^2\right) > F^2 > 0$. The CP metric describes a conformally anti-selfdual Einstein space[16] with scalar curvature normalized to $R=-12$. It is endowed with a triplet of SU(2) selfdual 2-forms J^x (complex structures) which are covariantly constant with an SU(2) connection ω^x [22]:

$$\begin{aligned} J = J^x\, i\sigma^x = &\frac{i}{F^2}\left(\left(\rho^2(F_\rho^2+F_\eta^2)-\frac{1}{4}F^2\right)\frac{\mathrm{d}\rho\wedge\mathrm{d}\eta}{\rho^2}+\alpha\wedge\beta\right)\sigma_1 \\ &+\frac{i}{F^2}\left((\rho F_\rho - i\sigma_1\rho F_\eta)(\alpha+i\sigma_1\beta)-\frac{1}{2}F(\alpha-i\sigma_1\beta)\right)\wedge\frac{\mathrm{d}\rho+i\sigma_1\mathrm{d}\eta}{\rho}\sigma_2\,, \\ \omega = \omega^x\, i\sigma^x = &\frac{i}{F}\left(F_\eta\mathrm{d}\rho-\left(\frac{1}{2}F+\rho F_\rho\right)\frac{\mathrm{d}\eta}{\rho}\right)\sigma_1+\frac{i}{F}(\alpha+i\sigma_1\beta)\sigma_2\,, \end{aligned} \tag{3.5}$$

and the identities (A.2), (A.3) are satisfied.

On the metric (3.2), Calderbank and Pedersen [22] write:[17] "*Any selfdual Einstein metric of nonzero scalar curvature with two linearly independent commuting Killing fields arises locally in this way (i.e., in a neighbourhood of any point, it is of the form (3.2) up to a constant multiple).*" We will see that a slight inaccuracy in this statement allows for an exception which is of fundamental importance in our subject.

The two Killing vectors of the CP metric are by construction $\xi_1 = \partial_\varphi$ and $\xi_2 = \partial_\psi$. Using eq. (2.29), the triplets of Killing prepotentials (or moment maps) are

$$P_1^x = \frac{1}{\kappa^2\sqrt{\rho}\,F}\begin{pmatrix}0\\-\rho\\ \eta\end{pmatrix}, \qquad P_2^x = \frac{1}{\kappa^2\sqrt{\rho}\,F}\begin{pmatrix}0\\0\\1\end{pmatrix}. \tag{3.6}$$

The standard vierbein one-forms of the metric (3.2) are

$$\begin{aligned} e^0 &= \frac{\sqrt{4\rho^2(F_\rho^2+F_\eta^2)-F^2}}{2F}\frac{\mathrm{d}\rho}{\rho}\,, & e^1 &= \frac{\sqrt{4\rho^2(F_\rho^2+F_\eta^2)-F^2}}{2F}\frac{\mathrm{d}\eta}{\rho}\,, \\ e^2 &= \frac{(F-2\rho F_\rho)\alpha-2\rho F_\eta\beta}{F\sqrt{4\rho^2(F_\rho^2+F_\eta^2)-F^2}}\,, & e^3 &= \frac{(F+2\rho F_\rho)\beta-2\rho F_\eta\alpha}{F\sqrt{4\rho^2(F_\rho^2+F_\eta^2)-F^2}}\,. \end{aligned} \tag{3.7}$$

We will need the corresponding symplectic vielbeins $f^{iA}{}_u$ obtained from relations

$$\mathrm{d}s^2 = \delta_{mn}\, e^m e^n = \varepsilon_{ij}\Omega_{AB}\, f^{iA}{}_u f^{jB}{}_v\,\mathrm{d}q^u\mathrm{d}q^v \tag{3.8}$$

or

$$f^{iA}{}_u = \frac{1}{\sqrt{2}}\left(e_u^0\,\varepsilon \pm i\, e_u^x\,\varepsilon\,\sigma^x\right)^{iA}, \qquad x=1,2,3\,, \tag{3.9}$$

and we have checked that the $f^{iA}{}_u$'s satisfy eq. (2.30). We will use the + sign below.

[16] An Einstein metric with anti-selfdual Weyl curvature.
[17] It is the point (ii) of their main theorem 1.1.

JHEP08(2018)045

3.2 Fermion shifts

Consider a generic dyonic gauging of the two isometries, described by:

$$\Theta_I{}^a = \begin{pmatrix} g_0 & g_1 \\ 0 & g_2 \\ 0 & 0 \\ 0 & -g_3 \end{pmatrix}, \qquad a = 1,2, \tag{3.10}$$

where the embedding tensor $\Theta_I{}^a$ is compatible with the locality condition (2.28) and $g_{0,1,2,3}$ are the gauge couplings. We use a prepotential frame and formulas (2.5) apply. The corresponding Kähler potential is given in eq. (2.15) for a single z:

$$\mathcal{K} = -\ln \mathcal{Y}, \qquad \mathcal{Y} = 2(f + \bar{f}) - (z - \bar{z})(f_z - \overline{f}_{\bar{z}}) . \tag{3.11}$$

In order to evaluate the fermion shifts given in (2.32), we use the results of section 3.1. Defining

$$\widetilde{c} = g_1 + g_2 z + i g_3 f_z + g_0 \eta, \tag{3.12}$$

we find

$$\begin{aligned} S^{ij} &= -\frac{1}{\kappa} e^{\mathcal{K}/2} \left(g_0 P_1^2 \delta^{ij} + i \left((g_1 + g_2 z + i g_3 f_z) P_2^3 + g_0 P_1^3 \right) (\sigma_1)^{ij} \right) \\ &= -\frac{e^{\mathcal{K}/2}}{\kappa^3 \sqrt{\rho} F} \left(-g_0 \rho\, \delta^{ij} + i \widetilde{c} (\sigma_1)^{ij} \right) \end{aligned} \tag{3.13}$$

for the gravitino shift, and

$$\begin{aligned} W_z^{ij} &= -\frac{1}{\kappa} e^{\mathcal{K}/2}\, \Theta_I{}^a P_a^{ij} \nabla_z U^I = i\, e^{\mathcal{K}/2} \frac{(g_2 + i g_3 f_{zz})}{\kappa^3 \sqrt{\rho} F} (\sigma_1)^{ij} - \mathcal{K}_z S^{ij} , \\ N^i{}_A &= -\frac{e^{\mathcal{K}/2}}{\kappa \sqrt{2\rho} F \sqrt{4\rho^2 (F_\rho^2 + F_\eta^2) - F^2}} (\rho A_2 \sigma_2 + A_3 \sigma_3)^{iA} , \\ A_2 &= -g_0 F + 2(\widetilde{c} F_\eta + g_0 \rho F_\rho) , \qquad A_3 = \widetilde{c} F + 2\rho(-g_0 \rho F_\eta + \widetilde{c} F_\rho) \end{aligned} \tag{3.14}$$

for the shifts of spin-$\frac{1}{2}$ fermions.[18]

3.3 Partial breaking in flat space

The first condition for partial breaking is certainly that the ground state does not lead to two goldstinos. The determinants of W_z^{ij} and $N^i{}_A$ should vanish at the ground state. Cancelling the determinant of W_z^{ij} requires

$$i g_2 - g_3 \langle f_{zz} \rangle = \mp \langle (\pm i \widetilde{c} + g_0\, \rho) \mathcal{K}_z \rangle \tag{3.15}$$

with zero eigenvector[19]

$$\widehat{\epsilon}_i(x) = \begin{pmatrix} \mp 1 \\ 1 \end{pmatrix} v(x), \tag{3.16}$$

[18] The fermion shifts have dimension mass^3 (S and W) or mass^1 (N).

[19] The $\pm$ or $\mp$ signs are correlated between the various equations. The parameter function $v(x)$ has dimension mass$^{-1/2}$.

JHEP08(2018)045

and this $\hat{\epsilon}_i$ is also eigenvector of S^{ij} with

$$\langle S^{ij}\rangle\,\hat{\epsilon}_j = \left\langle \frac{\mathrm{e}^{K/2}}{\kappa^3\sqrt{\rho}\,F} \right\rangle \langle g_0\rho \pm i\tilde{c}\rangle \hat{\epsilon}_i. \tag{3.17}$$

The eigenvalue should vanish for a Minkowski ground state:

$$g_0\langle\rho\rangle = \mp i\langle\tilde{c}\rangle \neq 0 \qquad\qquad \text{(Minkowski)} \tag{3.18}$$

turning condition (3.15) into

$$g_2 + ig_3\langle f_{zz}\rangle = 0 \qquad\qquad \text{(Minkowski)}, \tag{3.19}$$

and then to

$$\left\langle W_z^{ij}\right\rangle = -\left\langle \mathcal{K}_z S^{ij}\right\rangle \qquad\qquad \text{(Minkowski)}. \tag{3.20}$$

The determinant of $N^i{}_A$ turns out to be proportional to

$$\left(\rho^2 A_2\right)^2 + A_3^2 = 8\,\tilde{c}^2\rho F(F_\rho \pm iF_\eta), \tag{3.21}$$

using the Minkowski conditions (3.18) and (3.15). The conditions for the positivity of the CP metric, $\rho, F, F_\rho^2 + F_\eta^2 > 0$ and condition (3.18) imply that $N^i{}_A$ does not have a zero eigenvalue. Hence, t*he partial breaking of $\mathcal{N} = 2$ supersymmetry in Minkowski spacetime is excluded whenever the hypermultiplet can be described in the CP field coordinates and metric.* According to ref. [22], this would be always the case.

There is an apparent contradiction between this conclusion and the known existence [10] of a partial breaking on the SO(4,1)/SO(4) hypermultiplet in Minkowski spacetime. We will see shortly that for this quaternion-Kähler space, and only for this space, there exists a pair of isometries for which the coordinates used by Calderbank and Pedersen [22] *do not* exist. This is the earlier quoted exception, leading to a statement of *uniqueness* for partial breaking with a single multiplet and two gauged isometries.

3.4 Partial breaking in AdS

The obstruction found for partial breaking into Minkowski spacetime does not exist for AdS ground states. Partial breaking in this case requires at the first place eq. (3.15). With this condition, the gaugino and gravitino shifts read

$$\begin{aligned}
\langle W_z^{ij}\rangle &= -\left\langle \frac{g_0\sqrt{\rho}\,\mathrm{e}^{\mathcal{K}/2}}{\kappa^3 F}\,\mathcal{K}_z \right\rangle \begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}, \\
\langle S^{ij}\rangle &= \left\langle \frac{\mathrm{e}^{\mathcal{K}/2}}{\kappa^3\sqrt{\rho}\,F} \right\rangle \left[\mp i\langle\tilde{c}\rangle \begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix} + \langle g_0\rho \pm i\tilde{c}\rangle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right]
\end{aligned} \tag{3.22}$$

at the ground state. For the unbroken supersymmetry parameter $\hat{\epsilon}$ (the zero eigenvector of $\langle W^{ij}\rangle$),

$$\delta\,\psi_\mu^i = \left\langle \frac{\mathrm{e}^{\mathcal{K}/2}}{2\kappa\sqrt{\rho}\,F}(g_0\rho \pm i\tilde{c}) \right\rangle \gamma_\mu \hat{\epsilon}^i + \cdots, \tag{3.23}$$

and the cosmological constant is

$$\Lambda = -3 \left\langle \frac{e^{\mathcal{K}}}{\kappa^2 \rho F^2} \left(g_0^2 \rho^2 + |\widetilde{c}|^2 \right) \right\rangle . \tag{3.24}$$

The Minkowski condition (3.18) which cancels Λ does not apply and the second condition for partial breaking is that $\widehat{\epsilon}$ is also a zero eigenvector of the hyperino shift matrix (3.14):

$$\langle \rho A_2 \rangle = \mp i \langle A_3 \rangle . \tag{3.25}$$

Solutions to eqs. (3.15) and (3.25) would lead to stable AdS ground states.[20]

An example. We can realize the above conditions for $\mathcal{N}=1$ AdS vacua in a specific example. We consider for this a CP metric with $F_\eta = 0$, i.e.

$$F = \frac{1}{2}\rho^{3/2} - \sigma \rho^{-1/2} , \qquad \sigma = \text{constant} . \tag{3.26}$$

This metric has extended isometry Heisenberg $\ltimes$ U(1) and it describes the scalar manifold of the universal hypermultiplet in type II strings, including the one-loop perturbative corrections, as obtained in ref. [16]. The case $\sigma = 0$ is the tree-level $\mathrm{SU}(2,1)/\mathrm{SU}(2)\times\mathrm{U}(1)$.

From the expressions of A_2 and A_3 in eq. (3.14), one obtains:

$$A_2 = g_0(-F + 2\rho F_\rho) = g_0 \left(\rho^{3/2} + 2\,\sigma \rho^{-1/2} \right) , \tag{3.27}$$

$$A_3 = \widetilde{c}(F + 2\rho F_\rho) = 2\,\widetilde{c}\rho^{3/2}, \tag{3.28}$$

and thus the condition (3.25) implies:

$$\mathrm{Re}\,\widetilde{c} = 0 , \qquad \mathrm{Im}\,\widetilde{c} = \pm g_0 \left(\frac{\rho}{2} + \frac{\sigma}{\rho} \right) , \tag{3.29}$$

where we dropped the symbols of expectation values. On the other hand, condition (3.15) yields

$$- g_2 + g_3\,\mathrm{Im}\, f_{zz} = \pm g_0 \rho\, \mathrm{Im}\, \mathcal{K}_z , \qquad \mathrm{Re}\,\widetilde{c} = -g_3\,\mathrm{Re}\, f_{zz} \pm g_0 \rho\, \mathrm{Re}\, \mathcal{K}_z . \tag{3.30}$$

It follows that there are four equations that can be solved for the four expectation values of ρ, η and z. Indeed, using the expression of $\widetilde{c}$ in eq. (3.12), one obtains:

$$g_0 \eta = -g_1 - g_2 \mathrm{Re}\, z + g_3\,\mathrm{Re}\, f_z, \tag{3.31}$$

$$\pm g_0 \rho\, \mathrm{Im}\, \mathcal{K}_z = g_2 - g_3\,\mathrm{Im}\, f_{zz}, \tag{3.32}$$

which can be used in the remaining two equations (right part of (3.29) and (3.31)) for determining z. For instance, for $g_3 = 0$, on finds the solution:

$$\begin{aligned} g_0 \eta &= -g_1 - g_2\,\mathrm{Re}\, z , \qquad & \pm g_0 \rho &= 2 g_2\,\mathrm{Im}\, z , \\ \mathrm{Re}\,\mathcal{K}_z &= \frac{1}{2} , & \mathrm{Im}\,\mathcal{K}_z &= -\frac{1}{2\,\mathrm{Im} z} , \end{aligned} \tag{3.33}$$

where the last two equations determine z, using the expression (3.11) for $\mathcal{K}$. Note that partial $\mathcal{N}=2$ supersymmetry breaking in AdS can be realised without introducing magnetic FI coupling terms ($g_3 = 0$).

[20] Their stability is carefully discussed in the appendix B of ref. [13]. An early example was given in ref. [12].

3.5 Hyperbolic space and Calderbank-Pedersen coordinates

We now come back to the issue met at the end of section 3.3. To resolve the case, we will show that the hyperbolic space cannot be described by a CP metric with shift isometries on (φ, ψ) generated by pairs of elements in the three-dimensional abelian subalgebra of $\mathrm{SO}(4,1)$.[21] In other words, CP coordinates do not exist for this case. To proceed, we simply compare the value of scalar quantities (independent on the choice of coordinates), calculated either in CP or in PT coordinates.

The Calderbank-Pedersen metric (3.2) has a well-defined pair of isometries and Killing vectors. Scalar quantities like those appearing in the identity (2.40) can then be calculated unambiguously,

$$k^{-}_{1uv}\, k_1^{-uv} = \frac{\rho^2+\eta^2}{\rho\, F^2}\,, \qquad k^{-}_{1uv}\, k_2^{-uv} = \frac{\eta}{\rho\, F^2}\,, \qquad k^{-}_{2uv}\, k_2^{-uv} = \frac{1}{\rho\, F^2}, \tag{3.34}$$

and the dependence on the quaternion-Kähler space is in the function $F(\rho,\eta)$ only.

Consider now the simplest quaternion-Kähler space,

$$\frac{\mathrm{Sp}(2,2)}{\mathrm{Sp}(2)\times\mathrm{Sp}(2)} \sim \frac{\mathrm{SO}(4,1)}{\mathrm{SO}(4)}. \tag{3.35}$$

This hyperbolic space admits coordinates in which the line element is

$$\mathrm{d}s^2 = \frac{1}{b_0^2}\left(\mathrm{d}b_0^2+\mathrm{d}b_1^2+\mathrm{d}b_2^2+\mathrm{d}b_3^2\right). \tag{3.36}$$

This is a conformally-flat space with $R_{uv} = -3\,h_{uv}$. The corresponding symplectic vierbeins

$$f_0^{iA} = -\frac{1}{\sqrt{2}\,b_0}\tau_1^{iA}, \qquad f_1^{iA} = -\frac{1}{\sqrt{2}\,b_0}\tau_2^{iA}, \qquad f_2^{iA} = -\frac{1}{\sqrt{2}\,b_0}\tau_3^{iA}, \qquad f_3^{iA} = \frac{1}{\sqrt{2}\,b_0}\varepsilon^{iA}, \tag{3.37}$$

where $(\tau_x)^{iA} = (i\varepsilon\sigma_x)^{iA}$, follow from their definition (2.30). The triplet of SU(2) self-dual two-forms J^x (complex structures)

$$\begin{aligned} J^1 &= \frac{1}{b_0^2}\left(\mathrm{d}b_0\wedge\mathrm{d}b_3+\mathrm{d}b_1\wedge\mathrm{d}b_2\right), \\ J^2 &= \frac{1}{b_0^2}\left(\mathrm{d}b_0\wedge\mathrm{d}b_2+\mathrm{d}b_3\wedge\mathrm{d}b_1\right), \\ J^3 &= -\frac{1}{b_0^2}\left(\mathrm{d}b_0\wedge\mathrm{d}b_1+\mathrm{d}b_2\wedge\mathrm{d}b_3\right) \end{aligned} \tag{3.38}$$

is covariantly constant up to the SU(2) connection ω^x

$$\omega^1 = -\frac{\mathrm{d}b_3}{b_0}\,, \qquad \omega^2 = -\frac{\mathrm{d}b_2}{b_0}\,, \qquad \omega^3 = \frac{\mathrm{d}b_1}{b_0}\,. \tag{3.39}$$

Conditions (A.2) and (A.3) are verified. We are interested in the Killing vectors of two translation isometries acting on b_2 and b_3:

$$\xi_1 = \partial_{b_2}\,, \qquad \xi_2 = \partial_{b_3}\,. \tag{3.40}$$

[21] See appendix B for a review on H_4 coordinates.

Their Killing prepotential triplets follow from eq. (2.29):

$$P_1 = -\frac{1}{\kappa^2 b_0}\begin{pmatrix}0\\1\\0\end{pmatrix}, \qquad P_2 = -\frac{1}{\kappa^2 b_0}\begin{pmatrix}1\\0\\0\end{pmatrix}. \tag{3.41}$$

In these coordinates and for these isometries, we find the following expression for the scalars appearing in eq. (2.40):

$$k^-_{1uv}\,k_1^{-uv} = \frac{1}{b_0^2}, \qquad k^-_{1uv}\,k_2^{-uv} = 0, \qquad k^-_{2uv}\,k_2^{-uv} = \frac{1}{b_0^2}. \tag{3.42}$$

The comparison with the generic values (3.34) obtained for the CP metric indicates that for these isometries, CP coordinates cannot be found.[22]

The origin of this obstruction is located in the derivation of the CP metric given in ref. [22]. This metric is a consequence of the Joyce description for anti-selfdual conformal metrics with a $\mathrm{U}(1)\times\mathrm{U}(1)$ symmetry [32], the Jones-Tod correspondence for four dimensional anti-selfdual spaces with at least one isometry [33], and the use of Przanowski-Tod (PT) theorem to determine which metrics, among the Joyce metrics, are Einstein spaces [30, 31, 34]. In short, to identify the Einstein representatives among the conformal structures with anti-selfdual Weyl tensor, one employs the PT form where the metric is generated by a function $\Psi(X,Y,Z)$ solving the continual Toda equation [30, 31, 34][23]

$$\Psi_{XX} + \Psi_{YY} + \left(\mathrm{e}^{\Psi}\right)_{ZZ} = 0. \tag{3.43}$$

The PT line element is then

$$\begin{aligned} \mathrm{d}s^2 &= \frac{1}{Z^2}\left[\frac{1}{U}(\mathrm{d}\psi+\omega)^2 + U\left(\mathrm{d}Z^2 + \mathrm{e}^{\Psi}\left(\mathrm{d}X^2+\mathrm{d}Y^2\right)\right)\right],\\ \mathrm{d}\omega &= U_X\,\mathrm{d}Y\wedge\mathrm{d}Z + U_Y\,\mathrm{d}Z\wedge\mathrm{d}X + \left(U\,\mathrm{e}^{\Psi}\right)_Z\mathrm{d}X\wedge\mathrm{d}Y\,,\\ 2U &= 2 - Z\,\Psi_Z\,. \end{aligned} \tag{3.44}$$

The quaternion-Kähler metric has one isometry acting on the fourth coordinate ψ and generated by ∂_ψ. The simplest solution is of course

$$\Psi = C = \text{constant}, \qquad U = 1, \qquad \mathrm{d}\omega = 0, \qquad \omega = \mathrm{d}g(X,Y,Z), \tag{3.45}$$

for which

$$\mathrm{d}s^2 = \frac{1}{Z^2}\left[\mathrm{d}(\psi+g)^2 + \mathrm{d}Z^2 + \mathrm{e}^{C}\left(\mathrm{d}X^2+\mathrm{d}Y^2\right)\right]. \tag{3.46}$$

This is the metric (3.36) with $b_0 = Z$, $b_1 = \mathrm{e}^{C/2}X$, $b_2 = \mathrm{e}^{C/2}Y$, $b_3 = \psi + g$ and the isometry of the PT metric shifts b_3.

In order to make contact with the CP metric, we assume the existence, in the PT description, of a second isometry generated by ∂_φ. The case of primary interest for us is a shift isometry acting on b_1 or b_2. We choose a translation isometry of the Y coordinate.

[22] A similar conclusion, technically more involved though, derives from comparing the inner products of the two isometries, i.e. $\xi_a^u \xi_b^v h_{uv}$.

[23] As usual, indices indicate derivatives.

Assume then that $\varphi = Y$ and that Ψ does not depend on Y. Finding the CP coordinates is possible using a transformation due to Ward [35]:[24]

$$\begin{aligned}
&\left(X,\, Z;\, \mathrm{e}^{\Psi(X,Z)}\right) &&\Longrightarrow && (\rho,\, \eta;\, V(\rho,\eta))\,, \\
&X = V_\eta\,, \qquad 2\,Z = \rho\, V_\rho\,, && && \frac{1}{4}\,\rho^2 = \mathrm{e}^{\Psi}\,, \\
&\Psi_{XX} + \left(\mathrm{e}^{\Psi}\right)_{ZZ} = 0 &&\Longrightarrow && \frac{1}{\rho}\,(\rho V_\rho)_\rho + V_{\eta\eta} = 0\,,
\end{aligned} \tag{3.47}$$

resulting in the CP metric (3.2), with $F = \sqrt{\rho}\, V_\rho$. This transformation is clearly incompatible with a constant Ψ. For this hypermultiplet manifold and for this choice of second isometry with Killing vector ∂_Y, CP coordinates ρ and η do not exist and the argument against partial breaking proved in the previous section does not hold. The constancy of the Toda potential is at the origin of this exception.

With SO(4, 1) isometry, the hyperbolic space has a variety of other inequivalent pairs of commuting isometries. For these pairs, the corresponding Toda potentials are not constant and CP coordinates do exist. Some examples of CP coordinates for other isometries of the hyperbolic space are described in appendix B.

4 Partial breaking and the APT model

4.1 The SO(4, 1)/SO(4) model

Ferrara, Girardello and Porrati (FGP) [7, 10] have shown that partial breaking occurs on the simplest quaternion-Kähler space for one hypermultiplet, SO(4, 1)/SO(4), with two gauged translation isometries. Explicitly, coordinates (3.36) with

$$\mathrm{d}s^2 = \frac{1}{b_0^2}\left(\mathrm{d}b_0^2 + \mathrm{d}b_1^2 + \mathrm{d}b_2^2 + \mathrm{d}b_3^2\right), \qquad \mathcal{L}_{\text{kin.}} = -\frac{e}{2(\kappa b_0)^2}(\partial_\mu b^u)(\partial^\mu b^u), \tag{4.1}$$

and Killing vectors

$$\xi_1 = \partial_{b_2}\,, \qquad \xi_2 = \partial_{b_3} \tag{4.2}$$

are used for constructing the $\mathcal{N} = 2$ supergravity lagrangian. In ref. [10] they first worked in the *non-prepotential* frame described in section 2, eqs. (2.13) and (2.14). Then they reworked the example in a generic frame with arbitrary prepotential function $f(z)$ [7].

Our objective in this section is to complete the description of the model by showing explicitly that the $\mathcal{N} = 2$ supergravity theory (at finite κ then) admits a stable ground state with partial breaking which continuously deforms to the APT model in the gravity-decoupling limit $\kappa \to 0$. This can be seen as deriving *off-shell* the APT lagrangian as the $\kappa \to 0$ limit of the SO(4, 1)/SO(4) supergravity lagrangian.[25]

Using the embedding tensor (3.10), the prepotential frame (2.5) leading to Kähler potential (2.15) and coordinates b^u with metric (3.36) for the hypermultiplet, the supergravity

[24] See appendix C for details.

[25] Following for instance ref. [36]. Although the statement exists in the literature, we have not found an explicit construction with an appropriate use of the concept of prepotential frame.

JHEP08(2018)045

potential reads:

$$\mathcal{V} = \frac{\mathrm{e}^{\mathcal{K}}}{\kappa^4\, b_0^2 \mathcal{K}_{z\bar{z}}} \left[\left(g_0^2 + |c|^2 \right) \left(-\mathcal{K}_{z\bar{z}} + \mathcal{K}_z \mathcal{K}_{\bar{z}} \right) + |c_z|^2 + \bar{c} c_z \mathcal{K}_{\bar{z}} + c \bar{c}_z \mathcal{K}_z \right] \tag{4.3}$$

with c defined as

$$c = -i(g_1 + g_2 z + i g_3 f_z), \tag{4.4}$$

and $c_z = -ig_2 + g_3 f_{zz}$. Since hypermultiplet scalars only appear in the prefactor b_0^{-2}, the ground state of the potential, in order to escape the runaway of b_0, requires Minkowski geometry, $\langle \mathcal{V} \rangle = 0$. In ref. [10], the authors consider the particular case $g_1 = g_2 = 0$, $f(z) \sim z$, $\mathcal{K} = -\ln(z + \bar{z})$ and then $\mathcal{V} \equiv 0$.

Notice that the scalar potential (4.3) vanishes if

$$\langle c_z \rangle = 0 \qquad \Longrightarrow \qquad g_3 \langle f_{zz} \rangle = i g_2. \tag{4.5}$$

Since $\langle f_{zz} \rangle$ is imaginary, $\langle \mathcal{K}_{z\bar{z}} \rangle = \langle \mathcal{K}_z \mathcal{K}_{\bar{z}} \rangle$ and $\langle \mathcal{V} \rangle = 0$.

4.2 $\mathcal{N} = 1$ Minkowski vacua

The fermion shifts (2.32) induced by this gauging read

$$S^{ij} = \frac{\mathrm{e}^{\mathcal{K}/2}}{\kappa^3 b_0} \left(g_0\, \mathbb{I}_2 + c\, \sigma_3 \right)^{ij}, \qquad N^i{}_A = \frac{\mathrm{e}^{\mathcal{K}/2}}{\kappa \sqrt{2}\, b_0} \left(g_0\, \sigma_3 + c\, \mathbb{I}_2 \right)^{iA} \tag{4.6}$$

for gravitinos and hyperinos. They verify the relation[26]

$$S^{ik}\, \overline{S}_{jk} = \frac{2}{\kappa^4} N^i_A \overline{N}_j{}^A. \tag{4.7}$$

For gauginos,

$$W_z^{ij} = -\kappa^{-1} \mathrm{e}^{\mathcal{K}/2}\, \Theta_I{}^a P_a^{ij} \nabla_z U^I = -\mathrm{e}^{\mathcal{K}/2} \frac{c_z}{\kappa^3 b_0} (\sigma_3)^{ij} - \mathcal{K}_z\, S^{ij}, \tag{4.8}$$

where we have used

$$\Theta_I{}^a \partial_z U^I = \begin{pmatrix} 0 \\ i c_z \end{pmatrix}. \tag{4.9}$$

The conditions for partial breaking are then easily stated. To have a common zero eigenvector for W^{ij} and $N^i{}_A$, we need a dyonic (electric *and* magnetic) gauging with $g_3 \neq 0 \neq g_0$ and

$$\langle g_3 f_z - i g_2 z \rangle = \pm g_0 + i g_1, \qquad g_3 \langle f_{zz} \rangle = i g_2. \tag{4.10}$$

The first condition $\langle c \rangle = \pm g_0$ leads to a zero eigenvector $\widehat{\epsilon}$ of $N^i{}_A$ while the second condition $\langle c_z \rangle = 0$ ensures that the same $\widehat{\epsilon}$ is a zero eigenvector of W_z^{ij}. This second condition for partial breaking also implies $\langle S^{ij} \rangle \widehat{\epsilon}_j = 0$ and $\langle \mathcal{V} \rangle = 0$ and then partial breaking can only exist in Minkowski spacetime. The conditions (4.10) define an $\mathcal{N} = 1$ supersymmetric stable ground state. In ref. [10] where $g_1 = g_2 = f_{zz} = 0$, these conditions reduce to $g_3 = \pm g_0 \neq 0$.

[26] As matrices, $\kappa^2 S = \sqrt{2} \sigma_3 N$.

Solving the conditions for partial breaking commonly impose, for a given choice of $f(z)$, particular values or relations on the coupling constants. For instance, a linear $f = z$, as used in ref. [10], has partial breaking only if $g_0 = \pm g_3 \neq 0$, $g_1 = g_2 = 0$. The conditions may be impossible to solve: $f = z^2$ forces all g_i to be zero (but this example is irrelevant since the Kähler metric $\mathcal{K}_{z\bar{z}} \equiv 0$). For a generic prepotential $f(z)$, one usually finds that two couplings are determined in terms of the other two.

The spectrum of the partially broken theory includes $\mathcal{N} = 1$ supergravity ($2_{\mathrm{B}} + 2_{\mathrm{F}}$ on-shell states), a massive $\mathcal{N} = 1$ gravitino multiplet (gravitino, the two spin-one fields, one fermion, $6_{\mathrm{B}} + 6_{\mathrm{F}}$), a massless chiral multiplet ($2_{\mathrm{B}} + 2_{\mathrm{F}}$) and a chiral multiplet with the scalar z and mass proportional to the free parameter $\langle f_{zzz}\rangle$, precisely as in the APT model, see below. The four hypermultiplet scalars are massless (two are Goldstone bosons) and the mass matrix reduces to z only with

$$\langle \mathcal{V}_{z\bar{z}} \rangle = \left\langle \frac{g_3^4 \mathcal{Y}}{4\kappa^4 g_0^2 b_0^2} \, |f_{zzz}|^2 \right\rangle \geqslant 0 \,, \tag{4.11}$$

where $\mathcal{Y}$ is defined through the Kähler potential (3.11):

$$\mathcal{K} = -\ln \mathcal{Y} \,, \qquad \mathcal{Y} = -\frac{i(z-\bar{z})(2g_1 + g_2(z+\bar{z}))}{g_3} + 2(f + \bar{f}) > 0 \,. \tag{4.12}$$

Hence, the mass of the scalar z (and of its fermion partner) is given by

$$m_z^2 = \kappa^2 \left\langle \frac{\mathcal{V}_{z\bar{z}}}{\mathcal{K}_{z\bar{z}}} \right\rangle = \left\langle \frac{g_3^6 \mathcal{Y}^3}{16\,\kappa^2 g_0^4 b_0^2} \, |f_{zzz}|^2 \right\rangle \tag{4.13}$$

since $\mathcal{K}_{z\bar{z}} = \frac{4g_0^2}{g_3^2 \mathcal{Y}^2}$.

At the $\mathcal{N} = 1$ ground state, the value of the hypermultiplet scalar $\langle b_0 \rangle$ is an arbitrary parameter. From the gravitino shift matrix[27] or from the expression of the scalar potential however, the mass of the massive gravitino scales as

$$m_{\frac{3}{2}} \sim \langle \kappa b_0 \rangle^{-1}, \tag{4.14}$$

and the theory has two order parameters, $\langle b_0 \rangle$ and $\langle f_{zzz} \rangle$ for the massive gravitino and chiral (with z) multiplets respectively.

In order to discuss the gravity-decoupling limit $\kappa \to 0$ of the supergravity theory and make contact with the APT model, we first redefine the hypermultiplet scalars ($\langle b_0 \rangle \neq 0$):[28]

$$b_0 = \langle b_0 \rangle (1 + \kappa \tilde{\mu}\, \widetilde{b}_0) \,, \qquad b_i = \kappa \tilde{\mu} \langle b_0 \rangle \, \widetilde{b}_i \,, \qquad i = 1,2,3, \tag{4.15}$$

where $\tilde{\mu}$ is a mass scale (and $\kappa\tilde{\mu} \sim \frac{\tilde{\mu}}{M_{\mathrm{P}}}$ is dimensionless). The hypermultiplet kinetic terms are then

$$\mathcal{L}_{\mathrm{hyper}} = -\frac{e\,\tilde{\mu}^2}{2\left(1 + \kappa\tilde{\mu}\widetilde{b}_0\right)^2} \, \delta_{uv} (\partial_\mu \widetilde{b}^u)(\partial^\mu \widetilde{b}^v), \tag{4.16}$$

and in the limit $\kappa \to 0$, the kinetic metric is the trivial hyper-Kähler $h_{uv} = \tilde{\mu}^2 \delta_{uv}$.[29]

[27] Which in a Minkowski ground state is proportional to the mass matrix, $S^{ij} \sim \kappa^{-2} m^{ij}_{\frac{3}{2}}$.

[28] This is a simplistic use of the procedure described in refs. [17, 18, 37, 38].

[29] We could as well define dimension-one fields with $\tilde{\mu}\widetilde{b}^u \to \widetilde{b}^u$.

JHEP08(2018)045

For the vector multiplet kinetic term, we need as $\kappa \to 0$

$$-\frac{e}{\kappa^2}\,\mathcal{K}_{z\overline{z}}(\partial_\mu z)(\partial^\mu \overline{z}) \qquad \longrightarrow \qquad -(i\overline{\mathcal{F}}_{\overline{x}\overline{x}} - i\mathcal{F}_{xx})(\partial_\mu x)(\partial^\mu \overline{x}), \tag{4.17}$$

where $\mathcal{F}(x)$ is the dimension-two prepotential of the rigid $\mathcal{N}=2$ theory and x is a dimension-one scalar. In other words, we need

$$\frac{1}{\kappa^2}\,\mathcal{K}(z,\overline{z}) = -\frac{1}{\kappa^2}\,\ln\mathcal{Y} \qquad \longrightarrow \qquad -i\overline{x}\mathcal{F}_x + ix\overline{\mathcal{F}}_{\overline{x}} + g(x) + \overline{g}(\overline{x}) \tag{4.18}$$

and the Kähler potential of the rigid theory will be $\widehat{\mathcal{K}}(x,\overline{x}) = -i\overline{x}\mathcal{F}_x + ix\overline{\mathcal{F}}_{\overline{x}}$. Following ref. [7], this is obtained from the formal κ expansion,

$$f(z) = \frac{1}{4} + \lambda\kappa\tilde{\mu}\,z + \kappa^2\left[i\tilde{\mu}^2\widehat{F}(z) + \frac{1}{4}\tilde{\mu}^2(\lambda+\overline{\lambda})z^2\right] + \mathcal{O}(\kappa^3\tilde{\mu}^3), \tag{4.19}$$

and the definition

$$\mathcal{F}(x) = \tilde{\mu}^2\,\widehat{F}\left(\frac{x}{\tilde{\mu}}\right), \tag{4.20}$$

with $\mathcal{F}_x = \tilde{\mu}\,\widehat{F}_z$ and $\mathcal{F}_{xx} = \widehat{F}_{zz}$. The arbitrary complex number λ will get a precise value later on.

With the rescaling (4.15) of the hypermultiplet scalars, a corresponding rescaling of the Killing vectors, or equivalently a (first) rescaling of the coupling constants, is needed:

$$g_i = \kappa\tilde{\mu}\langle b_0\rangle\,\widetilde{g}_i\,, \tag{4.21}$$

leading to the scalar potential

$$\mathcal{V} = \mu^4\,\frac{e^{\mathcal{K}}}{\left(1+\kappa\mu\widetilde{b}_0\right)^2}\left[-\frac{1}{\kappa^2\mu^2}\left(\widetilde{g}_0^2 + |c|^2\right) + \frac{1}{\kappa^2\mu^2\mathcal{K}_{z\overline{z}}}\left(\widetilde{g}_0^2\mathcal{K}_z\mathcal{K}_{\overline{z}} + |c_z + c\mathcal{K}_z|^2\right)\right], \tag{4.22}$$

where c and c_z are expressed in terms of $\widetilde{g}_i$ (instead of g_i), $c = -i(\widetilde{g}_1 + \widetilde{g}_2 z) + \widetilde{g}_3 f_z$ and $c_z = -i\widetilde{g}_2 + \widetilde{g}_3 f_{zz}$.

Before expanding in powers of κ, we perform a second redefinition of the gauge couplings,

$$\widetilde{g}_0 = \kappa\tilde{\mu}\,\widehat{g}_0\,, \qquad \widetilde{g}_1 = \kappa\tilde{\mu}\,\widehat{g}_1\,, \qquad \widetilde{g}_2 = (\kappa\tilde{\mu})^2\,\widehat{g}_2\,, \qquad \widetilde{g}_3 = \widehat{g}_3\,. \tag{4.23}$$

The leading terms in the quantities c, c_z and $\mathcal{K}_z$ appearing in the potential (4.22) are then

$$\begin{aligned} &c = [\widehat{g}_3\lambda - i\widehat{g}_1]\,\kappa\tilde{\mu} + \mathcal{O}(\kappa^2\tilde{\mu}^2), \qquad c_z = \left[-i\widehat{g}_2 + 2\,(\mathrm{Re}\,\lambda)^2\widehat{g}_3 + i\widehat{g}_3\mathcal{F}_{xx}\right]\kappa^2\tilde{\mu}^2 + \mathcal{O}(\kappa^2\tilde{\mu}^2), \\ &\mathcal{K}_z = -2\,\mathrm{Re}\,\lambda\,\kappa\tilde{\mu} + \mathcal{O}(\kappa^2\tilde{\mu}^2), \end{aligned} \tag{4.24}$$

and, to leading order in κ, the potential reads

$$\begin{aligned} \mathcal{V} &= \frac{\tilde{\mu}^4}{\widehat{\mathcal{K}}_{x\overline{x}}}\left[4(\mathrm{Re}\,\lambda)^2\widehat{g}_0^2 + \left|-\widehat{g}_2 + 2\,\widehat{g}_1\mathrm{Re}\,\lambda - 2\,\widehat{g}_3\,\mathrm{Re}\,\lambda\mathrm{Im}\,\lambda + \widehat{g}_3\,\mathcal{F}_{xx}\right|^2\right] - C \\ &= \frac{1}{2\,\mathrm{Im}\,\mathcal{F}_{xx}}\left[\zeta^2 + |m^2 + M^2\mathcal{F}_{xx}|^2\right] - C \\ &= \frac{1}{2\,\mathrm{Im}\,\mathcal{F}_{xx}}\left|m^2 - i\zeta + M^2\mathcal{F}_{xx}\right|^2 + \zeta\,M^2 - C, \end{aligned} \tag{4.25}$$

where

$$\begin{aligned} m^2 &= -(\widehat{g}_2 - 2\,\widehat{g}_1 \mathrm{Re}\,\lambda + 2\,\widehat{g}_3\,\mathrm{Re}\lambda\,\mathrm{Im}\lambda)\tilde{\mu}^2, & M^2 &= \widehat{g}_3\tilde{\mu}^2, \\ \zeta &= 2\,\mathrm{Re}\,\lambda\widehat{g}_0\,\tilde{\mu}^2, & C &= \tilde{\mu}^4\widehat{g}_0^2 + \tilde{\mu}^4|\widehat{g}_3\lambda - i\widehat{g}_1|^2. \end{aligned} \tag{4.26}$$

The scalar potential of a globally supersymmetric theory is not expected to have an irrelevant additive constant and we cancel $\zeta M^2 - C$ by choosing

$$\lambda = \frac{1}{\widehat{g}_3}\,(\widehat{g}_0 + i\widehat{g}_1), \tag{4.27}$$

which also implies

$$c = \widehat{g}_0\,\kappa\tilde{\mu} + \mathcal{O}(\kappa^2\tilde{\mu}^2) = \widetilde{g}_0 + \mathcal{O}(\kappa^2\tilde{\mu}^2). \tag{4.28}$$

This is the leading term in the first condition (4.10) for partial breaking (related to the gravitino and hyperino shift matrices). The shift matrix for canonically normalized (mass$^{\frac{3}{2}}$ dimension) gauginos Λ^i becomes

$$\delta\,\Lambda^i = \mathcal{W}_x^{ij}\,\epsilon_j + \cdots\,, \qquad\qquad \mathcal{W}_x^{ij} = \frac{\tilde{\mu}^2}{2\,\mathrm{Im}\,\mathcal{F}_{xx}}\begin{pmatrix} 4\,\widehat{g}_0^2\,\widehat{g}_3^{\,-1} - c_z & 0 \\ 0 & c_z \end{pmatrix}, \tag{4.29}$$

where $\mathcal{K}_z$, c_z and c are given in eqs. (4.24) and (4.28).[30]

Up to here, the analysis has been off-shell only. We now expect that the second condition (4.10) for partial breaking, which indicates that only one gaugino is a goldstino, follows from the minimum of the potential, which is at[31]

$$\widehat{g}_3\langle\mathcal{F}_{xx}\rangle = \widehat{g}_2 + 2i\,\mathrm{Re}\,\lambda\widehat{g}_0 = \widehat{g}_2 + 2i\,\widehat{g}_0^2/\widehat{g}_3. \tag{4.30}$$

This vacuum equation is also the leading order term of $\langle c_z\rangle = 0$, the second condition for partial breaking (4.10). And for $\langle c_z\rangle = 0$, the gaugino shift matrix (4.29) has one zero eigenvalue. At the ground state, the vector multiplet metric is

$$2\,\langle\mathrm{Im}\,\mathcal{F}_{xx}\rangle = 4\,\frac{\widehat{g}_o^2}{\widehat{g}_3^2}, \tag{4.31}$$

and the deformation parameter of the supersymmetry variation in the goldstino direction is then

$$\delta\Lambda_{\mathrm{goldstino}} = M^2 + \cdots = \widehat{g}_3\,\tilde{\mu}^2 + \cdots\,. \tag{4.32}$$

The coupling constant $\widehat{g}_3\tilde{\mu}^2$ is the magnetic FI term at the origin of the partial breaking. The $\mathcal{N} = 2$ multiplet splits in a massless $N = 1$ Maxwell, including the goldstino, and a chiral $\mathcal{N} = 1$ multiplet with mass

$$M_x^2 = \left\langle \frac{\widehat{g}_3^6\mu^4}{16\widehat{g}_0^4}|\mathcal{F}_{xxx}|^2 \right\rangle, \tag{4.33}$$

as expected from eq. (4.13).

In conclusion, we have shown that this $\mathcal{N} = 2$ supergravity theory possesses for all values of κ an $\mathcal{N} = 1$ ground state which coincides in the limit $\kappa \to 0$ with the APT lagrangian and its $\mathcal{N} = 1$ vacuum.

[30] The shift matrices $\kappa^2 S^{ij}$ and $N^i{}_A$ vanish when $\kappa \to 0$ and hyperinos decouple from the goldstino.

[31] Metric positivity requires $\mathrm{Im}\,\mathcal{F}_{xx} > 0$.

JHEP08(2018)045

4.3 $\mathcal{N}=0$ Minkowski vacua

The scalar potential (4.3) can also be written

$$e^{-1}\mathcal{V} = \frac{1}{\kappa^4\, b_0^2 \mathcal{K}_{z\bar{z}}\mathcal{Y}^2}\Big[2\left(g_0^2+|c|^2\right)\mathrm{Re}\, f_{zz} + \mathcal{Y}|c_z|^2 - \bar{c}c_z\mathcal{Y}_{\bar{z}} - c\bar{c}_z\mathcal{Y}_z\Big]\,. \tag{4.34}$$

Non-supersymmetric vacua will then follow by solving

$$\partial_z h = \partial_{\bar{z}} h = h = 0, \qquad h(z,\bar{z}) = 2\left(g_0^2+|c|^2\right)\mathrm{Re}\, f_{zz} + \mathcal{Y}|c_z|^2 - \bar{c}c_z\mathcal{Y}_{\bar{z}} - c\bar{c}_z\mathcal{Y}_z, \tag{4.35}$$

supplemented by stability conditions and the existence of two goldstinos. Since h is real, eqs. (4.35) give three conditions for the two real components of z. Hence, for a given $f(z)$ one expects at least one non-trivial condition on the gauge coupling constants: once $\langle z\rangle$ is fixed by $\partial_z h = 0$, the number $\langle\mathcal{V}\rangle$ must vanish to avoid runaway in b_0.

Since $c_{zz} = g_3\, f_{zzz}$, with the definition (3.11) of $\mathcal{Y}$, it is immediate that $\langle f_{zzz}\rangle = 0$ solves $\partial_z h = 0$. We have already observed that $\langle c_z\rangle = 0$ leads to $h=0$. Hence, $\langle f_{zzz}\rangle = \langle c_z\rangle = 0$ is a solution of conditions (4.35). Since $\langle c_z\rangle = 0$ also implies that the gaugino shift matrix has a zero eigenvector, the determinant of $\langle N^i{}_A\rangle$ should be nonzero in an $\mathcal{N}=0$ ground state:

$$g_0 \neq \pm\langle c\rangle. \tag{4.36}$$

This happens in the FGP model [10] with $f(z)=z$, $g_1=g_2=0$ and then $c=g_3$: all $\langle z\rangle$ are stable ground states since $\mathcal{V}\equiv 0$, the generic ground state has $\mathcal{N}=0$ and partial breaking occurs when $g_0=\pm g_3$. Hence we may think that partial-breaking solutions are surrounded (in the parameter space of the solutions of a model) by $\mathcal{N}=0$ solutions. However, assuming

$$f(z) = f_0 + f_1\, z + f_2\, z^2\,, \qquad\qquad f_{0,1,2}\in\mathbb{C}\,, \tag{4.37}$$

leads to the scalar potential

$$\begin{aligned} &\mathcal{V} = \frac{\mathcal{C}}{\kappa^4 b_0^2}, \qquad\qquad \mathcal{C} = \frac{(g_0^2+|A_1|^2)\mathrm{Re}f_2 + |A_2|^2\mathrm{Re}f_0 - \mathrm{Re}f_1\mathrm{Re}(A_1\bar{A}_2)}{(\mathrm{Re}f_1)^2 - 4\mathrm{Re}f_0\,\mathrm{Re}f_2}\,, \\ &A_1 = g_1 + ig_3 f_1\,, \qquad A_2 = g_2 + 2ig_3 f_2 = i\,c_z\,. \end{aligned} \tag{4.38}$$

Parameters should be such that the constant $\mathcal{C}$ vanishes to avoid a runaway in b_0. The choice $c_z = 0$ with $g_3 f_2 \neq 0$ leads to $\mathcal{N}=0$ ground states for arbitrary $\langle z\rangle$ but the supplementary condition for an $\mathcal{N}=1$ vacuum is never verified.

Working out conditions (4.35) leads to two distinct classes of Minkowski vacua:

i. All solutions with $\langle f_{zzz}\rangle \neq 0$ and $\langle h\rangle = 0$ are $\mathcal{N}=1$ vacua already described in eqs. (4.10).[32]

ii. Solutions with $\langle f_{zzz}\rangle = 0$ and $\langle h\rangle = 0$ are generically $\mathcal{N}=0$ vacua.

Stability of the $\mathcal{N}=0$ ground states is provided in terms of the mass matrix for the six real scalars b_u and z. The non-trivial second derivatives of the potential are $\langle\mathcal{V}_{z\bar{z}}\rangle$ which vanishes with $\langle f_{zzz}\rangle = 0$ and $\langle\mathcal{V}_{zz}\rangle$ which is controlled by the fourth derivative of f. The vacuum is then unstable except if $\langle f^{(n)}\rangle = 0$ for all $n\geqslant 3$ and this leads us naturally to the choice (4.37).

[32]For a proof, see appendix D.

JHEP08(2018)045

5 Outlook

In summary, our motivation was to classify spontaneous (partial) supersymmetry breaking in the minimal case of $\mathcal{N} = 2$ supergravity, containing one hypermultiplet and one vector multiplet. The former could describe the universal dilaton of type II superstrings compactified on a Calabi-Yau threefold, while the latter should gauge together with the $\mathcal{N} = 2$ graviphoton two commuting isometries of the hypermultiplet quaternion-Kähler manifold, which is necessary in order to obtain a massive $\mathcal{N} = 1$ spin-3/2 multiplet.

The analysis can be done in a general way, since a four-dimensional quaternionic manifold with a two-torus isometry can be put in the Calderbank-Pedersen metric form [22]. To our surprise, using this approach we found a no-go theorem on the existence of $\mathcal{N} = 1$ Minkowski vacua, which would also hold for any number of abelian vector multiplets. This result seems in conflict with the well-known example of the hyperbolic space $\mathrm{SO}(4,1)/\mathrm{SO}(4)$ [7]. However, we proved that the hyperbolic space cannot be written in a Calderbank-Pedersen form, where its torus symmetry lies within the three-dimensional abelian subalgebra of $\mathrm{SO}(4,1)$. We furthermore showed that it is easy to obtain $\mathcal{N} = 1$ vacua of partially broken supersymmetry in AdS space.

Finally, we revisited the hyperbolic space for gaugings within the three-dimensional Abelian subalgebra of $\mathrm{SO}(4,1)$, while for the scalars of the vector sector we considered a generic holomorphic prepotential. We worked out the details for a generic gauging leading to a supergravity theory with potential (4.3) and possessing $\mathcal{N} = 1$ or $\mathcal{N} = 0$ Minkowski vacua for all values of κ. For the $\mathcal{N} = 1$ vacua, we also worked out their off-shell gravity-decoupling limit, and obtained the APT lagrangian [1].

Some open questions remain to be answered, which are outside of our present scope. Regarding the gauged isometries, on the one hand, one may consider gauging isometries of the special-Kähler manifold of vector multiplets, or (part of) the $\mathrm{SU}(2)_R$ R-symmetry with the compensating hypermultiplet. On the other hand, one may study the effect of more hypermultiplets, for which however explicit and general metrics for quaternion-Kähler spaces with isometries are not available.

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A Four-dimensional quaternionic manifolds with isometries

Consider a four-dimensional quaternionic space, described by an Einstein metric with anti-selfdual Weyl curvature

$$W_{xyrw} + \frac{1}{2}\,\varepsilon_{xyuv}\,W^{uv}{}_{rw} = 0\,, \qquad R_{uv} = -3\,h_{uv}\,, \tag{A.1}$$

normalized with $R = -12$. This space is endowed with a triplet of SU(2) self-dual complex structures J^x_{uv}, which are covariantly constant modulo an SU(2) one-form connection ω^x

$$\nabla_w J^x_{uv} + \varepsilon^{xyz}\,\omega^y_w J^z_{uv} = 0\,. \tag{A.2}$$

The complex structures J^x_{uv} are normalized to satisfy:

$$(J^x)_u{}^r\,(J^y)_r{}^v = -\delta^{xy}\,\delta^v_u - \varepsilon^{xyz}\,(J^z)_u{}^v\,, \quad (J^x)_u{}^v\,(J^x)_w{}^r = h_{uw}\,g^{vr} - \delta^r_u\,\delta^v_w + \varepsilon_{uw}{}^{vr}\,. \tag{A.3}$$

Assume that the quaternionic space has some isometries generated by $\xi_a = \xi^u_a\partial_u$. As a consequence of the Bianchi identity for the Riemann tensor, condition (A.1) leads to

$$\nabla_w k^+_{auv} = 2P^+_{uvwr}\xi^r_a\,, \tag{A.4}$$

in terms of the (anti)-selfdual covariant derivatives

$$\begin{aligned} k^\pm_{auv} &= P^\pm_{uv}{}^{wr}\nabla_w\xi_{ar}\,, \\ P^\pm_{uv}{}^{wr} &= \frac{1}{2}\left(\delta^{wr}_{uv} \pm \frac{1}{2}\,\varepsilon_{uv}{}^{wr}\right), \qquad \delta^{wr}_{uv} := \frac{1}{2}\left(\delta^w_u\delta^r_v - \delta^r_u\delta^w_v\right). \end{aligned} \tag{A.5}$$

These (anti)-selfdual covariant derivatives obey the following identities

$$\begin{aligned} &h^{uv}\left(k^\pm_{aru}k^\pm_{bwv} + k^\pm_{bru}k^\pm_{awv}\right) = \frac{1}{2}\,h_{rw}\,k^\pm_a\cdot k^\pm_b\,, \qquad && k^\pm_a\cdot k^\pm_b = h^{rw}h^{uv}k^\pm_{aru}k^\pm_{bwv}\,, \\ &h^{uv}\left(k^\pm_{aru}k^\mp_{bwv} - k^\mp_{bru}k^\pm_{awv}\right) = 0\,, && h^{rw}h^{uv}\,k^\pm_{aru}k^\mp_{bwv} = 0 \end{aligned} \tag{A.6}$$

valid for any four-dimensional metric.[33]

B The hyperbolic space and its Calderbank-Pedersen coordinates

B.1 The hyperbolic space in global and Poincaré coordinates

The SO(4,1) isometry algebra of the hyperbolic space H_4 obtained as SO(4,1)/SO(4) includes six compact SO(4) generators X_{uv} and four noncompact SO(4,1)/SO(4) generators $Y_u = X_{u5}$, with $\eta_{55} = -1 = -\eta_{uu}$. It has a three-dimensional abelian subalgebra related to noncompactness. In the standard notation or the SO(4,1) algebra,[34]

$$[X_{i4} + X_{i5}, X_{j4} + X_{j5}] = -(\eta_{44} + \eta_{55})X_{ij} = 0, \qquad i,j = 1,2,3. \tag{B.1}$$

The commuting $r_i = X_{i4} + Y_i$ form a vector of $SO(3) \subset SO(4)$.

[33] They follow from SO(4) group theory.
[34] The same would hold for $X_{i4} - X_{i5}$.

We can describe H_4 in global coordinates. In this set of coordinates the SO(4) acts linearly and the line element takes the form

$$\mathrm{d}s^2 = \frac{4\,\mathrm{d}x^u \mathrm{d}x^u}{(1 - x^v x^v)^2}\,, \qquad x^u = (x_1, x_2, x_3, x_4)\,. \tag{B.2}$$

Its ten isometry generators are:

$$\begin{aligned} &\text{Generators of SO(4)}: && X_{uv} = x^v \partial_u - x^u \partial_v\,, \\ &\text{Generators of SO(4,1)/SO(4)}: && Y_u = \frac{4 + x^v x^v}{4}\partial_u - \frac{1}{2}\,x^u x^v \partial_v\,, \end{aligned} \tag{B.3}$$

where $\partial_u = \frac{\partial}{\partial x^u}$. The three-dimensional abelian subalgebra is generated by

$$r_1 = X_{14} + Y_1\,, \qquad r_2 = X_{24} + Y_2\,, \qquad r_3 = X_{34} + Y_3\,. \tag{B.4}$$

In these coordinates, the SO(4) generators act as simple linear variations but the action of the commuting r_i is more involved. The curvature is directly related to the SO(4,1)-invariant quantity $1 - x^v x^v$, and these coordinates are then convenient for describing the (flat or rigid) gravity-decoupling limit.

An alternative coordinate system is the Poincaré patch $b^u = (b_0, b_1, b_2, b_3)$. The metric takes the form (3.36)

$$\mathrm{d}s^2 = \frac{\mathrm{d}b_0^2 + \mathrm{d}b_1^2 + \mathrm{d}b_3^2 + \mathrm{d}b_3^2}{b_0^2}\,. \tag{B.5}$$

In these coordinates, the generators of the three-dimensional abelian subalgebra act as translations of b^i:

$$r_i = \frac{\partial}{\partial b^i}. \tag{B.6}$$

The two sets of coordinates are related by

$$b_0 = \frac{4 - x^u x^u}{4(1 + x^4) + x^u x^u}\,, \qquad b_i = \frac{4x^i}{4(1 + x^4) + x^u x^u}\,, \qquad i = 1, 2, 3\,. \tag{B.7}$$

B.2 Calderbank-Pedersen coordinates

The isometry algebra SO(4,1) admits a variety of pairs of commuting generators and for each pair, according to ref. [22], there should exist CP coordinates ρ, η, φ, ψ. Examples of (inequivalent) pairs are:

i. A pair of isometries in the three-dimensional abelian subalgebra, for instance r_1 and r_2.

ii. The Cartan subalgebra of SO(4), chosen as the compact generators X_{12} and X_{34}, or a compact and a non compact SO(4,1) generator, like X_{12} and $Y_4 = X_{45}$.

iii. A compact generator of SO(4) and one of the r_i's, for instance X_{23} and r_1.

In each case, there are equivalent choices obtained by either SO(4) or SO(3) rotations. The case (iii) is only one example of pairing one SO(4) generator with any generator in the SO(2,1) algebra commuting with it.

We have shown in section 3.5 that CP coordinates do not exist for the case (i). We here show how CP coordinates can be derived for cases (ii) and (iii).

JHEP08(2018)045

Case (ii) — the Cartan algebra of SO(4). This is easily analyzed in coordinates where the SO(4) has a linear action, i.e. coordinates (B.2). The commuting (compact) Killing vectors are rotations in planes (12) and (34)

$$\xi_1 = x_2 \partial_{x_1} - x_1 \partial_{x_2} \,, \qquad \xi_2 = x_4 \partial_{x_3} - x_3 \partial_{x_4} \,. \tag{B.8}$$

We next identify $\xi_{1,2}$ with the Killing vectors of the CP metric or with linear combinations of them, and use the identity (3.34) for recognizing the change of coordinates. There are actually several possibilities and we focus on two cases, following ref. [22].

- The identification $(\partial_\varphi, \partial_\psi) = (\xi_1, \xi_2)$ leads to

$$\begin{aligned} (r_1 + i r_2)^2 &= \frac{\eta + 1 - i\rho}{\eta - 1 - i\rho} \,, \qquad r_1^2 = x_1^2 + x_2^2 \,, \quad r_2^2 = x_3^2 + x_4^2 \,, \\ F(\rho, \eta) &= \frac{1}{2\sqrt{\rho}} \left(\sqrt{\rho^2 + (\eta+1)^2} - \sqrt{\rho^2 + (\eta-1)^2} \right) . \end{aligned} \tag{B.9}$$

- Choosing instead $(\partial_\varphi, \partial_\psi) = (\xi_1 + \xi_2, \xi_1 - \xi_2)$, we obtain

$$\rho = 2 r_1 r_2 \,, \qquad \eta = r_1^2 - r_2^2 \,, \qquad F(\rho, \eta) = \frac{1}{2\sqrt{\rho}} \left(\sqrt{\rho^2 + \eta^2} - 1 \right) . \tag{B.10}$$

Choosing instead $\xi_1 = X_{12}$, $\xi_2 = Y_4 = X_{45}$ and using coordinates b^u, the Killing vectors are

$$\xi_1 = b_2 \, \partial_{b_1} - b_1 \, \partial_{b_2} \,, \qquad \xi_2 = -b_0 \, \partial_{b_0} - b_1 \, \partial_{b_1} - b_2 \, \partial_{b_2} - b_3 \, \partial_{b_3} \,. \tag{B.11}$$

Working as above we obtain:

$$\begin{aligned} \rho &= r \frac{\sqrt{b_0^2 + r^2 + b_3^2}}{r^2 + b_3^2} \,, \quad \eta = -\frac{b_0 b_3}{r^2 + b_3^2} \,, \quad r^2 = b_1^2 + b_2^2 \,, \\ F &= \frac{2^{1/4} \eta}{\sqrt{\rho} \left(t^2 + (1 - \rho^2 - \eta^2) t - 2\eta^2 \right)^{1/4}} \,, \quad t = \sqrt{(\rho+1)^2 + \eta^2} \sqrt{(\rho-1)^2 + \eta^2} \,. \end{aligned} \tag{B.12}$$

In any case, since CP coordinates exist, gauging these isometries does not lead to partial breaking.

Case (iii) — r_1 and X_{23}. This case is more easily analyzed in coordinates b^u where r_1 is a translation of b^1:

$$\xi_1 = \partial_{b_1} \,, \qquad \xi_2 = b_3 \partial_{b_2} - b_2 \partial_{b_3} \,. \tag{B.13}$$

We again consider two cases:

- With $(\partial_\varphi, \partial_\psi) = (\xi_1, \xi_2)$, we obtain

$$\rho = \frac{r}{b_0^2 + r^2} \,, \quad \eta = \frac{b_0}{b_0^2 + r^2} \,, \quad F(\rho, \eta) = \frac{\eta}{\sqrt{\rho(\rho^2 + \eta^2)}} \,, \tag{B.14}$$

  where $r^2 = b_2^2 + b_3^2$.

JHEP08(2018)045

- The choice $(\partial_\varphi, \partial_\psi) = (\xi_2, \xi_1)$ leads to

$$\rho = r\,, \qquad \eta = b_0\,, \qquad F(\rho,\eta) = \frac{\eta}{\sqrt{\rho}}\,. \tag{B.15}$$

Again, CP coordinates exist and these isometries do not induce partial breaking.

Finally, for completeness and out of curiosity, we present the CP form of the sphere SO(5)/SO(4) (which cannot describe a hypermultiplet), where all pairs of commuting isometries are equivalent to X_{12} and X_{34}. In coordinates where

$$\mathrm{d}s^2 = \frac{4\,\mathrm{d}x^u\mathrm{d}x^u}{(1+x^vx^v)^2}\,, \qquad \xi_1 = x_2\partial_{x_1} - x_1\partial_{x_2}\,, \qquad \xi_2 = x_4\partial_{x_3} - x_3\partial_{x_4}\,, \tag{B.16}$$

we again consider two choices of identification [22]:

- Now $(\partial_\varphi, \partial_\psi) = (\xi_1, \xi_2)$ leads to

$$\begin{aligned}(r_1 + ir_2)^2 &= \frac{\eta + 1 - i\rho}{\eta - 1 - i\rho}\,,\\ F(\rho,\eta) &= \frac{1}{2\sqrt{\rho}}\left(\sqrt{\rho^2 + (\eta+1)^2} + \sqrt{\rho^2 + (\eta-1)^2}\right),\\ r_1^2 = x_1^2 + x_2^2\,, &\qquad r_2^2 = x_3^2 + x_4^2\,.\end{aligned} \tag{B.17}$$

- For $(\partial_\varphi, \partial_\psi) = (\xi_1 + \xi_2, \xi_1 - \xi_2)$ we obtain

$$\rho = 2\,r_1r_2\,, \qquad \eta = r_1^2 - r_2^2\,, \qquad F(\rho,\eta) = \frac{1}{2\sqrt{\rho}}\left(\sqrt{\rho^2+\eta^2} + 1\right). \tag{B.18}$$

C Ward transformation

Assume that we have a solution $V(\rho,\eta)$ of the equation

$$\frac{1}{\rho}\,(\rho V_\rho)_\rho + V_{\eta\eta} = 0, \tag{C.1}$$

where indices denote derivatives with respect to η or ρ. A further derivative with respect to ρ leads to

$$\frac{\partial^2 F}{\partial\rho^2} + \frac{\partial^2 F}{\partial\eta^2} = \frac{3F}{4\rho^2} \qquad \text{with} \qquad F(\rho,\eta) = \sqrt{\rho}\,V_\rho, \tag{C.2}$$

and $F(\rho,\eta)$ generates via eq. (3.2) a quaternion-Kähler metric in Calderbank-Pedersen coordinates. Coordinates (ρ,η) can be traded for (X,Z) by a double Legendre transformation:

$$V(\rho,\eta) - X\eta - 2\,Z\ln\rho = -K(X,Z). \tag{C.3}$$

This implies firstly

$$\rho V_\rho = 2\,Z, \qquad V_\eta = X, \qquad \eta = K_X, \qquad 2\,\ln\rho = K_Z. \tag{C.4}$$

Secondly

$$\frac{\partial Z}{\partial \rho} = \frac{1}{2}\left(\rho V_\rho\right)_\rho, \qquad \frac{\partial Z}{\partial \eta} = \frac{\rho}{2} V_{\rho\eta}, \qquad \frac{\partial X}{\partial \rho} = V_{\rho\eta}, \qquad \frac{\partial X}{\partial \eta} = V_{\eta\eta}, \tag{C.5}$$

and

$$\frac{\partial \rho}{\partial X} = \frac{\rho}{2} K_{XZ}, \qquad \frac{\partial \rho}{\partial Z} = \frac{\rho}{2} K_{ZZ}, \qquad \frac{\partial \eta}{\partial X} = K_{XX}, \qquad \frac{\partial \eta}{\partial Z} = K_{XZ}. \tag{C.6}$$

As usual, $\frac{\partial x^i}{\partial x^j} = \delta^i_j$ for each set of coordinates delivers the relations between the second derivatives of V and K. Using then eq. (C.1), the relevant equation appears to be

$$K_{XX} + \frac{\rho^2}{4} K_{ZZ} = 0 \tag{C.7}$$

as the "Legendre partner" of eq. (C.1). Define finally

$$\Psi(X,Z) = \ln\left(\frac{1}{4}\rho^2\right), \qquad \mathrm{e}^\Psi = \frac{1}{4}\rho^2 = \frac{1}{4}\mathrm{e}^{K_Z}. \tag{C.8}$$

The relations induced by the Legendre transformation and eq. (C.7) lead to Toda equation for Ψ:

$$\Psi_{XX} + \left(\mathrm{e}^\Psi\right)_{ZZ} = 0. \tag{C.9}$$

This procedure has been elaborated by Ward in ref. [35] and used in the derivation of the CP metric [22]. It allows in particular to find CP coordinates for a quaternion-Kähler metric with two isometries expressed in PT coordinates, for a given Toda solution Ψ.

The case where Ψ is a constant is clearly excluded.

D A proof

In section 4.3 on $\mathcal{N}=0$ vacua of the SO(4, 1)/SO(4) model, we claim that *all solutions of* $\partial_z h = h = 0$ *with* $f_{zzz} \neq 0$ *are* $\mathcal{N}=1$ *vacua.*[35] We give here a proof of this statement.

Recall that, for a given prepotential function $f(z)$,

$$\mathcal{Y} = 2\left(f + \overline{f}\right) - (z - \overline{z})\left(f_z - \overline{f}_{\overline{z}}\right), \qquad c = -i(g_1 + g_2 z) + g_3 f_z, \qquad g_3 \neq 0. \tag{D.1}$$

Starting with

$$\begin{aligned} h &= \left(g_0^2 + |c|^2\right)\left(f_{zz} + \overline{f}_{\overline{zz}}\right) + \mathcal{Y}|c_z|^2 - \overline{c}c_z\mathcal{Y}_{\overline{z}} - c\overline{c}_{\overline{z}}\mathcal{Y}_z\,, \\ \partial_z h &= f_{zzz}\Big[g_0^2 + c\overline{c} + g_3\,\overline{c}_{\overline{z}}\,\mathcal{Y} - g_3\,\overline{c}\,\mathcal{Y}_{\overline{z}} + (z-\overline{z})c\overline{c}_{\overline{z}}\Big] \\ &= f_{zzz}\Big[(g_0 + \overline{c})(g_0 - \overline{c}) + \overline{c}_{\overline{z}}\left[g_3\mathcal{Y} + (z-\overline{z})(c - \overline{c})\right]\Big] \end{aligned} \tag{D.2}$$

and assuming $f_{zzz} \neq 0$, one finds the factorization:

$$c_z\,(\partial_z\,h)\,{f_{zzz}}^{-1} + \overline{c}_{\overline{z}}\,(\partial_{\overline{z}}\,h)\,{\overline{f}_{\overline{zzz}}}^{-1} - g_3\,h = c_z\,\overline{c}_{\overline{z}}\Big[g_3\,\mathcal{Y} + (z-\overline{z})(c-\overline{c})\Big]. \tag{D.3}$$

This quantity should vanish for a solution of $\partial_z h = h = 0$. The solutions are either $c_z = 0$ or $g_3\,\mathcal{Y} + (z-\overline{z})(c-\overline{c}) = 0$ and in both cases $\partial_z\,h = 0$ requires $c = \pm g_0$.

[35]In order to avoid cluttering in the formulas, we systematically omit $\langle \ldots \rangle$ in this appendix.

- If $c_z = 0$ and $c = \pm g_0$ the vacuum has $\mathcal{N} = 1$ supersymmetry: the two conditions for partial breaking (4.10) are fulfilled.
- If $c_z \neq 0$, the vacuum state would be at $g_3 \mathcal{Y} + (z - \bar{z})(c - \bar{c}) = 0$ and $c = \pm g_0 = \bar{c}$. Then $\mathcal{Y} = e^{-\mathcal{K}} = 0$, which is excluded.

Hence, *Minkowski $\mathcal{N} = 0$ vacua with $f_{zzz} \neq 0$ do not exist.*

References

[1] I. Antoniadis, H. Partouche and T.R. Taylor, *Spontaneous breaking of $\mathcal{N} = 2$ global supersymmetry*, *Phys. Lett.* **B 372** (1996) 83 [hep-th/9512006] [INSPIRE].

[2] I. Antoniadis, J.-P. Derendinger and C. Markou, *Nonlinear $\mathcal{N} = 2$ global supersymmetry*, *JHEP* **06** (2017) 052 [arXiv:1703.08806] [INSPIRE].

[3] U. Lindström and M. Roček, *Scalar Tensor Duality and $\mathcal{N} = 1$, $\mathcal{N} = 2$ Nonlinear σ-models*, *Nucl. Phys.* **B 222** (1983) 285 [INSPIRE].

[4] B. de Wit and A. Van Proeyen, *Potentials and Symmetries of General Gauged $\mathcal{N} = 2$ Supergravity: Yang-Mills Models*, *Nucl. Phys.* **B 245** (1984) 89 [INSPIRE].

[5] E. Cremmer et al., *Vector Multiplets Coupled to $\mathcal{N} = 2$ Supergravity: SuperHiggs Effect, Flat Potentials and Geometric Structure*, *Nucl. Phys.* **B 250** (1985) 385 [INSPIRE].

[6] B. de Wit, P.G. Lauwers and A. Van Proeyen, *Lagrangians of $\mathcal{N} = 2$ Supergravity-Matter Systems*, *Nucl. Phys.* **B 255** (1985) 569 [INSPIRE].

[7] S. Ferrara, L. Girardello and M. Porrati, *Spontaneous breaking of $\mathcal{N} = 2$ to $\mathcal{N} = 1$ in rigid and local supersymmetric theories*, *Phys. Lett.* **B 376** (1996) 275 [hep-th/9512180] [INSPIRE].

[8] S. Cecotti, L. Girardello and M. Porrati, *Two into one won't go*, *Phys. Lett.* **B 145** (1984) 61 [INSPIRE].

[9] S. Cecotti, L. Girardello and M. Porrati, *Constraints on partial super-Higgs*, *Nucl. Phys.* **B 268** (1986) 295 [INSPIRE].

[10] S. Ferrara, L. Girardello and M. Porrati, *Minimal Higgs branch for the breaking of half of the supersymmetries in $\mathcal{N} = 2$ supergravity*, *Phys. Lett.* **B 366** (1996) 155 [hep-th/9510074] [INSPIRE].

[11] S. Cecotti, L. Girardello and M. Porrati, *Some Features of SUSY Breaking in $N = 2$ Supergravity*, *Phys. Lett.* **B 151** (1985) 367 [INSPIRE].

[12] S. Cecotti, L. Girardello and M. Porrati, *An Exceptional $N = 2$ Supergravity With Flat Potential and Partial Superhiggs*, *Phys. Lett.* **B 168** (1986) 83 [INSPIRE].

[13] J. Louis, P. Smyth and H. Triendl, *Spontaneous $\mathcal{N} = 2$ to $\mathcal{N} = 1$ Supersymmetry Breaking in Supergravity and Type II String Theory*, *JHEP* **02** (2010) 103 [arXiv:0911.5077] [INSPIRE].

[14] J. Louis, P. Smyth and H. Triendl, *The $\mathcal{N} = 1$ Low-Energy Effective Action of Spontaneously Broken $\mathcal{N} = 2$ Supergravities*, *JHEP* **10** (2010) 017 [arXiv:1008.1214] [INSPIRE].

JHEP08(2018)045

[15] S. Ferrara and S. Sabharwal, *Quaternionic Manifolds for Type II Superstring Vacua of Calabi-Yau Spaces*, *Nucl. Phys.* **B 332** (1990) 317 [INSPIRE].

[16] I. Antoniadis, R. Minasian, S. Theisen and P. Vanhove, *String loop corrections to the universal hypermultiplet*, *Class. Quant. Grav.* **20** (2003) 5079 [hep-th/0307268] [INSPIRE].

[17] N. Ambrosetti, I. Antoniadis, J.-P. Derendinger and P. Tziveloglou, *The Hypermultiplet with Heisenberg Isometry in $\mathcal{N} = 2$ Global and Local Supersymmetry*, *JHEP* **06** (2011) 139 [arXiv:1005.0323] [INSPIRE].

[18] I. Antoniadis, J.-P. Derendinger, P.M. Petropoulos and K. Siampos, *Heisenberg symmetry and hypermultiplet manifolds*, *Nucl. Phys.* **B 905** (2016) 293 [arXiv:1512.06418] [INSPIRE].

[19] B. de Wit, H. Samtleben and M. Trigiante, *On Lagrangians and gaugings of maximal supergravities*, *Nucl. Phys.* **B 655** (2003) 93 [hep-th/0212239] [INSPIRE].

[20] B. de Wit, H. Samtleben and M. Trigiante, *Magnetic charges in local field theory*, *JHEP* **09** (2005) 016 [hep-th/0507289] [INSPIRE].

[21] D. Cassani, S. Ferrara, A. Marrani, J.F. Morales and H. Samtleben, *A Special road to AdS vacua*, *JHEP* **02** (2010) 027 [arXiv:0911.2708] [INSPIRE].

[22] D.M.J. Calderbank and H. Pedersen, *Selfdual Einstein metrics with torus symmetry*, *J. Diff. Geom.* **60** (2002) 485 [math/0105263] [INSPIRE].

[23] B. Craps, F. Roose, W. Troost and A. Van Proeyen, *What is special Kähler geometry?*, *Nucl. Phys.* **B 503** (1997) 565 [hep-th/9703082] [INSPIRE].

[24] D.Z. Freedman and A. Van Proeyen, *Supergravity*, Cambridge University Press, Cambridge (2012).

[25] A. Ceresole, R. D'Auria, S. Ferrara and A. Van Proeyen, *Duality transformations in supersymmetric Yang-Mills theories coupled to supergravity*, *Nucl. Phys.* **B 444** (1995) 92 [hep-th/9502072] [INSPIRE].

[26] S. Ferrara and L. Maiani, *An Introduction to supersymmetry breaking in extended supergravity*, in *5th SILARG Symposium on Relativity, Supersymmetry and Cosmology*, Bariloche, Argentina, January 4-11 1985, CERN-TH-4232/85 [INSPIRE].

[27] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara and P. Fré', *General matter coupled $\mathcal{N} = 2$ supergravity*, *Nucl. Phys.* **B 476** (1996) 397 [hep-th/9603004] [INSPIRE].

[28] L. Andrianopoli et al., *$\mathcal{N} = 2$ supergravity and $\mathcal{N} = 2$ superYang-Mills theory on general scalar manifolds: Symplectic covariance, gaugings and the momentum map*, *J. Geom. Phys.* **23** (1997) 111 [hep-th/9605032] [INSPIRE].

[29] J. Bagger and E. Witten, *Matter Couplings in $\mathcal{N} = 2$ Supergravity*, *Nucl. Phys.* **B 222** (1983) 1 [INSPIRE].

[30] M. Przanowski, *Killing vector fields in selfdual, Euclidean Einstein spaces with Lambda not equal 0*, *J. Math. Phys.* **32** (1991) 1004 [INSPIRE].

[31] P. Tod, *The* SU(∞)*-Toda field equation and special four-dimensional metrics*, in *Geometry and physics*, J.E. Andersen et al. eds., Lecture Notes in Pure and Applied Mathematics 184, Marcel Dekker, Inc., New York (1997) [INSPIRE].

[32] D. Joyce, *Explicit construction of selfdual 4-manifolds*, *Duke Math. J.* **77** (1995) 519.

[33] P. Jones and K. Tod, *Minitwistor spaces and Einstein-Weyl spaces*, *Class. Quant. Grav.* **2** (1985) 565 [INSPIRE].

[34] P. Tod, *A Note on Riemannian anti-self-dual Einstein metrics with symmetry*, hep-th/0609071 [INSPIRE].

[35] R.S. Ward, *Einstein-Weyl spaces and* SU(∞) *Toda fields*, *Class. Quant. Grav.* **7** (1990) L95 [INSPIRE].

[36] L. Andrianopoli, P. Concha, R. D'Auria, E. Rodriguez and M. Trigiante, *Observations on BI from $\mathcal{N} = 2$ Supergravity and the General Ward Identity*, *JHEP* **11** (2015) 061 [arXiv:1508.01474] [INSPIRE].

[37] I. Antoniadis, J.-P. Derendinger, P.M. Petropoulos and K. Siampos, *Isometries, gaugings and $\mathcal{N} = 2$ supergravity decoupling*, *JHEP* **11** (2016) 169 [arXiv:1611.00964] [INSPIRE].

[38] S. Alexandrov, S. Banerjee and P. Longhi, *Rigid limit for hypermultiplets and five-dimensional gauge theories*, *JHEP* **01** (2018) 156 [arXiv:1710.10665] [INSPIRE].

JHEP08(2018)045