

Topological Structures in the Standard Model at High T

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(FubiniFest, 24-26 February 1994, Torino ITALY. MIT-CTP-2285.)

We meet to wish our dear colleague Sergio Fubini all the best on his sixty-fifth birthday. Moreover, behind the formality of this calendrically significant instant, there is also our timeless expression of great affection and thanks to Sergio: affection for his sympathetic character and thanks for his activities in our profession, both within the scientific framework and in the broader social context. But for those of us from MIT there is a special feeling of gratitude towards Sergio, and that is because in the late sixties and early seventies he was with us and helped shape what can now be seen as the most recent golden age of physics, thereby establishing at MIT a tradition that still flourishes today. You have to appreciate the moment: Steven Weinberg was on the faculty completing the standard model, while Gabriele Veneziano and Sergio were inventing what proved to be the physics of the future — string theory. These people have since departed from our University, but Sergio has kept the legacy vital by visiting us — not as frequently as we would have liked — and by encouraging continuing contact with his wonderful and talented Italian compatriot physicists. The people who came from Italy to enrich our department are too numerous to list, but theirs is an ongoing presence, formalized recently by an agreement with the INFN, and the well-spring of all this good fortune is Sergio Fubini, whom we all thank.

Sergio is now gone from MIT, but I am certain he wants to be informed of activity there, so I shall describe one project, with the hope that it meets his criteria of simplicity and symmetry.

These days, as high energy particle colliders become unavailable for testing speculative theoretical ideas, physicists are looking to other environments that may provide extreme conditions where theory confronts physical reality. One such circumstance may arise at high temperature T , which perhaps can be attained in heavy ion collisions or in astrophysical settings. It is natural therefore to examine the high-temperature behavior of the standard model, and here I shall report on recent progress in constructing the high- T limit of QCD.

* This work is supported in part by funds provided by the U. S. Department of Energy (D.O.E.) under contract #DE-AC02-76ER03069.

In studying a field theory at finite temperature, the simplest approach is the so-called imaginary-time formalism. We continue time to the imaginary interval $[0, 1/iT]$ and consider bosonic (fermionic) fields to be periodic (anti-periodic) on that interval. Perturbative calculations are performed by Feynman rules as at zero temperature, except that in the conjugate energy-momentum, Fourier-transformed space, the energy variable p^0 (conjugate to the periodic time variable) becomes discrete — it is $2\pi nT$, (n integer) for bosons. From this one immediately sees that at high temperature — in the limiting case, at infinite temperature — the time-direction disappears, because the temporal interval shrinks to zero. Also only zero-energy processes survive, since “non-vanishing energy” necessarily means high energy owing to the discreteness of the energy variable $p^0 \sim 2\pi nT$, and therefore all modes with $n \neq 0$ decouple at large T . In this way a Euclidean three-dimensional field theory becomes effective at high temperatures and describes essentially static processes.

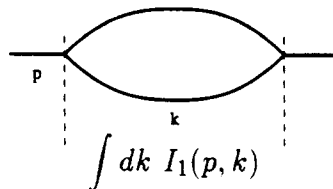
While all this is quick and simple, it may be physically inadequate. First of all, frequently one is interested in non-static processes in real time, so complicated analytic continuation from imaginary time needs to be made. Also one may wish to study amplitudes where the real external energy is neither large nor zero, even though virtual internal energies are high.

Large T Feynman graphs with external legs carrying limited amounts of energy and internal lines characterized by large momenta because T is large have been dubbed “hard thermal loops.” In fact they are a very important feature of high temperature QCD, because they necessarily arise in a resummed perturbative expansion.^{1,2}

Here is the argument. Consider a one-loop amplitude $\Pi_1(p)$,

$$\Pi_1(p) \equiv \int dk I_1(p, k) ,$$

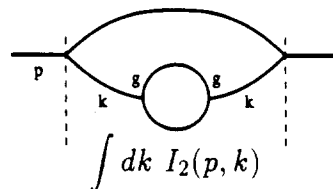
given by the graph in the figure.

$$\begin{aligned} \Pi_1(p) &= \text{---} \overset{p}{\text{---}} \text{---} \text{---} \text{---} \text{---} \\ &\equiv \int dk I_1(p, k) \end{aligned}$$


Compare this to a two-loop amplitude $\Pi_2(p)$,

$$\Pi_2(p) \equiv \int dk I_2(p, k) ,$$

in which Π_1 is an insertion, as in the figure below.

$$\begin{aligned} \Pi_2(p) &= \text{---} \overset{p}{\text{---}} \text{---} \text{---} \text{---} \text{---} \\ &\equiv \int dk I_2(p, k) \end{aligned}$$


Following Pisarski,¹ I estimate the relative importance of Π_2 to Π_1 by the ratio of their integrands,

$$\frac{\Pi_2}{\Pi_1} \sim \frac{I_2}{I_1} = g^2 \frac{\Pi_2(k)}{k^2} ,$$

Here g is the coupling constant, and the k^2 in the denominator reflects that we are considering a massless particle as in QCD. Clearly the $k^2 \rightarrow 0$ limit is relevant to the question whether the higher order graph can be neglected relative to the lower order one. Because one finds that for small k and large T , $\Pi_2(k)$ behaves as T^2 , the ratio Π_2/Π_1 is $g^2 T^2/k^2$. Thus when k is $O(gT)$ or smaller the two-loop amplitude is not negligible compared to the one-loop amplitude. Thus graphs with “soft” external momenta [$O(gT)$ or smaller] have to be included as insertions in higher order calculations.

These so-called “hard thermal loops,” *i.e.* the high-temperature limits of real-time Feynman graphs with finite external momenta, have become the object of much study, which culminated with the discovery (Braaten, Pisarski, Frenkel, Taylor)² of a remarkable simplicity in their structure. Specifically, the generating functional for hard thermal loops with only external gauge field legs, in an $SU(N)$ gauge theory containing N_F fermion species of the fundamental representation is found (i) to be proportional to $(N + \frac{1}{2}N_F)$, (ii) to behave as T^2 at high temperature, and (iii) to be gauge invariant.

$$\begin{array}{c} \text{A wavy line with a circle} + \text{A wavy line with a circle and a wavy line} + \text{A wavy line with a circle and two wavy lines} + \text{A wavy line with a circle and three wavy lines} + \dots \\ \Gamma_{\text{HTL}}(U^{-1} A U + U^{-1} dU) = \Gamma_{\text{HTL}}(A) \end{array} = (N + \frac{1}{2}N_F) \frac{g^2 T^2}{12\pi} \Gamma_{\text{HTL}}(A)$$

A further kinematical simplification in Γ_{HTL} has also been established. To explain this we define two light-like four-vectors Q_{\pm}^{μ} depending on a unit three-vector \hat{q} , pointing in an arbitrary direction.

$$Q_{\pm}^{\mu} = \frac{1}{\sqrt{2}}(1, \pm \hat{q})$$

$$\hat{q} \cdot \hat{q} = 1 , \quad Q_{\pm}^{\mu} Q_{\pm\mu} = 0 , \quad Q_{\pm}^{\mu} Q_{\mp\mu} = 1$$

Coordinates and potentials are projected onto Q_{\pm}^{μ}

$$x^{\pm} \equiv x_{\mu} Q_{\pm}^{\mu} , \quad \partial_{\pm} \equiv Q_{\pm}^{\mu} \frac{\partial}{\partial x^{\mu}} , \quad A_{\pm} \equiv A_{\mu} Q_{\pm}^{\mu}$$

The additional fact that is now known is that (iv) after separating an ultralocal contribution from Γ_{HTL} , the remainder may be written as an average over the angles of \hat{q} of a functional W that depends only on A_{+} ; also this functional is non-local only on the two-dimensional x_{\pm} plane, and is ultralocal in the remaining directions, perpendicular to the x_{\pm} plane.

[“Ultralocal” means that any potentially non-local kernel $k(x, y)$ is in fact a δ -function of the difference $k(x, y) = \delta(x - y)$.]

$$\Gamma_{\text{HTL}}(A) = 2\pi \int d^4x A_0^a(x)A_0^a(x) + \int d\Omega_{\hat{q}} W(A_+)$$

These results are established in perturbation theory, and a perturbative expansion of $W(A_+)$, *i.e.* a power series in A_+ , exhibits the above mentioned properties. A natural question is whether one can sum the series, *i.e.* obtain an expression for $W(A_+)$.

Important progress on this problem was made when it was observed (Taylor, Wong)³ that the gauge-invariance condition can be imposed infinitesimally, whereupon it leads to a functional differential equation for $W(A_+)$, which is best presented as

$$\begin{aligned} \frac{\partial}{\partial x^+} \frac{\delta}{\delta A_+^a} \left[W(A_+) + \frac{1}{2} \int d^4x A_+^b(x)A_+^b(x) \right] \\ - \frac{\partial}{\partial x^-} [A_+^a] + f^{abc} A_+^b \frac{\delta}{\delta A_+^c} \left[W(A_+) + \frac{1}{2} \int d^4x A_+^d(x)A_+^d(x) \right] = 0 \end{aligned}$$

In other words we seek a quantity, call it

$$S(A_+) \equiv W(A_+) + \frac{1}{2} \int d^4x A_+^a(x)A_+^a(x) ,$$

which is a functional on a two-dimensional manifold $\{x^+, x^-\}$, depends on a single functional variable A_+ , and satisfies

$$\partial_1 \frac{\delta}{\delta A_1^a} S - \partial_2 A_1^a + f^{abc} A_1^b \frac{\delta}{\delta A_1^c} S = 0$$

$$\text{“1”} \equiv x^+ , \quad \text{“2”} \equiv -x^- , \quad A_1^a \equiv A_+^a$$

Another suggestive version of the above is gotten by defining $A_2^a \equiv \frac{\delta S}{\delta A_1^a}$. Then we need to solve

$$\partial_1 A_2^a - \partial_2 A_1^a + f^{abc} A_1^b A_2^c = 0 .$$

To solve the functional equation and produce an expression for $W(A_+)$, we now turn to a completely different corner of physics, and that is Chern-Simons theory.

The Chern-Simons term is a peculiar gauge theoretic topological structure that can be constructed in odd dimensions, and here we consider it in 3-dimensional space-time.

$$I_{\text{CS}} \propto \int d^3x \epsilon^{\alpha\beta\gamma} \text{Tr} (\partial_\alpha A_\beta A_\gamma + \frac{2}{3} A_\alpha A_\beta A_\gamma)$$

This object was introduced into physics over a decade ago, and since that time it has been put to various physical and mathematical uses. Indeed one of our originally stated motivations for studying the Chern-Simons term was its possible relevance to high-temperature gauge

theory.⁴ Here following Efraty and Nair,⁵ we shall employ the Chern-Simons term for a determination of the hard thermal loop generating functional, Γ_{HTL} .

Since it is the space-time integral of a density, I_{CS} may be viewed as the action for a quantum field theory in (2+1) dimensional space-time, and the corresponding Lagrangian would then be given by a two-dimensional, spatial integral of a Lagrange density.

$$I_{\text{CS}} \propto \int dt L_{\text{CS}}$$

$$L_{\text{CS}} \propto \int d^2x \left(A_2^a \dot{A}_1^a + A_0^a F_{12}^a \right)$$

I have separated the temporal index (0) from the two spatial ones (1,2) and have indicated time differentiation by an over dot. F_{12}^a is the non-Abelian field strength, defined on a two-dimensional plane.

$$F_{12}^a = \partial_1 A_2^a - \partial_2 A_1^a + f^{abc} A_1^b A_2^c$$

Examining the Lagrangian, we see that it has the form

$$L \sim p\dot{q} - \lambda H(p, q)$$

where A_2^a plays the role of p , A_1^a that of q , F_{12} is like a Hamiltonian and A_0 is like the Lagrange multiplier λ , which forces the Hamiltonian to vanish; here A_0^a enforces the vanishing of F_{12}^a .

$$F_{12}^a = 0$$

The analogy instructs us how the Chern-Simons theory should be quantized.

We postulate equal-time commutation relations, like those between p and q .

$$[A_1^a(\mathbf{r}), A_2^b(\mathbf{r}')] = i \delta^{ab} \delta(\mathbf{r} - \mathbf{r}')$$

In order to satisfy the condition enforced by the Lagrange multiplier, we demand that F_{12}^a , operating on “allowed” states, annihilate them.

$$F_{12}^a | \rangle = 0$$

This equation can be explicitly presented in a Schrödinger-like representation for the Chern-Simons quantum field theory, where the state is a functional of A_1^a . The action of the operators A_1^a and A_2^a is by multiplication and functional differentiation, respectively.

$$| \rangle \sim \Psi(A_1^a)$$

$$A_1^a | \rangle \sim A_1^a \Psi(A_1^a)$$

$$A_2^a | \rangle \sim \frac{1}{i} \frac{\delta}{\delta A_1^a} \Psi(A_1^a)$$

This of course is just the field theoretic analog of the quantum mechanical situation where states are functions of q , the q operator acts by multiplication, and the p operator by differentiation. In the Schrödinger representation, the condition that states be annihilated by F_{12}^a

$$\left(\partial_1 A_2^a - \partial_2 A_1^a + f_{abc} A_1^b A_2^c \right) | \rangle = 0$$

leads to a functional differential equation.

$$\left(\partial_1 \frac{1}{i} \frac{\delta}{\delta A_1^a} - \partial_2 A_1^a + f_{abc} A_1^b \frac{1}{i} \frac{\delta}{\delta A_2^c} \right) \Psi(A_1^a) = 0$$

If we define S by $\Psi = e^{iS}$ we get equivalently

$$\partial_1 \frac{\delta}{\delta A_1^a} S - \partial_2 A_1^a + f_{abc} A_1^b \frac{\delta}{\delta A_2^c} S = 0$$

This equation comprises the entire content of Chern-Simons quantum field theory. S is the Chern-Simons eikonal, which gives the exact wave functional owing to the simple dynamics of the theory. Also the above eikonal equation is recognized to be precisely the equation for the hard thermal loop generating functional.

The gained advantage is that “acceptable” Chern-Simons states, *i.e.* solutions to the above functional equations, had been constructed long ago,⁶ and one can now take over those results to the hard thermal loop problem. One knows from the Chern-Simons work that Ψ and S are given by a 2-dimensional fermionic determinant, *i.e.* by the Polyakov-Wiegman expression. While these are not described by very explicit formulas, many properties are understood, and the hope is that one can use these properties to obtain further information about high-temperature QCD processes.

For example one can compute the induced current $j_{\text{induced}}^\mu \sim \frac{\delta \Gamma_{\text{HTL}}}{\delta A_\mu^a}$, and use this as a source in the Yang-Mills equation, thereby obtaining a non-Abelian generalization of the Kubo equation, which governs the response of the hot gluonic plasma to external disturbances.⁷

$$D_\mu F^{\mu\nu} = \frac{m^2}{2} j_{\text{induced}}^\nu$$

$$m = gT \sqrt{\frac{N + N_F/2}{3}}$$

From the known properties of the fermionic determinant — hard thermal loop generating functional — one can show that j_{induced}^μ is given by

$$j_{\text{induced}}^\mu = \int \frac{d\Omega_{\hat{q}}}{4\pi} \left\{ Q_+^\mu \left(a_-(x) - A_-(x) \right) + Q_-^\mu \left(a_+(x) - A_+(x) \right) \right\}$$

where a_\pm are solutions to the equations

$$\partial_+ a_- - \partial_- A_+ + [A_+, a_-] = 0$$

$$\partial_+ A_- - \partial_- a_+ + [a_+, A_-] = 0$$

Evidently j_{induced}^μ , as determined by the above equations, is a non-local and non-linear functional of the vector potential A_μ .

An alternative, equivalent derivation of the induced current has been given by Blaizot and Iancu,⁸ directly from the QCD field equations. Their argument may be succinctly put in the language of the composite effective action⁹ and makes use of two approximations. The composite action is truncated at the one loop (semi-classical) level — two-particle irreducible graphs are omitted. This comprises the first, dynamical approximation. Then, in the second, kinematical approximation, the stationary conditions on the one-loop action are shown to lead to the gauge invariance equation for Γ_{HTL} .

In the Abelian case, everything commutes and linearizes. One can determine a_\pm in terms of A_\mp .

$$a_\pm = \frac{\partial_\pm}{\partial_\mp} A_\mp$$

Incidentally, this formula exemplifies the kinematical simplicity, mentioned above, of hard thermal loops: the nonlocality of $1/\partial_\pm$ is entirely in $\{x^+, x^-\}$ plane. With the above form for a_\pm inserted into the Kubo equation, the solution can be constructed explicitly. It coincides with the results obtained by Silin long ago, on the basis of the Boltzmann-Vlasov equation,¹⁰ and one sees that ours is the non-Abelian generalization of that physics. In particular m is recognized as the gauge invariant Debye screening length.

At the present time the non-Abelian equations are under further investigation. It has been possible to find local expressions for the current in the static case^{9,11} and in the position-independent case.¹¹

$$\begin{aligned} \frac{m^2}{2} j_{\text{induced}}^\mu &= (-m^2 A_0, \mathbf{0}) && \text{(static)} \\ \frac{m^2}{2} j_{\text{induced}}^\mu &= (0, -\frac{1}{3} m^2 \mathbf{A}) && \text{(position-independent)} \end{aligned}$$

The non-Abelian Kubo equation may then be solved, but the physical relevance of the solutions is unclear. In particular the static solutions are not solitons, since their energy is infinite.

A much more interesting result is due to Blaizot and Iancu.¹² They abstract from the Silin solution the plane-wave *Ansatz* $A_\mu(x) = A_\mu(x \cdot p)$ where p is a constant 4-vector and they determine explicitly the induced current associated with non-Abelian plane waves. In terms of the above, this corresponds to

$$a_\pm = \frac{Q_\pm \cdot p}{Q_\mp \cdot p} A_\mp$$

The physics of all these solutions, as well as of other, still undiscovered ones, remains to be elucidated, and I invite any of you to join in this interesting task.

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