

# COHERENT AND INCOHERENT BUCKET FOR A BEAM LOADED RF SYSTEM

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For the case of a beam-loaded radio-frequency acceleration system, we have found the equation for coherent oscillation by comparing the bunch centroid with the synchronous particle; and we have found the equation for incoherent oscillation by comparing motion of an individual particle with the bunch centroid. For the case of single particle motion, stability depends on the synchronous phase  $\phi_s$ . Stability of coherent motion depends upon the generator induced partial voltage, whereas stability of incoherent motion depends upon the total cavity voltage including the beam induced voltage component. With beam loading, the stability of coherent motion depends upon  $\phi_s + \psi + \phi_g$ , with tuning angle  $\psi$  and generator current phase  $\phi_g$ , and the coherent bucket is deformed. With beam loading, the stability of incoherent motion depends upon  $\phi_b + \phi_c$ , with beam current phase  $\phi_b$  and perturbed cavity phase  $\phi_c$ . For the case that the centroid is coincident with the synchronous particle, the incoherent bucket is found to be identical with the case of no beam loading.

**KEY WORDS:** Beam loading, coherent, collective effects, impedances, instability, particle dynamics, radio-frequency devices

## 1 INTRODUCTION

The effect of the rf electric fields in a synchrotron is to accelerate the beam and to longitudinally focus beam particles into bunches. Individual particles and also whole bunches may perform non-linear oscillations; these motions are called synchrotron oscillations. The rf accelerating cavities are driven both by the generator current and the beam current; and the effect of the beam on the cavity gap voltage is known as ‘beam loading’. As a result of this beam loading, the coherent motion of a bunch taken as a whole may differ considerably from the incoherent motion of an individual charged particle. The usual sinusoidal choice of the rf waveform leads to a phase-space separatrix which bounds the oscillations, called the bucket.

In this note, we shall find the incoherent bucket for individual particles within the bunch, and the coherent bucket for rigid oscillations of the bunch centroid. The cavity dynamical response shall be taken as quasi-static, corresponding to the case that the period of synchrotron oscillations is much longer than the cavity filling time.

Linear stability analysis<sup>1,2</sup> shows the frequency  $\Omega$  of small amplitude rigid dipole oscillations to scale according to

$$\Omega^2 = \Omega_s^2 [1 - I_b \sin 2\psi / 2I_0 \cos \phi_s] ,$$

where  $\Omega_s$  is the synchrotron frequency,  $I_b/I_0 \cos \phi_s$  is the beamload ratio, and the other quantities appearing will be defined below. This scaling suggests that the coherent bucket shrinks to zero at some threshold beam current  $I_b$ , but begs the questions: “is the bucket merely scaled in height, or is the shape distorted?” In the following non-linear analysis, we shall find that the incoherent bucket is little affected by beam loading, but the coherent bucket size and shape has a complicated dependence on cavity detuning.

## 2 COHERENT BUCKET

We shall suppose that the synchrotron period is much greater than the cavity time constant; so that cavity dynamical effects can be ignored and the cavity response is virtually instantaneous. This premise affords great simplicity but also implies that observations of a real beam may differ somewhat from the idealization presented here. We shall take it for granted that the cavity is detuned in the correct sense to avoid the dynamical Robinson instability.

We shall represent currents and voltages by phasor quantities in the complex plane. The energy increment received by a bunch is then the scalar product of the beam current  $\mathbf{I}_b$  vector and the cavity voltage  $\mathbf{V}_c$  vector, that is the product of their magnitudes and the cosine of the angle formed between the vectors. Let  $\mathcal{R}[\dots]$  indicate the real part of a complex quantity. Consider the general case of phasors  $\mathbf{B} = B e^{i\phi_B}$  and  $\mathbf{A} = A e^{i\phi_A}$ ; their scalar product is

$$AB \cos(\phi_B - \phi_A) = \mathcal{R}[\mathbf{A}\mathbf{B}^*] = \mathcal{R}[\mathbf{A}^*\mathbf{B}] .$$

### 2.1 Coherent Bucket Stability Criterion

Throughout this paper we shall need to distinguish between steady state vector quantities, to be indicated by a superscript zero, and perturbed vectors which shall carry no superscripts.

Let the cavity gap voltage be  $\mathbf{V}_c = V_c e^{j\phi_c}$  and the cavity complex impedance be  $R \cos \psi e^{j\psi}$  where  $R$  is the shunt resistance and  $\psi$  is the detuning angle. We shall adopt the phase convention used by Pedersen:<sup>3</sup> that the steady state voltage phasor  $\mathbf{V}_c^0$  is aligned with the positive real axis, in which case  $\phi_c$  becomes zero. Let the generator current driving the cavity be a fixed vector  $\mathbf{I}_g^0 = I_g e^{j\phi_g}$  which is chosen, for given detuning and beam current, to produce the correct cavity voltage in the steady state. Let the beam image current be  $\mathbf{I}_b = I_b e^{-j(\pi/2 + \phi_b)}$  where  $\phi_b$  is the phase of the bunch centroid measured from the zero crossing of the voltage waveform. The disposition of the phasors is sketched in Figure 1.

The absolute energy change  $\Delta E$  of the bunch per cavity crossing is the scalar product

$$\begin{aligned} \Delta E(\phi_b) &= \mathcal{R}[\mathbf{V}_c \mathbf{I}_b^*] = \mathcal{R}[R \cos \psi e^{+j\psi} (\mathbf{I}_g^0 + \mathbf{I}_b) \mathbf{I}_b^*] (\tau_{\text{rev}}/n) \\ &= R \cos \psi [\cos \psi |I_b|^2 - I_g I_b \sin(\psi + \phi_g + \phi_b)] (\tau_{\text{rev}}/n) . \end{aligned}$$

Here  $\tau_{\text{rev}}$  is the revolution period.

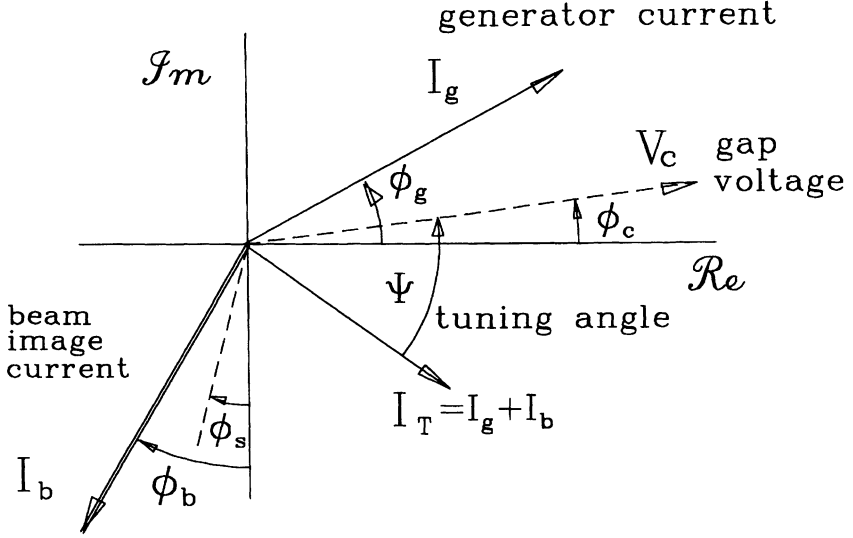


FIGURE 1: Schematic of current and voltage phasors.

The energy change  $\Delta E(\phi_s)$  of a bunch which arrives at the synchronous phase  $\phi_s$  is obtained by putting  $\phi_b = \phi_s$ . The motion will be unstable if changes in bunch phase produce no change in energy increment, that is if  $\Delta E(\phi_b) - \Delta E(\phi_s) = 0$ . In the limit of small oscillations, this implies the instability condition

$$\left. \frac{d}{d\phi_b} \Delta E(\phi_b) \right|_{\phi_b = \phi_s} = 0 .$$

Hence arises the condition

$$\cos(\psi + \phi_g + \phi_s) = 0 \quad \text{or} \quad \psi + \phi_g + \phi_s = \pi/2, 3\pi/2, \dots \quad (1)$$

The physical interpretation of this condition is given immediately below.

The cavity voltage can be decomposed into parts due to the generator current and the beam current, that is  $\mathbf{V}_c = \mathbf{V}_g + \mathbf{V}_b$  where  $\mathbf{V}_g = R \cos \psi e^{j\psi} I_g e^{j\phi_s}$ . The sum  $\psi + \phi_g$  is the phase of that part of the cavity voltage which is due to the generator current. At the stability limit, the generator induced voltage leads the cavity voltage by  $\pi/2 - \phi_s$ . Now the steady state beam current leads the cavity voltage by  $\pi/2 - \phi_s$ . Consequently, at the instability limit Equation (1) the generator induced voltage phasor and the steady state ( $\phi_b = \phi_s$ ) beam current phasor are in phase. In this case, the bunch sits on the crest of the sinusoidal-shaped generator-induced voltage waveform, and the power limited Robinson instability is attributed to loss of phase-focusing for the coherent motion.

One might well ask “why does not the instability depend on the total cavity voltage?” The answer is subtle. The absolute energy increment the bunch receives depends on the total cavity voltage  $\mathbf{V}_c$ . However, if we compare the energy the bunch receives when arriving at two different phases  $\phi_{b,1}$  and  $\phi_{b,2}$ , then the difference of absolute values depends on the partial voltage  $\mathbf{V}_g$ . Now, phase-focusing depends on the relative differences of energy increment, and so depends only on  $\mathbf{V}_g$ .

## 2.2 Equation for Coherent Oscillation

For the single particle motion, it is the phase difference between the particle and the zero crossing of  $\mathbf{V}_c^0$  which is important. However, for the coherent motion, it is the phase difference between the bunch centroid and the zero crossing of  $\mathbf{V}_g$  which is important. Furthermore, in the absence of any feedbacks,  $\psi$  and  $\phi_g$  are constants; and so  $\mathbf{I}_g$  or  $\mathbf{V}_g$  can serve as reference phasors. Accordingly, we define new variables

$$\phi'_b = \phi_b + \psi + \phi_g , \quad (2)$$

$$\phi'_s = \phi_s + \psi + \phi_g . \quad (3)$$

Angle  $\phi'_s$  for coherent motion plays an analogous rôle to the synchronous phase  $\phi_s$  for single particle motion; motion is unstable if either is equal to  $\pi/2$ . Whereas  $\pi/2 - \phi_s$  is the steady state phase difference between  $-\mathbf{I}_b$  and  $\mathbf{V}_c$ ,  $\pi/2 - \phi'_s$  is the steady state phase difference between  $-\mathbf{I}_b$  and  $\mathbf{V}_g$ . Hence, if  $\phi_s = \pi/2$  then  $-\mathbf{I}_b^0$  and  $\mathbf{V}_c^0$  are in phase, while  $\phi'_s = \pi/2$  implies  $-\mathbf{I}_b^0$  and  $\mathbf{V}_g^0$  are in phase.

2.2.1 *Steady state* The cavity voltage is

$$\mathbf{V}_c^0 = R \cos \psi e^{+j\psi} [\mathbf{I}_g^0 + \mathbf{I}_b^0] . \quad (4)$$

Let us define the steady state voltage phasor to be purely real, that is  $\mathbf{V}_c^0 = V_0$  (i.e.  $\phi_c = 0$  and  $V_c = V_0$ ). The steady state bunch phase is  $\phi_b = \phi_s$ , so  $\mathbf{I}_b^0 = -jI_b e^{-j\phi_s}$ . Comparing real and imaginary parts of Equation (4) we find the equations

$$V_0/R \cos \psi = I_g \cos(\psi + \phi_g) + I_b \sin(\psi - \phi_s)$$

$$0 = I_g \sin(\psi + \phi_g) - I_b \cos(\psi - \phi_s) .$$

We may solve these simultaneous equations for  $I_b$  and  $I_g$ . Let  $I_0 = V_0/R$  be the generator current if the cavity were operating on resonance and with no beam. The beam current modulus is given by

$$\frac{I_b}{I_0} \cos \psi = \frac{\sin(\psi + \phi_g)}{\cos(\phi_s + \phi_g)} = \frac{\sin(\phi'_s - \phi_s)}{\cos(\psi - \phi'_s)} . \quad (5)$$

The generator current modulus is given by

$$\frac{I_g}{I_0} \cos \psi = \frac{\cos(\psi - \phi_s)}{\cos(\phi_s + \phi_g)} = \frac{\cos(\psi - \phi_s)}{\cos(\psi - \phi'_s)}. \quad (6)$$

The ratio of current moduli is given by

$$\frac{I_b}{I_g} = \frac{\sin(\psi + \phi_g)}{\cos(\psi - \phi_s)} = \frac{\sin(\phi'_s - \phi_s)}{\cos(\psi - \phi_s)}.$$

It will be useful to define the factor  $U \equiv (I_g/I_0) \cos \psi$ . For the normal mode of operation with  $0 \leq \psi < \pi/2$  and  $I_g > 0$  the function  $U$  is always greater than zero, but is very small for large tuning angles. Setting  $\phi'_s = \pi/2$  we find  $U = \cos(\psi - \phi_s)/\sin \psi$  at the instability threshold beam current.

**2.2.2 Non steady state** Let the bunch contain  $N_b$  particles of charge  $q$ . For brevity, let  $E_b = E(\phi_b)$  and  $E_s = E_s(\phi_s)$  be the bunch centroid energy and the synchronous energy, respectively. The rate of relative energy change obeys the equation

$$\tau \frac{d}{dt} (E_b - E_s) = -R \cos \psi I_g I_b [\sin(\psi + \phi_g + \phi_b) - \sin(\psi + \phi_g + \phi_s)] / N_b, \quad (7)$$

Now the bunch current is  $I_b = N_b q / \tau_{\text{rev}}$  and the synchrotron has  $n$  radio-frequency cavities. Hence the acceleration rate is

$$\tau_{\text{rev}} \frac{d}{dt} (E_b - E_s) = -nqR \cos \psi I_g [\sin \phi'_b - \sin \phi'_s]. \quad (8)$$

We substitute the constant value of  $I_g$  from Equation (6) and replace  $I_0 R$  with  $V_0$ .

Suppose  $h$  is the harmonic number. In the absence of any frequency error, the rate of change of phase is

$$\frac{d}{dt} \phi_b = \frac{d}{dt} \phi'_b = \frac{2\pi h}{\tau_{\text{rev}}} \eta \frac{(E_b - E_s)}{E_s} \quad \text{where} \quad \eta = \left[ \alpha_p - \frac{1}{\gamma_s^2} \right], \quad (9)$$

and  $E_s = \gamma_s m_0 c^2$  and  $\alpha_p$  is the momentum compaction factor. After combining Equations (8) and (9), the equation for coherent oscillations is

$$\tau_{\text{rev}} \frac{d}{dt} \left[ \frac{E_s \tau_{\text{rev}}}{h \eta} \frac{d}{dt} \phi'_b \right] = -nqV_0 U [\sin \phi'_b - \sin \phi'_s] \quad \text{where} \quad U = \frac{\cos(\psi - \phi_s)}{\cos(\psi - \phi'_s)}. \quad (10)$$

Apart from a prefactor  $U$  which alters the bucket height, this is just the equation for synchrotron oscillations about a stable phase angle  $\phi'_s$  with respect to the zero crossing of the  $\mathbf{V}_g$  waveform. Immediately from Equation (10) we recognize that for given  $\phi_s$ , curves such that  $\phi'_s(\psi, I_b/I_g) = \text{constant}$  are curves of constant bucket length and (apart from a height scale factor) constant shape.

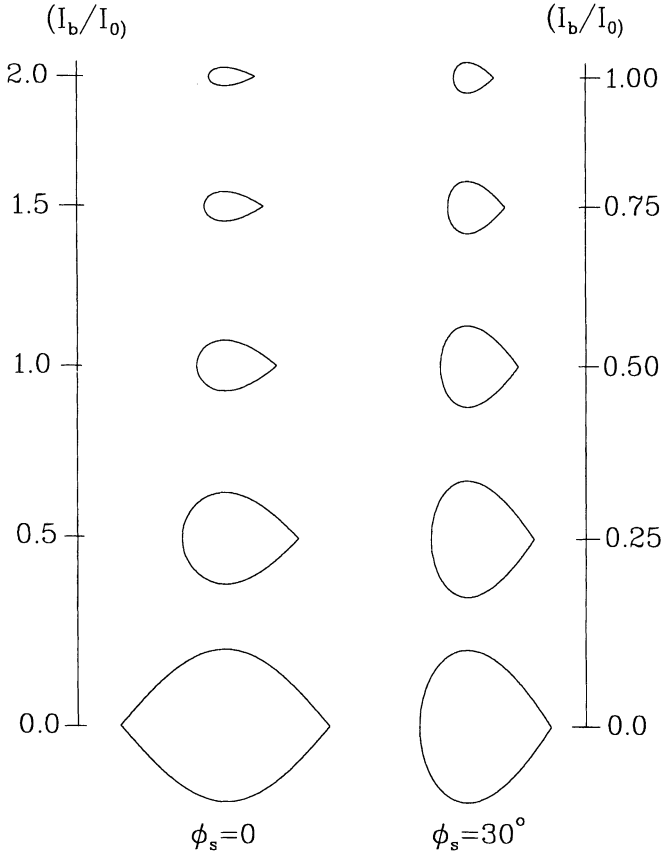


FIGURE 2: Coherent bucket shape versus beam loading parameter.

### 2.3 Coherent Stability Limits

In general, the beam-loaded coherent bucket behaves like a moving-bucket ( $\phi'_s \neq 0$ ), even when the beam is not accelerating ( $\phi_s = 0$ ). Consequently, the phase and energy extent of stable oscillations diminishes as the beam loading increases, and falls to zero as does  $\phi'_s \rightarrow \pi/2$ . This variation of bucket shape is sketched in Figure 2 for the special case that generator current and steady state cavity voltage are in phase (i.e.  $\phi_g = 0$ ).

**2.3.1 Small oscillations** It is interesting to consider the limit of stability of small amplitude oscillations. We substitute  $\phi'_s = \pi/2$  into the Equations (5) and (6) for beam and generator current, to find

$$\frac{I_b}{I_0} = \frac{2 \cos \phi_s}{\sin(2\psi)} \quad \text{and} \quad \frac{I_g}{I_0} = \frac{2 \cos(\psi - \phi_s)}{\sin(2\psi)}. \quad (11)$$

The threshold beam current value is, of course, identical with power-limited instability threshold derived by Robinson.<sup>1</sup>

**2.3.2 Large oscillations** The important consequence of the deformation of the coherent bucket by beam loading is that a beam may be unstable for large amplitude coherent synchrotron oscillations well before the Robinson limit is reached. The nature of the large amplitude motion is determined by  $\phi'_s$ , but is modified by the scale factor  $\sqrt{U}$ . These quantities are plotted in Figures 3a through 3d as functions of beam load ratio  $I_b/(I_0 \cos \phi_s)$  and tuning angle  $\psi$  for a variety of synchronous phase angles  $\phi_s = 0, 15^\circ, 30^\circ, 45^\circ$ . As a general trend, we see that for a given coherent stable phase  $\phi'_s$ , there is a slight advantage in using a larger tuning angle; because this allows a larger beam load ratio. However, this will be at the cost of reduced coherent bucket height.

The curve  $U = 1$  has a special significance: below it, bucket height falls more rapidly than we should guess from  $\phi'_s$  alone; and above the curve, bucket height is boosted. The curve  $U = 1$  is given by the solutions of  $\psi - \phi_s = \pm(\phi_s + \phi_g)$ . The solution  $\phi_g = -\psi$  implies  $\phi'_s = \phi_s$  and generates the case of zero beam current ( $I_b = 0$ ). The solution  $\phi_g = \psi - 2\phi_s$  implies the curve of values

$$I_b/I_0 = 2 \sin(\psi - \phi_s) / \cos \psi . \quad (12)$$

When this curve is plotted in the  $I_b/I_0$  versus  $\psi$  plane, then combinations of  $I_b$  and  $\psi$  above the curve correspond to  $U > 1$ , while combinations below correspond to  $U < 1$ . It is evident from the figures, that coherent bucket height is less reduced at small tuning angles.

## 2.4 Coherent Bucket Shape

The separatrix for coherent motion is called the coherent bucket and is the stability limit for large amplitude motions. The bucket coordinates are the bunch energy with respect to the synchronous energy (i.e.  $E_b - E_s$ ) and the bunch centroid phase ( $\phi'_b$ ) with respect to the zero crossing of  $\mathbf{V}_g^0$ . The centre of phase motion is  $\phi'_s$ , or  $\phi_s$  with respect to the zero crossing of the cavity waveform  $\mathbf{V}_c^0$ . The maximum extent of oscillations about  $\phi_s$  is  $\pi - 2\phi'_s$ . The relation of these phases to the bucket extent is made clear in Figure 4.

It will be useful to consider the parameters  $\tau_{\text{rev}}$ ,  $E_s$  and  $\eta$  to be slowly varying and to introduce the synchrotron angular frequency  $\Omega_s$  defined by:

$$\Omega_s^2 = q n V_0 h \eta / \tau_{\text{rev}}^2 E_s . \quad (13)$$

Equation (10) is derivable from the function:

$$(\phi'_b)^2 - 2\Omega_s^2 U \{ \cos \phi'_b + \cos \phi'_s + \sin \phi'_s [\phi'_b + \phi'_s - \pi] \} = 0 , \quad (14)$$

whose condition also gives the bounding stable path in phase space, or bucket. The bucket half-height is given by

$$(\phi'_b)_{\text{max}} = \Omega_s \sqrt{2^* U} [2 \cos \phi'_s - \sin \phi'_s (\pi - 2\phi'_s)]^{1/2} ,$$

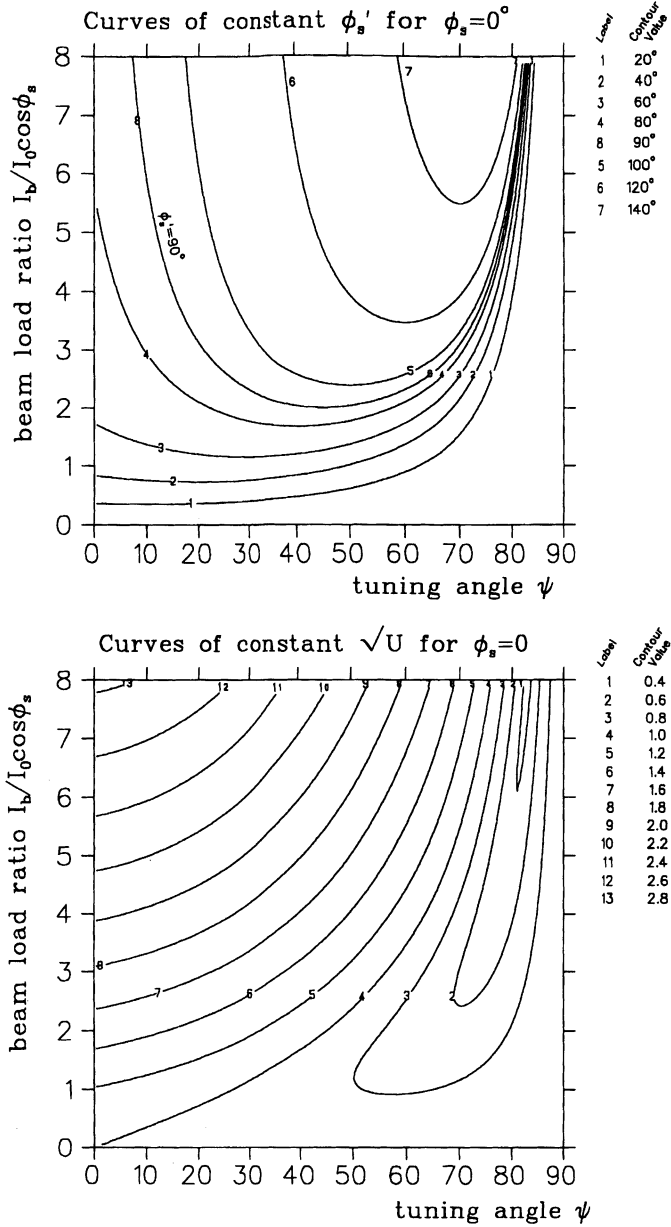


FIGURE 3a: Coherent bucket parameters  $\phi_s'$  and  $U$  as a function of beam load and tuning.



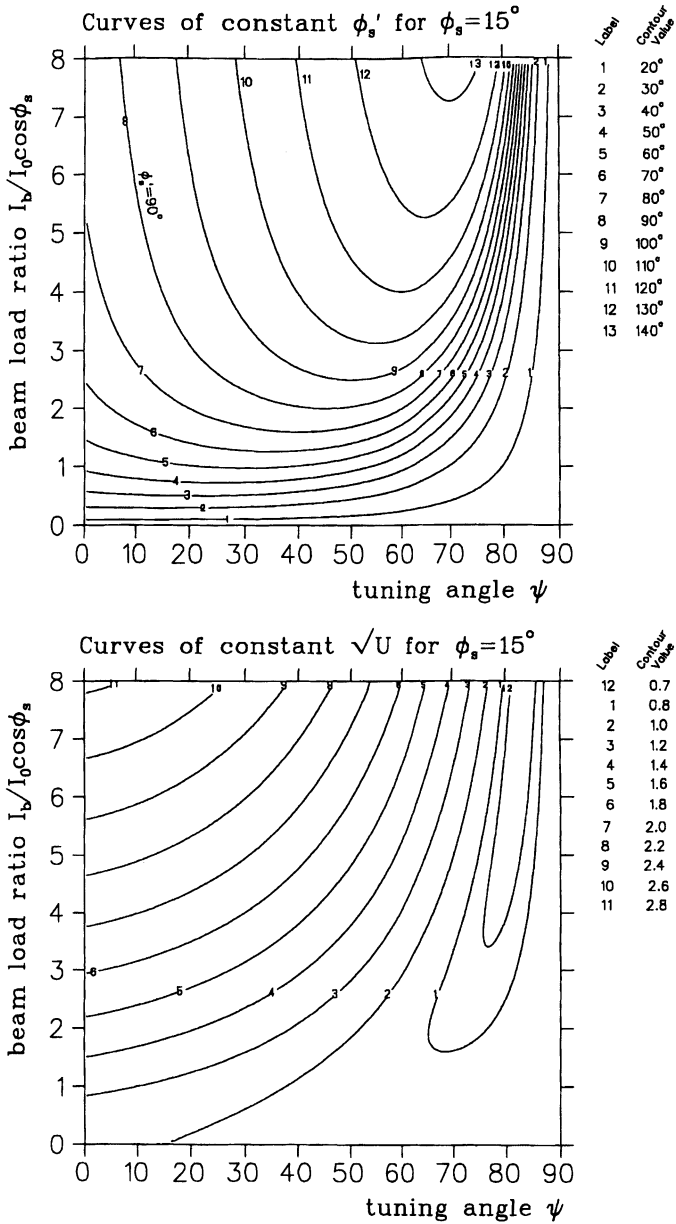


FIGURE 3b: Coherent bucket parameters  $\phi_s'$  and  $U$  as a function of beam load and tuning.

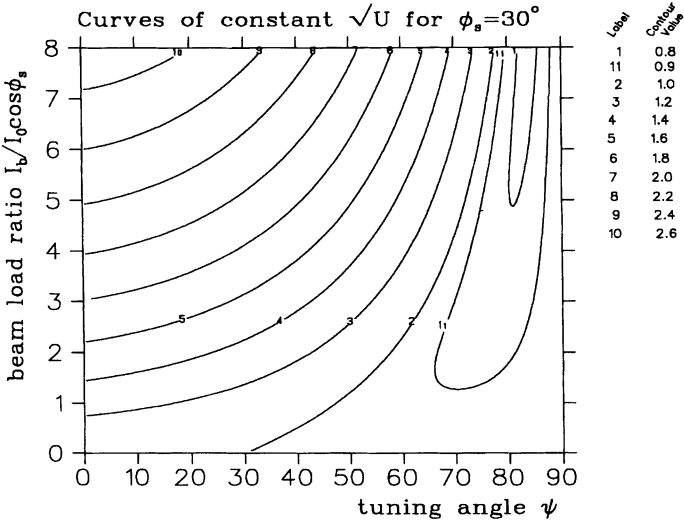
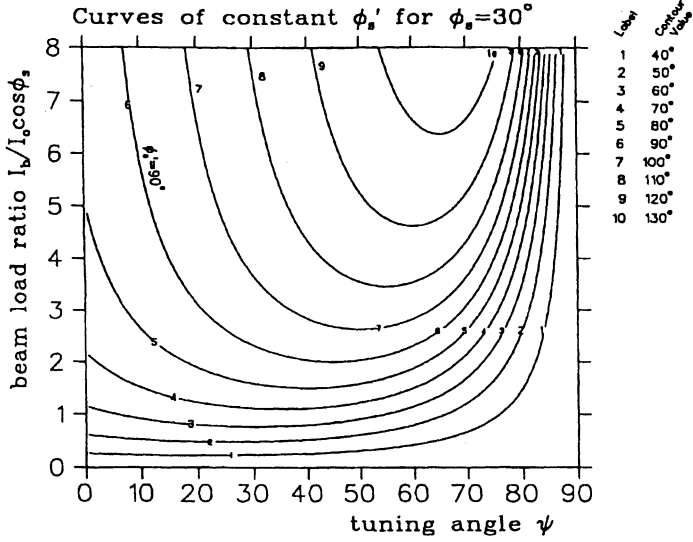


FIGURE 3c: Coherent bucket parameters  $\phi_s'$  and  $U$  as a function of beam load and tuning.

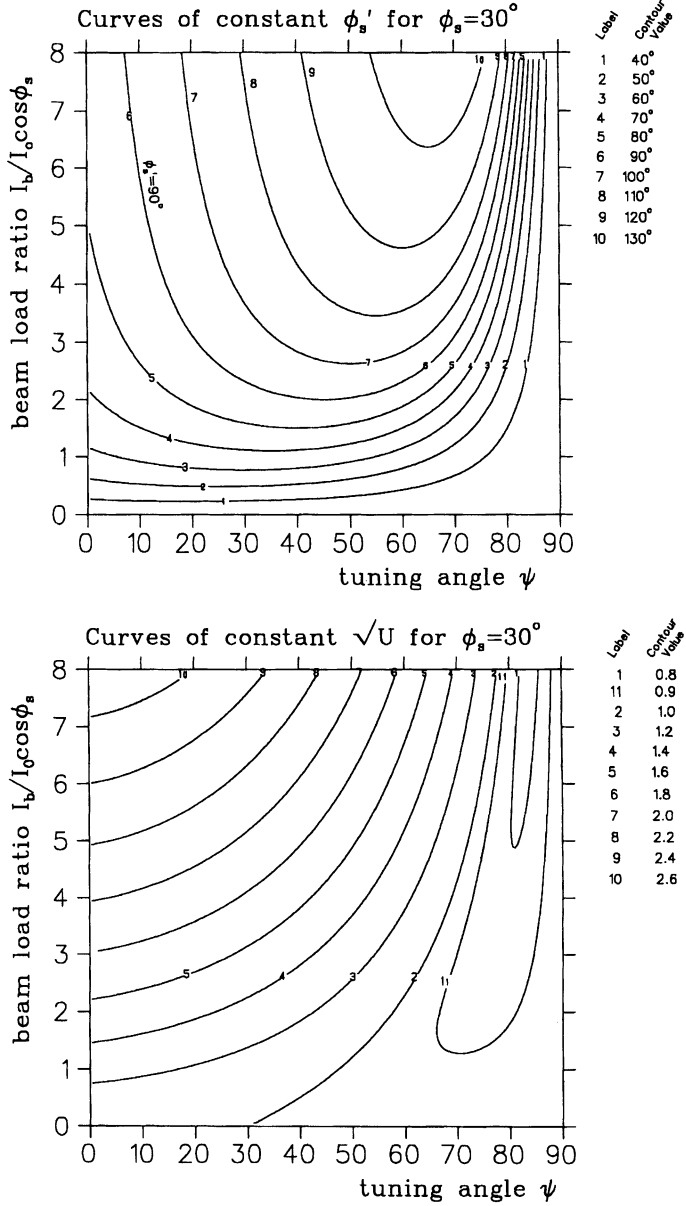


FIGURE 3d: Coherent bucket parameters  $\phi_s'$  and  $U$  as a function of beam load and tuning.

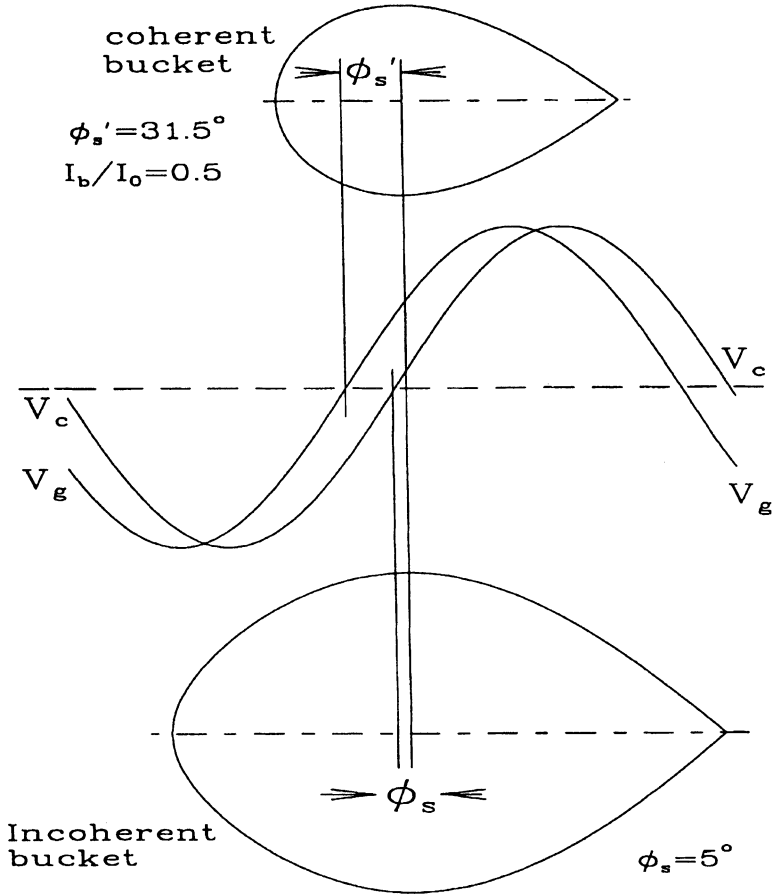


FIGURE 4: Relation of phases and buckets with respect to the voltage waves  $V_g$  and  $V_c$ .

where for given  $\psi$  and  $\phi_s$ , the values of  $\phi_s'$  and  $U$  are determined from Equations (5) and (6) respectively.

Note, close to threshold, the coherent bucket given by Equation (14) will be correct even if the terms of our initial assumption, that cavity time constant is small compared to the synchrotron period, are not fulfilled; because the coherent oscillation frequency tends toward zero at threshold irrespective of  $\Omega_s$ .

### 3 INCOHERENT BUCKET

We shall derive the bounding bucket for incoherent motions of individual particles with respect to the bunch centroid. Our derivation does not assume that the bunch centroid is

located at the synchronous phase angle. Instead, the derivation manifests the fact that under conditions of beam loading the incoherent bucket moves with the bunch centroid.

### 3.1 Incoherent Bucket Stability Criterion

Let the phase of an individual particle with respect to the bunch centre be  $\phi$  such that the phase with respect to the zero crossing of unperturbed voltage  $\mathbf{V}_c^0$  is  $\phi_b + \phi$ . We suppose the individual particle charge to be so small, compared with that of the bunch, that its motion does not affect the disposition of the beam image current phasor  $\mathbf{I}_b$ .

If the bunch phase is not equal to  $\phi_s$ , then the cavity voltage will deviate from the steady state value; let the new value be  $\mathbf{V}_c = V_c e^{j\phi_c}$  where the phase  $\phi_c$  depends on  $\phi_b$ . Let the individual particle contribute a current phasor  $\delta I = -jq e^{-j(\phi_b + \phi)} \tau_{\text{rev}}$  where  $q$  is the charge. Thus, the individual particle energy change at the accelerating cavity is the scalar product

$$\begin{aligned} \Delta E(\phi_b + \phi) &= \mathcal{R}[\mathbf{V}_c \delta I^*] \tau_{\text{rev}} \\ &= \mathcal{R}[jq V_c e^{j(\phi_c + \phi_b + \phi)}] \\ &= -q V_c \sin(\phi_b + \phi_c + \phi) . \end{aligned}$$

The motion will be unstable if changes in particle phase produce no change in energy increment, that is if  $\Delta E(\phi_b + \phi) - \Delta E(\phi_b) = 0$ . In the limit of small oscillations, this implies the instability condition

$$\left. \frac{d}{d\phi} \Delta E(\phi_b + \phi) \right|_{\phi=0} = 0 .$$

Hence arises the condition

$$\cos(\phi_b + \phi_c) = 0 \quad \text{or} \quad \phi_b + \phi_c = \pi/2, 3\pi/2, \dots \quad (15)$$

The incoherent motion is unstable when the perturbed beam current  $-\mathbf{I}_b$  and the perturbed cavity voltage  $\mathbf{V}_c$  are in phase. This corresponds to the case that the bunch sits on the crest of the perturbed cavity voltage waveform. In fact, the condition  $\phi_b + \phi_c = \pi/2$  is quite hard to arrange, particularly at large beamload values. This is because as  $\phi_b$  increases, so  $\phi_c$  decreases.

The cavity phase  $\phi_c$  is a function of the beam phase  $\phi_b$ , and so one must find a self-consistent solution of the instability condition. The perturbed cavity voltage is given by

$$V_c e^{j\phi_c} = R \cos \psi e^{j\psi} [\mathbf{I}_g^0 + \mathbf{I}_b] .$$

Now at the incoherent stability limit Equation (15)  $\phi_c = \pi/2 - \phi_b$ . We substitute this value, multiply throughout by  $\exp(+j\phi_b)$  and compare real parts to obtain:

$$0 = I_g \cos(\psi + \phi_g + \phi_b) + I_b \sin \psi .$$

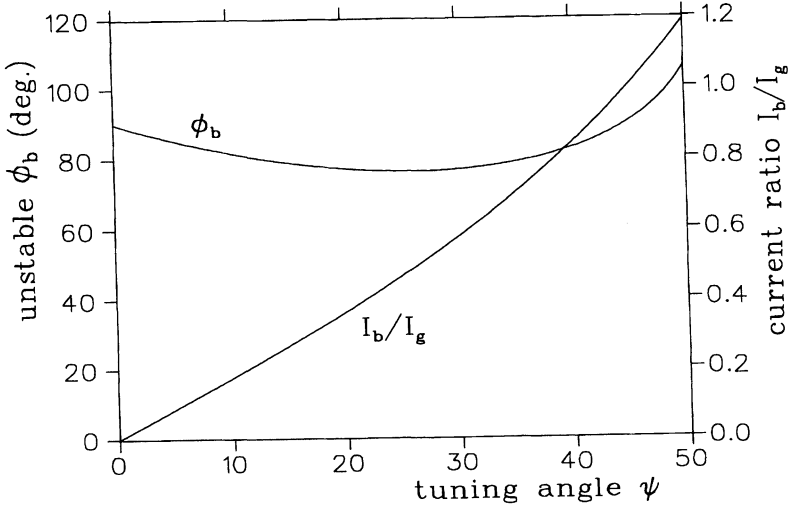


FIGURE 5: Values of the bunch centroid phase  $\phi_b$  which cause the small amplitude incoherent motion to be unstable, and the corresponding value of the current ratio, both as a function of tuning.

Now,  $I_g$  and  $I_b$  are determined by the steady state equation

$$0 = I_g \sin(\psi + \phi_g) - I_b \cos(\psi - \phi_s) .$$

Consequently, if unstable solutions  $\phi_b$  exist, then they must satisfy

$$-\cos(\psi + \phi_g + \phi_b) = \frac{\sin \psi \sin(\psi + \phi_g)}{\cos(\psi - \phi_s)} \leq 1 .$$

Clearly  $\pi/2 < \psi + \phi_g + \phi_b < 3\pi/2$  for instability to occur. Moreover, at sufficiently large values of the detuning angle  $\psi \geq \hat{\psi}$  no solutions  $\phi_b$  can be found which satisfy  $\phi_b + \phi_c = \pi/2$ . The angle  $\hat{\psi}$  is the solution of the equality

$$\sin \psi \sin(\psi + \phi_g) = \cos(\psi - \phi_s) .$$

For detuning angles  $\psi \geq \hat{\psi}$  the small amplitude incoherent motion is always stable. As an example, Figure 5 sketches the dependence of the unstable  $\phi_b$  and  $I_b/I_g$  as a function of  $\psi$  for the special case of  $\phi_s = 0$  and  $\phi_g = 0$ ; for which case  $\hat{\psi} = 51.83^\circ$ . In general, the solutions correspond to large values of  $\phi_b$ ; possibly outside the coherent bucket.

### 3.2 Equation for Incoherent Oscillation

For single particle motion, it is the phase difference between the particle and  $\mathbf{V}_c^0$  which counts. However, for incoherent motion, it is the phase difference between the individual

particle and the perturbed cavity voltage  $V_c$  which matters. Accordingly, we define a new variable

$$\tilde{\phi}_s = \phi_b + \phi_c . \quad (16)$$

Here  $\pi/2 - \tilde{\phi}_s$  is the phase difference between the perturbed beam current  $-I_b$  and the perturbed cavity voltage  $V_c$ . If  $\phi_b = \phi_s$ , then  $\tilde{\phi}_s = \phi_s$ . Angle  $\tilde{\phi}_s$  for the incoherent motion plays a similar rôle to  $\phi_s$  in single particle formulations of the phase motion; whenever either is equal to  $\pi/2$ , the motion is unstable.

*3.2.1 Non steady state* The relative energy change of an individual particle with respect to the bunch centroid is given by:

$$\Delta E(\phi_b + \phi) - \Delta E(\phi_b) = -q V_c [\sin(\tilde{\phi}_s + \phi) - \sin(\tilde{\phi}_s)] .$$

So the problem has been reduced to that of finding  $V_c$  and  $\tilde{\phi}_s$ . In fact, it is sufficient to find  $V_c \sin \tilde{\phi}_s$  and  $V_c \cos \tilde{\phi}_s$  because the the energy change may be written

$$\Delta E(\phi_b + \phi) - \Delta E(\phi_b) = -q V_c [\sin \phi \cos \tilde{\phi}_s + (\cos \phi - 1) \sin \tilde{\phi}_s] .$$

The perturbed cavity voltage is given by

$$V_c e^{j\phi_c} = V_c = R \cos \psi e^{j\psi} [\Gamma_g^0 - j I_b e^{-j\phi_b}] .$$

The form we want is

$$V_c e^{j\tilde{\phi}_s} = V_c e^{j\phi_b} = R \cos \psi e^{j\psi} [\Gamma_g^0 e^{j\phi_b} - j I_b] . \quad (17)$$

Comparing real and imaginary parts yields

$$V_c \cos \tilde{\phi}_s = R \cos \psi [I_g \cos(\psi + \phi_g + \phi_b) + I_b \sin \psi]$$

$$V_c \sin \tilde{\phi}_s = R \cos \psi [I_g \sin(\psi + \phi_g + \phi_b) - I_b \cos \psi] .$$

Now, if we set  $\phi_b = \phi_s$  we shall generate an almost identical pair of equations but with  $V_c = V_0$  and  $\tilde{\phi}_s = \phi_s$ . Eliminating  $I_b$  from these two sets of equations, and substituting  $\phi'_s = \phi_s + \psi + \phi_g$  and  $\phi'_b = \phi_b + \psi + \phi_g$ , we find the forms

$$V_c \sin \tilde{\phi}_s = V_0 \{ \sin \phi_s + U [\sin \phi'_b - \sin \phi'_s] \}$$

$$V_c \cos \tilde{\phi}_s = V_0 \{ \cos \phi_s + U [\cos \phi'_b - \cos \phi'_s] \} , \quad (18)$$

where the function  $U(\phi'_s)$  is equal to  $(I_g/I_0) \cos \psi$ .

For brevity we shall write  $E_q = E(\phi_b + \phi)$  to mean the energy of a particle with phase  $\phi$  with respect to the bunch centre, and  $E_s = E(\phi_s)$  to mean the energy of the particle which always arrives at the synchronous phase. The equation for phase advance is

$$\frac{d}{dt}(\phi_b + \phi) = \frac{2\pi h}{\tau_{\text{rev}}} \eta \frac{(E_q - E_s)}{E_s} \quad \text{but} \quad \frac{d}{dt}\phi_b = \frac{2\pi h}{\tau_{\text{rev}}} \eta \frac{(E_b - E_s)}{E_s},$$

and so upon subtraction of these two equations we find

$$\frac{d}{dt}\phi = \frac{2\pi h}{\tau_{\text{rev}}} \eta \frac{(E_q - E_b)}{E_s}.$$

Finally, after incrementing the energy through  $n$  cavities and replacing the turn derivative with a time derivative, the equation for incoherent oscillations is

$$\begin{aligned} -\tau_{\text{rev}} \frac{d}{dt} \left[ \frac{E_s \tau_{\text{rev}}}{h\eta} \frac{d}{dt} \phi \right] &= nq V_0 \sin \phi \{ \cos \phi_s + U[\cos \phi'_b - \cos \phi'_s] \} \\ &\quad + nq V_0 (\cos \phi - 1) \{ \sin \phi_s + U[\sin \phi'_b - \sin \phi'_s] \} \\ &= nq V_c [\sin(\tilde{\phi}_s + \phi) - \sin \tilde{\phi}_s], \end{aligned} \quad (19)$$

where  $\tilde{\phi}_s$  and  $V_c$  are obtained from Equation (18) as

$$\tan \tilde{\phi}_s = \frac{\sin \phi_s + U[\sin \phi'_b - \sin \phi'_s]}{\cos \phi_s + U[\cos \phi'_b - \cos \phi'_s]}, \quad (20)$$

and

$$\begin{aligned} (V_c/V_0)^2 &= 1 + 2U[\sin \phi_s (\sin \phi'_b - \sin \phi'_s) + \cos \phi_s (\cos \phi'_b - \cos \phi'_s)] \\ &\quad + 2U^2[1 - \cos(\phi'_b - \phi'_s)]. \end{aligned} \quad (21)$$

Equation (19) is that for synchrotron oscillations about a stable phase angle  $\tilde{\phi}_s$ .

### 3.3 Quasi-Static Approximation for Incoherent Motion

In general Equation (19) is a complicated equation to solve; particularly as  $\phi_b(t)$  is time dependent and has the effect of forcing the motion in  $\phi$ . However, we note that when the bunch centroid is coincident with the synchronous phase, and  $V_c = V_0$ ,  $\tilde{\phi}_s = \phi_s$ , then the equation immediately reduces to the simple form

$$\tau_{\text{rev}} \frac{d}{dt} \left[ \frac{E_s \tau_{\text{rev}}}{h\eta} \frac{d}{dt} \phi \right] = -nq V_0 [\sin(\phi_s + \phi) - \sin(\phi_s)]. \quad (22)$$



Note, the complete absence of  $\phi'_s$  from Equation (22) implies that the incoherent bucket and the incoherent synchrotron frequency are completely unaffected by beam loading for the case  $\phi_b = \phi_s$ . By continuity we should expect this to remain approximately true for small excursions  $\phi_b \neq \phi_s$ . Now when  $\phi'_s \approx \pi/2$ ,  $(E_b - E_s)$  must be small if the coherent motion remains within the coherent bucket, and so the slip rate  $d\phi_b/dt$  must also be small. Consequently, motion of the bunch centroid will appear 'frozen' compared with the individual particle motion, and the value of  $\phi_b$  can be taken as approximately constant during an oscillation of  $\phi$ .

Note, it is not sufficient to say that close to the condition  $\phi'_s = \pi/2$ , the coherent synchrotron frequency approaches zero implies the motion is frozen, because the bunch will drift at rate depending on  $E_b - E_s$  when there is no longitudinal focusing.

*3.3.1 Incoherent motion near to the Robinson stability limit* By continuity, we should expect incoherent motion in the vicinity of  $\phi'_s = \pi/2$  to be similar to the special case  $\phi'_s \equiv \pi/2$  which is simpler to treat. After the substitutions  $\phi'_s = \pi/2$  and  $\phi'_b = \pi/2 + (\phi_b - \phi_s)$ , Equations (21) and (20) take the form

$$(V_c/V_0)^2 = 1 + 2U_0[\sin(2\phi_s - \phi_b) - \sin \phi_s] + 2U_0^2[1 - \cos(\phi_b - \phi_s)]$$

and

$$\tan \tilde{\phi}_s = \frac{\sin \phi_s + U_0[\cos(\phi_b - \phi_s) - 1]}{\cos \phi_s - U_0 \sin(\phi_b - \phi_s)},$$

with  $U_0 = \cos(\psi - \phi_s)/\sin \psi$ .

For the case  $\phi_b = \phi_s$  we recover the previous limiting results that  $V_c = V_0$  and  $\tilde{\phi}_s = \phi_s$ . Hence, even at the limit of stability for coherent motions, the beam stays bunched. For small displacements  $|\phi_b - \phi_s| \ll 1$  we find that

$$(V_c/V_0)^2 \approx 1 - (\phi_b - \phi_s)2U_0 \cos \phi_s \quad \text{and} \quad \tan \tilde{\phi}_s \approx \tan \phi_s [1 + (\phi_b - \phi_s)U_0/\cos \phi_s].$$

Hence, the incoherent bucket height and shape are only slightly distorted when the bunch moves away from the synchronous phase.

The dynamical behaviour at threshold is complicated but, ultimately, self stabilizing. Suppose, initially, that  $\phi'_s = \pi/2$ ,  $\phi_b = \phi_s$  and  $E_b \neq E_s$ . With zero coherent bucket, the bunch starts to wander; but in doing so the incoherent bucket is slightly perturbed and particles start to redistribute thus slightly deforming the bunch shape. Now the current modulus  $I_b$  depends on the bunch shape, and so will decrease as the bunch deforms; but this will bring the beam current below the coherent instability threshold and so a small coherent bucket forms which captures the bunch centroid. The bunch centroid ceases to wander, and further distortion of the incoherent bucket stops also. Consequently, the beam now becomes stable with respect to both coherent and incoherent motions.

Far below threshold, we cannot say that  $\phi_b$  and  $E_b$  are slowly changing compared with  $\phi$  and  $E_q$ . However, with small beam current, the modulation of the cavity voltage induced

by the coherent motion will be small; and so one should expect little modification of the incoherent bucket in this regime also.

### 3.4 Incoherent Bucket Shape

The separatrix for incoherent motion is the incoherent bucket. The bucket coordinates are phase and energy with respect to the bunch centroid, that is  $\phi$  and  $(E_q - E_b)$ . The stable point of the motion ( $\dot{\phi} = 0$ ) is located at  $\phi_b(t)$  with respect to the zero crossing of the unperturbed waveform  $\mathbf{V}_c^0$ , that is the incoherent motion is 'pinned' to the bunch central phase. The incoherent bucket length and shape, however, depends on  $\tilde{\phi}_s$ . The maximum extent of oscillations about  $\phi_b$  is  $\hat{\phi} = \pi - 2\tilde{\phi}_s$ .

Eq. (19) is derivable from the function:

$$(\dot{\phi})^2 - 2\Omega_s^2(V_c/V_0) \left\{ \cos(\tilde{\phi}_s + \phi) + \cos \tilde{\phi}_s + \sin \tilde{\phi}_s[\phi + 2\tilde{\phi}_s - \pi] \right\} = 0, \quad (23)$$

whose condition also gives the bounding stable path in phase space, or bucket. The bucket half-height is given by

$$(\dot{\phi})|_{\max} = \Omega_s \sqrt{2V_c/V_0} [2\cos \tilde{\phi}_s - \sin \tilde{\phi}_s(\pi - 2\tilde{\phi}_s)]^{1/2}.$$

## 4 CONCLUSION

For the case of a beam-loaded radio-frequency acceleration system, we have found the coherent bucket by comparing motion of the bunch centroid with the synchronous particle. In general, the bucket is distorted and large amplitude motions may be unstable well before the Robinson limit is reached.

We have found the general incoherent oscillation equation by comparing motion of an individual particle with the bunch centroid. For the case  $\phi_b = \phi_s$  and  $E_b = E_s$ , the incoherent bucket is always identical with the case of no beam loading. For the case of extreme beam loading  $\phi'_s \approx \pi/2$ , and  $\phi_b \approx \phi_s$  and  $E_b \approx E_s$ , one may define a quasi-static incoherent bucket; and this is found to be slightly distorted.

For the case of no beam loading, the stability of longitudinal motions depends on the synchronous phase  $\phi_s$ . With beam loading we have found that stability of coherent motion depends upon the angle  $\phi'_s = \phi_s + \psi + \phi_g$ , and the stability of incoherent motion depends upon the angle  $\tilde{\phi}_s = \phi_b + \phi_c$ . Stability of coherent motion depends upon the partial voltage  $\mathbf{V}_g$ , whereas stability of incoherent motion depends upon the total cavity voltage  $\mathbf{V}_c$ .

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