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COHERENT AND INCOHERENT BUCKET FOR A BEAM LOADED RF SYSTEM

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For the case of a beam-loaded radio-frequency acceleration system, we have found the coherent oscillation equation by comparing the bunch centroid with the synchronous particle; and we have found the incoherent oscillation equation by comparing motion of a single particle with the bunch centroid.

For the case of single particle motion, stability depends on the synchronous phase ϕ_s. Stability of coherent motion depends upon the generator induced partial voltage, whereas stability of incoherent motion depends upon the total cavity voltage including the beam induced voltage component. With beam loading, the stability of coherent motion depends upon $\phi_s + \psi + \phi_g$, with tuning angle ψ and generator current phase ϕ_g , and the coherent bucket is deformed. With beam loading, the stability of incoherent motion depends upon $\phi_b + \phi_c$, with beam current phase ϕ_b and perturbed cavity phase ϕ_c . For the case that the centroid is coincident with the synchronous particle, the incoherent bucket is found to be identical with the case of no beam loading.

KEY WORDS: beam loading, instability, coherent

1. INTRODUCTION

The effect of the rf electric fields in a synchrotron is to accelerate the beam and to longitudinally focus beam particles into bunches. Individual particles and also whole bunches may perform non-linear oscillations; these motions are called synchrotron oscillations. The rf accelerating cavities are driven both by the generator current and the beam current; and the effect of the beam on the cavity gap voltage is known as 'beam loading'. As a result of this beam loading, the coherent motion of a bunch taken as a whole may differ considerably from the incoherent motion of a charged single particle. The usual sinusoidal choice of the rf waveform leads to a phase-space separatrix which bounds the oscillations, usually called the bucket.

In this note, we shall find the incoherent bucket for single particles within the bunch, and the coherent bucket for rigid oscillations of the bunch centroid. The cavity dynamical response shall be taken as quasi-static, corresponding to the case that the period of synchrotron oscillations is much longer than the cavity filling time.

Linear stability analysis[1,2] shows the frequency of small amplitude rigid dipole oscillations to scale according to

$$\Omega^2 = \Omega_s^2[1 - I_b \sin 2\psi / 2I_0 \cos\phi_s] ,$$

where the quantities appearing will be defined below. This suggests that the coherent bucket shrinks to zero at some threshold beam current I_b, but begs the questions: "is the bucket merely scaled in height, or is the shape distorted?" In the following non-linear analysis, we shall find that the incoherent bucket is little affected by beam loading, but the coherent bucket size and shape has a complicated dependence on cavity detuning.

2. COHERENT BUCKET

We shall suppose, that the synchrotron period is much greater than the cavity time constant; so that cavity dynamical effects can be ignored and the cavity response is virtually instantaneous. This premise affords great simplicity but, also implies that observations of a real beam may differ somewhat from the idealization presented here. We shall take it for granted that the cavity is detuned in the correct sense to avoid the dynamical Robinson instability.

We shall represent currents and voltages by phasor quantities in the complex plane. The energy increment received by a bunch is then the vector scalar product of the beam current $\mathbf{I}_b$ and the cavity voltage $\mathbf{V}_c$, that is the product of their magnitudes and the cosine of the angle formed between the vectors. Let $\mathcal{R}[\ldots]$ indicate the real part of a complex quantity. Consider the general case of phasors $\mathbf{B} = \mathrm{B}e^{i\phi_B}$ and $\mathbf{A} = \mathrm{A}e^{i\phi_A}$; their scalar product is

$$AB\cos(\phi_B - \phi_A) = \mathcal{R}[\mathbf{AB}^*] = \mathcal{R}[\mathbf{A}^*\mathbf{B}] .$$

2.1. Coherent Bucket Stability Criterion

Let the cavity gap voltage be $\mathbf{V}_c = \mathrm{V}_c e^{j\phi_c}$ and the cavity complex impedance be $R\cos\psi e^{j\psi}$ where R is the shunt resistance and ψ is the detuning angle. We

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shall adopt the phase convention used by Pedersen[3]: that the steady state voltage phasor is aligned with the positive real axis, in which case ϕ_c becomes zero. Let the generator current driving the cavity be a <u>fixed</u> vector $\mathbf{I}_g^0 = \mathbf{I}_g e^{j\phi_s}$ which is chosen, for given detuning and beam current, to produce the correct cavity voltage in the steady state. Let the beam image current be $\mathbf{I}_b = \mathbf{I}_b e^{-j(\pi/2+\phi_b)}$ where ϕ_b is the phase of the bunch centroid measured from the zero crossing of the voltage waveform. The disposition of the phasors is sketched in Figure 1.

The absolute energy change ΔE of the bunch per cavity crossing is the scalar product

$$\begin{aligned} c\Delta E(\phi_b) &= \mathcal{R}[\mathbf{V}_c \mathbf{I}_b^*] = \mathcal{R}[R\cos\psi e^{+j\psi}(\mathbf{I}_g^0 + \mathbf{I}_b)\mathbf{I}_b^*] \\ &= R\cos\psi\,[\cos\psi\,|I_b|^2 - I_g I_b \sin(\psi + \phi_g + \phi_b)] \,. \end{aligned}$$

Here c is the speed of light, and we have assumed an ultra-relativistic beam (i.e. $\beta = 1$).

The energy change $\Delta E(\phi_s)$ of a bunch which arrives at the synchronous phase ϕ_s is obtained by putting $\phi_b = \phi_s$. The motion will be unstable if changes in bunch phase produce no change in energy increment, that is if $\Delta E(\phi_b) - \Delta E(\phi_s) = 0$. In the limit of small oscillations, this implies the instability condition

$$\frac{d}{d\phi_b}\Delta E(\phi_b)\Big|_{\phi_b=\phi_s} = 0 \,.$$

Hence arises the condition

$$\cos(\psi + \phi_g + \phi_s) = 0 \quad \text{or} \quad \psi + \phi_g + \phi_s = \pi/2\,,\ 3\pi/2\,, \ldots \tag{1}$$

The physical interpretation of this condition is given immediately below.

The cavity voltage can be decomposed into parts due to the generator current and the beam current, that is $\mathbf{V}_c = \mathbf{V}_g + \mathbf{V}_b$ where $\mathbf{V}_g = R\cos\psi e^{j\psi} I_g e^{j\phi_s}$. The sum $\psi + \phi_g$ is the phase of that part of the cavity voltage which is due to the generator current. At the stability limit, the generator induced voltage leads the cavity voltage by $\pi/2 - \phi_s$. Now the steady state beam current leads the cavity voltage by $\pi/2 - \phi_s$. Consequently, at the instability limit Equation (1) the generator induced voltage phasor and the steady state ($\phi_b = \phi_s$) beam current phasor are in phase. In this case, the bunch sits on the crest of the sinusoidal-shaped generator-induced voltage waveform, and the power limited Robinson instability is attributed to loss of phase-focusing for the coherent motion.

One might well ask "why does not the instability depend on the total cavity voltage?" The answer is subtle. The absolute energy increment the bunch receives depends on the total cavity voltage $\mathbf{V}_c$. However, if we compare the energy the bunch receives when arriving at two different phases $\phi_{b,1}$ and $\phi_{b,2}$, then the difference of absolute values depends on the partial voltage $\mathbf{V}_g$. Now, phase-focusing depends on the relative differences of energy increment, and so depends only on $\mathbf{V}_g$.

2.2. *Equation for Coherent Oscillation*

For the single particle motion, it is the phase difference between the particle and the zero crossing of $\mathbf{V}_c^0$ which is important. However, for the coherent motion, it is the phase difference between the bunch and the zero crossing of $\mathbf{V}_g$ which is important. Furthermore, in the absence of any feedbacks, ψ and ϕ_g are constants; and so $\mathbf{I}_g$ or $\mathbf{V}_g$ can serve as reference phasors. Accordingly, we define new variables

$$\phi_b' = \phi_b + \psi + \phi_g \,, \tag{2}$$
$$\phi_s' = \phi_s + \psi + \phi_g \,. \tag{3}$$

Angle ϕ_s' for coherent motion plays an analogous rôle to the synchronous phase ϕ_s for single particle motion; motion is unstable if either is equal to $\pi/2$. Whereas ϕ_s is the steady state phase difference between $-\mathbf{I}_b$ and $\mathbf{V}_c$, ϕ_s' is the steady state phase difference between $-\mathbf{I}_b$ and $\mathbf{V}_g$. Hence, if $\phi_s = \pi/2$ then $-\mathbf{I}_b^0$ and $\mathbf{V}_c^0$ are in phase, while $\phi_s' = \pi/2$ implies $-\mathbf{I}_b^0$ and $\mathbf{V}_g$ are in phase.

2.2.1 Steady state The cavity voltage is

$$\mathbf{V}_c^0 = R\cos\psi\, e^{+j\psi}[\mathbf{I}_g^0 + \mathbf{I}_b^0] \,. \tag{4}$$

Let us define the steady state voltage phasor to be purely real, that is $\mathbf{V}_c^0 = V_0$ (i.e. $\phi_c = 0$ and $V_c = V_0$). The steady state bunch phase is $\phi_b = \phi_s$, so $\mathbf{I}_b^0 = -jI_b e^{-j\phi_s}$. Comparing real and imaginary parts of Equation (4) we find the equations

$$\begin{aligned} V_0/R\cos\psi &= I_g\cos(\psi + \phi_g) + I_b\sin(\psi - \phi_s) \\ 0 &= I_g\sin(\psi + \phi_g) - I_b\cos(\psi - \phi_s) \,. \end{aligned}$$

We may solve these simultaneous equations for I_b and I_g. Let $I_0 = V_0/R$ be the generator current if the cavity were operating on resonance and with no beam. The beam current modulus is given by

$$\frac{I_b}{I_0}\cos\psi = \frac{\sin(\psi + \phi_g)}{\cos(\phi_s + \phi_g)} = \frac{\sin(\phi_s' - \phi_s)}{\cos(\psi - \phi_s')} \,. \tag{5}$$

The generator current modulus is given by

$$\frac{I_g}{I_0}\cos\psi = \frac{\cos(\psi - \phi_s)}{\cos(\phi_s + \phi_g)} = \frac{\cos(\psi - \phi_s)}{\cos(\psi - \phi_s')} \,. \tag{6}$$

The ratio of current moduli is given by

$$\frac{I_b}{I_g} = \frac{\sin(\psi + \phi_g)}{\cos(\psi - \phi_s)} = \frac{\sin(\phi_s' - \phi_s)}{\cos(\psi - \phi_s)} \,.$$

It will be useful to define the factor $U \equiv (I_g/I_0)\cos\psi$. For the normal mode of operation with $0 \le \psi < \pi/2$ and $I_g > 0$ the function U is always greater than zero, but is very small for large tuning angles. Setting $\phi_s' = \pi/2$ we find $U = \cos(\psi - \phi_s)/\sin\psi$ at the instability threshold beam current.

2.2.2 Non steady state For brevity, let $E_b = E(\phi_b)$ and $E_s = E_s(\phi_s)$ be the bunch centroid energy and the synchronous energy, respectively. The rate of relative energy change obeys the equation

$$c\,\tau_{rev}\frac{d}{dt}(E_b - E_s) = -R\cos\psi\,\mathbf{I}_g\mathbf{I}_b[\sin(\psi + \phi_g + \phi_b) - \sin(\psi + \phi_g + \phi_s)] \,. \tag{7}$$

Now the bunch contains N_b particles of charge q (such that $I_b = N_b qc$) and the synchrotron has n radio-frequency cavities. Hence the acceleration rate is

$$\tau_{rev}\frac{d}{dt}(E_b - E_s) = -nqR\cos\psi\, I_g[\sin\phi_b' - \sin\phi_s'] \,.$$

We substitute the constant value of I_g from Equation (5) and replace I_0R with V_0.

Suppose h is the harmonic number. In the absence of any frequency error, the rate of change of phase is

$$\frac{d}{dt}\phi_b = \frac{d}{dt}\phi_b' = \frac{2\pi h}{\tau_{rev}}\eta\frac{(E_b - E_s)}{E_s} \quad \text{where} \quad \eta = \left[\alpha_p - \frac{1}{\gamma_s^2}\right] , \tag{8}$$

and $E_s = \gamma_s m_0 c^2$ and α_p is the momentum compaction factor. Combining the results Equations (7) and (8), the equation for coherent oscillations is

$$\tau_{rev}\frac{d}{dt}\left[\frac{E_s\tau_{rev}}{h\,\eta}\frac{d}{dt}\phi_b'\right] = -nqV_0U[\sin\phi_b' - \sin\phi_s'] \quad \text{where} \quad U = \frac{\cos(\psi - \phi_s)}{\cos(\psi - \phi_s')} . \tag{9}$$

Apart from a prefactor U which alters the bucket height, this is just the equation for synchrotron oscillations about a stable phase angle ϕ_s' with respect to the zero crossing of the V_g waveform. Immediately from Equation (9) we recognize that for given ϕ_s, curves such that $\phi_s'(\psi, I_b/I_g)$ = constant are curves of constant bucket length and (apart from a height scale factor) constant shape.

2.3. Coherent Stability Limits

In general, the beam-loaded coherent bucket behaves like a moving-bucket ($\phi_s' \neq 0$), even when the beam is not accelerating ($\phi_s = 0$). Consequently, the phase and energy extent of stable oscillations diminishes as the beam loading increases and falls to zero as does $\phi_s' \to \pi/2$. This variation of bucket shape is sketched in Figure 2 for the special case that generator current and steady state cavity voltage are in phase (i.e. $\phi_g = 0$).

2.3.1 Small oscillations It is interesting to consider the limit of stability of small amplitude oscillations. We substitute $\phi_s' = \pi/2$ into the Equation (5) for beam and generator current, to find

$$\frac{I_b}{I_0} = \frac{2\cos\phi_s}{\sin(2\psi)} \quad \text{and} \quad \frac{I_g}{I_0} = \frac{2\cos(\psi - \phi_s)}{\sin(2\psi)} . \tag{10}$$

The threshold beam current value is, of course, identical with power-limited instability threshold derived by Robinson.[1]

2.3.2 Large oscillations The important consequence of the deformation of the coherent bucket by beam loading is that a beam may be unstable for large amplitude coherent synchrotron oscillations well before the Robinson limit is reached. The nature of the large amplitude motion is determined by ϕ_s', but is modified by the scale factor $\sqrt{U}$. These quantities are plotted in Figures 4a through 4d as functions of

beam load ratio $I_b/(I_0\cos\phi_s)$ and tuning angle ψ for a variety of synchronous phase angles ϕ_s = 0, 15°, 30°, 45°. As a general trend, we see that for a given coherent stable phase ϕ_s', there is a slight advantage in using a larger tuning angle; because this allows a larger beam load ratio. However, this will be at the cost of reduced coherent bucket height.

The curve $U = 1$ has a special significance: below it, bucket height falls more rapidly than we should guess from ϕ_s' alone; and above the curve, bucket height is boosted. The curve $U = 1$ is given by the solutions of $\psi - \phi_s = \pm(\phi_s + \phi_g)$. The solution $\phi_g = -\psi$ implies $\phi_s' = \phi_s$ and generates the case of zero beam current ($I_b = 0$). The solution $\phi_g = \psi - 2\phi_s$ implies the curve of values

$$I_b/I_0 = 2\sin(\psi - \phi_s)\big/\cos\psi . \tag{11}$$

When this curve is plotted in the I_b/I_0 versus ψ plane, then combinations of I_b and ψ above the curve correspond to $U > 1$, while combinations below correspond to $U < 1$. It is evident from the figures, that coherent bucket height is less reduced at small tuning angles.

2.4. Coherent Bucket Shape

The separatrix for coherent motion is called the coherent bucket and is the stability limit for large amplitude motions. The bucket coordinates are the bunch energy with respect to the synchronous energy $(E_b - E_s)$ and the bunch centroid phase (ϕ_b') with respect to the zero crossing of V_g^0. The centre of phase motion is ϕ_s' or ϕ_s with respect to the cavity waveform V_c^0. The maximum extent of oscillations about ϕ_s is $\pi - 2\phi_s'$. The relation of these phases to the bucket extent is made clear in Figure 3.

It will be useful to consider the parameters τ_{rev}, E_s and η to be slowly varying and to introduce the angular frequency Ω_s defined by:

$$\Omega_s^2 = q\,n\,V_0\,h\,\eta\big/\tau_{rev}^2 E_s . \tag{12}$$

Equation (9) is derivable from the function:

$$(\dot\phi_b')^2 - 2\Omega_s^2 U\left\{\cos\phi_b' + \cos\phi_s' + \sin\phi_s'[\phi_b' + \phi_s' - \pi]\right\} = 0 . \tag{13}$$

which condition also gives the bounding stable path in phase space, or bucket. The bucket half-height is given by

$$(\dot\phi_b')|_{max} = \Omega_s\sqrt{2U}[2\cos\phi_s' - \sin\phi_s'(\pi - 2\phi_s')]^{1/2} ,$$

where for given ψ and ϕ_s, the values of ϕ_s' and U are determined from expressions Equations (5) and (6) respectively.

Note, close to threshold, the coherent bucket given by Equation (13) will be correct even if the terms of our initial assumption, that cavity time constant is small compared to the synchrotron period, are not fulfilled; because the coherent oscillation frequency tends toward zero at threshold irrespective of Ω_s.

3. INCOHERENT BUCKET

We shall derive the bounding bucket for incoherent motions of single particles with respect to the bunch centroid. Our derivation does not assume that the bunch centroid is located at the synchronous phase angle. Instead, the derivation manifests the fact that under conditions of beam loading the incoherent bucket moves with the bunch centroid.

3.1. *Incoherent Bucket Stability Criterion*

Let the phase of an individual particle with respect to the bunch centre be ϕ such that the phase with respect to the zero crossing (of unperturbed voltage) is $\phi_b + \phi$. We suppose the individual particle charge to be so small, compared with that of the bunch, that its motion does not affect the disposition of the beam image current phasor $\mathbf{I}_b$.

If the bunch phase is not equal to ϕ_s , then the cavity voltage will deviate from the steady state value; let the new value be $\mathbf{V}_c = V_c e^{j\phi_c}$ where the phase ϕ_c depends on ϕ_b . Let the single particle contribute a current phasor $\delta I = -jqce^{-j(\phi_b+\phi)}$ where q is the charge. Thus, the single particle energy change at the accelerating cavity is the scalar product

$$
\begin{aligned}
c\Delta E(\phi_b + \phi) &= \mathcal{R}[\mathbf{V}_c \,\delta I^*] \\
&= \mathcal{R}[jqcV_c\, e^{j(\phi_c+\phi_b+\phi)}] \\
&= -q\,c\,V_c \sin(\phi_b + \phi_c + \phi)\ .
\end{aligned}
$$

The motion will be unstable if changes in particle phase produce no change in energy increment, that is if $\Delta E(\phi_b + \phi) - \Delta E(\phi_b) = 0$. In the limit of small oscillations, this implies the instability condition

$$\frac{d}{d\phi}\Delta E(\phi_b + \phi)\Big|_{\phi=0} = 0\ .$$

Hence arises the condition

$$\cos(\phi_b + \phi_c) = 0 \quad \text{ or } \quad \phi_b + \phi_c = \pi/2\,,\ 3\pi/2\,, \ldots \tag{14}$$

The incoherent motion is unstable when the perturbed beam current $-\mathbf{I}_b$ and the perturbed cavity voltage $\mathbf{V}_c$ are in phase. This corresponds to the case that the bunch sits on the crest of the perturbed cavity voltage waveform.

Of course, the cavity phase ϕ_c is a function of the beam phase ϕ_b , and so one must find a self-consistent solution of the instability condition. The perturbed cavity voltage is given by

$$V_c e^{j\phi_c} = R\cos\psi e^{j\psi}[\mathbf{I}_g^0 + \mathbf{I}_b]\ .$$

Now at the incoherent stability limit Equation (14) $\phi_b = \pi/2 - \phi_c$ and so $\mathbf{I}_b = -I_b e^{j\phi_c}$. We substitute this value, multiply throughout by $\exp(-j\phi_c)$ and compare imaginary parts to obtain:

$$0 = I_g \sin(\psi + \phi_g - \phi_c) - I_b \sin\psi\ .$$

Now, I_g and I_b are determined by the steady state equation

$$0 = I_g \sin(\psi + \phi_g) - I_b \cos(\psi - \phi_s)\ .$$

Consequently, if unstable solutions $\hat{\phi}_c$ exist, then they must satisfy

$$\frac{\sin(\psi + \phi_g - \hat{\phi}_c)}{\sin(\psi + \phi_g)} = \frac{\sin\psi}{\cos(\psi - \phi_s)}\ .$$

In general, the solutions correspond to large values of ϕ_b ; possibly outside the coherent bucket.

3.2. *Equation for Incoherent Oscillation*

For single particle motion, it is the phase difference between the particle and $\mathbf{V}_c^0$ which counts. However, for incoherent motion, it is the phase difference between the particle and the perturbed cavity voltage $\mathbf{V}_c$ which matters. Accordingly, we define a new variable

$$\tilde{\phi}_s = \phi_b + \phi_c\ . \tag{15}$$

Here $\tilde{\phi}_s$ is the phase difference between the perturbed beam current $-\mathbf{I}_b$ and the (zero crossing of) perturbed cavity voltage $\mathbf{V}_c$. If $\phi_b = \phi_s$, then $\tilde{\phi}_s = \phi_s$. Angle $\tilde{\phi}_s$ for the incoherent motion plays a similar rôle to ϕ_s in single particle formulations of the phase motion; whenever either is equal to $\pi/2$, the motion is unstable.

3.2.1 Non steady state The relative energy change with respect to the bunch centroid is given by:

$$\Delta E(\phi_b + \phi) - \Delta E(\phi_b) = -qV_c[\sin(\tilde{\phi}_s + \phi) - \sin(\tilde{\phi}_s)]\ .$$

So the problem has been reduced to that of finding V_c and $\tilde{\phi}_s$. In fact, it is sufficient to find $V_c \sin\tilde{\phi}_s$ and $V_c\cos\tilde{\phi}_s$ because the the energy change may be written

$$\Delta E - \Delta E_b = -qV_c[\sin\phi\cos\tilde{\phi}_s + (\cos\phi - 1)\sin\tilde{\phi}_s]\ .$$

The perturbed cavity voltage is given by

$$V_c e^{j\phi_c} = \mathbf{V}_c = R\cos\psi e^{j\psi}[\mathbf{I}_g^0 - jI_b e^{-j\phi_b}]\ .$$

The form we want is

$$V_c e^{j\tilde{\phi}_s} = \mathbf{V}_c e^{j\phi_b} = R\cos\psi e^{j\psi}[\mathbf{I}_g^0 e^{j\phi_b} - jI_b]\ . \tag{16}$$

Comparing real and imaginary parts yields

$$
\begin{aligned}
V_c\cos\tilde{\phi}_s &= R\cos\psi[I_g\cos(\psi + \phi_g + \phi_b) + I_b\sin\psi] \\
V_c\sin\tilde{\phi}_s &= R\cos\psi[I_g\sin(\psi + \phi_g + \phi_b) - I_b\cos\psi]\ .
\end{aligned}
$$

Now, if we set $\phi_b = \phi_s$ we shall generate an almost identical pair of equations but with $V_c = V_0$ and $\tilde{\phi}_s = \phi_s$. Eliminating I_b from these two sets of equations, and substituting $\phi_s' = \phi_s + \psi + \phi_g$ and $\phi_b' = \phi_b + \psi + \phi_g$, we find the forms

$$\begin{aligned} V_c \sin\tilde{\phi}_s &= V_0 \{\sin\phi_s + U[\sin\phi_b' - \sin\phi_s']\} \\ V_c \cos\tilde{\phi}_s &= V_0 \{\cos\phi_s + U[\cos\phi_b' - \cos\phi_s']\} , \end{aligned} \tag{17}$$

where the function $U(\phi_s')$ is equal to $(I_g/I_0)\cos\psi$.

For brevity we shall write $E_q = E(\phi_b + \phi)$ to mean the energy of a particle with phase ϕ with respect to the bunch centre, and $E_s = E(\phi_s)$ to mean the energy of the particle which always arrives at synchronous phase. The equation for phase advance is

$$\frac{d}{dt}(\phi_b + \phi) = \frac{2\pi h}{\tau_{rev}}\eta\frac{(E_q - E_s)}{E_s} \quad \text{but} \quad \frac{d}{dt}\phi_b = \frac{2\pi h}{\tau_{rev}}\eta\frac{(E_b - E_s)}{E_s} ,$$

and so upon subtraction of these two equations we find

$$\frac{d}{dt}\phi = \frac{2\pi h}{\tau_{rev}E_s}\eta\frac{(E_q - E_b)}{E_s} .$$

Finally, after incrementing the energy through n cavities and replacing the turn derivative with a time derivative, the equation for incoherent oscillations is

$$\begin{aligned} -\tau_{rev}\frac{d}{dt}\left[\frac{E_s\tau_{rev}}{h\eta}\frac{d}{dt}\phi\right] &= nqV_0\sin\phi\{\cos\phi_s + U[\cos\phi_b' - \cos\phi_s']\} \\ &\quad + nqV_0(\cos\phi - 1)\{\sin\phi_s + U[\sin\phi_b' - \sin\phi_s']\} \\ &= nqV_c[\sin(\tilde{\phi}_s + \phi) - \sin\tilde{\phi}_s] , \end{aligned} \tag{18}$$

where $\tilde{\phi}_s$ and V_c are obtained from Equation (17).

$$\tan\tilde{\phi}_s = \frac{\sin\phi_s + U[\sin\phi_b' - \sin\phi_s']}{\cos\phi_s + U[\cos\phi_b' - \cos\phi_s']} , \tag{19}$$

$$(V_c/V_0)^2 = 1 + 2U[\sin\phi_s(\sin\phi_b' - \sin\phi_s') + \cos\phi_s(\cos\phi_b' - \cos\phi_s')] + 2U^2[1 - \cos(\phi_b' - \phi_s')] . \tag{20}$$

Equation (18) is that for synchrotron oscillations about a stable phase angle $\tilde{\phi}_s$.

3.3. *Quasi-Static Approximation for Incoherent Motion*

In general Equation (18) is a complicated equation to solve; particularly as $\phi_b(t)$ is time dependent and has the effect of forcing the motion in ϕ . However, we note that when the bunch centroid is coincident with the synchronous phase, and $V_c = V_0$, $\tilde{\phi}_s = \phi_s$, then the equation immediately reduces to the simple form

$$\tau_{rev}\frac{d}{dt}\left[\frac{E_s\tau_{rev}}{h\eta}\frac{d}{dt}\phi\right] = -nqV_0[\sin(\phi_s + \phi) - \sin(\phi_s)] . \tag{21}$$

Note, the complete absence of ϕ_s' from Equation (21) implies that the incoherent bucket and the incoherent synchrotron frequency are completely unaffected by beam loading for the case $\phi_b = \phi_s$. By continuity we should expect this to remain approximately true for small excursions $\phi_b \neq \phi_s$. Now when $\phi_s' \approx \pi/2$, $(E_b - E_s)$ must be small if the coherent motion remains within the coherent bucket, and so the slip rate $d\phi_b/dt$ must also be small. Consequently, motion of the bunch centroid will appear 'frozen' compared with the individual particle motion, and the value of ϕ_b can be taken as approximately constant during an oscillation of ϕ .

Note, it is not sufficient to say that close to the condition $\phi_s' = \pi/2$, the coherent synchrotron frequency approaches zero implies the motion is frozen, because the bunch will drift at rate depending on $E_b - E_s$ when there is no longitudinal focusing.

3.3.1 Incoherent motion near to the Robinson stability limit By continuity, we should expect incoherent motion in the vicinity of $\phi_s' = \pi/2$ to be similar to the special case $\phi_s' \equiv \pi/2$ which is simpler to treat. After the substitutions $\phi_s' = \pi/2$ and $\phi_b' = \pi/2 + (\phi_b - \phi_s)$, Equations (20) and (19) take the form

$$(V_c/V_0)^2 = 1 + 2U_0[\sin(2\phi_s - \phi_b) - \sin\phi_s] + 2U_0^2[1 - \cos(\phi_b - \phi_s)]$$

and

$$\tan\tilde{\phi}_s = \frac{\sin\phi_s + U_0[\cos(\phi_b - \phi_s) - 1]}{\cos\phi_s - U_0\sin(\phi_b - \phi_s)}$$

with $U_0 = \cos(\psi - \phi_s)/\sin\psi$.

For the case $\phi_b = \phi_s$ we recover the previous limiting results that $V_c = V_0$ and $\tilde{\phi}_s = \phi_s$. Hence, even at the limit of stability for coherent motions, the beam stays bunched. For small displacements $|\phi_b - \phi_s| \ll 1$ we find that

$$(V_c/V_0)^2 \approx 1 - (\phi_b - \phi_s)2U_0\cos\phi_s \quad \text{and} \quad \tan\tilde{\phi}_s \approx \tan\phi_s[1 + (\phi_b - \phi_s)U_0/\cos\phi_s] .$$

Hence, the incoherent bucket height and shape are only slightly distorted when the bunch moves away from the synchronous phase.

The dynamical behaviour at threshold is complicated but, ultimately, self stabilizing. Suppose, initially, that $\phi_s' = \pi/2$, $\phi_b = \phi_s$ and $E_b \neq E_s$. With zero coherent bucket, the bunch starts to wander; but in doing so the incoherent bucket is slightly perturbed and particles start to redistribute thus slightly deforming the bunch shape. Now the current modulus I_b depends on the bunch shape, and so will decrease as the bunch deforms; but this will bring the beam current below the coherent instability threshold and so a small coherent bucket forms which captures the bunch centroid. The bunch centroid ceases to wander, and further distortion of the incoherent bucket stops also. Consequently, the beam now becomes stable with respect to both coherent and incoherent motions.

Far below threshold, we cannot say that ϕ_b and E_b are slowly changing compared with ϕ and E_q . However, with small beam current, the modulation of the cavity voltage induced by the coherent motion will be small; and so one should expect little modification of the incoherent bucket in this regime also.

3.4. *Incoherent Bucket Shape*

The separatrix for incoherent motion is the incoherent bucket. The bucket coordinates are phase and energy with respect to the bunch centroid, that is ϕ and

$(E_q - E_b)$. The stable point of the motion ($\phi = 0$) is located at $\phi_b(t)$ with respect to the unperturbed waveform V_c^0 , that is the incoherent motion is 'pinned' to the bunch central phase. The incoherent bucket length and shape, however, depends on $\tilde{\phi}_s$. The maximum extent of oscillations about ϕ_b is $\hat{\phi} = \pi - 2\tilde{\phi}_s$.
Equation (18) is derivable from the function:

$$(\dot{\phi})^2 - 2\Omega_s^2(V_c/V_0)\left\{\cos(\tilde{\phi}_s + \phi) + \cos\tilde{\phi}_s + \sin\tilde{\phi}_s[\phi + 2\tilde{\phi}_s - \pi]\right\} = 0 , \quad (22)$$

which condition also gives the bounding stable path in phase space, or bucket. The bucket half-height is given by

$$(\dot{\phi})|_{max} = \Omega_s\sqrt{2V_c/V_0}[2\cos\tilde{\phi}_s - \sin\tilde{\phi}_s(\pi - 2\tilde{\phi}_s)]^{1/2} .$$

4. CONCLUSION

For the case of a beam-loaded radio-frequency acceleration system, we have found the coherent bucket by comparing motion of the bunch centroid with the synchronous particle. In general, the bucket is distorted and large amplitude motions may be unstable well before the Robinson limit is reached.

We have found the general incoherent oscillation equation by comparing motion of a single particle with the bunch centroid. For the case $\phi_b = \phi_s$ and $E_b = E_s$, the incoherent bucket is always identical with the case of no beam loading. For the case of extreme beam loading $\phi_s' \approx \pi/2$, and $\phi_b \approx \phi_s$ and $E_b \approx E_s$, one may define a quasi-static incoherent bucket; and this is found to be slightly distorted.

For the case of no beam loading, the stability of longitudinal motions depends on the synchronous phase ϕ_s. With beam loading we have found that stability of coherent motion depends upon the angle $\phi_s' = \phi_s + \psi + \phi_g$, and the stability of incoherent motion depends upon the angle $\tilde{\phi}_s = \phi_b + \phi_c$. Stability of coherent motion depends upon the partial voltage V_g , whereas stability of incoherent motion depends upon the total cavity voltage V_c .

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3. F. Pedersen: *Beam loading effects in PS Booster*, IEEE Trans. Nuc. Sci. Vol.22, 1975.

<u>figure captions</u>

Fig.1 Schematic of current and voltage phasors.

Fig.2 Coherent bucket shape versus beam loading parameter.

Fig.3 Relation of phases and buckets w.r.t. voltage waves V_g and V_c.

Fig.4a Coherent bucket parameters ϕ_s' and U as a function of beam load and tuning.

Fig.4b Coherent bucket parameters ϕ_s' and U as a function of beam load and tuning.

Fig.4c Coherent bucket parameters ϕ_s' and U as a function of beam load and tuning.

Fig.4d Coherent bucket parameters ϕ_s' and U as a function of beam load and tuning.

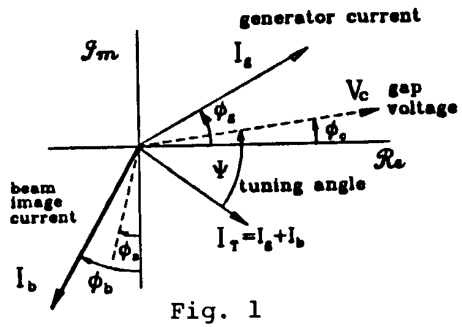


Fig. 1

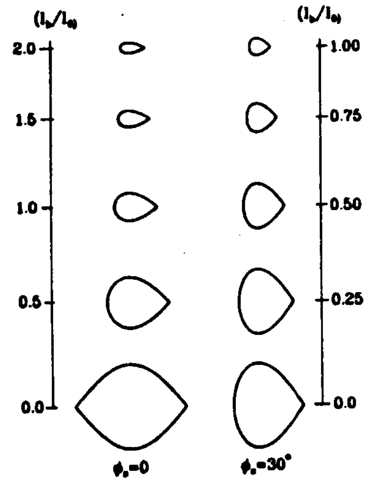


Fig 2

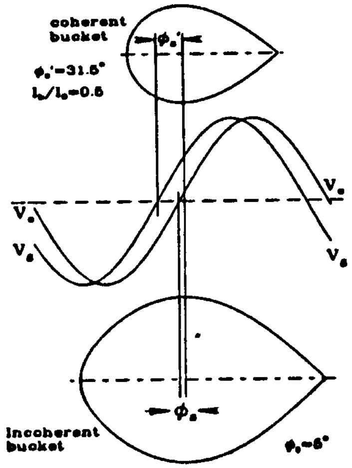
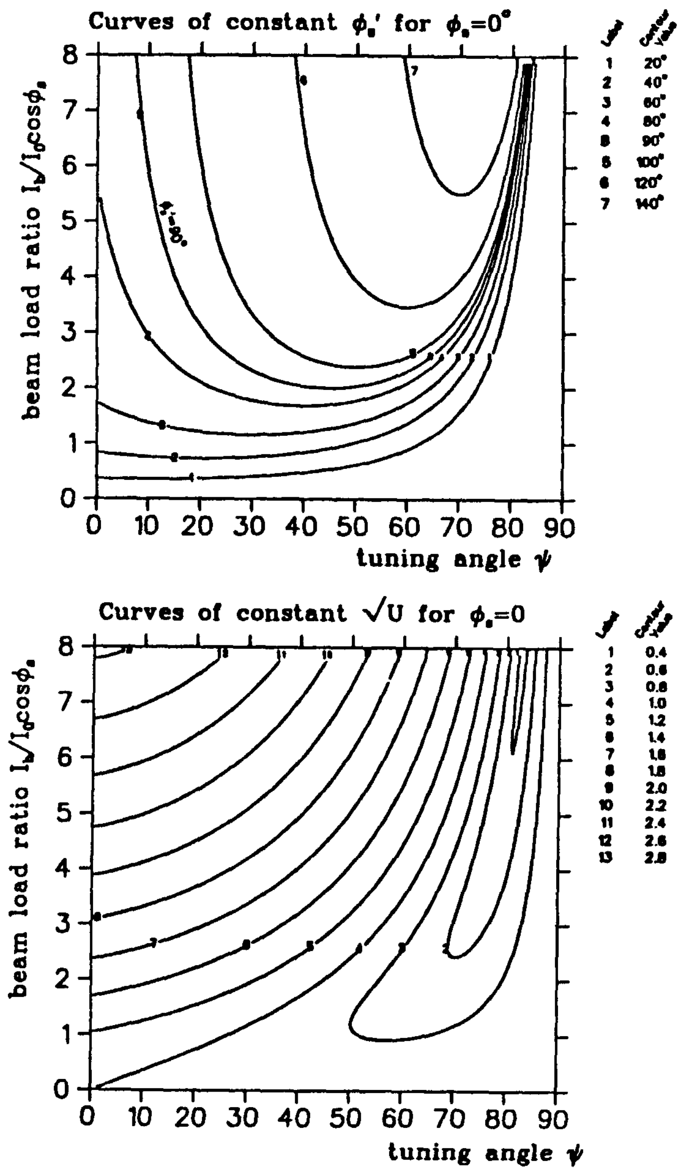
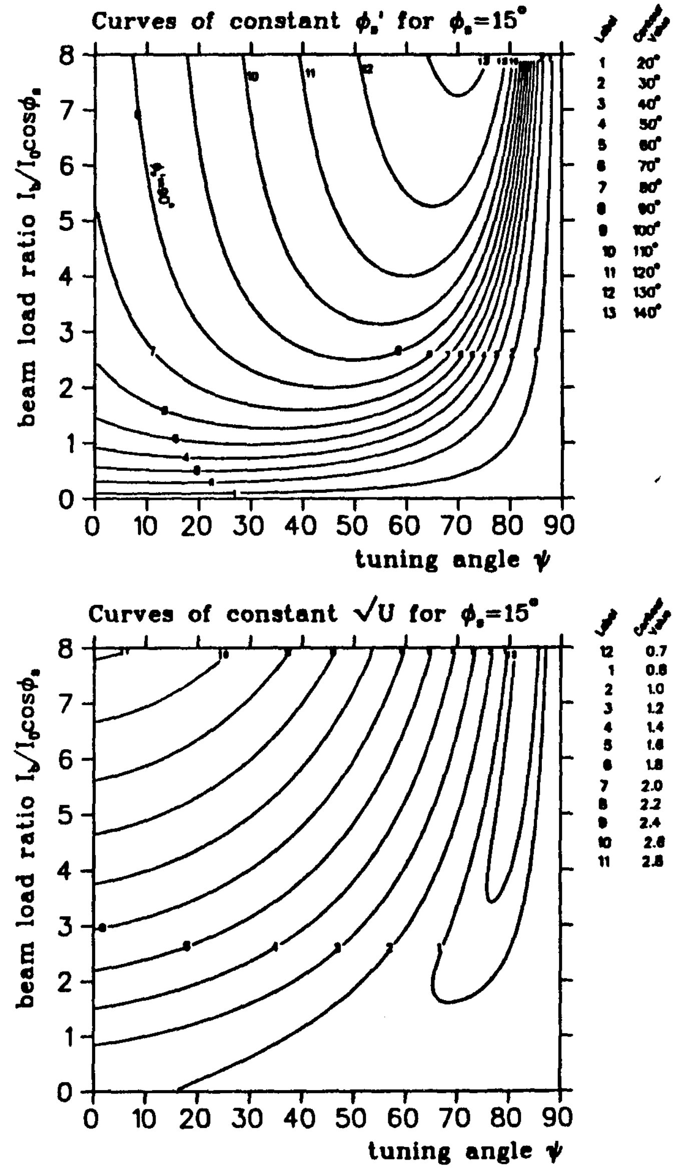


Fig. 3



Curves of constant ϕ_s' for $\phi_s=0°$
beam load ratio $I_b/I_0\cos\phi_s$
tuning angle ψ
Curves of constant $\sqrt{U}$ for $\phi_s=0$
beam load ratio $I_b/I_0\cos\phi_s$
tuning angle ψ

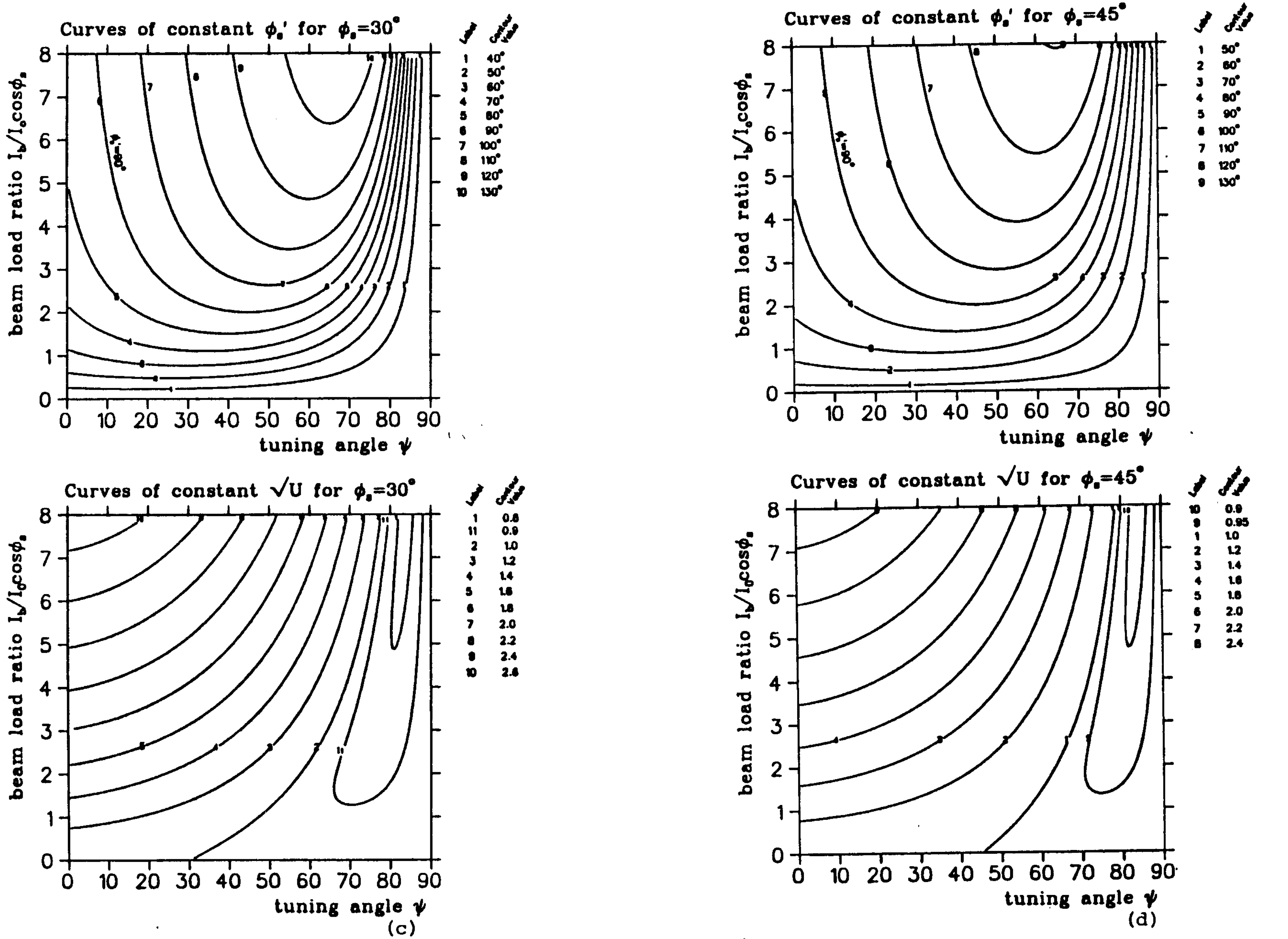
(a)



Curves of constant ϕ_s' for $\phi_s=15°$
beam load ratio $I_b/I_0\cos\phi_s$
tuning angle ψ
Curves of constant $\sqrt{U}$ for $\phi_s=15°$
beam load ratio $I_b/I_0\cos\phi_s$
tuning angle ψ

(b)

Fig. 4



Curves of constant ϕ_s' for ϕ_s=30°
beam load ratio $I_b/I_o\cos\phi_s$
tuning angle ψ
Label / Contour Value
1 40°
2 50°
3 60°
4 70°
5 80°
6 90°
7 100°
8 110°
9 120°
10 130°
ϕ_s'=90°
Curves of constant ϕ_s' for ϕ_s=45°
beam load ratio $I_b/I_o\cos\phi_s$
tuning angle ψ
Label / Contour Value
1 50°
2 60°
3 70°
4 80°
5 90°
6 100°
7 110°
8 120°
9 130°
ϕ_s'=90°
Curves of constant $\sqrt{U}$ for ϕ_s=30°
beam load ratio $I_b/I_0\cos\phi_s$
tuning angle ψ
Label / Contour Value
1 0.8
11 0.9
2 1.0
3 1.2
4 1.4
5 1.6
6 1.8
7 2.0
8 2.2
9 2.4
10 2.6
(c)
Curves of constant $\sqrt{U}$ for ϕ_s=45°
beam load ratio $I_b/I_0\cos\phi_s$
tuning angle ψ
Label / Contour Value
10 0.9
9 0.95
1 1.0
2 1.2
3 1.4
4 1.6
5 1.8
6 2.0
7 2.2
8 2.4
(d)

Fig. 4 (Cont'd.)