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DEPENDENCE OF HOLOMORPHICITY OF THE GAUGE COUPLING CONSTANT ON THE MASS MATRIX IN SUSY THEORIES

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ABSTRACT

We investigate the dependence of holomorphicity of the gauge coupling constant function on the mass matrix at one- and two-loop levels in supersymmetric theories. Gauge invariance puts constraints on the mass matrix. These constraints at one-loop level lead us to three cases of mass matrix that require different ways of regulating the infrared contributions: massive, pseudo massive and intrinsically massless. The first two give rise to a holomorphic gauge coupling constant function whereas the last one does not. Two-loop contributions to super QED and super Yang-Mills theory are calculated using the super background field method and their dependence on the mass matrix is found to fall under the same three cases as at the one-loop level. Remarks concerning the general nature of this result to all orders in perturbation theory are included. Making use of our two-loop results we also verify the holomorphicity of the Wilson coupling based on general arguments of Shifman and Vainshtein.

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I. INTRODUCTION

Recently Dixon *et al* [1] have calculated one-loop threshold correction to $1/g^2$, g being the gauge coupling constant, in orbifold vacua of the heterotic string and in a particular class of renormalizable $N = 1$ supersymmetric (SUSY) theories. They find that this correction is non-holomorphic in its field dependence. Shifman and Vainshtein [2] have discussed case of super QED and show that non-holomorphicity of gauge couplings arises at two-loop level. To make this more explicit and to show what entails for the definition of the effective gauge vacuum angle, consider the action for $N = 1$ supergravity coupled matter and gauge fields [3]:

$$\begin{aligned} \mathcal{A} = &\int d^4x d^4\theta\, E[\Phi(S, \bar{S}e^{2V}) + \mathrm{Re}(\frac{1}{\mathcal{R}}P(S))] \\ &+ \int d^4x d^4\theta\, E\, \mathrm{Re}(\frac{1}{\mathcal{R}}f_{ab}(S)W^aW^b). \end{aligned} \tag{1.1}$$

Indices a and b, in general, indicate different group sectors. E is the superspace determinant. $\mathcal{R}$ is the super curvature. $S = \varphi + \theta\chi + \theta\theta z$ is the chiral superfield, where φ is the bosonic component, χ is the fermionic component and z is an auxiliary field. V is the super gauge field of the group G and one of its bosonic components is the normal gauge field A_μ. $W = \lambda + F_{\mu\nu}\sigma^{\mu\nu}\bar{\theta} + D\theta$ is the the super field strength, where λ is the gaugino field, $F_{\mu\nu}$ is the normal gauge field strength and D is an auxiliary field. $P(S)$ is the superpotential. Φ is an arbitrary real function. $f_{ab}(S)$ are the gauge coupling functions. Here the coupling functions $f_{ab}(S)$ are chiral and hence are analytic function of S (but not $\bar{S}$). In other words, $f_{ab}(S)$ are holomorphic functions of S. As the fermionic part of the action is uniquely determined by its bosonic part, we concentrate on the latter. The bosonic part of the Lagrangian density is given by

$$\begin{aligned} e^{-1}\mathcal{L}_B =& R + \left(\frac{1}{4g^2(\varphi,\bar\varphi)}\right)_{ab} F^a_{\mu\nu}F^{b\mu\nu} + i\frac{\Theta_{ab}(\varphi,\bar\varphi)}{32\pi^2}F^a_{\mu\nu}\tilde F^{b\mu\nu} \\ &+ \frac{1}{2}G_{i\bar j}(\varphi,\bar\varphi)D^\mu\varphi^i D_\mu\bar\varphi^{\bar j} + V(\varphi,\bar\varphi). \end{aligned} \tag{1.2}$$

Here, R is the canonical gravitational curvature, e is the determinant of the space-time metric, D is the covariant derivative, $G_{i\bar j}(\varphi,\bar\varphi)$ is the metric for the scalar fields φ^i, $V(\varphi,\bar\varphi)$ is the potential for them, F is the gauge field strength and $\tilde F$ is its dual. $g_{ab}(\varphi,\bar\varphi)$ are the gauge coupling functions, which can be written as

$$\left(\frac{1}{g^2(\varphi,\bar\varphi)}\right)_{ab} = \mathrm{Re}f_{ab}(\varphi). \tag{1.3}$$

$\Theta_{ab}(\varphi,\bar\varphi)$ are the so-called gauge vacuum angles the derivatives of which give the couplings of axions to gauge field; they are given by

$$\frac{\Theta_{ab}(\varphi,\bar\varphi)}{8\pi^2} = \mathrm{Im}f_{ab}(\varphi). \tag{1.4}$$

Considering the global super gauge theory (SGT) as a derivative of this local supergravity, the couplings in the SGT should also be holomorphic functions of $\langle\varphi^i\rangle$, where $\langle\varphi^i\rangle$ stand for the vacuum expectation values of scalar fields φ^i. When loop corrections are included, Θ_{ab} cannot be obtained directly from Feynman diagrams, because $F^a_{\mu\nu}\tilde F^{b\mu\nu}$ is a total derivative. But one can obtain Θ_{ab} from $\frac{\partial\Theta_{ab}}{\partial\varphi^i}$. We need the following integrability conditions

$$\frac{\partial}{\partial\langle\varphi^j\rangle}\left\{\frac{\partial\Theta_{ab}}{\partial\varphi^i}\right\}_{\text{effective}} = \frac{\partial}{\partial\langle\varphi^i\rangle}\left\{\frac{\partial\Theta_{ab}}{\partial\varphi^j}\right\}_{\text{effective}}, \tag{1.5}$$

at all loop levels to have well-defineded Θ_{ab}. These conditions are only true if holomorphicity holds at all loop levels. Hence it is desirable to have holomorphic $f(\varphi)$ at all loops in order to define the effective Θ_{ab}'s.

We note that if the gauge coupling function $1/g^2(\varphi,\bar\varphi)$ is the real part of a holomorphic function, then $\Theta(\varphi,\bar\varphi)/8\pi^2$ is the imaginary part of the same holomorphic function, the function being $f = \frac{1}{g^2} + i\frac{\Theta}{8\pi^2}$. So it is sufficient to study only the dependence of $1/g^2$ on

the mass matrix in order to determine the holomorphic property of the f-function. But it is found in Ref. [1] that the one-loop threshold correction to $1/g^2$ in SUSY theories is not the real part of a holomorphic function. They attribute this to the presence of the infrared divergence. Motivated by this Derendinger *et al.* [4] have constructed a new supergravity theory in which the coupling functions are non-holomorphic even at tree level; this theory is non-local at tree level.

Our analysis shows that the holomorphicity of $1/g^2$ at one-loop and two-loop levels depends on the structure of the mass matrix, $M = M(\langle\varphi\rangle)$, which in turn is representation dependent. In Sec. II, we study the holomorphic property at one-loop level and emphasize the role of the representation of the mass matrix. Sec. III concentrates on the two-loop holomorphicity for the super QED. We investigate the two-loop holomorphicity for the super Yang-Mills theory in Sec. IV. In Sec. V we verify the holomorphicity of $1/g_W^2$, g_W being the Wilson coupling constant [2]. Sec. VI contains concluding remarks. A brief report of the result is contained in Ref. [5] and many details can be found in Ref. [6].

II. HOLOMORPHICITY AT THE ONE-LOOP LEVEL AND THE MASS MATRIX

A. One-loop calculation

In this section we study the holomorphic property of the gauge coupling constant at the one-loop level in a supersymmetric gauge theory (SGT) coupled to matter. To do this, we need to have an explicit expression for the dependence of the gauge coupling constant on the mass matrix. Previously, the one-loop correction to the gauge coupling constant has been calculated in many places [7]. However, the purpose of all these calculations is to find the β-function, and they are performed in the dimensional regularization scheme or in the zero mass matrix case. For our holomorphicity study, the whole β-function calculation is not necessary; we need only the part that depends on the mass matrix. Furthermore, all calculations have to be carried out in four dimensional space-time since holomorphicity is a four dimensional property. (There is no definition of the Θ-angle in any other space-time dimension.) Therefore, it is necessary for us to redo the one-loop calculation and demonstrate how the holomorphic or non-holomorphic dependence arises.

Super Feynman propagators can be derived from action. Let us start with the general action [7]

$$\begin{aligned} \mathcal{A} = &-\frac{1}{4g^2}\,\mathrm{Tr}\int d^4x d^4\theta [(e^{-V}D^\alpha e^V)\bar{D}^2(e^{-V}D_\alpha e^V)] \\ &+\int d^4x d^4\theta\, \bar{\phi}^{\mathrm{T}} e^V \phi + \int d^4x\,[\,d^2\theta\, P(\phi) + \mathrm{h.c.}]. \end{aligned} \tag{2.1}$$

Here the trace "Tr" is taken on the gauge group; θ is the spinor coordinate, and $d^4\theta$ is the integration in the full spinor space while $d^2\theta$ is in the chiral spinor space; V is the gauge vector super field; ϕ is the matter chiral super field; $P(\phi)$ is the super potential; and D_α and $\bar{D}_{\dot\alpha}$ are covariant spinor derivatives. A renormalizable super potential can have up to the fourth order in ϕ, but since we are only concerned about the mass matrix dependence, we focus on the second order term in $P(\phi)$, the mass term $-\frac{1}{2}\phi^{\mathrm{T}}M\phi$, where M is the mass matrix and superscript "T" is the symbol for the "transpose" of a matrix. The mass matrix cannot be arbitrary. A gauge invariant mass term in action (2.1) must satisfy the constraints

$$T^{\sigma\mathrm{T}}M + MT^\sigma = 0, \tag{2.2}$$

where T^σ are the generators of the gauge group. The consequences of these constraints are given in the next subsection. The needed Faddeev-Popov ghost action [8] is

$$\mathcal{A}_{\mathrm{ghost}} = \mathrm{Tr}\int d^4x d^4\theta \Big[\bar{c}'c - c'\bar{c} + \frac{1}{2}(c' + \bar{c}')[V, c+\bar{c}] + \cdots\Big]$$

Here ghosts c and c' are chiral ($\bar{D}_{\dot\alpha}c = \bar{D}_{\dot\alpha}c' = 0$), while $\bar{c}$ and $\bar{c}'$ are antichiral ($D_\alpha\bar{c} = D_\alpha\bar{c}' = 0$). The gauge fixing term is

$$\mathcal{A}_{\mathrm{gf}} = -\frac{1}{2\alpha g^2}\,\mathrm{Tr}\int d^4x d^4\theta D^2 V \bar{D}^2 V,$$

α being the gauge parameter. For simplicity, we choose the Feynman gauge, $\alpha = 1$. The quadratic part of the action in V is given by (see Appendix A)

$$\mathcal{A}_{2V} = -\frac{1}{4g^2}\,\mathrm{Tr}\int d^4x d^4\theta V\partial^2 V,$$

from which the propagator for the gauge field follows (see Appendix B):

$$\langle V^\sigma(1)V^\rho(2)\rangle = -\frac{ig^2}{\partial^2}\delta_{12}\delta^{\sigma\rho}, \tag{2.3}$$

where we define δ-function $\delta_{12} \equiv \delta^4(\theta_1 - \theta_2)\delta^4(x_1 - x_2)$. The propagators for super matter fields ϕ and $\bar{\phi}$ are given by (see Appendix C):

$$\begin{aligned} \langle\phi(1)\phi^{\mathrm{T}}(2)\rangle &= M^\dagger \frac{i}{\partial^2 - MM^\dagger}\,\bar{D}_1^2\delta_{12} \\ \langle\bar{\phi}(1)\bar{\phi}^{\mathrm{T}}(2)\rangle &= \frac{i}{\partial^2 - MM^\dagger}M\; D_1^2\delta_{12} \\ \langle\phi(1)\bar{\phi}^{\mathrm{T}}(2)\rangle &= \frac{i}{\partial^2 - M^\dagger M}\,\bar{D}_1^2 D_1^2\delta_{12} \\ \langle\bar{\phi}(1)\phi^{\mathrm{T}}(2)\rangle &= \frac{i}{\partial^2 - MM^\dagger}\,D_1^2\bar{D}_1^2\delta_{12}. \end{aligned} \tag{2.4}$$

The ghosts cannot contribute a dependence on the mass matrix as indicated later and hence we do not need to write their propagators explicitly. Feynman rules for vertex operators can be read off from the action (2.1) in a straightforward manner. We denote by Z_V and Z_g the renormalization constants for V-wave function and for gauge coupling g, respectively. As shown in Ref. [7], Z_g is obtained by the relation

$$Z_g Z_V^{\frac{1}{2}} = 1.$$

Therefore, it is sufficient to calculate Z_V in order to get Z_g.

There are eight diagrams that contribute to the self-energy of V at the one-loop level, as shown in Fig. 1.

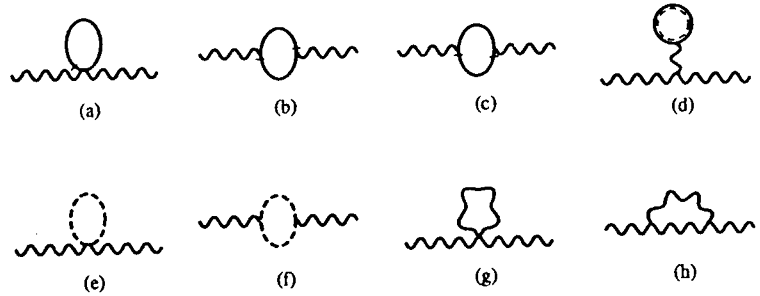

FIG. 1. Self-energy of the gauge field

The dashed line stands for the ghost propagator and we use symbol "▬▬▬" to indicate that it can be a super matter, super gauge or super ghost propagator. The tadpole contribution from Fig. 1(d), can be shown to be zero, and the contributions from Figs. 1(e), 1(f), 1(g) and 1(h) do not concern us, since they do not have a dependence on the mass matrix. So only Figs. 1(a), 1(b) and 1(c) need be considered. In the calculation, we use the constraints $T^{\sigma T}M + MT^{\sigma} = 0$. Then these three diagrams together give

$$\Delta\mathcal{A}_V = \frac{1}{4}\,\mathrm{Tr}\int\frac{d^4q d^4\theta_1}{(2\pi)^4} V_1^{\sigma}(-p)T^{\sigma}T^{\rho}\frac{D_1^{\alpha}\bar{D}_1^2 D_{1\alpha}}{(q^2+M^{\dagger}M)[(q+p)^2+M^{\dagger}M]}V_1^{\rho}(p).$$

The trace here is taken on both the matrix of group generators and the mass matrix. However, we can decompose this trace into a product of two: the trace on the matrix of group generators and the trace on the mass matrix. This is achieved by adopting the result given in Appendix D:

$$\mathrm{Tr}\left(T^{\sigma}T^{\rho}M^{\dagger}M\cdots M^{\dagger}M\right) = \frac{1}{d_R}\,\mathrm{Tr}\left(T^{\sigma}T^{\rho}\right)\times\mathrm{Tr}\left(M^{\dagger}M\cdots M^{\dagger}M\right)$$

d_R being the dimension of representation R. Now, the expression for $\Delta\mathcal{A}_V$ can be rewritten as

$$\Delta\mathcal{A}_V = \frac{1}{4d_R}\,\mathrm{Tr}\int d^4\theta_1 V_1^{\sigma}(-p)T^{\sigma}T^{\rho}D_1^{\alpha}\bar{D}_1^2 D_{1\alpha}V_1^{\rho}(p)$$
$$\times\,\mathrm{Tr}\int\frac{d^4q}{(2\pi)^4}\frac{1}{(q^2+M^{\dagger}M)[(q+p)^2+M^{\dagger}M]}.$$

By comparing the coefficient of the V^2 term in the above equation with the coefficient of the

V^2 term in Eq. (2.1), we identify wave-function renormalization constant for V:

$$Z_V = 1 - ig^2\frac{T_R}{2d_R}\,\mathrm{Tr}\int\frac{d^4q}{(2\pi)^4}\frac{1}{(q^2+M^{\dagger}M)[(q+p)^2+M^{\dagger}M]},$$

where the trace $\mathrm{Tr}(T^{\sigma}T^{\rho}) = T_R\delta^{\sigma\rho}$ has been used. In this paper we take $T_R = 2$ [7]. Now recalling the relation $Z_V^{\frac{1}{2}}Z_g = 1$, we obtain the one-loop correction to $1/g^2$, up to terms independent of M,

$$\Delta\frac{1}{g^2} = -i\frac{T_R}{2d_R}\,\mathrm{Tr}\int\frac{d^4q}{(2\pi)^4}\frac{1}{(q^2+M^{\dagger}M)[(q+p)^2+M^{\dagger}M]}.$$

For our holomorphic study, we can take external momentum $p \to 0$ for simplicity. To regularize the ultraviolet divergence, we need to have a regulator to deal with ultraviolet divergences. Usually the dimensional regularization scheme is used to accomplish this. However, since we are here studying the holomorphic property, which is a four dimensional phenomenon, we have to limit ourselves to four dimensional space-time. Hence the dimensional regularization scheme does not serve our purpose. In fact, the natural choice in this situation is to use the Pauli-Villars [9] regularization scheme for matter fields. (The ultraviolet regulator for other fields are not needed, since the regulator for matter fields curbs all ultraviolet divergences as far as this study is concerned.) This regularization scheme is performed by the propagator replacement

$$\frac{1}{k^2+M^{\dagger}M} \to \left(\frac{1}{k^2+M^{\dagger}M} - \frac{1}{k^2+\Lambda^{\dagger}\Lambda}\right), \tag{2.5}$$

where Λ is the matrix of ultraviolet momentum cutoff. Implementing this in Eq. (2.5), we get

$$\Delta\frac{1}{g^2} = -i\frac{T_R}{2d_R}\,\mathrm{Tr}\int\frac{d^4q}{(2\pi)^4}\frac{(\Lambda^{\dagger}\Lambda)^2}{(q^2+M^{\dagger}M)^2(q^2+\Lambda^{\dagger}\Lambda)^2}. \tag{2.6}$$

This is an expression in Minkowski space. This integral is ultravioletly finite, but its infrared property depends on the mass matrix M and also on the regulator Λ. The gauge invariance gives constraints on both mass matrix and ultraviolet cutoff: $T^{\sigma T}M + MT^{\sigma} = 0$ and

$T^{\sigma T}\Lambda + \Lambda T^{\sigma} = 0$. We need to know the consequences of these constraints. For different representations of the gauge group, we have different representations of the mass matrix. We discuss the representations of the mass matrix in the next section before we integrate Eq. (2.6) and study its holomorphic property.

B. The mass matrix

The holomorphicity of the gauge coupling is dependent on the representation of the mass matrix. In the following discussion, we focus on non-Abelian groups. But as we see, results for Abelian groups can be found trivially.

It follows from Eq. (2.2) (see Appendix D) that if the representation R of the gauge group is irreducible, then the mass matrix M is either trivially zero or all its modes are massive ($\det M \neq 0$). Thus, to have a general mass matrix containing both massive and massless modes, we have to go to a reducible representation. In general, although a reducible representation can contain real, pseudo-real and complex types of irreducible representations, as shown in Appendix D, different types of representations are trivially decoupled. Therefore, we can study each type of representation separately without loss of generality.

A real or pseudo-real representation R of the gauge group can be simplified to have the form (see Appendix D)

$$G(R) = \mathrm{diag}\{\overbrace{G_r,\ G_r,\ \ldots,\ G_r}^{l}\}, \tag{2.7}$$

where the submatrix G_r is an irreducible $n \times n$ real or pseudo-real representation. We assume that we have a total of l G_r's. The conditions (2.2) put constraints on the mass matrix, and the solution to the constraints takes the following form

$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1l} \\ a_{21} & a_{22} & \cdots & a_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ a_{l1} & a_{l2} & \cdots & a_{ll} \end{pmatrix} \otimes J.$$

Here $n \times n$ dimensional matrix J is given by $G_r = J G_r^* J^{-1}$, and a_{ij}'s could be arbitrary complex numbers. For the real representation, we have $J = I$, the unit matrix, and $a_{ij} = a_{ji}$,

so that the whole mass matrix is symmetric. For the pseudo-real representation, we have $J^2 = -1$ and $J^{\mathrm{T}} = -J$. The matrix elements $a_{ij} = -a_{ji}$ so that we have an overall symmetric mass matrix.

A complex representation R of the gauge group can be transformed to have the form

$$G(R) = \mathrm{diag}\{\overbrace{G_c,\ G_c,\ \ldots,\ G_c}^{l},\ \overbrace{G_c^*,\ G_c^*,\ \ldots,\ G_c^*}^{\bar{l}}\}, \tag{2.8}$$

where G_c is an irreducible $n \times n$ complex representation and G_c^* is its complex conjugate. We assume that we have a total of l G_c's and $\bar{l}$ G_c^*'s, and we call l as the number of n-dimensional families of chiral fields and $\bar{l}$ as the number of $\bar{n}$-dimensional antifamilies of chiral fields. l and $\bar{l}$ are not necessarily equal to each other. For this complex representation, the gauge invariant constraints require the mass matrix taking the form

$$M = \begin{pmatrix} & 0 & & b_{1l+1} & \cdots & b_{1l+\bar{l}} \\ & & & \vdots & \ddots & \vdots \\ & & & b_{ll+1} & \cdots & b_{ll+\bar{l}} \\ b_{l+11} & \cdots & b_{l+1l} & & & \\ \vdots & \ddots & \vdots & & 0 & \\ b_{l+\bar{l}1} & \cdots & b_{l+\bar{l}l} & & & \end{pmatrix} \otimes I,$$

where the matrix elements b_{ij} are arbitrary complex numbers with $b_{ij} = b_{ji}$, and I is an $n \times n$ unit matrix.

Massive case: For all the three cases of real, pseudo-real and complex representations, the mass matrix M may have zero eigenvalues, or massless modes. Depending on whether the mass matrix has massless modes or not, the integral in Eq. (2.6) behaves differently. A mass matrix is called *massive* if all its eigenvalues are non-zero. For the massive mass matrix, since M^{-1} exists, M naturally serves as an infrared regulator in the calculation of the one-loop correction to $1/g^2$.

Pseudo massive and intrinsically massless cases: For a mass matrix with at least one zero mode (MMWZ), we have to distinguish different cases. For a real or pseudo-real representation, we may have a mass matrix with massless modes. But due to the arbitrariness of the matrix elements a_{ij} (except for the symmetric conditions), we can always perturb them (*i.e.*, change them by infinitesimal amounts) so that all the modes in the perturbed matrix M_p are massive. Since we can manage to have the perturbed mass term gauge invariant, this perturbed mass matrix can be used as the infrared regulator for the matter sector. We call this type of mass matrix ***pseudo massive***. For a complex representation, there are two different types of MMWZ: (i) The type when the number l of G_c in $G(R)$ is equal to the number $\bar{l}$ of G_c^* in $G(R)$, or in other words, the numbers of families and antifamilies are balanced. We can easily see that this case is similar to the real case discussed above, and we can perturb the mass matrix such that M_p is massive. This is, again, the so-called pseudo massive case. (ii) The type when $l \neq \bar{l}$, *i.e.*, the numbers of families and antifamilies are unbalanced. In this case, we can see that the rank of the matrix (*i.e.*, the number of non-zero modes) is smaller than the dimension of the matrix, $(l+\bar{l})\cdot n$. Thus, we have some massless modes no matter how we perturb the matrix elements b_{ij}. Since in this case, the perturbed mass matrix can never be massive, we call the mass matrix *intrinsically massless*. This only happens for the complex representation. For the pseudo massive case, we can perturb matrix elements a_{ij} or b_{ij}, so that M_p has no zero modes and we can adopt it as the infrared regulator. Since M_p^{-1} exists, we can pick the ultraviolet cutoff Λ so that Λ^{-1} exists. On the other hand, in the intrinsically massless case, M_p^{-1} does not exist and cannot be used as an infrared regulator; hence we have to deal with the infrared divergence with a momentum cutoff.

C. Holomorphicity of $1/g^2$

Now we perform the integration in Eq. (2.6). For a massive matrix M, M^{-1} exists and so does Λ^{-1}. Rescale q^2 by

$$q^2 \rightarrow \frac{q^2}{\Lambda^\dagger \Lambda}.$$

Eq. (2.6) becomes

$$\Delta\frac{1}{g^2} = \frac{T_R}{2d_R}\,\mathrm{Tr}\int\frac{q^2dq^2}{16\pi^2}\frac{1}{(q^2+a^\dagger a)^2(q^2+1)^2}, \tag{2.9}$$

where we have moved into Euclidean space by replacing q_0 with iq_4, and defined $a \equiv \frac{M}{\Lambda}$. The integration gives

$$\begin{aligned}\Delta\frac{1}{g^2} &= -\frac{T_R}{2d_R}\,\mathrm{Tr}\left(\frac{1}{16\pi^2}\right)\ln(a^\dagger a)\\ &= -\frac{T_R}{2d_R}\,\mathrm{Tr}\left(\frac{1}{16\pi^2}\right)\ln\left(\frac{M^\dagger}{\Lambda^\dagger}\right) - \frac{T_R}{2d_R}\,\mathrm{Tr}\left(\frac{1}{16\pi^2}\right)\ln\left(\frac{M}{\Lambda}\right),\end{aligned} \tag{2.10}$$

where higher order terms in $(a^\dagger a)$ and contributions independent of $(a^\dagger a)$ have been ignored. The r.h.s. of Eq. (2.10) is the real part of

$$-\frac{T_R}{d_R}\,\mathrm{Tr}\left(\frac{1}{16\pi^2}\right)\ln\frac{M}{\Lambda},$$

which is holomorphic in M $(= M(\langle\varphi\rangle))$.

For the pseudo massive case, we add a perturbative matrix ϵ to M by defining $M_p = M + \epsilon$, and adopt M_p as the infrared regulator while keeping the perturbed mass term gauge invariant. We get

$$\Delta\frac{1}{g^2} = -\frac{T_R}{2d_R}\,\mathrm{Tr}\left(\frac{1}{16\pi^2}\right)\ln\left(\frac{M_p^\dagger M_p}{\Lambda^\dagger\Lambda}\right).$$

This is, again, the real part of a holomorphic function as long as $\epsilon \neq 0$. Hence, the holomorphic dependence on the mass is still true for the pseudo massive case. Therefore, we conclude that, at one-loop level, MMWZ does not necessarily mean non-holomorphicity for $1/g^2$; and holomorphicity holds for both the massive and pseudo massive cases.

Finally, we discuss the intrinsically massless mass matrix case. In this case, we have a complex representation with $l \neq \bar{l}$, in other words, the number of the families for chiral fields, is not equal to the number of the antifamilies. We know that, because of the unbalanced numbers of families and antifamilies, we cannot perturb the mass matrix M so that it becomes massive while the mass term is still gauge invariant. This means that the perturbed mass matrix cannot be used as the infrared regulator. Therefore, to integrate Eq. (2.6), we have to put in an infrared momentum cutoff, p_0. This results in

$$\Delta\frac{1}{g^2} = -\frac{T_R}{2d_R}\lim_{p_0\to 0}\mathrm{Tr}\left(\frac{1}{16\pi^2}\right)\Big[\ln\left(p_0^2 + M^\dagger M\right) + \text{const.}\Big], \qquad (2.11)$$

where "const." stands for terms that are independent of M. Since we cannot factorize $p_0^2 + M^\dagger M$, the above expression is not the real part of a holomorphic function. An example of this is the SGT based on the E_6 group with two 27-dimensional families of chiral fermions and one $\bar{27}$-dimensional antifamily considered in ref. [1]. Here the mass matrix is

$$\begin{pmatrix} 0 & 0 & \langle\varphi_1\rangle \\ 0 & 0 & \langle\varphi_2\rangle \\ \langle\varphi_1\rangle & \langle\varphi_2\rangle & 0 \end{pmatrix}. \qquad (2.12)$$

The calculated one-loop correction to $1/g^2$ is given by [1]

$$\Delta\frac{1}{g^2} = -\frac{6}{16\pi^2}\Big[\ln\left[|\langle\varphi^1\rangle|^2 + |\langle\varphi^2\rangle|^2 + O(p_0^2)\right] + \text{const.}\Big]. \qquad (2.13)$$

Since this $\Delta\frac{1}{g^2}$ cannot be expressed as the sum of a holomorphic function of $\langle\varphi^1\rangle$ and $\langle\varphi^2\rangle$ and its complex conjugate, one-loop correction is not holomorphic.

Thus the above analysis shows that the one-loop correction to $1/g^2$ is holomorphic for the massive and pseudo massive mass matrix cases and it is non-holomorphic for the intrinsically massless mass matrix case. The latter case arises for a complex group representation with unbalanced numbers of families and antifamilies.

III. TWO-LOOP HOLOMORPHICITY IN SUPER QED

We first study the two-loop corrections to $1/g^2$ for the simple Abelian case of super QED with a reducible representation for ϕ having $l = \bar{l} = 1$, and then consider a general representation. This would facilitate the general discussion for a non-Abelian group given in the next section. The use of the super background field method (SBFM) simplifies two-loop calculations. We first briefly review SBFM.

A. Super background field method

In the background field method (BFM) each field is split into a background part and a quantum part, and then all the quantum fields are functionally integrated out. The background fields are kept untouched; and we obtain an effective action for the background fields. The gauge field is split up in such a way that the action is both background and quantum gauge invariant. The gauge fixing term is chosen to be background gauge invariant. Therefore, the final effective action for the background fields is guaranteed to be background gauge invariant. Furthermore, to maintain both the initial action and the final effective action to be background gauge invariant, the following condition has to be satisfied [10]:

$$Z_V^{\frac{1}{2}} Z_g = 1.$$

This means that in BFM, one only needs to calculate Z_V in order to get the correction to $1/g^2$. This makes things much simpler.

In the super background field method (SBFM) [8], we, again, split fields into background parts and quantum parts and then integrate out the quantum parts. The difference of SBFM from the regular BFM is that the former method is supersymmetric and its background-quantum splitting is, as we will show, nonlinear. For convenience, we use subscript "t" for the "total" field, to distinguish it from the background and quantum fields. We split the vector super field V_t into the quantum part V and the background part Ω, according to

$$e^{V_t} = e^{\Omega} e^{V} e^{-\bar{\Omega}}.$$

This is the only way to maintain the action to be both background and quantum gauge invariant [11]. Accordingly, the background covariant derivatives are given by

$$\mathcal{D}^\alpha = e^{-\Omega} D^\alpha e^{\Omega}, \quad \bar{\mathcal{D}}^{\dot\alpha} = e^{-\bar\Omega} \bar{D}^{\dot\alpha} e^{\bar\Omega},$$

with the super background strengths defined as

$$W_\alpha \equiv -\frac{1}{2i}[\bar{\mathcal{D}}^{\dot\alpha}, \{\bar{\mathcal{D}}_{\dot\alpha}, \mathcal{D}_\alpha\}], \quad \bar{W}_{\dot\alpha} \equiv -\frac{1}{2i}[\mathcal{D}^\alpha, \{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\alpha}\}].$$

The background chiral super field ϕ and antichiral super field $\bar{\phi}$ are defined by

$$\bar{\mathcal{D}}^{\dot{\alpha}}\phi = 0, \quad \mathcal{D}^{\alpha}\bar{\phi} = 0,$$

respectively. They can be related with the original "total" fields by

$$\phi = e^{-\bar{\Omega}}\phi_t, \quad \bar{\phi} = e^{-\Omega}\bar{\phi}_t.$$

After the splittings, the action in Eq. (2.1) can be rewritten as [8]

$$\begin{aligned}\mathcal{A} = &-\frac{1}{4g^2}\mathrm{Tr}\int d^4x d^4\theta \left(e^{-V}\mathcal{D}^{\alpha}e^{V}\right)[\bar{\mathcal{D}}^{\dot{\alpha}}, \{\bar{\mathcal{D}}_{\dot{\alpha}}, e^{-V}\mathcal{D}_{\alpha}e^{V}\}] \\ &+\int d^4x d^4\theta\, \bar{\phi}^{\mathrm{T}}e^{V}\phi - \frac{1}{2}\int d^4x\,[d^2\theta\,\phi^{\mathrm{T}}M\phi + \mathrm{h.c.}],\end{aligned} \tag{3.1}$$

in terms of background covariant derivatives and commutators. This action has both quantum and background gauge invariance. The background invariant gauge fixing term (in Feynman gauge) is given by [8]

$$\mathcal{A}_{\mathrm{gf}} = -\frac{1}{4g^2}\mathrm{Tr}\int d^4x d^4\theta \left(V[\mathcal{D}^2, [\bar{\mathcal{D}}^2, V]] + V[\bar{\mathcal{D}}^2, [\mathcal{D}^2, V]]\right).$$

This term is background gauge invariant, but not quantum gauge invariant. By taking the background fields to be zero, we come back to the expressions for the regular action and regular gauge fixing term, which have been given in the previous section. Now the Feynman rules for the quantum fields V and ϕ follow.

B. Super QED in the simplest representation

Before we go to the general representation of the mass matrix in super QED, we first discuss the simplest case where $l = \bar{l} = 1$. This is the representation of super QED that has

been frequently used since it is simple and physical. The representation of the U(1) group is of the form [8]:

$$G(R) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix},$$

Then the representation of the mass matrix has to take the form of

$$M = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}, \tag{3.2}$$

in order to make the mass term gauge invariant. The matter field is denoted as

$$\phi = \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix}.$$

The super gauge field, V_R, has the form

$$e^{V_R} = \begin{pmatrix} e^{V} & 0 \\ 0 & e^{-V} \end{pmatrix}.$$

The action can be easily simplified as

$$\begin{aligned}\mathcal{A} = &-\frac{1}{2g^2}\int d^4x d^4\theta\, D^{\alpha}V\bar{D}^2D_{\alpha}V \\ &+\int d^4x d^4\theta\,[\bar{\phi}_+\phi_+ + \bar{\phi}_-\phi_- + V(\bar{\phi}_+\phi_+ - \bar{\phi}_-\phi_-) + \frac{1}{2}V^2(\bar{\phi}_+\phi_+ + \bar{\phi}_-\phi_-) + \cdots] \\ &-\left[m\int d^4x d^2\theta\,\phi_+\phi_- + \mathrm{h.c.}\right].\end{aligned} \tag{3.3}$$

The background invariant gauge fixing term is

$$\mathcal{A}_{\mathrm{gf}} = -\frac{1}{4g^2}\mathrm{Tr}\int d^4x d^4\theta \left(V_R[\mathcal{D}^2, [\bar{\mathcal{D}}^2, V_R]] + V_R[\bar{\mathcal{D}}^2, [\mathcal{D}^2, V_R]]\right).$$

The covariant derivative $\mathcal{D}^{\alpha}$ can be written as $D^{\alpha} - i\Gamma^{\alpha}$, where Γ^{α} are the super connections.

From the commuting property of the U(1) group we can easily prove that

$$\mathcal{A}_{gf} = -\frac{1}{2g^2}\int d^4x d^4\theta\, V(D^2\bar{D}^2 + \bar{D}^2 D^2)V.$$

Adding $\mathcal{A}$ and $\mathcal{A}_{gf}$ together, we get

$$\begin{aligned}\mathcal{A} + \mathcal{A}_{gf} = &-\frac{1}{2g^2}\int d^4x d^4\theta\, V\Box_0 V \\ &+\int d^4x d^4\theta\,[\bar{\phi}_+\phi_+ + \bar{\phi}_-\phi_- + V(\bar{\phi}_+\phi_+ - \bar{\phi}_-\phi_-) + \frac{1}{2}V^2(\bar{\phi}_+\phi_+ + \bar{\phi}_-\phi_-) + \cdots] \\ &-\left[m\int d^4x d^2\theta\,\phi_+\phi_- + \text{h.c.}\right].\end{aligned} \tag{3.4}$$

From the quadratic part of the action, we can construct the Feynman propagators for matter fields $\phi_\pm$ and $\bar{\phi}_\pm$, and gauge field V. The super matter propagators are (see Appendix C)

$$\begin{aligned}\langle\bar{\phi}_\pm(1)\phi_\pm(2)\rangle &= \frac{i}{\Box_+ - m^*m}D_1^2\bar{D}_1^2\delta_{12}, \\ \langle\phi_+(1)\phi_-(2)\rangle &= \frac{im^*}{\Box_- - m^*m}\bar{D}^2\delta_{12}, \\ \langle\bar{\phi}_+(1)\bar{\phi}_-(2)\rangle &= \frac{im}{\Box_+ - m^*m}D^2\delta_{12}.\end{aligned} \tag{3.5}$$

The super gauge propagator is

$$\langle V(1)V(2)\rangle = -\frac{i}{\Box_0}\delta_{12}. \tag{3.6}$$

In the above equations

$$\begin{aligned}\Box_+ &\equiv \Box_0 - i\Gamma^a\partial_a - \frac{1}{2}i(\partial^a\Gamma_a) - \frac{1}{2}\Gamma^a\Gamma_a - \frac{1}{2}i(D^\alpha W_\alpha) - iW^\alpha D_\alpha, \\ \Box_- &\equiv \Box_0 - i\Gamma^a\partial_a - \frac{1}{2}i(\partial^a\Gamma_a) - \frac{1}{2}\Gamma^a\Gamma_a - \frac{1}{2}i(\bar{D}^{\dot\alpha}\bar{W}_{\dot\alpha}) - i\bar{W}^{\dot\alpha}\bar{D}_{\dot\alpha},\end{aligned}$$

where the super background connections Γ^a are defined by $\mathcal{D}^a = D^a - i\Gamma^a$, with $a \equiv \alpha\dot\alpha$. The background field strengths in super QED are

$$W_\alpha \equiv -\frac{1}{2i}[\bar{\mathcal{D}}^{\dot\alpha}, \{\bar{\mathcal{D}}_{\dot\alpha}, \mathcal{D}_\alpha\}] = \bar{D}^2\Gamma_\alpha, \quad \bar{W}_{\dot\alpha} \equiv -\frac{1}{2i}[\mathcal{D}^\alpha, \{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\alpha}\}] = D^2\bar{\Gamma}_{\dot\alpha}.$$

For loop calculations in the SBFM, the first step is to find all possible vacuum diagrams for quantum fields, and then expand the quantum propagators for V and ϕ fields

around their external background fields. Any two-loop vacuum diagrams that do not have a dependence on the mass matrix will not concern us. There are four relevant diagrams as shown in Fig. 2. We use the symbol "......" to indicate the propagator of a quantum matter field, quantum gauge field or quantum ghost field. The wavy lines stand for the propagators of the quantum gauge fields. The straight solid lines stand for the propagators of the quantum matter fields.

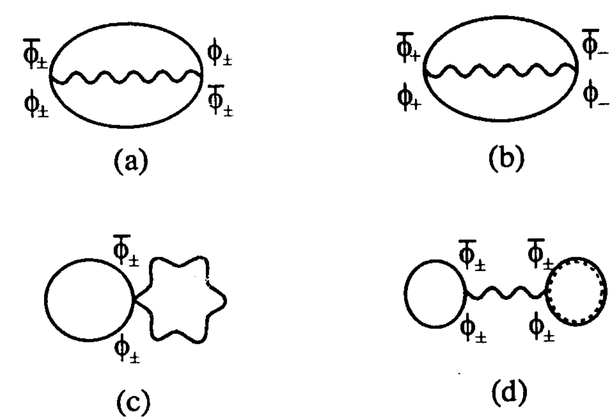

FIG. 2. Four two-loop vacuum Feynman diagrams

For each diagram, we write down quantum matter fields, ϕ and $\bar{\phi}$, explicitly to indicate thei locations in the diagram. Each of the solid line in Fig. 2 stands for a propagator proportiona to either $\frac{1}{\Box_+ - m^*m}$, or $\frac{1}{\Box_- - m^*m}$. Since the external fields Γ, $\bar{\Gamma}$, W and $\bar{W}$ are hidden in th propagators $\frac{1}{\Box_+ - m^*m}$ and $\frac{1}{\Box_- - m^*m}$, we need to expand the propagators to get the explici dependence on the external fields. To do the expansion for $\frac{1}{\Box_+ - m^*m}$, we need to use th expression for $\Box_+$

$$\Box_+ = \Box_0 - i\Gamma^a\partial_a - \frac{1}{2}i(\partial^a\Gamma_a) - \frac{1}{2}\Gamma^a\Gamma_a - \frac{1}{2}i(D^\alpha W_\alpha) - iW^\alpha D_\alpha,$$

and its commutation with D_α

$$[D_\alpha, \Box_+] = \bar{W}^{\dot\alpha}D_{\alpha\dot\alpha} + \frac{1}{2}D_{\alpha\dot\alpha}\bar{W}^{\dot\alpha} - i(D_\alpha W^\beta)D_\beta + \frac{i}{2}D^2 W_\alpha.$$

The expansion of $\frac{1}{\Box_+ - m^*m}$ is straightforward but tedious. By keeping terms up to secon

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order in Γ, $\bar{\Gamma}$, W and $\bar{W}$, we have the following result

$$
\begin{aligned}
\frac{1}{\Box_+ - m^*m} = {}& \\
& \frac{1}{\Box_0 - m^*m} + \frac{1}{\Box_0 - m^*m}\left[i\Gamma^a\partial_a + \frac{i}{2}(\partial^a\Gamma_a)\right]\frac{1}{\Box_0 - m^*m} \\
& + \frac{1}{\Box_0 - m^*m}\left[\frac{i}{2}(D^\alpha W_\alpha)\right]\frac{1}{\Box_0 - m^*m} + \frac{1}{\Box_0 - m^*m}\frac{1}{2}\Gamma^a\Gamma_a\frac{1}{\Box_0 - m^*m} \\
& - \frac{1}{\Box_0 - m^*m} iW^\alpha \frac{1}{\Box_0 - m^*m}\left[\bar{W}^{\dot\alpha}D_{\alpha\dot\alpha} + (D_{\alpha\dot\alpha}\bar{W}^{\dot\alpha})\right]\frac{1}{\Box_0 - m^*m} \\
& + \frac{1}{\Box_0 - m^*m}\left[i\Gamma^a\partial_a + \frac{i}{2}(\partial^a\Gamma_a)\right]\frac{1}{\Box_0 - m^*m}\left[i\Gamma^b\partial_b + \frac{i}{2}(\partial^b\Gamma_b)\right]\frac{1}{\Box_0 - m^*m} \\
& + \frac{1}{\Box_0 - m^*m}\left[i\Gamma^a\partial_a + \frac{i}{2}(\partial^a\Gamma_a)\right]\frac{1}{\Box_0 - m^*m}\frac{i}{2}(D^\alpha W_\alpha)\frac{1}{\Box_0 - m^*m} \\
& + \frac{1}{\Box_0 - m^*m}\frac{i}{2}(D^\alpha W_\alpha)\frac{1}{\Box_0 - m^*m}\left[i\Gamma^a\partial_a + \frac{i}{2}(\partial^a\Gamma_a)\right]\frac{1}{\Box_0 - m^*m} \qquad (3.7)\\
& + \frac{1}{\Box_0 - m^*m}\frac{i}{2}(D^\alpha W_\alpha)\frac{1}{\Box_0 - m^*m}\frac{i}{2}(D^\beta W_\beta)\frac{1}{\Box_0 - m^*m} + \frac{1}{\Box_0 - m^*m} iW^\alpha \frac{1}{\Box_0 - m^*m} D_\alpha \\
& + \frac{1}{\Box_0 - m^*m} iW^\alpha \frac{1}{\Box_0 - m^*m} i(D_\alpha W^\beta)\frac{1}{\Box_0 - m^*m} D_\beta \\
& + \frac{1}{\Box_0 - m^*m}\left[i\Gamma^a\partial_a + \frac{i}{2}(\partial^a\Gamma_a)\right]\frac{1}{\Box_0 - m^*m} iW^\alpha \frac{1}{\Box_0 - m^*m} D_\alpha \\
& + \frac{1}{\Box_0 - m^*m}\frac{i}{2}(D^\alpha W_\alpha)\frac{1}{\Box_0 - m^*m} iW^\beta \frac{1}{\Box_0 - m^*m} D_\beta \\
& + \frac{1}{\Box_0 - m^*m} iW^\alpha \frac{1}{\Box_0 - m^*m}\left[i\Gamma^a\partial_a + \frac{i}{2}(\partial^a\Gamma_a)\right]\frac{1}{\Box_0 - m^*m} D_\alpha \\
& + \frac{1}{\Box_0 - m^*m} iW^\alpha \frac{1}{\Box_0 - m^*m}\frac{i}{2}(D^\beta W_\beta)\frac{1}{\Box_0 - m^*m} D_\alpha \\
& + \frac{1}{\Box_0 - m^*m} iW^\alpha \frac{1}{\Box_0 - m^*m} iW^\beta \frac{1}{\Box_0 - m^*m} D_\beta D_\alpha,
\end{aligned}
$$

where we have used the relation $(D^2 W_\alpha) = -i(D_{\alpha\dot\alpha}\bar{W}^{\dot\alpha})$. We can see that this expansion only depends on W^α, $\bar{W}^{\dot\alpha}$ and Γ^a, and their derivatives, but not on Γ^α and $\bar{\Gamma}^{\dot\alpha}$ [8] (notice that $\Gamma^a = \bar{\Gamma}^a = \Gamma^{\alpha\dot\alpha}$). The expansion for $\frac{1}{\Box_- - m^*m}$ is similar, except that we need to replace D^α by $\bar{D}^{\dot\alpha}$ and W^α by $\bar{W}^{\dot\alpha}$. But as we will see, it is not necessary to write down the expansion terms for $\frac{1}{\Box_- - m^*m}$.

We now argue that Figs. 2(b) and 2(c) give zero contribution. For Fig. 2(b), the form of derivatives acting on δ_{12} can be one of the following three:

$$
\begin{gathered}
\delta_{12}\bar{D}_1^2\delta_{12}D_1{}^2\delta_{12}, \\
\delta_{12}D_{1\alpha}\bar{D}_1^2\delta_{12}D_1{}^2\delta_{12}, \\
\delta_{12}D_{1\alpha}D_{1\beta}\bar{D}_1^2\delta_{12}D_1{}^2\delta_{12}.
\end{gathered}
$$

Using the following properties for δ-function and commutator of D and $\bar{D}$,

$$\delta_{12}\bar{D}_1^2 D_1{}^2\delta_{12} = \delta_{12}D_1{}^2\bar{D}_1^2\delta_{12} = \delta_{12}, \qquad (3.8)$$

$$\delta_{12}D_1{}^m\bar{D}_1^n\delta_{12} = 0, \quad \text{for} \quad n + m < 4, \qquad (3.9)$$

$$\{D_\alpha,\ \bar{D}_{\dot\alpha}\} = iD_{\alpha\dot\alpha}, \qquad (3.10)$$

we can see that there are not enough number of D's and $\bar{D}$'s (–there need to be eight) to make Fig. 2(b) non-zero. Similarly, for Fig. 2(c), the form of derivatives acting on δ_{11} can be one of the following three:

$$
\begin{gathered}
\delta_{11}\bar{D}_1^2 D_1{}^2\delta_{11}, \\
\delta_{11}D_{1\alpha}\bar{D}_1^2 D_1{}^2\delta_{11}, \\
\delta_{11}D_{1\alpha}D_{1\beta}\bar{D}_1^2 D_1{}^2\delta_{11}.
\end{gathered}
$$

Here, we denote δ_{11} for $\lim_{2\to 1}\delta_{12}$. Again, there are not enough number of D's and $\bar{D}$'s to make the diagram non-vanishing.

Now we show that Fig. 2(d) gives no contribution. Second order terms in Γ, W and $\bar{W}$, give two kinds of contributions: (i) one of the two loops has two external legs, but the other one has none; (ii) each of the two loops has one external leg. Case (i) represents a tadpole diagram, and on the loop without external leg, the trace over the group generator, $\mathrm{Tr}(T^\alpha)$, gives a vanishing result; case (ii) represents a non-1PI diagram, which does not contribute to two-loop Z_V.

The only non-zero contribution is due to Fig. 2(a). Its contribution is

$$I = \frac{1}{2i} i^2 \langle V(1)V(2)\rangle \langle \bar\phi_+(1)\phi_+(2)\rangle \langle \bar\phi_+(2)\phi_+(1)\rangle + \frac{1}{2i} i^2 \langle V(1)V(2)\rangle \langle \bar\phi_-(1)\phi_-(2)\rangle \langle \bar\phi_-(2)\phi_-(1)\rangle .$$

Plugging the expressions for the matter and gauge propagators, we obtain

$$I = i\int d^4x_1 d^4\theta_1 d^4x_2 d^4\theta_2 \frac{i}{\Box_+ - m^* m} D_1^2 \bar D_1^2 \delta_{12} \frac{(-i)}{\Box_0}\delta_{12} \cdot \frac{i}{\Box_+ - m^* m} D_2^2 \bar D_2^2 \delta_{21}. \qquad (3.11)$$

What we need to do now is to expand the quantum matter propagator $\frac{1}{\Box_+ - m^* m}$ using Eq. (3.7) in the integral. Graphically, there are three different types of terms in the expansion of Eq. (3.11)as shown in Fig. 3. Here the curled lines stand for the external background fields and the arrows in the propagators stand for the direction of momentum flow.

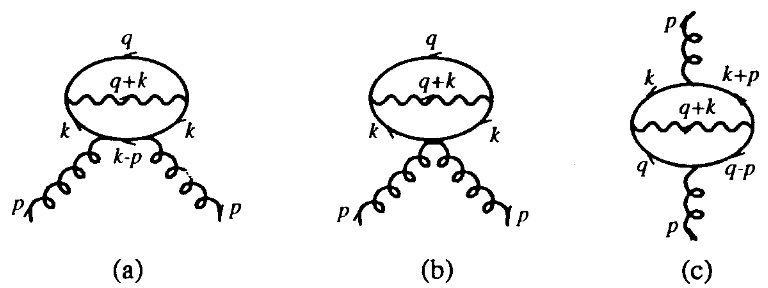

FIG. 3. The three diagrams represent three types expansions of Eq. (3.11)

Many individual terms in Eq. (3.7) may give a non-zero contribution upon their use in Eq. (3.11), but the sum of the terms with "naked" derivatives is zero. This is proved as follows. The terms with naked derivatives can contribute to Eq. (3.11) a term of the form

$$\delta_{12} D_{1\alpha} \bar D_1^2 D_1{}^2 \delta_{12} \bar D_2^2 D_2{}^2 \delta_{12},$$

or

$$\delta_{12} D_{1\alpha} D_{1\beta} \bar D_1^2 D_1{}^2 \delta_{12} \bar D_2^2 D_2{}^2 \delta_{12}.$$

By using the commutation in Eq. (3.10), and the δ-function properties in Eqs. (3.8) and (3.9), we can see that these two types of contributions do not survive. Hence, the non-vanishing contribution to Fig. 2(a) comes from those terms in Eq. (3.7) that do not have naked derivatives. Collecting all contributions of terms pf the type represented by Fig. 3(a), and integrating the spinor coordinate over one vertex, we obtain

$$\begin{aligned} I_a = &-2\int d^4x d^4x' d^4\theta \frac{1}{\Box_0 - m^* m}\cdot\frac{1}{\Box_0}\cdot\frac{1}{\Box_0 - m^* m} \\ &\times \left[i\Gamma^a\partial_a + \frac{i}{2}(\partial^a\Gamma_a) + \frac{i}{2}(D^\alpha W_\alpha)\right] \\ &\times \frac{1}{\Box_0 - m^* m}\left[i\Gamma^b\partial_b + \frac{i}{2}(\partial^b\Gamma_b) + \frac{i}{2}(D^\beta W_\beta)\right]\frac{1}{\Box_0 - m^* m} \\ &-2\int d^4x d^4x' d^4\theta \frac{1}{\Box_0 - m^* m}\cdot\frac{1}{\Box_0}\cdot(-1)\cdot\frac{1}{\Box_0 - m^* m} iW^\alpha \frac{1}{\Box_0 - m^* m} \\ &\times \left[\bar W^{\dot\alpha} D_{\alpha\dot\alpha} + (D_{\alpha\dot\alpha}\bar W^{\dot\alpha})\right]\frac{1}{\Box_0 - m^* m}, \end{aligned} \qquad (3.12)$$

where the factor 2 comes from two symmetric situations. Similarly, all contributions of the terms represented by Fig. 3(b) can be expressed as

$$I_b = -2\int d^4x d^4x' d^4\theta \frac{1}{\Box_0 - m^* m}\cdot\frac{1}{\Box_0}\cdot\frac{1}{\Box_0 - m^* m}\cdot\frac{1}{2}\Gamma^a\Gamma_a\frac{1}{\Box_0 - m^* m}, \qquad (3.13)$$

and those represented by Fig. 3(c) as

$$\begin{aligned} I_c = &-2\int d^4x d^4x' d^4\theta \frac{1}{\Box_0}\cdot\frac{1}{\Box_0 - m^* m}\cdot\frac{1}{\Box_0 - m^* m} \\ &\times \left[i\Gamma^a\partial_a + \frac{i}{2}(\partial^a\Gamma_a) + \frac{i}{2}(D^\alpha W_\alpha)\right]\frac{1}{\Box_0 - m^* m}\cdot\frac{1}{\Box_0 - m^* m} \\ &\times \left[i\Gamma^b\partial_b + \frac{i}{2}(\partial^b\Gamma_b) + \frac{i}{2}(D^\beta W_\beta)\right]\frac{1}{\Box_0 - m^* m}. \end{aligned} \qquad (3.14)$$

The above I_a, I_b and I_c are in Minkowski space. As usual, we transform to momentum space by using the momentum assignments given in Fig. 3. We work in Euclidean space, by replacing q_0 with iq_4 and k_0 with ik_4. If the original mass matrix has a massless mode, then we must have $m = m^* = 0$. For this massless case, the dependence of $1/g^2$ on the mass matrix is trivially holomorphic, although we have to deal with the infrared divergence. If the original mass matrix contains no massless mode at all, then m and m^* must be non-zero. In this case, we have to do the integrations to get the actual m-dependence of $1/g^2$.

As in the one-loop case, we work in four dimensional space-time and use the Pauli-Villars regularization scheme to deal with the ultraviolet divergence of the matter fields:

$$(k^2 + m^* m)^{-1} \to (k^2 + m^* m)^{-1} - (k^2 + \Lambda^2)^{-1},$$

where the ultraviolet cutoff $\Lambda \to \infty$. The integrals are done by rescaling the momenta

$$q \to q\Lambda, \quad k \to k\Lambda, \quad p \to p\Lambda,$$

and are expressed in terms of parameter $a = m/\Lambda$. As

$$\left(\frac{1}{k^2 + m^* m} - \frac{1}{k^2 + \Lambda^2}\right) \to \frac{1}{\Lambda^2(k^2 + a^* a)(k^2 + 1)},$$

some integrals have a prefactor $1/\Lambda$ or $1/\Lambda^2$ and hence can be eliminated right away. The non-zero integrals have $\Gamma\Gamma$ external fields. Since we are only concerned about holomorphicity, in these integrals we take external momentum $p \to 0$ for simplicity.

The sum of contributions terms I_a, I_b and I_c is given by [6]

$$I = I_a + I_b + I_c = \frac{1}{2}\Gamma^a(0)\Gamma_a(0)\left(\frac{1}{16\pi^2}\right)^2 \left[\ln(a^* a) + 1 + \frac{\pi^2}{6}\right]. \tag{3.15}$$

Examining the result, we find that the second order logarithmically divergent term $[\ln(a^* a)]^2$ cancels out leaving only the first order $\ln(a^* a)$ term. This is crucial for the holomorphicity of the gauge coupling constant as we will see later.

C. General representations in super QED

In a general representation in which l and $\bar{l}$ are arbitrary, the mass matrix does not take the simple form as in Eq. (3.2). The calculations are similar to those of previous subsection. Here also the second order logarithmic divergence cancels out, and only the first order logarithmic divergence is left. The final integrational result depends on the representation of the mass matrix:

a) if the representation of the mass matrix is massive or pseudo massive, we have

$$I = \frac{1}{4d_R}\,\mathrm{Tr}[\Gamma^a(0)\Gamma_a(0)]\left(\frac{1}{16\pi^2}\right)^2 \mathrm{Tr}\left[\ln\left(\frac{M^\dagger M}{\Lambda^\dagger\Lambda}\right) + 1 + \frac{\pi^2}{6}\right], \tag{3.16}$$

which is of the same form as Eq. (3.15).

b) if the representation of the mass matrix is intrinsically massless, we need to use the momentum cutoff to regulate the infrared divergence. Hence

$$I = \frac{1}{4d_R}\,\mathrm{Tr}[\Gamma^a(0)\Gamma_a(0)]\left(\frac{1}{16\pi^2}\right)^2 \mathrm{Tr}\left[\ln(p_0^2 + M^\dagger M) + \mathrm{const.}\right], \tag{3.17}$$

where "const." stands for the terms that do not depend on the mass matrix M. In Eqs. (3.16) and (3.17) "Tr" stands for the trace over both mass matrix M and Λ, and group generators T^a. But as indicated earlier this trace can be decomposed into a product of two, the trace over the mass matrix and the trace over the matrix of group generators.

D. Holomorphicity of $1/g^2$

The expressions (3.16) and (3.17) give the two-loop corrections to the gauge action in respective cases. We write it in the standard form

$$\frac{1}{4g^2}\,\mathrm{Tr}\int d^4x d^2\theta\, W^a W_a, \tag{3.18}$$

in order to get the correction to $1/g^2$. For this purpose, we use the following identity

$$\int d^4x d^4\theta\, \Gamma^a\Gamma_a = -2\int d^4x d^2\theta\, W^a W_a + \text{total derivative}. \tag{3.19}$$

From this we immediately obtain $\Delta\frac{1}{g^2}$. a) For the massive and pseudo massive cases, the two-loop correction to $1/g^2$ is

$$\Delta\frac{1}{g^2} = -\frac{T_R}{d_R}\left(\frac{g}{16\pi^2}\right)^2 \mathrm{Tr}\left[\ln\left(\frac{M^\dagger M}{\Lambda^\dagger\Lambda}\right) + 1 + \frac{\pi^2}{6}\right], \tag{3.20}$$

up to terms that do not depend on $M^\dagger$ and M, or negligible. This correction is the real part of the holomorphic function

$$-\frac{2T_R}{d_R}\left(\frac{g}{16\pi^2}\right)^2 \mathrm{Tr}\left[\ln\left(\frac{M}{\Lambda}\right) + \frac{1}{2} + \frac{\pi^2}{12}\right].$$

Therefore, we conclude that the two-loop correction to the gauge coupling constant is holomorphic for the cases of massive and pseudo massive mass matrices. The simplest representation with $l = \bar{l} = 1$ and $m \neq 0$ is a special case of this. b) For the intrinsically massless case (representation with unbalanced numbers of families and antifamilies), we have

$$\Delta\frac{1}{g^2} = -\frac{T_R}{d_R}\left(\frac{g}{16\pi^2}\right)^2 \mathrm{Tr}\left[\ln\left(p_0^2 + M^\dagger M\right) + \mathrm{const.}\right]. \tag{3.21}$$

Since the infrared momentum cutoff is nonzero, $1/g^2$ is clearly not the real part of any holomorphic function of M.

IV. TWO-LOOP HOLOMORPHICITY IN SUPER YANG-MILLS THEORY

In super Yang-Mills theory, the action has the same form as that for the super QED in the previous section, Eq. (3.1), except that the gauge group is now non-Abelian. The mass term again can be written as $\phi^{\mathrm{T}} M \phi$, where, in general, the mass matrix M can always be chosen to be symmetric. Again we impose the constraint $T^{a\mathrm{T}} M + M T^a = 0$ in order to have the action gauge invariant. Here the T^a's are the gauge group generators in the representation R. In the rest of the section, we calculate explicitly the two-loop dependence of $1/g^2$ on the mass matrix, by using the SBFM. The basic calculational procedure is very similar to that in the super QED case, but because of the non-commuting nature of the gauge group, the calculations are more complicated. But as at one-loop level, here also a representation of the mass matrix can give rise to massive, pseudo massive or intrinsically massless cases.

The background gauge covariant space-time derivative is denoted by $\mathcal{D}_{\alpha\dot{\alpha}}$ $(= \mathcal{D}_\mu \sigma^\mu_{\alpha\dot{\alpha}})$, and the spinor derivatives are denoted by $\mathcal{D}_\alpha$ and $\bar{\mathcal{D}}_{\dot{\alpha}}$. They are related by

$$i\mathcal{D}_{\alpha\dot{\alpha}} = \{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}}\}.$$

In this background invariant representation, we write the action as [8]

$$\begin{aligned}\mathcal{A} = &-\frac{1}{4g^2}\,\mathrm{Tr}\int d^4x d^4\theta\,(e^{-V}\mathcal{D}^\alpha e^V)\,[\bar{\mathcal{D}}^{\dot{\alpha}}, \{\bar{\mathcal{D}}_{\dot{\alpha}}, e^{-V}\mathcal{D}_\alpha e^V\}] \\ &+ \int d^4x d^4\theta\,\bar{\phi}^{\mathrm{T}} e^V \phi - \frac{1}{2}\int d^4x\,[d^2\theta\,\phi^{\mathrm{T}} M \phi + \mathrm{h.c.}].\end{aligned} \tag{4.1}$$

Here, V is the quantum gauge field and ϕ is the quantum matter field. The background gauge fields are hidden in the covariant derivatives. More specifically, the background spinor connection terms Γ^α and $\bar{\Gamma}^{\dot{\alpha}}$ are given by $\mathcal{D}^\alpha = D^\alpha - i\Gamma^\alpha$ and $\bar{\mathcal{D}}^{\dot{\alpha}} = \bar{D}^{\dot{\alpha}} - i\bar{\Gamma}^{\dot{\alpha}}$, respectively, where D^α and $\bar{D}^{\dot{\alpha}}$ are the regular non-background covariant derivatives. A similar relation exists for the space-time connection $\Gamma^{\alpha\dot{\alpha}}$. The background field strengths W_α and $\bar{W}_{\dot{\alpha}}$ are defined by

$$\begin{aligned}W_\alpha &\equiv -\frac{1}{2i}[\bar{\mathcal{D}}^{\dot{\alpha}}, \{\bar{\mathcal{D}}_{\dot{\alpha}}, \mathcal{D}_\alpha\}], \\ \bar{W}_{\dot{\alpha}} &\equiv -\frac{1}{2i}[\mathcal{D}^\alpha, \{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}}\}].\end{aligned}$$

To get the Feynman propagators for ϕ $(\bar{\phi})$ and V fields, we should find, from action (4.1), terms quadratic in ϕ $(\bar{\phi})$ and V, respectively. For the gauge field V, we need to expand e^V in the action and keep only the terms that are in the second order of V. This yields

$$\mathcal{A}_V = -\frac{1}{4g^2}\,\mathrm{Tr}\int d^4x d^4\theta\, V(-\underline{\mathcal{D}}^\alpha \underline{\bar{\mathcal{D}}}^2 \underline{\mathcal{D}}_\alpha + \underline{\mathcal{D}}^\alpha \underline{W}_\alpha)V.$$

For simplicity, we have used $\underline{\mathcal{D}}$ for the commutator $[\mathcal{D},\ \}$, $\underline{\bar{\mathcal{D}}}$ for $[\bar{\mathcal{D}},\ \}$, $\underline{W}$ for $[W,\ \}$, and $\underline{\Box}$ for $[\Box,\ \}$. Here, we define the commutator "[, }" by $[A, B\} = AB - BA$, if at least one of the two variables, A and B, is bosonic, or $[A, B\} = AB + BA$, if both of the two variables are fermionic. The background invariant gauge fixing term is given by

$$\mathcal{A}_{gf} = -\frac{1}{4g^2}\,\mathrm{Tr}\int d^4x d^4\theta\, V(\underline{\mathcal{D}}^2\bar{\underline{\mathcal{D}}}^2 + \bar{\underline{\mathcal{D}}}^2\underline{\mathcal{D}}^2)V.$$

Now adding this to $\mathcal{A}_V$, we can show that

$$\mathcal{A}_V + \mathcal{A}_{gf} = -\frac{1}{4g^2}\,\mathrm{Tr}\int d^4x d^4\theta\, V(\underline{\Box} - i\underline{W}^\alpha\underline{\mathcal{D}}_\alpha - i\bar{\underline{W}}^{\dot\alpha}\bar{\underline{\mathcal{D}}}_{\dot\alpha})V. \qquad (4.2)$$

We define

$$\hat{\Box} \equiv \Box - iW^\alpha\mathcal{D}_\alpha - i\bar{W}^{\dot\alpha}\bar{\mathcal{D}}_{\dot\alpha},$$

where the operator $\Box$ is defined by

$$\Box \equiv \frac{1}{2}\mathcal{D}^{\alpha\dot\alpha}\mathcal{D}_{\alpha\dot\alpha},$$

and

$$\hat{\underline{\Box}} \equiv \underline{\Box} - i\underline{W}^\alpha\underline{\mathcal{D}}_\alpha - i\bar{\underline{W}}^{\dot\alpha}\bar{\underline{\mathcal{D}}}_{\dot\alpha}.$$

Then, we obtain the propagator for the gauge field V (see Appendix B)

$$\langle V_1^\sigma V_2^\rho\rangle = -i\left(\frac{g^2}{\hat{\Box}_A}\right)^{\sigma\rho}\delta_{12},$$

where the subscript "A" indicates that the group generators are in the adjoint representation.

The action involving ϕ and $\bar\phi$ fields is given by

$$\mathcal{A}_\phi = \int d^4x d^4\theta\, \bar\phi^{\mathrm{T}}\phi + \int d^4x d^4\theta\, \bar\phi^{\mathrm{T}}V\phi + \frac{1}{2}\int d^4x d^4\theta\, \bar\phi^{\mathrm{T}}V^2\phi - \frac{1}{2}\int d^4x\, [d^2\theta\, \phi^{\mathrm{T}}M\phi + \mathrm{h.c.}], \qquad (4.3)$$

up to second order in V. From this action, we can derive the following propagators for ϕ and $\bar\phi$ fields (see Appendix C for details)

$$\begin{aligned}
\langle\phi(1)\phi^{\mathrm{T}}(2)\rangle &= M^\dagger\frac{i}{\Box_+ - MM^\dagger}\bar{\mathcal{D}}_1^2\delta_{12},\\
\langle\bar\phi(1)\bar\phi^{\mathrm{T}}(2)\rangle &= \frac{i}{\Box_- - MM^\dagger}M\,\mathcal{D}_1^2\delta_{12},\\
\langle\phi(1)\bar\phi^{\mathrm{T}}(2)\rangle &= \frac{i}{\Box_+ - M^\dagger M}\bar{\mathcal{D}}_1^2\mathcal{D}_1^2\delta_{12},\\
\langle\bar\phi(1)\phi^{\mathrm{T}}(2)\rangle &= \frac{i}{\Box_- - MM^\dagger}\mathcal{D}_1^2\bar{\mathcal{D}}_1^2\delta_{12},
\end{aligned} \qquad (4.4)$$

where $\Box_+$ and $\Box_-$ are defined by (see Appendix A)

$$\bar{\mathcal{D}}^2\mathcal{D}^2\phi \equiv \Box_+\phi, \qquad \mathcal{D}^2\bar{\mathcal{D}}^2\bar\phi \equiv \Box_-\bar\phi.$$

The solutions for $\Box_+$ and $\Box_-$ are given by

$$\Box_+ = \Box - iW^\alpha\mathcal{D}_\alpha - \frac{1}{2}i\{\mathcal{D}^\alpha, W_\alpha\},$$

and

$$\Box_- = \Box - i\bar{W}^{\dot\alpha}\bar{\mathcal{D}}_{\dot\alpha} - \frac{1}{2}i\{\bar{\mathcal{D}}^{\dot\alpha}, \bar{W}_{\dot\alpha}\}.$$

At two-loop level, the ghost action does not contribute any dependence on the mass matrix to the gauge coupling constant. Since we are studying holomorphicity, and our major concern is the dependence of the gauge coupling constant on the mass matrix, we can ignore the contribution from ghosts. Also notice that, although the mass matrix M here is actually a background covariant one, as proved in Appendix A, it equals the original " bare" mass matrix M_0. So we can use the same symbol M for both. We now look for all non-vanishing two-loop contributions to the gauge action. Up to second order in V, we have two vertices that contain matter fields: $i\bar\phi_1^{\mathrm{T}}V_1\phi_1$ and $\frac{i}{2}\bar\phi_1^{\mathrm{T}}V_1^2\phi_1$. From these vertices, we have the following quantum correction in the order of g^2

$$
\begin{aligned}
I =& \frac{1}{2i}\left\langle (i\bar{\phi}_1^{\rm T} V_1 \phi_1)(i\bar{\phi}_2^{\rm T} V_2 \phi_2)\right\rangle_{\rm connected} + \frac{1}{2i}\left\langle (i\bar{\phi}_1^{\rm T} V_1^2 \phi_1)\right\rangle_{\rm connected} \\
=& \frac{1}{2i}\,{\rm Tr}\left[\left\langle V_1^\sigma V_2^\rho\right\rangle T^\sigma \left\langle i\phi_1 \bar{\phi}_2^{\rm T}\right\rangle T^\rho \left\langle i\phi_2 \bar{\phi}_1^{\rm T}\right\rangle\right] \\
&+ \frac{1}{2i}\,{\rm Tr}\left[\left\langle V_1^\sigma V_2^\rho\right\rangle T^{\sigma{\rm T}} \left\langle i\bar{\phi}_1 \bar{\phi}_2^{\rm T}\right\rangle T^\rho \left\langle i\phi_2 \phi_1^{\rm T}\right\rangle\right] \\
&+ \frac{1}{2i}\,{\rm Tr}\left[\left\langle V_1^\sigma V_1^\rho\right\rangle T^\sigma T^\rho \left\langle i\phi_1 \bar{\phi}_1^{\rm T}\right\rangle\right],
\end{aligned}
\tag{4.5}
$$

where we have only included the connected graphs. Graphically, I can be represented by three Feynman diagrams as shown in Fig. 4.

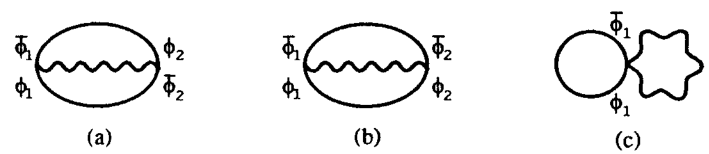

FIG. 4. Two-loop vacuum Feynman diagrams in non-Abelian gauge theory

The dependence on background fields W, $\bar{W}$ and Γ is contained in the quantum propagators.

Now plugging in the super matter and gauge propagators, we have

$$
\begin{aligned}
I =& -\frac{1}{2}\left(\frac{1}{\widehat{\Box}_A}\right)^{\sigma\rho} \delta_{12}\,{\rm Tr}\left[T^\sigma \frac{1}{\Box_+ - M^\dagger M}\,(\bar{\mathcal{D}}_1^2 \mathcal{D}_1^2 \delta_{12}) T^\rho \frac{1}{\Box_+ - M^\dagger M}\,(\bar{\mathcal{D}}_2^2 \mathcal{D}_2^2 \delta_{12})\right] \\
&+ \frac{1}{2}\left(\frac{1}{\widehat{\Box}_A}\right)^{\sigma\rho} \delta_{12}\,{\rm Tr}\left[T^{\sigma{\rm T}} \frac{1}{\Box_+ - M M^\dagger}\,(\mathcal{D}_1^2 \delta_{12}) T^{\rho{\rm T}} M M^\dagger \frac{1}{\Box_+ - M M^\dagger}\,(\bar{\mathcal{D}}_2^2 \delta_{12})\right] \\
&+ \frac{1}{2}\left(\frac{1}{\widehat{\Box}_A}\right)^{\sigma\rho} \delta_{11}\,{\rm Tr}\left[T^\sigma T^\rho \frac{1}{\Box_+ - M^\dagger M}\,(\bar{\mathcal{D}}_1^2 \mathcal{D}_1^2 \delta_{11})\right],
\end{aligned}
\tag{4.6}
$$

where the first, second and third terms of the equation correspond to Figs. 4(a), 4(b) and 4(c), respectively. To get contributions from each of the diagrams, we need to expand propagators $\frac{1}{\Box_+ - M^\dagger M}$ and $\frac{1}{\widehat{\Box}_A}$ in Eq. (4.6). In the non-Abelian super gauge theory, the expansion of $\frac{1}{\Box_+ - M^\dagger M}$ is very much like the one in the super QED case, Eq. (3.7), except

that we need to replace m by M, m^* by $M^\dagger$, D by $\mathcal{D}$ and $\bar{D}$ by $\bar{\mathcal{D}}$. Also, we need to expand operator $\frac{1}{\widehat{\Box}_A}$ using

$$
\widehat{\Box}_A = \Box_A - i W_A^\alpha \mathcal{D}_{A\alpha} - i \bar{W}_A^{\dot\alpha} \bar{\mathcal{D}}_{A\dot\alpha},
$$

and

$$
[\mathcal{D}_{A\alpha}, \widehat{\Box}_A] = \frac{1}{2}[\mathcal{D}_{A\alpha\dot\alpha}, \bar{W}_A^{\dot\alpha}] - i\{\mathcal{D}_{A\alpha}, W_A{}^\beta\}\mathcal{D}_{A\beta}.
$$

Similar to the δ-function properties in the super QED, we have

$$
\delta_{12}\bar{\mathcal{D}}^2\mathcal{D}^2\delta_{12} = \delta_{12}\mathcal{D}^2\bar{\mathcal{D}}^2\delta_{12} = \delta_{12}, \tag{4.7}
$$

$$
\delta_{12}\bar{\mathcal{D}}^m\mathcal{D}^n\delta_{12} = 0, \quad \text{for} \quad m + n < 4. \tag{4.8}
$$

Using the $\mathcal{D}$-properties we can show that Figs. 4(b) and 4(c) give zero contribution, and only Figs. 4(a) gives non-zero contribution. Fig. 4(a) represents

$$
I = -\frac{1}{2}\left(\frac{1}{\widehat{\Box}_A}\right)^{\sigma\rho} \delta_{12}\,{\rm Tr}\left[T^\sigma \frac{1}{\Box_+ - M^\dagger M}\,(\bar{\mathcal{D}}_1^2 \mathcal{D}_1^2 \delta_{12}) T^\rho \frac{1}{\Box_+ - M^\dagger M}\,(\bar{\mathcal{D}}_2^2 \mathcal{D}_2^2 \delta_{12})\right].
$$

The expansion of of propagators $\frac{1}{\Box_+ - M^\dagger M}$ and $\frac{1}{\widehat{\Box}_A}$ in I gives rise to different types of terms represented by Fig. 5. The curled lines stand for the external gauge fields. Compared with the the expansion in the Abelian case (Fig. 3), we can see that there are three more diagrams in the Abelian case. Since each external leg can be Γ, W or $\bar{W}$, or their derivative, we have many combinations. Most of them could be shown to give rise to zero contribution because of the $\mathcal{D}$-properties in Eqs. (4.7) and (4.8).

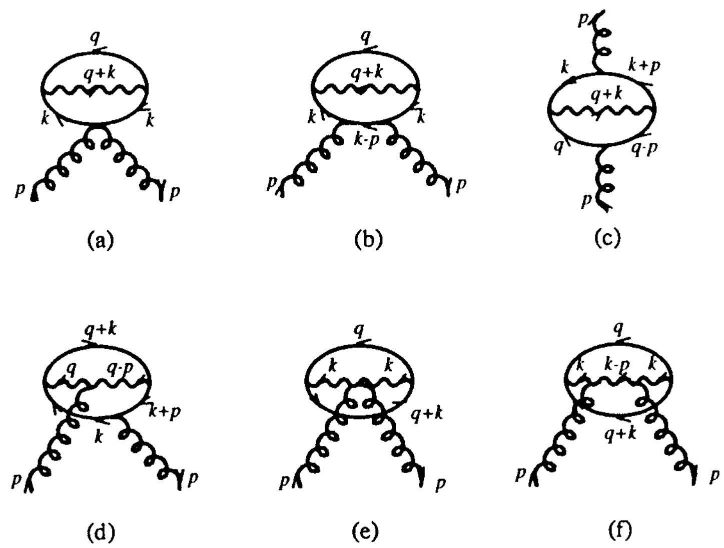

FIG. 5. Feynman diagrams in non-Abelian gauge theory

The integrals are evaluated using the Pauli-Villars regularization procedure for matter fields as in the case of super QED. Similarly we can identify their contribution to $1/g^2$. For massive and pseudo massive cases, for a representation with dimension d_R, we get

$$\Delta\frac{1}{g^2} = -\frac{T_R}{d_R}\left(\frac{g}{16\pi^2}\right)^2\left[(C_R + C_A)\,\mathrm{Tr}\ln\left(\frac{M^\dagger}{\Lambda^\dagger}\cdot\frac{M}{\Lambda}\right) + \mathrm{const.}\right]. \tag{4.9}$$

This is the real part of the holomorphic function

$$-\frac{2T_R}{d_R}\left(\frac{g}{16\pi^2}\right)^2\left[(C_R + C_A)\,\mathrm{Tr}\ln\left(\frac{M}{\Lambda}\right) + \mathrm{const.}\right], \tag{4.10}$$

where C_R is given by $\sum_\sigma T^\sigma T^\sigma = C_R I$, and C_A is C_R for the adjoint representation. Notice that the coefficient in front of the logarithmic function differs from that of the two-loop β-function the reason being that we have omitted the pure gauge or ghost contributions, which are not necessary for our purpose.

For the intrinsically massless case, we have

$$\Delta\frac{1}{g^2} = -\frac{T_R}{d_R}\left(\frac{g}{16\pi^2}\right)^2\left[(C_R + C_A)\,\mathrm{Tr}\ln\left(p_0^2 + M^\dagger M\right) + \mathrm{const.}\right]. \tag{4.11}$$

This has the same dependence on the mass matrix as Eq. (2.11) indicating that it is not the real part of a holomorphic function of M.

V. MASS MATRIX INDEPENDENCE OF $1/G_W^2$

In Ref. [2] general arguments have been made to define a "Wilson coupling" g_W (a coupling in the Wilsonian effective action) which is claimed to be holomorphic. In this section we explicitly verify this claim using our two-loop results.

In a general SQED case, generalizing the simple SQED expression for $1/g_W^2$ given in Ref [2], we have

$$\frac{1}{g_W^2} = \frac{1}{g^2} + \frac{T_R}{8\pi^2 d_R}\,\mathrm{Tr}\ln Z_\phi, \tag{5.1}$$

where Z_ϕ is the wave-function renormalization constant of the matter field. (Note that for the simple SQED $d_R = 2$.) The holomorphicity of $1/g_W^2$ implies that the r.h.s of the above equation should be independent of the mass matrix in the intrinsically massless case. This means that the mass-matrix dependence of the two-loop correction to $1/g^2$ should be canceled by the one-loop contribution to Z_ϕ. In fact that this is so can be explicitly verified using the one-loop result

$$Z_\phi = 1 + \frac{g^2}{32\pi^2}\ln(M^\dagger M + p_0^2), \tag{5.2}$$

from Ref [6] and the two-loop correction to $1/g^2$ in SQED from Eq. (3.21).

In the case of super Yang-Mills theory

$$\frac{1}{g_W^2} = \frac{1}{g^2} - \frac{C_A}{8\pi^2}\ln\frac{1}{g^2} + \frac{T_R}{8\pi^2 d_R}\,\mathrm{Tr}\ln Z_\phi. \tag{5.3}$$

At one-loop

$$Z_\phi = 1 + \frac{g^2 C_R}{32\pi^2}\ln(M^\dagger M + p_0^2). \tag{5.4}$$

Again we see that the mass matrix dependence of the second and third terms at one-loop level cancel the mass-matrix dependence of $1/g^2$ at two-loop level, making $1/g_W^2$ holomorphic.

VI. CONCLUSION

The form of the mass dependent corrections to $1/g^2$ in the one-loop and two-loop cases are identical. The reason is in the two-loop calculation, for all cases, the second order logarithmic terms, $[\ln(M^\dagger M)]^2$ or $[\ln(p_0^2 + M^\dagger M)]^2$, cancel out leaving only the first order $\ln(M^\dagger M)$ or $\ln(p_0^2 + M^\dagger M)$ terms. We expect this to be true for the higher loops. The work to prove this using Slavnov-Taylor-like identities is in progress. In conclusion, we have explicitly shown that, up to two-loops, the holomorphicity of the gauge coupling function depends on the representation of the mass matrix M constrained by Eq. (2.2), and in the massive and pseudo massive cases it is holomorphic and in the intrinsically massless case it is not. This is because in the first two cases one can use the mass matrix or perturbed mass matrix as a regulator for the infrared divergence whereas the intrinsically massless case requires an infrared momentum cutoff. We have explicitly verified the holomorphicity of $1/g_W^2$ in both SQED and super Yang-Mills theories, supporting the general arguments of Ref. [2].

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APPENDIX A SUPERSYMMETRY ALGEBRA

In this appendix, some useful aspects of supersymmetry algebra are given. We use the convention in Ref. [11], with the exception of the covariant derivative which is denoted by D instead of ∇.

A. Super Algebra in the Regular SUSY Theory

The covariant spinor derivatives D_α and $\bar{D}_{\dot\alpha}$ are given by

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\frac{1}{2}\bar\theta^{\dot\alpha} D_{\alpha\dot\alpha}, \quad \bar{D}_{\dot\alpha} = \frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i\frac{1}{2}\theta^\alpha D_{\alpha\dot\alpha},$$

where the upper and lower spinor indices can be mutually converted to each other by using $C_{\alpha\beta}$, the $SL(2,C)$ metric. The commutator of D_α and $\bar{D}_{\dot\alpha}$ is

$$\{D_\alpha, \bar{D}_{\dot\alpha}\} \equiv i D_{\alpha\dot\alpha},$$

where $D_{\alpha\dot\alpha}$ is the space-time derivative ($D_{\alpha\dot\alpha} = \sigma^\mu_{\alpha\dot\alpha} D_\mu$). Now denoting $\frac{1}{2}D^\alpha D_\alpha$ by D^2 and $\frac{1}{2}\bar{D}^{\dot\alpha}\bar{D}_{\dot\alpha}$ by $\bar{D}^2$, we record the following properties for the covariant derivatives [11]:

$$\frac{1}{2}D^{\alpha\dot\alpha}D_{\alpha\dot\alpha} = \Box = \partial^2,$$
$$[D_\alpha, \bar{D}^2] = -iD_{\alpha\dot\alpha}\bar{D}^{\dot\alpha}, \quad [\bar{D}_{\dot\alpha}, D^2] = -iD_{\alpha\dot\alpha}D^\alpha,$$
$$D^\alpha D_\beta = \delta^\alpha_\beta D^2, \quad \bar{D}^{\dot\alpha}\bar{D}_{\dot\beta} = \delta^{\dot\alpha}_{\dot\beta}\bar{D}^2, \quad D^2\bar{D}^2 D^2 = \Box D^2,$$
$$D^\alpha \bar{D}^2 D_\alpha = \bar{D}^{\dot\alpha} D^2 \bar{D}_{\dot\alpha},$$
$$\bar{D}^2 D^2 + D^2\bar{D}^2 - D^\alpha\bar{D}^2 D_\alpha = \Box,$$
$$[D^2, \bar{D}^2] = \Box + i\bar{D}_{\dot\alpha}D_\alpha D^{\alpha\dot\alpha} = -\Box - iD_\alpha\bar{D}_{\dot\alpha}D^{\alpha\dot\alpha},$$
$$2\bar{D}^2 D^2 + i\bar{D}_{\dot\alpha}D_\alpha D^{\alpha\dot\alpha} = \bar{D}^{\dot\alpha}D^2\bar{D}_{\dot\alpha}.$$

Defining $\theta^2 = \frac{1}{2}C_{\alpha\beta}\theta^\alpha\theta^\beta$, $\bar\theta^2 = \frac{1}{2}C_{\dot\alpha\dot\beta}\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta}$ and $\delta_{12} = (\theta_1 - \theta_2)^2(\bar\theta_1 - \bar\theta_2)^2$, we have the following identities

$$D^2\theta^2 = \bar D^2\bar\theta^2 = -1, \quad D_1^\alpha\delta_{12} = -D_2^\alpha\delta_{12},$$
$$\delta_{12}D^n\bar D^m\delta_{12} = 0, \quad \text{for } m+n<4,$$
$$\delta_{12}D^2\bar D^2\delta_{12} = \delta_{12}\bar D^2 D^2\delta_{12} = \delta_{12}.$$

B. Super Algebra in the Background Field Method

In the super background field theory, the background covariant derivatives $\mathcal{D}^\alpha \equiv e^{-\Omega}D^\alpha e^\Omega \equiv D^\alpha - i\Gamma^\alpha$, and $\bar{\mathcal{D}}^{\dot\alpha} \equiv e^{-\bar\Omega}\bar D^{\dot\alpha}e^{\bar\Omega} \equiv \bar D^{\dot\alpha} - i\bar\Gamma^{\dot\alpha}$, where Ω is the background gauge field and Γ's are the super connections. Like in the case of regular super algebra, we use the following definitions

$$\bar{\mathcal{D}}^2 \equiv \frac{1}{2}\bar{\mathcal{D}}^{\dot\alpha}\bar{\mathcal{D}}_{\dot\alpha}, \quad \mathcal{D}^2 \equiv \frac{1}{2}\mathcal{D}^\alpha\mathcal{D}_\alpha,$$

and

$$\mathcal{D}_{\alpha\dot\alpha} \equiv \frac{1}{i}\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\alpha}\} \equiv D_{\alpha\dot\alpha} - i\Gamma_{\alpha\dot\alpha}.$$

The background field strengths are defined by

$$W_\alpha \equiv -\frac{1}{2i}[\bar{\mathcal{D}}^{\dot\alpha}, \{\bar{\mathcal{D}}_{\dot\alpha}, \mathcal{D}_\alpha\}], \quad \bar W_{\dot\alpha} \equiv -\frac{1}{2i}[\mathcal{D}^\alpha, \{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\alpha}\}].$$

For future simplicity, we also define

$$\underline{\mathcal{D}} \equiv [\mathcal{D}, \ \}, \quad \underline{W} \equiv [W, \ \},$$

where the bracket [, } can be either bosonic or fermionic. In these symbols

$$\mathcal{D}_{\alpha\dot\alpha} = (\underline{\mathcal{D}}_\alpha\bar{\mathcal{D}}_{\dot\alpha}) = (\underline{\bar{\mathcal{D}}}_{\dot\alpha}\mathcal{D}_\alpha),$$

and

$$W_\alpha = -\frac{1}{2i}(\underline{\bar{\mathcal{D}}}^{\dot\alpha}(\underline{\bar{\mathcal{D}}}_{\dot\alpha}\mathcal{D}_\alpha)), \quad \bar W_{\dot\alpha} = -\frac{1}{2i}(\underline{\mathcal{D}}^\alpha(\underline{\mathcal{D}}_\alpha\bar{\mathcal{D}}_{\dot\alpha})).$$

Some properties given below are very important for the propagator expansions in

Appendices C and B:

$$\{\mathcal{D}^\alpha, W_\alpha\} + \{\bar{\mathcal{D}}^{\dot\alpha}, \bar W_{\dot\alpha}\} \equiv (\underline{\mathcal{D}}^\alpha W_\alpha) + (\underline{\bar{\mathcal{D}}}^{\dot\alpha}\bar W_{\dot\alpha}) = 0,$$
$$[\mathcal{D}_\alpha, \mathcal{D}_{\beta\dot\beta}] = C_{\alpha\beta}\bar W_{\dot\beta}, \quad [\bar{\mathcal{D}}_{\dot\alpha}, \mathcal{D}_{\beta\dot\beta}] = C_{\dot\alpha\dot\beta}W_\beta,$$

and

$$[\mathcal{D}_\alpha, \bar{\mathcal{D}}^2] = -i\mathcal{D}_{\alpha\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha} + iW_\alpha = -i\bar{\mathcal{D}}^{\dot\alpha}\mathcal{D}_{\alpha\dot\alpha} - iW_\alpha.$$

The background chiral and antichiral matter fields are denoted by ϕ and $\bar\phi$, respectively. We define the operators $\Box_\pm$ by

$$\bar{\mathcal{D}}^2\mathcal{D}^2\phi \equiv \Box_+\phi, \quad \mathcal{D}^2\bar{\mathcal{D}}^2\bar\phi \equiv \Box_-\bar\phi.$$

These operators can then be expressed as

$$\Box_+ = \Box - iW^\alpha\mathcal{D}_\alpha - \frac{1}{2}i\{\mathcal{D}^\alpha, W_\alpha\},$$

up to a term in the form of $f \cdot \bar{\mathcal{D}}$, and

$$\Box_- = \Box - i\bar W^{\dot\alpha}\bar{\mathcal{D}}_{\dot\alpha} - \frac{1}{2}i\{\bar{\mathcal{D}}^{\dot\alpha}, \bar W_{\dot\alpha}\},$$

up to a term in the form of $f \cdot \mathcal{D}$. Here f is an arbitrary function, and $\Box$ denotes $\frac{1}{2}\mathcal{D}^{\alpha\dot\alpha}\mathcal{D}_{\alpha\dot\alpha}$. It is obvious that

$$\mathcal{D}^2 f(\Box_+)\bar{\mathcal{D}}^2 = f(\Box_-)\mathcal{D}^2\bar{\mathcal{D}}^2, \quad \bar{\mathcal{D}}^2 f(\Box_-)\mathcal{D}^2 = f(\Box_+)\bar{\mathcal{D}}^2\mathcal{D}^2,$$

and

$$[\mathcal{D}_\alpha, \Box] = \frac{1}{2}\bar W^{\dot\alpha}\mathcal{D}_{\alpha\dot\alpha} + \frac{1}{2}\mathcal{D}_{\alpha\dot\alpha}\bar W^{\dot\alpha}.$$

The covariant derivative on M has to be defined as

$$(\mathcal{D}M) \equiv \mathcal{D}^{\mathrm{T}}M - M\mathcal{D}.$$

This is the consequence of the chain rule for the covariant derivative $\mathcal{D}$

$$(\mathcal{D}\bar\phi)^{\mathrm{T}}M\phi + \bar\phi^{\mathrm{T}}(\mathcal{D}M)\phi + \bar\phi^{\mathrm{T}}M(\mathcal{D}\phi) \equiv \mathcal{D}(\bar\phi^{\mathrm{T}}M\phi) \equiv D(\bar\phi^{\mathrm{T}}M\phi).$$

Then from the constraints $T^{\sigma\mathrm{T}}M + MT^\sigma = 0$, it is easy to see that $(\mathcal{D}M) = 0$. Similarly, we have $(\mathcal{D}M^\dagger) = 0$, $(\bar{\mathcal{D}}M) = 0$ and $(\bar{\mathcal{D}}M^\dagger) = 0$.

APPENDIX B SUPER GAUGE PROPAGATOR

In this appendix, we will derive the expression for the propagator of the super gauge field in the background field theory. The canonical supergauge field propagator is obtained by putting the background field equal to zero in the final result.

A. Propagator for the Gauge Field

As given in Eq. (4.2), the quadratic part of the gauge action plus gauge fixing term is

$$\mathcal{A}_V + \mathcal{A}_{gf} = -\frac{1}{4g^2}\,\mathrm{Tr}\int d^4x d^4\theta\, V(\underline{\Box} - i\underline{W}^\alpha \underline{\mathcal{D}}_\alpha - i\underline{\bar{W}}^{\dot\alpha}\underline{\bar{\mathcal{D}}}_{\dot\alpha})V, \tag{B.1}$$

where we have denoted $\underline{\mathcal{D}}$ for commutator $[\mathcal{D},\ \}$, and $\underline{W}$ for $[W,\ \}$, and $\underline{\Box}$ for $[\Box,\ \}$. In these notations, we have

$$(\underline{\mathcal{D}}V) \equiv [\mathcal{D}, V] = [D - i\Gamma, V] = (DV) - i[T^\lambda, T^\sigma]\Gamma^\lambda V^\sigma = (DV) - i f^{\lambda\sigma\rho} T^\rho \Gamma^\lambda V^\sigma,$$

where $f^{\lambda\sigma\rho}$ are the group structure constants and T^σ are the group generators in the representation R. Meanwhile, since in the adjoint representation, the $\rho\sigma$ component of the group generators T^λ can be written as $(T^\lambda_A)^{\rho\sigma} = f^{\lambda\sigma\rho}$, we have

$$(\underline{\mathcal{D}}V) = (DV) - iT^\rho \Gamma^{\rho\sigma}_A V^\sigma = T^\rho(\delta^{\rho\sigma} D - i\Gamma^{\rho\sigma}_A)V^\sigma.$$

Now defining $\mathcal{D}_A$ as the background covariant derivative in the adjoint representation and defining

$$\underline{V} \equiv \begin{pmatrix} V^1 \\ \vdots \\ V^d \end{pmatrix}, \quad \underline{T} \equiv \begin{pmatrix} T^1 \\ \vdots \\ T^d \end{pmatrix},$$

in the column matrix form, where d is the rank of the group, we can write

$$(\underline{\mathcal{D}}V) = \underline{T}^{\mathrm{T}}(\mathcal{D}_A \underline{V}).$$

It is straightforward to generalize this to

$$(\underline{\mathcal{D}}\underline{\mathcal{D}}\cdots\underline{\mathcal{D}}V) = \underline{T}^{\mathrm{T}}(\mathcal{D}_A\mathcal{D}_A\cdots\mathcal{D}_A\underline{V}).$$

Therefore, we have

$$\mathrm{Tr}(V\underline{\mathcal{D}}\underline{\mathcal{D}}\cdots\underline{\mathcal{D}}V) = \mathrm{Tr}(VT^\rho)(\mathcal{D}_A\mathcal{D}_A\cdots\mathcal{D}_A\underline{V})^\rho$$
$$= 2[\underline{V}^{\mathrm{T}}(\mathcal{D}_A\mathcal{D}_A\cdots\mathcal{D}_A\underline{V})],$$

and

$$\mathrm{Tr}\,(V(\underline{\Box} - i\underline{W}^\alpha\underline{\mathcal{D}}_\alpha - i\underline{\bar{W}}^{\dot\alpha}\underline{\bar{\mathcal{D}}}_{\dot\alpha})V)$$
$$= 2[\underline{V}^{\mathrm{T}}((\Box_A - iW_A{}^\alpha\mathcal{D}_{A\alpha} - i\bar{W}_A{}^{\dot\alpha}\bar{\mathcal{D}}_{A\dot\alpha})\underline{V})],$$

where we have used $\mathrm{Tr}(T^\sigma T^\rho) \equiv T_R\delta^{\sigma\rho}$ with the trace factor $T_R \equiv 2$.

Now adding the real source

$$\frac{1}{2}\,\mathrm{Tr}\int d^4x d^4\theta JV = \int d^4x d^4\theta \underline{J}^{\mathrm{T}}\underline{V},$$

to action (B.1), where $\underline{J} \equiv \begin{pmatrix} J^1 \\ \vdots \\ J^d \end{pmatrix}$, we obtain the logarithm of the partition function

$$\ln Z(J) = i\frac{1}{2}\int d^4x d^4\theta \underline{J}^{\mathrm{T}}(\Box_A - iW_A{}^\alpha\mathcal{D}_{A\alpha} - i\bar{W}_A{}^{\dot\alpha}\bar{\mathcal{D}}_{A\dot\alpha})^{-1}\underline{J}.$$

Obviously, the gauge propagator is

$$\langle \underline{V}\underline{V}^{\mathrm{T}}\rangle = -\frac{ig^2}{\Box_A - iW_A{}^\alpha\mathcal{D}_{A\alpha} - i\bar{W}_A{}^{\dot\alpha}\bar{\mathcal{D}}_{A\dot\alpha}} \equiv -\frac{ig^2}{\widehat{\Box}_A}.$$

APPENDIX C SUPER MATTER PROPAGATORS

In this appendix, we derive the expressions for the propagators of the super matter fields in super background field theory. This is done in Minkowski space. All the results can be applied to the regular SUSY theory by simply taking the background fields to be zero and replacing the background covariant derivatives with regular covariant derivatives.

A. Super Propagator for the Matter Fields

The quadratic action for the matter fields is given by

$$\mathcal{A}_\phi = \int d^4x d^4\theta\, \bar\phi^{\rm T}\phi - \frac{1}{2}\int d^4x\,[\,d^2\theta\, \phi^{\rm T} M\phi + {\rm h.c.}],$$

where ϕ and $\bar\phi$ are the background chiral and antichiral fields, respectively, and M is the mass matrix in the background field method. We denote the regular chiral and antichiral fields by ϕ_t and $\bar\phi_t$, respectively, then

$$\phi = e^{-\Omega}\phi_t, \quad \bar\phi = e^{-\Omega}\bar\phi_t,$$

where Ω and $\bar\Omega$ are the background fields, and

$$M = e^{\bar\Omega^{\rm T}} M_0 e^{\bar\Omega},$$

where M_0 is the mass matrix with no background field. We claim that M has to equal M_0. This is because the mass term has to be invariant under a gauge transformation and the right hand side of the above expression is just a special gauge transformation.

Converting the action into the full spinor space, we get

$$\mathcal{A}_\phi = \int d^4x d^4\theta \left(\bar\phi^{\rm T}\phi - \frac{1}{2}\phi^{\rm T} M\frac{1}{\Box_-}\mathcal{D}^2\phi - \frac{1}{2}\bar\phi^{\rm T} M^\dagger \frac{1}{\Box_+}\bar{\mathcal{D}}^2\bar\phi\right),$$

by using properties in the previous appendix. The spinor covariant derivatives are given by $\mathcal{D}^\alpha = e^{-\Omega}D^\alpha e^{\Omega}$, and $\bar{\mathcal{D}}^{\dot\alpha} = e^{-\bar\Omega}\bar D^{\dot\alpha}e^{\bar\Omega}$. Now adding background chiral source j and antichiral source $\bar j$ to this action [7], we have

$$\begin{aligned}
\mathcal{A}_\phi(j,\bar j) &= \int d^4x d^4\theta \left(\bar\phi^{\rm T}\phi - \frac{1}{2}\phi^{\rm T} M\frac{1}{\Box_-}\mathcal{D}^2\phi - \frac{1}{2}\bar\phi^{\rm T} M^\dagger \frac{1}{\Box_+}\bar{\mathcal{D}}^2\bar\phi\right) \\
&\quad + \int d^4x\, d^2\theta\, \phi^{\rm T} j + \int d^4x\, d^2\bar\theta\, \bar\phi^{\rm T}\bar j \\
&= \int d^4x d^4\theta \Big(\bar\phi^{\rm T}\phi - \frac{1}{2}\phi^{\rm T} M\frac{1}{\Box_-}\mathcal{D}^2\phi - \frac{1}{2}\bar\phi^{\rm T} M^\dagger \frac{1}{\Box_+}\bar{\mathcal{D}}^2\bar\phi \\
&\qquad + \phi^{\rm T}\frac{1}{\Box_-}\mathcal{D}^2 j + \bar\phi^{\rm T}\frac{1}{\Box_+}\bar{\mathcal{D}}^2\bar j\Big) \\
&= \int d^4x d^4\theta \left[\frac{1}{2}(\phi^{\rm T} \;\; \bar\phi^{\rm T})\, A \begin{pmatrix}\phi \\ \bar\phi\end{pmatrix} + \left(\phi^{\rm T}\frac{1}{\Box_-}\mathcal{D}^2 j + \bar\phi^{\rm T}\frac{1}{\Box_+}\bar{\mathcal{D}}^2\bar j\right)\right],
\end{aligned}$$

where A is the matrix

$$\begin{pmatrix} -M\frac{1}{\Box_-}\mathcal{D}^2 & 1 \\ 1 & -M^\dagger\frac{1}{\Box_+}\bar{\mathcal{D}}^2 \end{pmatrix}.$$

Now functionally integrating out the background chiral and antichiral fields, we have the logarithm of the partition function in the matrix product form

$$\ln Z(j,\bar j) = -\frac{i}{2}\int d^4x d^4\theta \left(\left(\frac{1}{\Box_-}\mathcal{D}^2 j\right)^{\rm T} \left(\frac{1}{\Box_+}\bar{\mathcal{D}}^2\bar j\right)^{\rm T}\right) A^{-1}\begin{pmatrix}\frac{1}{\Box_-}\mathcal{D}^2 j \\ \frac{1}{\Box_+}\bar{\mathcal{D}}^2\bar j\end{pmatrix}, \qquad \text{(C.1)}$$

where the inverse of the matrix A can be shown to be

$$A^{-1} = \begin{pmatrix} M^\dagger\frac{1}{\Box_+ - MM^\dagger}\bar{\mathcal{D}}^2 & 1 + M^\dagger\frac{1}{\Box_+}\bar{\mathcal{D}}^2 M\frac{1}{\Box_- - M^\dagger M}\mathcal{D}^2 \\ 1 + M\frac{1}{\Box_-}\mathcal{D}^2 M^\dagger\frac{1}{\Box_+ - MM^\dagger}\bar{\mathcal{D}}^2 & M\frac{1}{\Box_- - M^\dagger M}\mathcal{D}^2 \end{pmatrix}.$$

After simplifying Eq. (C.1), we have

$$\begin{aligned}
&\ln Z(j,\bar j) \\
&= -\frac{i}{2}\int d^4x d^4\theta\, (j^{\rm T}\;\; \bar j^{\rm T}) \begin{pmatrix} M^\dagger\frac{1}{\Box_-(\Box_- - MM^\dagger)}\mathcal{D}^2 & \frac{1}{\Box_- - M^\dagger M} \\ \frac{1}{\Box_+ - MM^\dagger} & \frac{1}{\Box_+(\Box_+ - MM^\dagger)}M\bar{\mathcal{D}}^2 \end{pmatrix}\begin{pmatrix} j \\ \bar j\end{pmatrix}.
\end{aligned}$$

By using the facts that

$$\frac{\delta j_1}{\delta j_2} = \bar{\mathcal{D}}^2 \delta_{12}, \quad \frac{\delta \bar{j}_1}{\delta \bar{j}_2} = \mathcal{D}^2 \delta_{12},$$

where $\delta_{12} = \delta^4(x_1 - x_2)\delta^4(\theta_1 - \theta_2)$, we obtain

$$\begin{aligned}
\langle \phi(1)\phi^{\mathrm{T}}(2) \rangle &= -\frac{\partial^2 \ln Z(j,\bar{j})}{\partial j_1^{\mathrm{T}} \partial j_2} = M^\dagger \frac{i}{\Box_+ - MM^\dagger} \bar{\mathcal{D}}_1^2 \delta_{12}, \\
\langle \bar{\phi}(1)\bar{\phi}^{\mathrm{T}}(2) \rangle &= -\frac{\partial^2 \ln Z(j,\bar{j})}{\partial \bar{j}_1^{\mathrm{T}} \partial \bar{j}_2} = \frac{i}{\Box_- - MM^\dagger} M \mathcal{D}_1^2 \delta_{12}, \\
\langle \phi(1)\bar{\phi}^{\mathrm{T}}(2) \rangle &= -\frac{\partial^2 \ln Z(j,\bar{j})}{\partial j_1^{\mathrm{T}} \partial \bar{j}_2} = \frac{i}{\Box_+ - M^\dagger M} \bar{\mathcal{D}}_1^2 \mathcal{D}_1^2 \delta_{12}, \\
\langle \bar{\phi}(1)\phi^{\mathrm{T}}(2) \rangle &= -\frac{\partial^2 \ln Z(j,\bar{j})}{\partial \bar{j}_1^{\mathrm{T}} \partial j_2} = \frac{i}{\Box_- - MM^\dagger} \mathcal{D}_1^2 \bar{\mathcal{D}}_1^2 \delta_{12}.
\end{aligned}$$

These are the four matter propagators in the super background field method.

APPENDIX D MASS MATRIX

A. Constraints on the Mass Matrix

In this appendix, we discuss the general representation of the mass matrix M, which is subject to the constraints from gauge invariance. We mainly discuss the mass matrix in the super Yang-Mills gauge theory, but the discussion is applicable to the super Abelian gauge theory as well.

R labels the representation of the gauge group G, and M denotes the mass matrix. The invariance of the mass term

$$\int d^2\theta \, \phi^{\mathrm{T}} M \phi + \text{h.c.}$$

under gauge transformation requires

$$G(R)^{\mathrm{T}} M G(R) = M.$$

Because finite representations of simple groups are unitary, we have

$$G(R)^{-1} = G(R)^\dagger, \quad MG(R) = G(R)^* M.$$

Indicating the group generators in this representation as T^α, we have

$$T^{\alpha\mathrm{T}} M + MT^\alpha = 0, \quad T^\alpha M^\dagger + M^\dagger T^{\alpha\mathrm{T}} = 0.$$

From Schur's lemmas [12], we know that if R is an irreducible representation, we must have either $M \equiv 0$, or $\det M \neq 0$. This means that for an irreducible representation, the mass matrix is either trivially zero, or all of its modes are massive.

To have a mass matrix with both massless modes and massive modes, the representation has to be reducible. The reducible representation R can be written as

$$G(R) = \begin{pmatrix} G_1 & 0 & \cdots & 0 \\ 0 & G_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G_l \end{pmatrix},$$

where G_i ($i = 1, \cdots, l$) are irreducible representations. Assuming the mass matrix in this representation has the form

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1l} \\ \vdots & \ddots & \vdots \\ M_{l1} & \cdots & M_{ll} \end{pmatrix},$$

where M_{ij} is a matrix with dimension $\dim G_i \times \dim G_j$, and using the relation $G(R)^{\mathrm{T}} M G(R) = M$, we have $G_i^{\mathrm{T}} M_{ij} G_j = M_{ij}$. From Schur's lemmas [12], we know that, if $\dim(G_i) \neq \dim(G_j)$, then $M_{ij} = 0$. This means that M can be decomposed into diagonal blocks, and for each block the corresponding G_i all have the same dimension. Since different blocks are trivially decoupled, we can assume the whole mass matrix M is one of such blocks, without loss of generality. In other words, we can assume that all G_i's ($i = 1, \cdots, l$) have the same dimension, $n \times n$. Furthermore, though different types of representations (like real, pseudo-real or complex) might have same dimension, they cannot mix. That is, the coupling between type i and type j, M_{ij}, is zero. Hence, we can investigate different types of representations separately.

B. Real and Pseudo-Real Representation

For a real or pseudo-real irreducible representation, G_i, we have

$$G_i = J^{-1} G_i^* J. \tag{D.1}$$

For a real representation, the matrix J equals I, the $n \times n$ unit matrix; and for a pseudo-real representation $J = -J^{\mathrm{T}}$ with $J^2 = -1$. For both cases, $J^{-1} = J^{\dagger}$. If M_{ij} is nonzero, we have

$$G_j = M_{ij}^{-1} G_i^* M_{ij}. \tag{D.2}$$

Comparing this with Eq. (D.1), we see that representations G_i and G_j are equivalent. By rotating sector i or j of the super matter field ϕ, we can have $G_i = G_j$, and $M_{ij} = a_{ij} J$, where a_{ij} are nonzero complex constants. Since this is true for all i and j, we can define $G_i = G_r$ for all i, where subscript "r" stands for "real" or "pseudo-real". The representation of the group then becomes

$$G(R) = \mathrm{diag}\{\overbrace{G_r,\ G_r,\ \ldots,\ G_r}^{l}\}, \tag{D.3}$$

and the mass matrix becomes

$$M = \begin{pmatrix} a_{11}J & \cdots & a_{1l}J \\ \vdots & \ddots & \vdots \\ a_{l1}J & \cdots & a_{ll}J \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1l} \\ \vdots & \ddots & \vdots \\ a_{l1} & \cdots & a_{ll} \end{pmatrix} \otimes J.$$

Furthermore, since we can always choose a symmetric mass matrix, we have $a_{ij} = a_{ji}$ for the real representation, and $a_{ij} = -a_{ji}$ for the pseudo-real representation.

C. Complex Representation

For a complex irreducible representation G_i, we must have $M_{ii} = 0$, otherwise, we will have

$$G_i = M_{ii}^{-1} G_i^* M_{ii},$$

which conflicts with the complex condition. For $j \neq i$, we have either $M_{ij} = 0$, or

$$G_j = M_{ij}^{-1} G_i^* M_{ij}.$$

This means that, if $M_{ij} \neq 0$, then representation G_j is equivalent to the complex conjugate of representation G_i. As in the case of a real or pseudo-real representation, we can rotate the sector i or j of the super matter field ϕ, so that we have exactly $G_i = G_j^*$. This implies $M_{ij} = b_{ij} I$, where I is an $n \times n$ unit matrix. Since this is true for all i and j, we can define $G_i = G_c$ for all i, where subscript "c" stands for "complex". The representation of the group then becomes

$$G(R) = \mathrm{diag}\{\overbrace{G_c,\ G_c,\ \ldots,\ G_c}^{l},\ \overbrace{G_c^*,\ G_c^*,\ \ldots,\ G_c^*}^{\bar{l}}\},$$

Notice that the number of irreducible complex representations is not necessarily equal to the number of their conjugates. Assuming the former number is l and the latter number is $\bar{l}$, we have the following form for the mass matrix

$$M = \begin{pmatrix} & 0 & & b_{1l+1}I & \ldots & b_{1l+\bar{l}}I \\ & & & \vdots & \ddots & \vdots \\ & & & b_{ll+1}I & \ldots & b_{ll+\bar{l}}I \\ b_{l+11}I & \ldots & b_{l+1l}I & & & \\ \vdots & \ddots & \vdots & & 0 & \\ b_{l+\bar{l}1}I & \ldots & b_{l+\bar{l}l}I & & & \end{pmatrix},$$

which can be reexpressed in terms of matrix product

$$M = \begin{pmatrix} & 0 & & b_{1l+1} & \ldots & b_{1l+\bar{l}} \\ & & & \vdots & \ddots & \vdots \\ & & & b_{ll+1} & \ldots & b_{ll+\bar{l}} \\ b_{l+11} & \ldots & b_{l+1l} & & & \\ \vdots & \ddots & \vdots & & 0 & \\ b_{l+\bar{l}1} & \ldots & b_{l+\bar{l}l} & & & \end{pmatrix} \otimes I.$$

Since the whole matrix M is chosen to be symmetric, we have $b_{ij} = b_{ji}$.

D. Trace Calculations

We now calculate the following typical trace:

$$\mathrm{Tr}\,\left(T^{\sigma}T^{\rho}(M^{\dagger}M)^{k}\right),$$

where k is an arbitrary integer. We show that the trace over mass matrix can be separated from the trace over the group generators.

For a real or pseudo-real representation, we have shown that

$$M = A \otimes J,$$

where

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1l} \\ \vdots & \ddots & \vdots \\ a_{l1} & \cdots & a_{ll} \end{pmatrix},$$

is an $l \times l$ dimensional matrix, and J, as we know, is an $n \times n$ dimensional matrix. From this we have

$$M^{\dagger} = A^{\dagger} \otimes J^{\dagger},$$

and therefore,

$$M^{\dagger}M = (A^{\dagger}A) \otimes I_{n\times n},$$

where $I_{n\times n}$ is an $n \times n$ dimensional unit matrix. Meanwhile, the group generator T^{σ} in

representation R can written as (see Eq. (D.3))

$$T^{\sigma} = I_{l\times l} \otimes T^{\sigma}_{r},$$

where $I_{l\times l}$ is an $l \times l$ dimensional unit matrix. We then have

$$\begin{aligned} &T^{\sigma}T^{\rho}(M^{\dagger}M)^{k} \\ &= (A^{\dagger}A)^{k} \otimes (T^{\sigma}_{r}T^{\rho}_{r}), \end{aligned}$$

and

$$\begin{aligned} &\mathrm{Tr}\,\left(T^{\sigma}T^{\rho}(M^{\dagger}M)^{k}\right) \\ &= \mathrm{Tr}\,(A^{\dagger}A)^{k} \cdot \mathrm{Tr}\,(T^{\sigma}_{r}T^{\rho}_{r}). \end{aligned}$$

Also because

$$\mathrm{Tr}\,(T^{\sigma}T^{\rho}) = l\,\mathrm{Tr}\,(T^{\sigma}_{r}T^{\rho}_{r}),$$
$$\mathrm{Tr}\,(M^{\dagger}M)^{k} = n\,\mathrm{Tr}\,(A^{\dagger}A)^{k},$$

we have

$$\begin{aligned} &\mathrm{Tr}\,\left(T^{\sigma}T^{\rho}(M^{\dagger}M)^{k}\right) \\ &= \frac{1}{d_R}\,\mathrm{Tr}\,(M^{\dagger}M)^{k} \cdot \mathrm{Tr}\,(T^{\sigma}T^{\rho}), \end{aligned}$$

where $d_R \equiv l \cdot n$ is the dimension of the representation.

For a complex representation, we have

$$M = \begin{pmatrix} 0 & B \\ B^{\mathrm{T}} & 0 \end{pmatrix} \otimes I_{n\times n},$$

where

$$B = \begin{pmatrix} b_{1l+1} & \cdots & b_{1l+\bar{l}} \\ \vdots & \ddots & \vdots \\ b_{ll+1} & \cdots & b_{ll+\bar{l}} \end{pmatrix},$$

is an $l \times \bar{l}$ dimensional matrix. Then,

$$M^{\dagger}M = \begin{pmatrix} (B^{\mathrm{T}\dagger}B^{\mathrm{T}}) \otimes I_{n\times n} & 0 \\ 0 & (B^{\dagger}B) \otimes I_{n\times n} \end{pmatrix}.$$

In this representation, the group generator T^σ can be written as

$$T^\sigma = \begin{pmatrix} I_{l\times l}\otimes T^\sigma_c & 0 \\ 0 & -I_{\bar{l}\times\bar{l}}\otimes (T^\sigma_c)^{\mathrm{T}} \end{pmatrix},$$

therefore,

$$\begin{aligned} &\mathrm{Tr}\,(T^\sigma T^\rho (M^\dagger M)^k) \\ &= \mathrm{Tr}\begin{pmatrix} (B^{\mathrm{T}\dagger}B^{\mathrm{T}})^k \otimes (T^\sigma_c T^\rho_c) & 0 \\ 0 & (B^\dagger B)^k \otimes ((T^\sigma_c)^{\mathrm{T}}(T^\rho_c)^{\mathrm{T}}) \end{pmatrix} \\ &= \left[\mathrm{Tr}\,(B^\dagger B)^k + \mathrm{Tr}\,(B^{\mathrm{T}\dagger}B^{\mathrm{T}})^k\right]\cdot \mathrm{Tr}\,(T^\sigma_c T^\rho_c). \end{aligned}$$

Also, since we have

$$\mathrm{Tr}\,(T^\sigma T^\rho) = (l+\bar{l})\,\mathrm{Tr}\,(T^\sigma_c T^\rho_c),$$

$$\mathrm{Tr}\,(M^\dagger M)^k = n\left[\mathrm{Tr}\,(B^\dagger B)^k + \mathrm{Tr}\,(B^{\mathrm{T}\dagger}B^{\mathrm{T}})^k\right],$$

we obtain

$$\begin{aligned} &\mathrm{Tr}\,(T^\sigma T^\rho (M^\dagger M)^k) \\ &= \frac{1}{d_R}\,\mathrm{Tr}\,(T^\sigma T^\rho)\cdot \mathrm{Tr}\,(M^\dagger M)^k, \end{aligned}$$

where $d_R \equiv (l+\bar{l})\cdot n$ is the dimension of the complex representation. This is of the same form as that for the real or pseudo-real representation.

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