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Polynomial Relations in the Centre of $\mathcal{U}_q(sl(N))$

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When the parameter of deformation q is a root of unity, the centre of $\mathcal{U}_q(sl(N))$ contains, besides the usual q-deformed Casimirs, a set of new generators which are basically the m-th powers of all the Cartan generators of $\mathcal{U}_q(sl(N))$. All these central elements are however not independent. In this letter, generalising the well-known case of $\mathcal{U}_q(sl(2))$, we write explicitely polynomial relations satisfied by the generators of the centre. Application to the parametrization of irreducible representations and to fusion rules are sketched.

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1. Symmetric polynomials

In the course of our study of Casimirs for $\mathcal{U}_q(sl(N))$ we shall encounter repeatedly special symmetric polynomials of N variables denoted in this section by $x_1, \cdots, x_N$. The special role played in the classical theory of Casimirs of $sl(N)$ by symmetric polynomials in N variables stems from the fact that the Weyl group of $sl(N)$ is the symmetric group on N letters. The Weyl group acts on the Cartan torus and on its Lie algebra $\Im$ and a well-known theorem of Harish–Chandra says that there is natural isomorphism between the centre of $\mathcal{U}(sl(N))$ and the Weyl-invariant elements of $\mathcal{U}(\Im)$. The more precise corresponding statements in the case of $\mathcal{U}_q(sl(N))$ when q is a root of unity will be given below.

The elementary symmetric polynomials $c_1, \cdots, c_N$ are defined by the identity

$$\prod_{i=1}^{N}(1 - tx_i) = 1 - c_1 t + c_2 t^2 - \cdots + (-1)^N c_N t^N \equiv G(t). \tag{1.1}$$

Hence for $i = 1, \cdots, N$

$$c_i = \sum_{1 \leq j_1 < \cdots < j_i \leq N} x_{j_1} \cdots x_{j_i} \tag{1.2}$$

and it is an old theorem attributed to Newton that any symmetric polynomial in $x_1, \cdots, x_N$ (with coefficients in a ring) is a polynomial in $c_1, \cdots, c_N$ with coefficients in the same ring.

The polynomials of interest for us in the sequel are generalisations of the elementary ones obtained by replacing the variables x_i by their m^{th} power. Hence we define $P_{i,m}^{(N)}(c_1, \cdots, c_N)$ for $i = 1, \cdots, N$ and $m = 1, 2, \cdots$ by the identity

$$\prod_{i=1}^{N}(1 - tx_i^m) = 1 - P_{1,m}^{(N)} t + P_{2,m}^{(N)} t^2 - \cdots + (-1)^N P_{N,m}^{(N)} t^N. \tag{1.3}$$

It is useful to have expressions displaying these polynomials directly in terms of the elementary symmetric polynomials c_i (and not in terms of the variables $x_1, \cdots, x_N$). A method that works nicely for fixed m is to remark that for any primitive m^{th} root of unity q

$$1 - t^m x_i^m = \prod_{l=1}^{m}(1 - q^l t x_i)\ , \tag{1.4}$$

from which we deduce that

$$\prod_{i=1}^{N}(1 - t^m x_i^m) = \prod_{l=1}^{m} G(q^l t)\ . \tag{1.5}$$

Finally we obtain the desired result

$$1 - P_{1,m}^{(N)} t^m + P_{2,m}^{(N)} t^{2m} - \cdots + (-1)^N P_{N,m}^{(N)} t^{Nm} = \prod_{l=1}^{m} G(q^l t). \tag{1.6}$$

This formula makes the computation of the $P_{i,m}^{(N)}$'s for reasonnable values of N and m tractable, at least with the help of a computer.

The polynomials $P_{1,m}^{(N)}$ will play a distinguished role in what follows. The generating function

$$\sum_{m=1}^{\infty} P_{1,m}^{(N)} \frac{t^m}{m} \tag{1.7}$$

is easy to express in terms of $c_1, \cdots, c_N$ because

$$-\log(1 - t x_i) = \sum_{m=1}^{\infty} x_i^m \frac{t^m}{m} \tag{1.8}$$

leading to

$$\sum_{m=1}^{\infty} P_{1,m}^{(N)} \frac{t^m}{m} = -\log G(t) \tag{1.9}$$

Let us end this section with some examples of these polymials. Note first that for our purpose we will have to consider only the particular case $c_N = 1$.

In the case of $\mathcal{U}_q(sl(2))$, we will need $P_{1,m}^{(2)}$, which is closely related to the m^{th} Chebitchev polynomial of the first kind.

In the case of $\mathcal{U}_q(sl(3))$ and $m = 5$, the polynomials of interest are

$$\begin{aligned} P_{1,5}^{(3)}(c_1, c_2) &= c_1^5 - 5c_1^3 c_2 + 5c_1 c_2^2 + 5c_1^2 - 5c_2 \\ P_{2,5}^{(3)}(c_1, c_2) &= c_2^5 - 5c_1 c_2^3 + 5c_1^2 c_2 + 5c_2^2 - 5c_1 \ . \end{aligned} \tag{1.10}$$

In the case of $\mathcal{U}_q(sl(4))$ and $m = 5$, we will need

$$\begin{aligned} P_{1,5}^{(4)}(c_1, c_2, c_3) &= c_1^5 - 5c_1^3 c_2 + 5c_1 c_2^2 + 5c_1^2 c_3 - 5c_2 c_3 - 5c_1 \\ P_{2,5}^{(4)}(c_1, c_2, c_3) &= c_2^5 - 5c_1 c_2^3 c_3 + 5c_1^2 c_2 c_3^2 + 5c_2^2 c_3^2 - 5c_1 c_3^3 + 5c_1^2 c_2^2 \\ &\quad - 5c_2^3 - 5c_1^3 c_3 - 5c_1 c_2 c_3 + 5c_3^2 + 5c_1^2 + 5c_2 \\ P_{3,5}^{(4)}(c_1, c_2, c_3) &= c_3^5 - 5c_2 c_3^3 + 5c_2^2 c_3 + 5c_1 c_3^2 - 5c_1 c_2 - 5c_3 \ . \end{aligned} \tag{1.11}$$

2. $\mathcal{U}_q(sl(N))$ at roots of unity

Let $\{\alpha_1, ..., \alpha_{N-1}\}$ be the set of simple roots of $sl(N)$. We define vectors ϵ_1,...,ϵ_N by $\alpha_i = \epsilon_i - \epsilon_{i+1}$ and $\sum_{i=1}^{N} \epsilon_i = 0$.

The "simply connected" quantum group $\mathcal{U}_q(sl(N))$ is defined by the generators e_i, and f_i, for $i = 1, ..., N-1$, and $k_{\pm\epsilon_i}$ for $i = 1, ..., N$ and the relations

$$\left\{\begin{aligned} k_{\beta_1} k_{\beta_2} &= k_{\beta_1+\beta_2} \ , \\ k_{\epsilon_i} e_j k_{\epsilon_i}^{-1} &= q^{\delta_{ij}-\delta_{i-1,j}} e_j \ , \\ k_{\epsilon_i} f_j k_{\epsilon_i}^{-1} &= q^{-\delta_{ij}+\delta_{i-1,j}} f_j \ , \\ [e_i, f_j] &= \delta_{ij} \frac{k_{\alpha_i} - k_{\alpha_i}^{-1}}{q - q^{-1}} \ , \\ [e_i, e_j] &= [f_i, f_j] = 0 \qquad \text{for} \qquad |i-j| \geq 2 \ , \\ e_i^2 e_{i\pm1} - (q+q^{-1}) & e_i e_{i\pm1} e_i + e_{i\pm1} e_i^2 = 0 \ , \\ f_i^2 f_{i\pm1} - (q+q^{-1}) & f_i f_{i\pm1} f_i + f_{i\pm1} f_i^2 = 0 \ , \end{aligned}\right. \tag{2.1}$$

Let $\mathcal{U}^0$ be the subalgebra generated by the k_{ϵ_i}'s, and $\mathcal{U}^+, \mathcal{U}^-$ the subalgebras generated by the e_i's, f_i's, respectively.

Two sets of quantum analogues of the roots vectors are inductively defined as

$$\left\{\begin{array}{ll} e_{i,i+1} = \tilde{e}_{i,i+1} \equiv e_i & \text{for } i = 1, ..., N-1 \\ e_{i,j+1} = e_{ij} e_j - q^{-1} e_j e_{ij} & \text{for } i < j \\ \tilde{e}_{i,j+1} = \tilde{e}_{ij} e_j - q e_j \tilde{e}_{ij} & \text{for } i < j \end{array}\right. \tag{2.2}$$

and idem for the f_{ij} and $\tilde{f}_{ij}$.

Quantum analogues of Poincaré–Birkhoff–Witt bases can be built with ordered monomials in these generators [1].

When q is not a root of unity, there exists a quantum analogue of Harish–Chandra theorem [2,3]: there exists an algebra isomorphism h from Z, the centre of $\mathcal{U}_q(sl(N))$, to the algebra of symmetric polynomials in the $k_{2\epsilon_i}$. This isomorphism h can be written as $h = \gamma^{-1} \circ h'$ with the following notations: h' is the projection on $\mathcal{U}^0$, within the direct sum $\mathcal{U} = \mathcal{U}^0 \oplus (\mathcal{U}^- \mathcal{U} + \mathcal{U}\mathcal{U}^+)$ with $\mathcal{U} \equiv \mathcal{U}_q(sl(N))$; γ is the automorphism of $\mathcal{U}^0$ given by $\gamma(k_{2\epsilon_i}) = q^{N+1-2i} k_{2\epsilon_i}$.

A set of generators of Z is given by

$$\{\mathcal{C}_i = h^{-1}(c_i(k_{2\epsilon_1}, ..., k_{2\epsilon_N}))\}_{i=1,...,N-1}. \tag{2.3}$$

An expanded expression for these generators (denoted there by $\tilde{c}_k$) appears in [4] in the form (up to slight changes of convention and normalization)

$$\mathcal{C}_i = q^{i(N-i)} \mathcal{N}_i(q^{-2})^{-1} \mathcal{N}_{N-i}(q^{-2})^{-1} \sum_{\sigma,\sigma' \in \mathcal{S}(N)} (-q^{-1})^{l(\sigma)+l(\sigma')} l^{(+)}_{\sigma_1\sigma'_1} ... l^{(+)}_{\sigma_i\sigma'_i} l^{(-)}_{\sigma_{i+1}\sigma'_{i+1}} ... l^{(-)}_{\sigma_N\sigma'_N} \ , \tag{2.4}$$

where $\mathcal{N}_i(x) = \prod_{n=1}^{i}(1+\cdots+x^{n-1})$, where $l(\sigma)$ is the length of the shortest expression of the permutation σ in terms of simple transpositions, and where

$$
\begin{aligned}
l_{ii}^{(+)} &= \left(l_{ii}^{(-)}\right)^{-1} = k_{\epsilon_i} \\
l_{ij}^{(+)} &= l_{ji}^{(-)} = 0 \qquad \text{for} \quad i > j \\
l_{ij}^{(+)} &= (q-q^{-1})(-1)^{j-i+1}\tilde{f}_{ij}k_{\epsilon_i} \qquad \text{for} \quad i < j \\
l_{ij}^{(-)} &= (q-q^{-1})(-1)^{j-i}k_{-\epsilon_i}\tilde{e}_{ij} \qquad \text{for} \quad i > j \;.
\end{aligned}
$$

The first and last of these Casimirs are explicitly given by

$$\mathcal{C}_1 = \sum_{i=1}^{N} q^{N+1-2i}k_{2\epsilon_i} + (q-q^{-1})^2 \sum_{1\le i<j\le N} (-1)^{j-i-1}q^{N+1-i-j}\tilde{f}_{ij}e_{ij}k_{\epsilon_i+\epsilon_j} \tag{2.5}$$

and

$$\mathcal{C}_{N-1} = \sum_{i=1}^{N} q^{-N-1+2i}k_{-2\epsilon_i} + (q-q^{-1})^2 \sum_{1\le i<j\le N} (-1)^{j-i-1}q^{-N-1+i+j}f_{ij}\tilde{e}_{ij}k_{-\epsilon_i-\epsilon_j} \tag{2.6}$$

When q is a root of unity, the image Z_1 of h is still a well-defined central sub-algebra of $\mathcal{U}_q\,(sl(N))$ [3], but it does not generate the whole centre. Let Z_0 be the sub-algebra of $\mathcal{U}_q\,(sl(N))$ generated by the elements f_{ij}^m, e_{ij}^m and $k_{m\epsilon_i}$. (We could also replace f_{ij} by $\tilde{f}_{ij}$, or e_{ij} by $\tilde{e}_{ij}$, this would lead to the same Z_0.) When m' is odd, these elements are central, and the centre Z of $\mathcal{U}_q\,(sl(N))$ is actually generated by Z_0 and Z_1 [3].

3. Relations in the centre of $\mathcal{U}_q\,(sl(N))$

Theorem: *If m' is odd, the following relations are satisfied in the centre of $\mathcal{U}_q\,(sl(N))$,*

$$
\begin{aligned}
P_{1,m}^{(N)}(\mathcal{C}_1,\ldots,\mathcal{C}_{N-1}) &= \sum_{i=1}^{N} q^{m(N+1)}k_{2m\epsilon_i} \\
&+ (q-q^{-1})^{2m} \sum_{1\le i<j\le N} (-1)^{m(j-i-1)}q^{m(N+1-i-j)}\tilde{f}_{ij}^m e_{ij}^m k_{m\epsilon_i+m\epsilon_j}
\end{aligned} \tag{3.1}
$$

and

$$
\begin{aligned}
P_{N-1,m}^{(N)}(\mathcal{C}_1,\ldots,\mathcal{C}_{N-1}) &= \sum_{i=1}^{N} q^{-m(N+1)}k_{-2m\epsilon_i} \\
&+ (q-q^{-1})^{2m} \sum_{1\le i<j\le N} (-1)^{m(j-i-1)}q^{m(-N-1+i+j)}f_{ij}^m \tilde{e}_{ij}^m k_{-m\epsilon_i-m\epsilon_j}
\end{aligned} \tag{3.2}
$$

Remark 1: Actually, all the powers of q are equal to 1 since m' is odd, but we conjecture that these formulæ remain true for even m'. [In this case, the term $\tilde{f}_{ij}^m$, e_{ij}^m and $k_{m\epsilon_i+m\epsilon_j}$ are not individually central, but their products are.]

Remark 2: To get the right hand sides of these relations, one simply replaces each term (including numerical factors) in the expression of $\mathcal{C}_1$ (resp. $\mathcal{C}_{N-1}$) by its m^{th} power. This remarkable relationship seems to hold between $P_{i,m}^{(N)}(\mathcal{C}_1,...,\mathcal{C}_{N-1})$ and $\mathcal{C}_i$ for the other values of i as well, if $\mathcal{C}_i$ is written in a suitable Poincaré–Birkhoff–Witt basis.

Proof:

a. We first apply the relations (2.1) and (2.2) in order to write (3.1) and (3.2) and the $\mathcal{C}_i$'s in the Poincaré–Birkhoff–Witt basis. Then

$$h\left(P_{1,m}^{(N)}(\mathcal{C}_1,...,\mathcal{C}_{N-1})\right) = P_{1,m}^{(N)}(h(\mathcal{C}_1),...,h(\mathcal{C}_{N-1})) = \sum_{i=1}^{N} q^{m(N+1)} k_{2m\epsilon_i} \tag{3.3}$$

(and the corresponding formula with $P_{N-1,m}^{(N)}$). This follows from the definitions of the first section. It appears then that this projection belongs to Z_0, and hence so does the whole result ([3] Prop. 6.3.c). This part of the proof also applies to $P_{i,m}^{(N)}(\mathcal{C}_1,...,\mathcal{C}_{N-1})$ for $1 < i < N-1$, whereas the second part is limited to the cases $i = 1$ or $i = N-1$.

b. We can then use considerations on the degrees of the monomials appearing in $P_{1,m}^{(N)}$ (and $P_{N-1,m}^{(N)}$) to complete the proof. The term of highest degree of $P_{1,m}^{(N)}$ (resp. $P_{N-1,m}^{(N)}$) is indeed $\mathcal{C}_1^m$ (resp. $\mathcal{C}_{N-1}^m$), and it is also the only term of degree m. According to the form of the $\mathcal{C}_i$ (2.4), only monomials of degree at least equal to m can contribute to non trivial terms belonging to Z_0: a necessary condition is indeed that the products of root vectors they contain correspond to an element of the root lattice R belonging to mR. For the same reason, the contribution of the monomial of degree m is precisely the second part of the right hand side of (3.1) (resp. (3.2)).

Relations (3.1), (3.2) differ, for $N > 2$, from the equation in the last remark of [3]. In particular, the degree of the polynomial is different. In the case of $\mathcal{U}_q(sl(2))$, the relation (3.1) was already given in [5].

4. Applications

a. Parametrization of generic irreducible representations:

We know from [6] that generic irreducible representations of $\mathcal{U}_q(sl(N))$ are characterized by the values of the central elements on them. Once the values of the elements of Z_0 are determined, a choice between m^{N-1} values for $\mathcal{C}_1,...,\mathcal{C}_N$ remain. A nice way to parametrize them is to write, for a representation ρ,

$$\rho(\mathcal{C}_i) = c_i(\zeta_1,...,\zeta_N) \tag{4.1}$$

with c_i defined in (1.2) and $\prod_{i=1}^{N}\zeta_i = 1$. (Note the absence of h^{-1}, by comparison with (2.3).) The m^{N-1} irreducible representations on which the elements of Z_0 take the same value simply correspond to the parameters

$$q^{p_1}\zeta_1, ..., q^{p_N}\zeta_N, \tag{4.2}$$

with $p_1, ..., p_N \in \mathbf{Z}$ and $\sum_1^N p_i = 0 \bmod m$. Since

$$\rho\left(P_{i,m}^{(N)}(\mathcal{C}_1, ..., \mathcal{C}_{N-1})\right) = c_i(\zeta_1^m, ..., \zeta_N^m) \tag{4.3}$$

for $1 \leq i \leq N-1$, these sets of parameters indeed correspond to the sets of solutions for the $\mathcal{C}_i$'s, to the system of $N-1$ equations including (3.1) and (3.2).
With this parametrization, the ζ_i become powers of q when the central elements e_{ij}^m, f_{ij}^m and $k_{2m\epsilon_i}$ take the values 0, 0 and 1 respectively. In this highly non-generic case, a finite number of irreducible representations is related to the same parametrization. These representations are q-deformations of classical representations.

b. Application to fusion rules:
We suggest that these relations and the above parametrization could help in the study of fusion of unrestricted (generic) irreducible representations of $\mathcal{U}_q(sl(N))$, as in [7] in the case of $\mathcal{U}(sl(2))$. The strategy would be the following: to evaluate the values of the elements of Z_0 in the tensor product of two irreducible representations (they are scalar); find then a solution for the parameters ζ_i compatible with these values. Then all the irreducible representations characterized by the parameters (4.2) should appear in the fusion rule, with multiplicity 1 in the case generic$\otimes$minimal–periodic, and with multiplicity $m^{(N-1)(N-2)/2}$ in the case generic$\otimes$generic.

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