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Dual formulation of cosmic strings and vortices

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We study four-dimensional systems of global, axion, and local strings. By using the path integral formalism, we derive the dual formulation of these systems, where Goldstone bosons, axions, and massive vector bosons are described by an antisymmetric tensor field, and strings appear as sources for this tensor field. We also show how a magnetic monopole attached to a local string is described in the dual formulation. We conclude with some remarks.

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I. INTRODUCTION

The four-dimensional field theories of a complex scalar field and other fields with an Abelian symmetry are known to have topological defects, such as global, axionic, or local strings in the broken symmetry phase. We derive the dual formulation of these theories by using the path integral formalism. The dual formulation has been extensively used to study the phase structure of these theories [1] and the dynamics of cosmic strings and superfluid vortices [2–7]. But the dual formulation of strings has been derived usually by using field equations [4] or canonical transformations, [5] making the whole situation somewhat unsatisfactory.

On the other hand, the dual formulation of the theory of a complex scalar field has been derived in the path integral formalism without including vortices [8]. Recently, the dual formulation of three-dimensional systems with vortices has been derived in the path integral formalism to study the dynamics of vortices in Chern-Simons Higgs systems [9]. Here we extend the idea of Ref. [9] to get the dual formulation of four-dimensional systems with vortices and strings. While there has been a large body of literature about the dual formulation, we feel our work is somewhat new and could be used to study quantum features of the vortex and string dynamics.

There are several advantages of the dual formulation of cosmic strings. As the interaction between strings and other fields is more explicit, one can understand the string evolution more clearly. A cosmic string or superfluid vortex moving on a background charge density would feel the so-called Magnus force. This Lorentz-type force can be seen directly in the dual formulation [6,9]. When the length scale of a string motion is larger than the string core size, one can obtain an effective action which describes the string dynamics and its interaction with low-energy modes.

The plan of this paper is as follows. In Sec. II we study the theory of a complex scalar field with a global U(1) symmetry. In the dual formulation, a global string appears as the source of an antisymmetric tensor field, which represents the Goldstone boson. In Sec. III we study the dual formulation of a theory where global strings appear as axionic strings. As shown in Ref. [10], the electromagnetic charge is not conserved in this theory without taking into account the chiral fermion zero modes on the string. In Sec. IV we study the Maxwell Higgs theory where there appear Nielsen-Olesen vortices [11]. We derive the dual formulation where the gauge field is integrated out and is not explicit. The antisymmetric field has a Higgs-like coupling, becoming massive. Magnetic monopoles attached to local strings are also included in this dual formulation. In Sec. V we conclude with some remarks.

II. GLOBAL STRINGS

We consider the theory of a complex scalar field $\phi=fe^{i\theta}/\sqrt{2}$, whose Lagrangian is given by

$$\mathcal{L}=\tfrac{1}{2}(\partial_\mu f)^2+\tfrac{1}{2}f^2(\partial_\mu\theta)^2-U(f) \;. \tag{2.1}$$

As the theory (1) is invariant under a global transformation, $\theta\to\theta+\text{const}$, there are a conserved current

$$j_\mu=f^2\partial_\mu\theta \tag{2.2}$$

and the corresponding global charge

$$Q=\int d^3r\, f^2\partial_0\theta \;. \tag{2.3}$$

The ground state of the energy functional lies in a broken symmetry phase because of either the potential or a background charge density.

To understand the quantum aspect, we use the generating functional

$$Z=\langle F|e^{-iHT}|I\rangle = \int [f\,df][d\theta]\bar\Psi_F \exp\left\{ i\int d^4x\,\mathcal{L}\right\}\Psi_I \;, \tag{2.4}$$

where $[f]\equiv\prod_x f(x)$ is the Jacobian factor for the radial

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coordinate of the scalar field. The initial and final wave functions $\Psi_{F,I}$ give the necessary boundary conditions.

A given field configuration in the path integral could contain strings, around each of which the value of the θ field changes by 2π times an integer. We can in principle split the θ field into two parts:

$$\theta(t,\mathbf{r})=\bar{\theta}(t,\mathbf{r})+\eta(t,\mathbf{r}) , \tag{2.5}$$

where $\bar{\theta}$ describes a given configuration of vortices and η represents single-valued fluctuations around the vortex configuration. The energy density and the complex scalar field ϕ should be single valued, or equivalently $\partial_\mu\bar{\theta}$ and $e^{i\bar{\theta}}$ should be so. Each elementary string is described by parametrized positions $\mathbf{q}_a(\sigma)$ or covariantly $q_a^\mu(\tau,\sigma)$, where $\sigma^\alpha=(\tau,\sigma)$ is the string world sheet coordinate. We choose σ so that $\bar{\theta}$ increases by 2π when one wraps the string in the direction of increasing σ with the right hand.

For a straight string along the z axis, we know that $(\partial_x\partial_y-\partial_y\partial_x)\bar{\theta}=2\pi\delta^2(\boldsymbol{\rho})$. By covariantizing it, we get the antisymmetric tensor vortex current

$$\begin{aligned}K^{\mu\nu}(x)&\equiv\epsilon^{\mu\nu\rho\sigma}\partial_\rho\partial_\sigma\bar{\theta}\\&=2\pi\sum_a\int d\tau\, d\sigma(\dot{q}_a^\mu q_a'^\nu-\dot{q}_a^\nu q_a'^\mu)\\&\qquad\times\delta^4(x^\rho-q_a^\rho(\tau,\sigma)) ,\end{aligned} \tag{2.6}$$

where the dot and prime indicate the differentiation by τ and σ, respectively. $K^{\mu\nu}$ is independent of the reparametrizations of σ^α up to a sign and satisfies the conservation law $\partial_\mu K^{\mu\nu}=0$. From Eq. (2.6), we can get

$$\begin{aligned}\partial_i\bar{\theta}(t,\mathbf{r})&=\epsilon_{ijk}\partial_j\int d^3s\frac{K^{0k}(t,\mathbf{s})}{4\pi|\mathbf{r}-\mathbf{s}|} ,\\ \partial_0\bar{\theta}(t,\mathbf{r})&=\epsilon_{ijk}\partial_i\int d^3s\frac{K^{jk}(t,\mathbf{s})}{8\pi|\mathbf{r}-\mathbf{s}|} ,\end{aligned} \tag{2.7}$$

by using a time-independent Green's function. By integrating Eq. (2.7), we get

$$e^{i\bar{\theta}(t,\mathbf{r})}=\exp\left\{i\int_{\mathbf{r}_0}^{\mathbf{r}}d\mathbf{s}\cdot\nabla\bar{\theta}(t,\mathbf{s})\right\} , \tag{2.8}$$

where $\mathbf{r}_0$ is a reference point. The exponent at the right-hand side of Eq. (2.8) is multivalued, but the exponential is single valued.

The measure for the θ variable becomes

$$[d\theta]=[d\bar{\theta}][d\eta]=[dq_a^\mu][d\eta] , \tag{2.9}$$

which means that we sum over single-valued fluctuations around a given configuration of strings and then sum over all possible string configurations. A typical string configuration would have the creation, annihilation, crossing, overlapping, and exchange of strings, making the Jacobian factor from $[d\bar{\theta}]$ to $[dq_a^\mu]$ rather complicated. The periodicity of the θ variable affects only the quantizations of both global charge and vorticity, due to the gradient term $(\partial_\mu\theta)^2$ in the Lagrangian.

In the generating functional Z, we can linearize the θ kinetic term by introducing an auxiliary vector field C^μ:

$$\begin{aligned}&\exp\left[i\int d^4x[\tfrac{1}{2}f^2(\partial_\mu\theta)^2]\right]\\&\quad=\int[f^{-4}dC^\mu]\exp\left[i\int d^4x\left\{-\frac{1}{2f^2}(C^\mu)^2+C^\mu\partial_\mu\bar{\theta}\right.\right.\\&\qquad\qquad\left.\left.+C^\mu\partial_\mu\eta\right\}\right] .\end{aligned} \tag{2.10}$$

As η is single valued, one can integrate over η in the standard way, leading to

$$\int[d\eta]\exp\left[i\int d^4x\, C^\mu\partial_\mu\eta\right]=\delta(\partial_\mu C^\mu) . \tag{2.11}$$

Now we introduce the dual antisymmetric tensor field $B_{\mu\nu}$ to satisfy

$$\begin{aligned}&\int[dC^\mu]\delta(\partial_\mu C^\mu)\cdots\\&\quad=\int[dC^\mu][dB_{\mu\nu}]\delta(C^\mu-\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma})\cdots\end{aligned} \tag{2.12}$$

and the dots denote the rest of the integrand. There would be an infinite gauge volume which can be taken care of later, but there is no nontrivial Jacobian factor as the change of variables is linear. By using the fact that

$$\epsilon^{\mu\nu\rho\sigma}(\partial_\mu\bar{\theta})\partial_\nu B_{\rho\sigma}=K^{\mu\nu}B_{\mu\nu} \tag{2.13}$$

up to a single-valued total derivative, we can integrate over C^μ, resulting in the dual Lagrangian

$$\mathcal{L}_D=\frac{1}{2}(\partial_\mu f)^2-U(f)+\frac{1}{12f^2}H_{\mu\nu\rho}^2+\frac{1}{2}B_{\mu\nu}K^{\mu\nu} , \tag{2.14}$$

where $H_{\mu\nu\rho}\equiv\partial_\mu B_{\nu\rho}+\partial_\nu B_{\rho\mu}+\partial_\rho B_{\mu\nu}$ is the field strength of $B_{\mu\nu}$. Note that the kinetic term for $B_{\mu\nu}$ has the standard normalization, e.g., $(\partial_0 B_{12})^2/2$, with a coupling constant f^2. The dual Lagrangian (2.14) is invariant under a local gauge symmetry, $B_{\mu\nu}\to B_{\mu\nu}+\partial_\mu\Lambda_\nu-\partial_\nu\Lambda_\mu$.

The resulting path integral becomes

$$Z=\int[f^{-3}df][dq_a^\mu][dB_{\mu\nu}]\bar{\Psi}_F\exp^{i\int d^4x\mathcal{L}_D}\Psi_I . \tag{2.15}$$

The gauge volume in $dB_{\mu\nu}$ could have been divided without making the above derivation complicated. The initial and final states should be rewritten in dual variables. The Goldstone boson is now described by $B_{\mu\nu}$ and strings appear as a source for $B_{\mu\nu}$.

The mass of vortices arises from the cloud of the $f,B_{\mu\nu}$ fields surrounding them. The variation of B_{0i} leads to Gauss's law

$$-\partial_j\left[\frac{1}{f^2}H_{0ij}\right]+K^{0i}=0 , \tag{2.16}$$

which dictates the field cloud around vortices. When the string of vorticity n is lying on the z axis, the f field would vanish at the string as one approaches from the x-y plane like $f\sim\rho^n$, where ρ is the radial distance on the plane. This can be seen directly from the f equation in the original formulation or from the f and $B_{\mu\nu}$ equations in the dual formulation. The classical relation between the original fields and dual fields can be found from the field equations from the Lagrangians at each step of the transformations. They are related to each other by

$$f^2\partial^\mu\theta=\tfrac{1}{6}\epsilon^{\mu\nu\rho\sigma}H_{\nu\rho\sigma} \ . \tag{2.17}$$

Let us consider now the string dynamics briefly. In the original formulation, the string motion is determined by the field equations for f,θ. In the dual formulation, there seems to be an additional equation of motion for strings derived directly from the variation of $q_a^\mu(\tau,\sigma)$. From the dual lagrangian (2.14), we get, from the variation of δq_a^ρ,

$$\delta\mathcal{L}_D=\sum_a \frac{1}{2}\int d\tau\, d\sigma\, H_{\mu\nu\rho}(q_a)(\dot q_a^\mu q_a'^\nu-\dot q_a^\nu q_a'^\mu)\delta q_a^\rho \ . \tag{2.18}$$

We know that $f^2\partial_\mu\theta$ vanishes at the string, and then Eq. (2.17) implies that the above variation vanishes. Thus the dynamics of strings is again determined by their surrounding field in the dual formulation. This is consistent with the picture that the kinetic energy of vortices originates entirely from the field cloud around them.

This leads naturally to the question of whether there is a low-energy effective action in terms of string positions which describes the string dynamics. As Goldstone bosons are massless, the effective action should describe both strings and Goldstone bosons. Here we are interested in physics whose energy scale is much less than the mass of the f field. Their assumption cannot be valid all the time as strings will annihilate each other. The effective action is usually given as the Nambu action for strings and the action for the antisymmetric tensor field [2–7]:

$$S_{\text{eff}}=\sum_a \mu_0\int d^2\sigma_a\sqrt{-\gamma} + \int d^4x\left\{\frac{1}{12v^2}H^2_{\mu\nu\rho}+B_{\mu\nu}K^{\mu\nu}\right\} , \tag{2.19}$$

where v is the asymptotic value of f, μ_0 is the bare string tension, and γ_a is the determinant of the induced metric on the string,

$$\gamma_{a\alpha\beta}=\frac{\partial q_a^\mu}{\partial\sigma^\alpha}\frac{\partial q_{a\mu}}{\partial\sigma^\beta} , \tag{2.20}$$

where v is the vacuum expectation value of f. The bare string mass per unit length μ_0 comes from the string core region of a size $1/m_f$. There is a logarithmically divergent term coming from the θ or $B_{\mu\nu}$ field. Note that the f field does not approach exponentially to its vacuum value at spatial infinity. For a straight string lying along the z axis, one can see, in the cylindrical coordinate (ρ,φ,z),

$$f\rightarrow v-\frac{1}{m_f\rho^2} , \tag{2.21}$$

as $\rho\rightarrow\infty$.

In addition, the action should be modified when there is a background charge density. The reason is that at low energy there is a sound wave of speed v_s rather than Goldstone bosons. There would be an effective action for sound waves and strings when strings move slower than the sound speed. In addition, there is a Magnus force on the string from the background charge. It would be interesting to see what happens when strings move faster than the sound speed.

III. AXIONIC STRINGS

Let us now consider the case where the scalar field describes an axion field. Here the theory is QED-like with an additional neutral scalar field and electrons get the mass via a chiral coupling. The global Abelian symmetry is broken. When the fermion is very massive, integration over a fermionic field introduces an effective interaction between the gauge field and the phase of the scalar field. For simplicity, we will consider the case where the gauge symmetry is Abelian. (Detailed aspects of anomalies, chiral zero modes, and bosonization ideas in the following discussion have appeared in Ref. [10].) The resulting interaction between an axion and two photons is given by

$$\mathcal{L}_{a\gamma\gamma}=-\frac{\theta}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \ . \tag{3.1}$$

This effective action is multivalued and not well defined at each string as θ loses its meaning.

The Lagrangian to consider is

$$\mathcal{L}_A=\frac{1}{2}(\partial_\mu f)^2+\frac{1}{2}f^2(\partial_\mu\theta)^2-U(f)-\frac{1}{4e^2}F^2_{\mu\nu}+Z^\mu\partial_\mu\theta , \tag{3.2}$$

where the Chern-Simons current is

$$Z^\mu=\frac{1}{8\pi^2}\epsilon^{\mu\nu\rho\sigma}A_\nu\partial_\rho A_\sigma , \tag{3.3}$$

satisfying $\partial_\mu Z^\mu=(1/32\pi^2)\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$. While the Lagrangian (3.2) is single valued, it is not gauge invariant as Z^μ is not. The current from the Lagrangian (3.2) is given by

$$J^\mu_A=-\frac{1}{8\pi^2}\epsilon^{\mu\nu\rho\sigma}\partial_\nu\theta F_{\rho\sigma}+\frac{1}{8\pi^2}K^{\mu\nu}A_\nu , \tag{3.4}$$

which is not conserved,

$$\partial_\mu J^\mu_A=-\frac{1}{16\pi^2}K^{\mu\nu}F_{\mu\nu} \ . \tag{3.5}$$

When the string lies along the z axis, $K^{0z}=2\pi\delta^2(\rho)$ and $\partial_\varphi\theta=1$ in the cylindrical coordinate. The current (3.4) becomes

$$J^\rho_A=-\frac{F_{0z}}{4\pi^2\rho},\quad J^z_A=-\frac{1}{4\pi}A_0\delta^2(\rho) \ . \tag{3.6}$$

For a uniform electric field along $\hat z$, there is a radial current moving to the string.

Chiral zero modes on strings should be taken into account to maintain the gauge invariance or the current conservation. The effect of these chiral zero modes can be seen more directly by the bosonization [10]. With the string metric $\gamma^{\alpha\beta}$ and the antisymmetric tensor field $\epsilon^{\alpha\beta}$, the Lagrangian for the chiral boson $\chi(\sigma^\alpha)$ on a string is given by

$$L_\chi=\frac{1}{2}\left[\partial_\alpha\chi-\frac{1}{2\sqrt{\pi}}A_\alpha\right]^2-\frac{1}{2\sqrt{\pi}}\chi\epsilon^{\alpha\beta}\partial_\alpha A_\beta+(\gamma^{\alpha\beta}-\epsilon^{\alpha\beta})\lambda_\alpha\left[\partial_\beta\chi-\frac{1}{2\sqrt{\pi}}A_\beta\right], \quad (3.7)$$

where the gauge field is evaluated at the string. The chiral Lagrangian is not invariant under the gauge transformation $\delta\chi=2\sqrt{\pi}\Lambda$ and $\delta A_\alpha=\partial_\alpha\Lambda$. (While it is trivial to introduce the world sheet metric on the string, we use the Cartesian coordinate.)

Let us put the string on the z axis and define the light cone variables on the string, $x^{\pm}=(1/\sqrt{2})(t\pm z)$ with $g_{+-}=g_{-+}=\epsilon_{+-}=-\epsilon^{+-}=1$. The field equations from Eq. (3.7) with $t=\tau$, $z=\sigma$ imply that λ_α can be chosen to be zero and that

$$\partial_-\chi-\frac{1}{2\sqrt{\pi}}A_-=0\ , \quad (3.8)$$

where $A_\pm=(A_t\pm A_z)/\sqrt{2}$. The electric current due to the chiral bosons is then

$$J_\chi^+=\frac{1}{4\pi}A_-,\quad J_\chi^-=-\frac{1}{\sqrt{\pi}}\partial_+\chi+\frac{1}{4\pi}A_+\ , \quad (3.9)$$

which is not conserved:

$$\partial_\alpha J_\chi^\alpha=\frac{1}{4\pi}\epsilon^{\alpha\beta}\partial_\alpha A_\beta\ . \quad (3.10)$$

For the axionic string lying along the z axis, the sum of currents from Eqs. (3.4) and (3.9) becomes

$$J^\rho=-\frac{1}{4\pi^2}\frac{1}{\rho}F_{0z}\ ,$$
$$J^+=0\ , \quad (3.11)$$
$$J^-=\left[-\frac{1}{\sqrt{\pi}}\partial_+\chi+\frac{1}{2\pi}A_+\right]\delta^2(\rho)\ ,$$

which is conserved due to the field equation for χ. Note that this current is chiral on the string. The combined action from Eqs. (3.2) and (3.7) is then gauge invariant.

Let us consider the dual formulation of this system. We take similar steps as in Sec. II. We first split θ into $\bar\theta$ and η, introduce C^μ, and integrate over η to get

$$\int[d\eta]\exp\left[i\int d^4x\,(C^\mu+Z^\mu)\partial_\mu\eta\right]=\delta(\partial_\mu(C^\mu+Z^\mu))\ . \quad (3.12)$$

We again introduce $B_{\mu\nu}$ to solve the δ function:

$$\delta(\partial_\mu(C^\mu+Z^\mu))=\int[dB_{\mu\nu}]\delta(C^\mu+Z^\mu-\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma})\cdots\ . \quad (3.13)$$

Integrating over C^μ leads to the dual Lagrangian

$$\mathcal{L}_{AD}=\frac{1}{2}(\partial_\mu f)^2-U(f)-\frac{1}{4e^2}F_{\mu\nu}^2+\frac{1}{12f^2}\tilde{H}_{\mu\nu\rho}^2+\frac{1}{2}B_{\mu\nu}K^{\mu\nu}\ , \quad (3.14)$$

where $\tilde{H}_{\mu\nu\rho}\equiv H_{\mu\nu\rho}-\epsilon_{\mu\nu\rho\sigma}Z^\sigma$. Even though the Chern-Simons current is not gauge invariant under the transformation $\delta A_\mu=\partial_\mu\Lambda$, the field strength $\tilde{H}_{\mu\nu\rho}$ can be made to be so if the $B_{\mu\nu}$ field is also transformed as

$$\delta B_{\mu\nu}=\frac{1}{8\pi^2}\Lambda F_{\mu\nu}\ . \quad (3.15)$$

The dual Lagrangian (3.14) is not invariant under this gauge transformation due to the last term.

We have also a chiral boson χ_a and a Lagrangian multiplier $\lambda_{a\alpha}$ with a Lagrangian $\mathcal{L}_{\chi_a}$ for each string q_a. When we add the action from Eq. (3.14) and actions for chiral fields on strings, the resulting action is gauge invariant. The generating functional has become

$$Z=\int[f^{-3}df\,dB_{\mu\nu}dq_a^\mu][d\chi_a d\lambda_{a\alpha}]\times\exp\left[i\left\{\int d^4x\,\mathcal{L}_{AD}+\sum_a\int d^2\sigma_a\mathcal{L}_{\chi_a}\right\}\right]. \quad (3.16)$$

The conserved electromagnetic current (3.10) in the dual formulation is given by

$$J^\mu=\frac{1}{16\pi^2f^2}\tilde{H}^{\mu\nu\rho}F_{\nu\rho}+\sum_a\int d^2\sigma\frac{\partial q_a^\mu}{\partial\sigma^\alpha}\frac{\gamma^{\alpha\beta}+\epsilon^{\alpha\beta}}{2}\times\left[-\frac{1}{\sqrt{\pi}}\partial_\beta\chi_a+\frac{1}{2\pi}A_\beta(q_a)\right]. \quad (3.17)$$

The relation between original fields and dual fields outside strings is given as

$$f^2\partial^\mu\theta=\tfrac{1}{6}\epsilon^{\mu\nu\rho\sigma}\tilde{H}_{\mu\rho\sigma}\ , \quad (3.18)$$

and Gauss's law from the variation of B_{i0} becomes

$$-\partial_j\left[\frac{1}{f^2}\tilde{H}_{0ij}\right]+K^{0i}=0\ . \quad (3.19)$$

Gauss's law from the variation of A_0 is given by

$$\partial_iF_{0i}+J^0=0\ . \quad (3.20)$$

The dual formulation of axionic strings has been used to study their interaction with the electromagnetic field [10].

There is an axial symmetry in the theory (3.2) coming from the shift of θ, which is broken by the anomaly. The gauge-dependent but conserved current is given by

$$J_\mu^5=f^2\partial_\mu\theta+Z_\mu=\tfrac{1}{6}\epsilon_\mu^{\nu\rho\sigma}H_{\nu\rho\sigma}\ . \quad (3.21)$$

The total chiral charge $Q_5=\int d^3x\,J_0^5=\int d^3x\,H_{123}$ is gauge invariant and well defined, say, if we put the system in a periodic box. Originally, there is a fermion contribution $\bar{\psi}\gamma_\mu\gamma_5\psi$ to the axial-vector current (3.20) and so there will be a contribution from the fermionic zero modes on axion strings. Naively, their contribution to Q_5 is proportional to the sum of winding numbers, $\sum_a \int d\sigma\, \partial_\sigma \chi_a$, and conserved separately until strings exchange, cross, or annihilate. We expect also that axionic strings would feel the presence of the axial-vector charge via a magnus force, even though the charge density is not gauge invariant. It deserves further investigation.

IV. LOCAL STRINGS

Let us now consider the dual formulation of the Maxwell Higgs systems. Some aspects have been studied in Refs. [12,13] and the last two papers of Ref. [6]. The Lagrangian is

$$\mathcal{L}_M = -\frac{1}{4e^2}F_{\mu\nu}^2 + \frac{\lambda}{8}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} + A_\mu J^\mu_{\rm ext} + \tfrac{1}{2}(\partial_\mu f)^2 + \tfrac{1}{2}f^2(\partial_\mu\theta + A_\mu)^2 - U(f) \; . \tag{4.1}$$

We assume that there are magnetic monopoles. The gauge field can be split into the monopole part $\bar{A}_\mu$ and the single-valued rest $\mathcal{A}_\mu$. Magnetic monopoles are described by $\bar{A}_\mu$ with a Dirac string or equivalently by the Wu-Yang construction. However, $\partial_i\theta + \bar{A}_i$ is gauge invariant and well defined except at $f=0$. Suppose that there is no point around a monopole where $f=0$. Then $\epsilon_{ijk}\partial_j(\partial_k\theta + \bar{A}_k) = B_i^{\rm mon}$ without a Dirac string, which is impossible. There should be a Dirac string for $\partial_i\theta + \bar{A}_i$ attached to the monopole along which f is zero and θ changes by 2π when one goes around it. This is exactly the cosmic string. In other words, the magnetic field from a magnetic monopole is shielded by the Meissner effect and is channeled to the spatial infinity by a cosmic string.

The Dirac string of $\partial_\mu\theta + A_\mu$ becomes a real local string attached to the magnetic monopole. The conserved string current is now

$$\epsilon^{\mu\nu\rho\sigma}\partial_\rho(\partial_\sigma\theta + A_\sigma) = K^{\mu\nu}_{\rm mon} + \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}F^{\rm mon}_{\rho\sigma} \; , \tag{4.2}$$

where the monopole field satisfies

$$\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\nu F^{\rm mon}_{\rho\sigma} = m^\mu \; , \tag{4.3}$$

where

$$m^\mu(x) = \sum_b 2\pi \int d\tau \frac{ds_b^\mu}{d\tau}\delta^4(x - s_b) \; ,$$

with monopole trajectories $s_b^\mu(\tau)$, and the string current $K^{\mu\nu}_{\rm mon}$ is given by Eq. (2.6) with an end point σ_0 of the internal coordinate, satisfying $q_a^\mu(\tau,\sigma_0) = s_a^\mu$. This string current $K^{\mu\nu}_{\rm mon}$ is no longer conserved. By applying ∂_ν to Eq. (4.2), we get $\partial_\nu K^{\mu\nu}_{\rm mon} = -m^\mu$. Here we consider monopoles of the minimum charge allowed by the Dirac quantization. For each magnetic monopole, there will be an attached string. If the magnetic monopoles are present in the system, strings in the configuration space can be closed, half-open with a monopole attached at one side, or open with monopole and antimonopole attached at each end.

After some steps similar to those in Sec. II, we get a Lagrangian

$$\mathcal{L}_1 = \frac{1}{2}(\partial_\mu f)^2 - U(f) - \frac{1}{4e^2}F_{\mu\nu}^2 + \frac{\lambda}{8}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} + \mathcal{A}_\mu J^\mu_{\rm ext} + \frac{1}{12f^2}H^2_{\mu\nu\rho} + \tfrac{1}{2}(K^{\mu\nu} + \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}F^{\rm mon}_{\rho\sigma})B_{\mu\nu} + \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\mu\mathcal{A}_\nu B_{\rho\sigma} \; , \tag{4.4}$$

where the string currents describe both open and closed strings and we dropped the interaction term between the monopoles and the external charge. In order to integrate over the single-valued gauge field $\mathcal{A}_\mu$, we introduce an antisymmetric tensor field $N_{\mu\nu}$ so that

$$\delta(F_{\mu\nu} - (\partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu) - F^{\rm mon}_{\mu\nu}) = \int [dN_{\mu\nu}]\exp\left\{ i\int d^4x \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}N_{\mu\nu}[F_{\mu\nu} - (\partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu) - F^{\rm mon}_{\mu\nu}]\right\} \tag{4.5}$$

Integration over $\mathcal{A}_\mu$ leads to a δ function, implying

$$\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\nu(N_{\rho\sigma} - B_{\rho\sigma}) - J^\mu_{\rm ext} = 0 \; , \tag{4.6}$$

which is consistent only if $\partial_\mu J^\mu_{\rm ext} = 0$. If $J^\mu_{\rm ext}$ has a dynamical origin so that it is not conserved identically, we cannot integrate over A_μ without getting multivalued $N_{\rho\sigma}$. With $\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\rho V^{\rm ext}_{\rho\sigma} = J^\mu_{\rm ext}$, we can solve the δ function (4.6) with

$$N_{\mu\nu} = B_{\mu\nu} + V_{\mu\nu} + V^{\rm ext}_{\mu\nu} \; , \tag{4.7}$$

where $V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$. A point external electric charge appears as a magnetic monopole in the $V^{\rm ext}_{\mu\nu}$ field. When there is a uniform background charge density, we can, for example, choose the gauge $V^{\rm ext}_{yz} = 2xJ^0_{\rm ext}$ or $V^{\rm ext}_{\theta\varphi} = 2rJ^0_{\rm ext}$. We change variables from $N_{\mu\nu}$ to V_μ, and then $[dN_{\mu\nu}] = [dV_\mu]$.

Now we can integrate over $F_{\mu\nu}$ and get the dual Lagrangian

$$\mathcal{L}_{MD} = \frac{1}{2}(\partial_\mu f)^2 - U(f) + \frac{1}{12f^2}H^2_{\mu\nu\rho} - \frac{e^2}{4(1+\lambda^2e^4)}\tilde{B}^2_{\mu\nu} - \frac{\lambda e^4}{8(1+\lambda^2e^4)}\epsilon^{\mu\nu\rho\sigma}\tilde{B}_{\mu\nu}\tilde{B}_{\rho\sigma} + \tfrac{1}{2}B_{\mu\nu}(K^{\mu\nu} + \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}F^{\rm mon}_{\rho\sigma}) \; , \tag{4.8}$$

where $\tilde{B}_{\mu\nu} \equiv B_{\mu\nu} + V_{\mu\nu} + V^{\rm ext}_{\mu\nu}$. Note that the dual Lagrangian is invariant under the gauge transformation $\delta B_{\mu\nu} = \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu$ and $\delta V_\mu = -\Lambda_\mu$. Here the conservation of the tensor (4.2) is crucial. One can see the above derivation is not affected even when the couplings e,λ are depending on spacetime, for example, describing axionic domain walls. When there is no Higgs field, we can just drop $f,B_{\mu\nu}$ from Eq. (4.8) and get a dual formulation of the Maxwell theory, where magnetic monopoles and electric charges have exchanged their pole. The generating functional is now

$$Z=\int[f^{-3}df][dB_{\mu\nu}][dq_a^\mu][dV_\mu]\bar{\Psi}_F \times\exp\left\{i\int d^4x\,\mathcal{L}_D\right\}\Psi_I\ . \qquad (4.9)$$

The massive vector boson is now described by $B_{\mu\nu}$. We could include a kinetic term for magnetic monopoles.

Gauss's law from the variation of B_{0i} is given by

$$-\partial_j\left[\frac{1}{f^2}H_{0ij}\right]+\frac{e^2}{1+\lambda^2e^4}\tilde{B}_{0i}-\frac{\lambda e^4}{2(1+\lambda^2e^4)}\epsilon_{ijk}\tilde{B}_{jk} +\tfrac{1}{2}\epsilon_{ijk}F^{\text{mon}}_{jk}+K^{0i}=0\ . \qquad (4.10)$$

Gauss's law from the variation of V_0 leads to

$$\frac{e^2}{1+\lambda^2e^4}\partial_i\tilde{B}_{0i}-\frac{\lambda e^4}{2(1+\lambda^2e^4)}\epsilon_{ijk}\partial_i\tilde{B}_{jk}=0\ , \qquad (4.11)$$

which is a consequence of Eq. (4.10). The constraint (4.10) should be satisfied by the field configuration around strings and monopoles. The classical relation between the original variables and dual variables is given by

$$f^2(\partial^\mu\theta+A^\mu)=\tfrac{1}{6}\epsilon^{\mu\nu\rho\sigma}H_{\nu\rho\sigma}\ ,$$

$$F_{\mu\nu}+F^{\text{mon}}_{\mu\nu}=\frac{e^2}{2(1+\lambda^2e^4)}\epsilon_{\mu\nu\rho\sigma}\tilde{B}^{\rho\sigma}-\frac{\lambda e^4}{1+\lambda^2e^4}\tilde{B}_{\mu\nu}\ . \qquad (4.12)$$

When there is a nonzero external background charge J^0_{ext}, the lowest-energy configuration would be such that this external charge is shielded completely by the Higgs field. In terms of the dual fields, there will be nonzero $H_{123}=f^2(\dot\theta+A_0)=-J^0_{\text{ext}}$.

There are two mass scales m_f,m_A when there is no background charge. When $m_f\gg m_A$, we expect an effective action for strings and the massive vector bosons, which would be given trivially by a simple generalization of Eq. (2.19). When there are magnetic monopoles, the effective action should include open strings with massive end points. When there is a nonzero background charge density, local strings would feel the Magnus force.

V. CONCLUSION

We found the path integral derivation of the dual formulation for various theories of strings and vortices. Goldstone bosons, axions, and massive vector bosons are described by an antisymmetric tensor field of rank 2, while cosmic strings appear as the source for this tensor field. While the dual formulation has been extensively used before, our derivation puts the dual formulation on a better footing and we hope this leads to a better understanding of the string or vortex dynamics. In addition, we should note that it is trivial to generalize our dual formulation in curved spacetime and Euclidean time. (In Euclidean time, there is a subtlety related to the conserved charges. See the second paper of Ref. [8].)

However, there are many questions arising as we have a more precise formulation to start with. While many low-energy effective actions for strings and vortices are known, it would be interesting to derive them from the dual formulation systematically. We may use the dual formulation to study the quantum dynamics of vortices. Some investigation of quantum fluid dynamics along this direction has been launched in Ref. [14]. While the dual Lagrangian does not look renormalizable, it should be so in a way if the original Lagrangian is so. How we can show this is not clear at this moment as the dual Lagrangian may lie in a strong-coupling regime. When magnetic monopoles are present around axionic strings, it has been known that there are interesting phenomena, such as electric charge transfer from monopoles to strings [15]. Chiral charge could be violated by magnetic monopoles, making the whole system quite rich. It would be interesting to study the interaction between magnetic monopoles, chiral charge, and axionic strings.

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