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The Dual Formulation of Cosmic Strings and Vortices†

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Abstract

We study four dimensional systems of global, axion and local strings. By using the path integral formalism, we derive the dual formulation of these systems, where Goldstone bosons, axions and massive vector bosons are described by antisymmetric tensor fields, and strings appear as a source for these tensor fields. We show also how magnetic monopoles attached to local strings are described in the dual formulation. We conclude with some remarks.

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1. Introduction

We study a class of four dimensional field theories of a complex scalar field and other fields with a global or local abelian symmetry. These theories have global, axionic or local strings as solutions in the symmetry broken phase. We derive the dual formulation of these theories by using the path integral. The dual formulation has been extensively used to study the phase structure of these theories,[1] and the dynamics of cosmic strings and superfluid vortices.[2∼7] However, the dual formulation of strings has been derived usually by using the field equations[4] or the canonical transformations,[5] making the whole situation somewhat unsatisfactory.

On the other hand, the dual formulation of the theory of a complex scalar field has been derived in the path integral when there is no vortex.[8] Recently, in path integral formalism we have derived the dual formulation of three dimensional systems of vortices to study the statistics of vortices in Chern-Simons Higgs systems.[9] Here we extend the idea of Ref. [9] to get the dual formulation of four dimensional systems with cosmic strings. While there have been a large literature about the dual formulation, we feel our work is somewhat new and could be used to study quantum feature of the string dynamics.

There are several advantages of the dual formulation of cosmic strings. As the interaction between strings and other fields is more explicit, one can understand the string evolution clearly. A string or superfluid vortex moving on a background charge density feels the so-called Magnus force. This Lorentz type force can be seen directly in the dual formulation.[6,9] When the length scale of a string motion is lower than the string core size, one can obtain an effective action which describes the string dynamics and its interaction with low energy modes.

The plan of this paper is as follows. In Sec. 2 we study the theory of a complex scalar field with a global abelian symmetry. In the dual formulation, a global string appears as the source of an antisymmetric tensor field, which represents the Goldstone boson. In Sec. 3, we study the dual formulation of a theory where global strings appear as axionic strings. As shown in Ref.[10] the electromagnetic charge is not conserved in this theory without taking into account the chiral fermion zero modes on the string. In Sec. 4, we study the Maxwell Higgs theory where there are Nielsen-Olsen vortices, or local strings.[11] We derive the dual formulation where the gauge field is integrated out and is not explicit. The antisymmetric fieldß has a Higgslike coupling which leads to its mass term. Magnetic monopoles attached to local strings are described in the dual formulation. In Sec.5, we conclude with some remarks.

2. Global Strings

We consider the theory of a complex scalar field $\phi = fe^{i\theta}/\sqrt{2}$, whose lagrangian is given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu f)^2 + \frac{1}{2}f^2(\partial_\mu\theta)^2 - U(f). \tag{2.1}$$

As the theory (1) is invariant under a global transformation, $\theta \rightarrow \theta +$ constant, there are conserved current,

$$j_\mu = f^2\partial_\mu\theta, \tag{2.2}$$

and global charge,

$$Q = \int d^3r f^2\partial_0\theta. \tag{2.3}$$

The ground state of the energy functional for the systems we consider is chosen to be a broken phase because of either the potential or a background charge density. The low energy mode is then given by the Goldstone bosons or sound wave.

To understand the quantum aspect, we use the generating functional

$$Z =< F|e^{-iHT}|I >= \int [fdf][d\theta]\overline{\Psi}_F \exp\{i\int d^3x\mathcal{L}\}\Psi_I, \tag{2.4}$$

where $[f] \equiv \prod_x f(x)$ is the Jacobian factor for the radial coordinate of the scalar field. The initial and final wave functions $\Psi_{F,I}$ give the necessary boundary conditions.

A given field configuration in the path integral could contain strings, around each of which the value of the θ field changes by 2π times an integer. We can in principle split the θ field into two parts,

$$\theta(t,\vec{r}) = \overline{\theta}(t,\vec{r}) + \eta(t,\vec{r}), \tag{2.5}$$

where $\overline{\theta}$ describes a given configuration of vortices and η represents singlevalued fluctuations around the vortex configuration. The energy density and the complex scalar field ϕ should be singlevalued, or equivalently $\partial_\mu\overline{\theta}$ and $e^{i\overline{\theta}}$ should be so. Each string is described by parameterized positions, $\vec{q}_a(\sigma)$ or covariantly $q_a^\mu(\tau,\sigma)$, where $\sigma^\alpha = (\tau,\sigma)$ is the string world sheet coordinate. We choose σ so that $\overline{\theta}$ increases by 2π when one wraps the string in the direction of increasing σ with the right hand.

For a straight string along z axis, we know that $(\partial_x\partial_y - \partial_y\partial_x)\overline{\theta} = 2\pi\delta^2(\vec{\rho})$. By covariantizing it, we get the antisymmetric tensor vortex current,

$$\begin{aligned} K^{\mu\nu}(x) &\equiv \epsilon^{\mu\nu\rho\sigma}\partial_\rho\partial_\sigma\overline{\theta} \\ &= 2\pi\sum_a\int d\tau d\sigma(\dot{q}_a^\mu q_a'^\nu - \dot{q}_a^\nu q_a'^\mu)\delta^4(x^\rho - q_a^\rho(\tau,\sigma)), \end{aligned} \tag{2.6}$$

where the dot and prime indicate the differentiation by τ and σ, respectively. $K^{\mu\nu}$ is independent of reparameterizations of σ^α up to sign and satisfies the conservation law, $\partial_\mu K^{\mu\nu} = 0$. From Eq.(2.6), we can get

$$\begin{aligned} \partial_i\overline{\theta}(t,\vec{r}) &= \epsilon_{ijk}\partial_j\int d^3s\ \frac{K^{0k}(t,\vec{s})}{4\pi|\vec{r}-\vec{s}|}, \\ \partial_0\overline{\theta}(t,\vec{r}) &= \epsilon_{ijk}\partial_i\int d^3s\ \frac{K^{jk}(t,\vec{s})}{8\pi|\vec{r}-\vec{s}|}, \end{aligned} \tag{2.7}$$

by using a time-independent Green function. By integrating Eq.(2.7), we get

$$e^{i\overline{\theta}(t,\vec{r})} = \exp\left\{i\int_{\vec{r}_0}^{\vec{r}} d\vec{s}\cdot\vec{\nabla}\overline{\theta}(t,\vec{s})\right\}, \tag{2.8}$$

where $\vec{\nabla}\overline{\theta}$ is given by Eq.(2.7) and $\vec{r}_0$ is a reference point. The exponent at the right hand side of Eq.(2.8) is multivalued but the exponential is singlevalued.

The measure for the θ variable becomes

$$[d\theta] = [d\overline{\theta}][d\eta] = [dq_a^\mu][d\eta], \tag{2.9}$$

which means that we sum over singlevalued fluctuations around a given configuration of strings and then sum over all possible string configurations. A typical string configuration would have the creation, annihilation, crossing, exchange of strings. The Jacobian factor from $[d\overline{\theta}]$ to $[dq_a^\mu]$ is complicated. The periodicity of the θ variable affects only the quantizations of both global charge and vorticity, due to the gradient term $(\partial_\mu\theta)^2$ in the lagrangian.

In the generating functional Z, we can linearize the θ kinetic term by introducing an auxiliary vector field C^μ,

$$\begin{aligned} &\exp i\int d^4x\big[\frac{1}{2}f^2(\partial_\mu\theta)^2\big] \\ &= \int[f^{-4}dC^\mu]\exp i\int d^4x\Big\{-\frac{1}{2f^2}(C^\mu)^2 + C^\mu\partial_\mu\overline{\theta} + C^\mu\partial_\mu\eta\Big\}. \end{aligned} \tag{2.10}$$

As η is singlevalued, one can integrate over η in the standard way, leading to

$$\int[d\eta]\exp\Big[i\int d^4x C^\mu\partial_\mu\eta\Big] = \delta(\partial_\mu C^\mu). \tag{2.11}$$

Now we introduce the dual antisymmetric tensor field $B_{\mu\nu}$ to satisfy

$$\int[dC^\mu]\delta(\partial_\mu C^\mu)... = \int[dC^\mu][dB_{\mu\nu}]\delta(C^\mu - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma})... \tag{2.12}$$

and the dots denote the rest of the integrand. There would be an infinite gauge volume which can be taken care of later, but there is no nontrivial Jacobian factor as the change of variables is linear. By using the fact that

$$\epsilon^{\mu\nu\rho\sigma}(\partial_\mu\overline{\theta})\partial_\nu B_{\rho\sigma} = K^{\mu\nu}B_{\mu\nu} \tag{2.13}$$

up to a singlevalued total derivative, we can integrate over C^μ, resulting in the lagrangian,

$$\mathcal{L}_D = \frac{1}{2}(\partial_\mu f)^2 - U(f) + \frac{1}{12f^2}H^2_{\mu\nu\rho} + \frac{1}{2}B_{\mu\nu}K^{\mu\nu}, \tag{2.14}$$

where $H_{\mu\nu\rho} \equiv \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$ is the field strength of $B_{\mu\nu}$. Note that the kinetic term for $B_{\mu\nu}$ has the standard normalization, e.g., $(\partial_0 B_{12})^2/2$. The dual lagrangian (2.14) is invariant under a local gauge symmetry, $B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu$.

The resulting path integral becomes

$$Z = \int [f^{-3}df][dq^\mu_a][dB_{\mu\nu}]\overline{\Psi}_F e^{i\int d^4x\mathcal{L}_D}\Psi_I. \tag{2.15}$$

One can now introduce the gauge fixing terms for $B_{\mu\nu}$. The initial and final states should be rewritten in dual variables. The Goldstone boson is now described by $B_{\mu\nu}$ and strings appear as a source for $B_{\mu\nu}$.

The mass of vortices arises from the cloud of the $f, B_{\mu\nu}$ fields surrounding them. The variation of B_{0i} will lead to a Gauss's law,

$$-\partial_j(\frac{1}{f^2}H_{0ij}) + K^{0i} = 0, \tag{2.16}$$

which would dictate the field cloud around vortices. When the string of vorticity n is lying on the z axis, the f field would vanish like $f \sim \rho^n$ as one approaches the string on the xy plane. This can be seen directly from the f equation in the original formulation, or from the f and $B_{\mu\nu}$ equations in the dual formulation. The classical relation between the original fields and dual fields can be found from the field equations from the lagrangians at each step of the transformations. They are related to each other by

$$f^2\partial^\mu\theta = \frac{1}{6}\epsilon^{\mu\nu\rho\sigma}H_{\nu\rho\sigma}. \tag{2.17}$$

Let us consider now the string dynamics briefly. In the original formulation, the string motion is determined by the field equations for f, θ. In the dual formulation, it seems that there is be an equation of motion for the string directly from the variation of $q_a^\mu(\tau, \sigma)$. This is not the case as we will see now. From the dual lagrangian (2.14), we get for the variation of δq_a^ρ

$$\delta\mathcal{L}_D = \sum_a \frac{1}{2}\int d\tau d\sigma H_{\mu\nu\rho}(q_a)(\dot{q}_a^\mu q_a'^\nu - \dot{q}_a^\nu q_a'^\mu)\delta q_a^\rho. \tag{2.18}$$

Since $f^2\partial_\mu\theta$ vanishes at the string, Eq.(2.17) implies that the above variation vanishes. The field equation obtained from the variation of the string position is trivial. In the dual formulation, the string motion is again governed by the field equations of $f, B_{\mu\nu}$. This is consistent with the picture that the kinetic energy of vortices arises entirely from the field cloud around it and so the dynamics of vortices should be determined by the field surrounding them.

This leads naturally to a question, whether there is an effective action in terms of string position which describes the string dynamics. We imagine the string dynamics whose energy scale is much lower than that of the string core scale or the mass of f. As the Goldstone boson is massless, we expects the effective action to describe both strings and Goldstone bosons. This assumption cannot be valid all the time as strings will annihilate each other. The effective action is usually given as the Nambu action for strings and the action for the antisymmetric tensor field,[2,3,4]

$$\begin{aligned} S_{eff} = &\sum_a \mu_0 \int d^2\sigma_a \sqrt{-\gamma} \\ &+ \int d^4x\{\frac{1}{12v^2}H_{\mu\nu\rho}^2 + B_{\mu\nu}K^{\mu\nu}\}, \end{aligned} \tag{2.19}$$

where v is the asymptotic value of f, μ_0 is the bare string tension, and γ_a is the determinant of the induced metric on the string,

$$\gamma_{a\,\alpha\beta} = \frac{\partial q_a^\mu}{\partial \sigma^\alpha} \frac{\partial q_{a\mu}}{\partial \sigma^\beta}, \tag{2.20}$$

where v is the vacuum expectation value of f. The bare string mass per unit length μ_0 comes from the string core region. A cutoff of scale m_f is necessary to make the string self-energy finite. Note that the f field does not approach exponentially to its vacuum value at the spatial infinity. For a straight string lying along the z axis, one can see in the cylindrical coordinate (ρ, φ, z),

$$f \to v - \frac{1}{m_f \rho^2}, \tag{2.21}$$

as $\rho \to \infty$. Since there is no sharp transition between core and outside regions, the bare mass density and the necessary cutoff are not clearly defined. Hence, the effective action (2.19) probably describes the string dynamics in order of magnitude and needs an improvement.

In addition the action should be modified when there is an background charge density. The reason is that there is a sound wave of speed v_s rather than a Goldstone mode at the low energy. There would be an effective action for sound wave and strings when strings move slower than the sound speed. In addition, there is a Magnus force on the string from the background charge. While these are intersting questions, it will not be pursued here.

3. Axionic Strings

Let us now consider the case where the scalar field describes an axion field. To achieve this we need a fermion whose mass term is chirally generated by our complex scalar field and a gauge boson which is coupled to the fermion, so that the global abelian symmetry is broken by the anomaly. When the fermion is very massive, integration over fermionic field introduces an effective interaction between the gauge field and the phase of complex scalar field. For simplicity, we will consider the case where the gauge symmetry is abelian. (The detail aspects of anomalies, chiral zero modes and bosonization ideas in the following discussion has appeared in Ref.[10]. We include the results in Ref.[10] to understand the dual formulation better.) The resulting interaction between axion and two photons is given by

$$\mathcal{L}_{a\gamma\gamma} = -\frac{\theta}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}. \tag{3.1}$$

This effective action is multivalued and not well defined at each string as θ loses its meaning.

The equivalent singlevalued lagrangian to consider is

$$\mathcal{L}_A = \frac{1}{2}(\partial_\mu f)^2 + \frac{1}{2}f^2(\partial_\mu\theta)^2 - U(f) - \frac{1}{4e^2}F_{\mu\nu}^2 + Z^\mu\partial_\mu\theta, \tag{3.2}$$

where the Chern-Simons current is

$$Z^\mu = \frac{1}{8\pi^2}\epsilon^{\mu\nu\rho\sigma}A_\nu\partial_\rho A_\sigma, \tag{3.3}$$

satisfying $\partial_\mu Z^\mu = \frac{1}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$. The lagrangian (3.2) is not gauge invariant as Z^μ is not. The current from the lagrangian (3.2) is given by

$$\begin{aligned} J_A^\mu &= \frac{\delta\mathcal{L}_A}{\delta A_\mu} \\ &= -\frac{1}{8\pi^2}\epsilon^{\mu\nu\rho\sigma}\partial_\nu\theta F_{\rho\sigma} + \frac{1}{8\pi^2}K^{\mu\nu}A_\nu, \end{aligned} \tag{3.4}$$

which is not conserved,

$$\partial_\mu J_A^\mu = -\frac{1}{16\pi^2} K^{\mu\nu} F_{\mu\nu}. \tag{3.5}$$

When the string lies along the z axis, $K^{0z} = 2\pi\delta^2(\vec{\rho})$ and $\partial_\varphi \theta = 1$ in the cylindrical coordinate. The current (2.4) becomes

$$\begin{aligned} J_A^\rho &= -\frac{F_{0z}}{4\pi^2 \rho}, \\ J_A^z &= -\frac{1}{4\pi} A_0 \delta^2(\rho). \end{aligned} \tag{3.6}$$

For a uniform electric field along $\hat{z}$, there is a radial current moving to the string.

It turns out there is an additional degree of freedom to solve this puzzle of gauge noninvariance. A chiral fermion zero mode lies along a string, leading to an current, J_χ, such that the total current, $J_A^\mu + J_\chi^\mu$, is conserved. The effect of this chiral fermion zero mode can be seen more directly by the bosonization.[10] With the string metric $\gamma^{\alpha\beta}$ and the antisymmetric tensor field $\epsilon^{\alpha\beta}$, the lagrangian for the chiral boson $\chi(\sigma^\alpha)$ on a string is given by

$$\begin{aligned} L_\chi = &\frac{1}{2}(\partial_\alpha \chi - \frac{1}{2\sqrt{\pi}} A_\alpha)^2 - \frac{1}{2\sqrt{\pi}} \chi \epsilon^{\alpha\beta} \partial_\alpha A_\beta \\ &+ (\gamma^{\alpha\beta} - \epsilon^{\alpha\beta}) \lambda_\alpha (\partial_\beta \chi - \frac{1}{2\sqrt{\pi}} A_\beta), \end{aligned} \tag{3.7}$$

where the gauge field is evaluated at the string. The chiral lagrangian is not invariant under the gauge transformation, $\delta\chi = 2\sqrt{\pi}\Lambda$ and $\delta A_\alpha = \partial_\alpha \Lambda$. (While it is trivial to introduce the world sheet metric on the string, let us use here the Cartesian coordinate.)

Let us put the string on the z axis, and define the light cone variables on the string, $x^\pm = \frac{1}{\sqrt{2}}(t \pm z)$ with $g_{+-} = g_{-+} = \epsilon_{+-} = -\epsilon^{+-} = 1$. The field equations from Eq.(3.7) with $t=\tau,\ z=\sigma$, imply that λ_α can be chosen to be zero and that

$$\partial_-\chi - \frac{1}{2\sqrt{\pi}}A_- = 0, \tag{3.8}$$

where $A_\pm = (A_t \pm A_z)/\sqrt{2}$. The electric current due to the chiral bosons becomes

$$\begin{aligned} J^+_\chi &= \frac{1}{4\pi}A_-, \\ J^-_\chi &= -\frac{1}{\sqrt{\pi}}\partial_+\chi + \frac{1}{4\pi}A_+, \end{aligned} \tag{3.9}$$

which is not conserved,

$$\partial_\alpha J^\alpha_\chi = \frac{1}{4\pi}\epsilon^{\alpha\beta}\partial_\alpha A_\beta. \tag{3.10}$$

For the axionic string lying along the z axis, the sum of currents from Eqs. (3.4) and (3.9) becomes

$$\begin{aligned} J^\rho &= -\frac{1}{4\pi^2}\frac{1}{\rho}F_{0z}, \\ J^+ &= 0, \\ J^- &= \left(-\frac{1}{\sqrt{\pi}}\partial_+\chi + \frac{1}{2\pi}A_+\right)\delta^2(\rho), \end{aligned} \tag{3.11}$$

which is conserved due to the field equation for χ. Note that the current is chiral on the string. The combined action from Eqs. (3.2) and (3.7) is then gauge invariant, and so has the conserved electromagnetic current (3.11).

Let us consider the dual formulation of this lagrangian. We take the similar steps as in Sec.2. We first split θ into $\overline{\theta}$ and η, and introduce C^μ. Integrate over η to get

$$\int [d\eta]\exp[i\int d^4x(C^\mu + Z^\mu)\partial_\mu\eta] = \delta(\partial_\mu(C^\mu + Z^\mu)). \tag{3.12}$$

We again introduce $B_{\mu\nu}$ to solve the delta function,

$$\delta(\partial_\mu(C^\mu + Z^\mu)) = \int [dB_{\mu\nu}]\delta(C^\mu + Z^\mu - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma})... \tag{3.13}$$

Integration over C^μ leads to the dual lagrangian,

$$\begin{aligned}\mathcal{L}_{AD} = \frac{1}{2}(\partial_\mu f)^2 - U(f) - \frac{1}{4e^2}F_{\mu\nu}^2 \\ + \frac{1}{12f^2}\tilde{H}_{\mu\nu\rho}^2 + \frac{1}{2}B_{\mu\nu}K^{\mu\nu},\end{aligned} \tag{3.14}$$

where $\tilde{H}_{\mu\nu\rho} \equiv H_{\mu\nu\rho} - \epsilon_{\mu\nu\rho\sigma}Z^\sigma$. Since the Chern-Simons current is not gauge invariant, the field strength $\tilde{H}_{\mu\nu\rho}$ is invariant under the electromagnetic gauge transformation $\delta A_\mu = \partial_\mu \Lambda$ only if the $B_{\mu\nu}$ field is also transformed as

$$\delta B_{\mu\nu} = \frac{1}{8\pi^2}\Lambda F_{\mu\nu}. \tag{3.15}$$

The dual lagrangian (3.14) is not invariant under this gauge transformation due to the last term.

In general, there will be a chiral boson χ_a, lagrangian multiplier $\lambda_{a\alpha}$ and lagrangian $\mathcal{L}_{\chi_a}$ for each string q_a. When we add the dual action from Eq. (3.14) and these chiral actions, the total action is gauge invariant. The generating functional has become

$$Z = \int [f^{-3}df dB_{\mu\nu}dq_a^\mu][d\chi_a d\lambda_{a\alpha}] \exp i\Big\{\int d^4x\mathcal{L}_{AD} + \sum_a \int d^2\sigma_a \mathcal{L}_{\chi_a}\Big\}. \tag{3.16}$$

The conserved electromagnetic current in the dual formulation is given by

$$\begin{aligned}J^\mu = & \frac{1}{16\pi^2 f^2}\tilde{H}^{\mu\nu\rho}F_{\nu\rho} \\ & + \sum_a \int d^2\sigma \frac{\partial q_a^\mu}{\partial\sigma^\alpha}\frac{(\gamma^{\alpha\beta}+\epsilon^{\alpha\beta})}{2}\left(-\frac{1}{\sqrt{\pi}}\partial_\beta\chi_a + \frac{1}{2\pi}A_\beta(q_a)\right).\end{aligned} \tag{3.17}$$

The relation between the original field and dual fields outside strings is given as

$$f^2\partial^\mu\theta = \frac{1}{6}\epsilon^{\mu\nu\rho\sigma}\tilde{H}_{\nu\rho\sigma}, \tag{3.18}$$

and Gauss's law from the variation of B_{i0} becomes

$$-\partial_j\left(\frac{1}{f^2}\tilde{H}_{0ij}\right) + K^{0i} = 0. \tag{3.19}$$

4. Local Strings

Let us now consider the dual formulation of the Maxwell Higgs systems. Some aspects have been studied in Ref.[12,13]. The lagrangian is

$$\begin{aligned}\mathcal{L}_M = & -\frac{1}{4e^2}F_{\mu\nu}^2 + \frac{\lambda}{8}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} + A_\mu J_{ext}^\mu \\ & + \frac{1}{2}(\partial_\mu f)^2 + \frac{1}{2}f^2(\partial_\mu\theta + A_\mu)^2 - U(f).\end{aligned} \tag{4.1}$$

We assume that there are magnetic monopoles. The gauge field can be spilted into the monopole part $\overline{A}_\mu$ and the single valued rest $\mathcal{A}_\mu$. The magnetic monopoles are described by the vector potential with a Dirac string or equivalently by the Wu-Yang construction. When $f \neq 0$, $\partial_i\theta + \overline{A}_i$ is gauge invariant and well defined. Suppose that there is no point around the monopole where $f = 0$. Then $\epsilon_{ijk}\partial_j(\partial_k\theta + \overline{A}_k) = B_i^{mon}$ without a Dirac string, which is impossible. There should be a line attached to the monopole along which f is zero and θ changes by 2π when one goes around it. This exactly the cosmicOA string. In the broken phase, the magnetic field from a magnetic monopole is shielded by the Meissner effect, and is channeled by a local string.

The Dirac string of $\partial_\mu\theta + A_\mu$ becomes a real local string attached to the magnetic monopole,

$$\epsilon^{\mu\nu\rho\sigma}\partial_\rho(\partial_\sigma\theta + A_\sigma) = K_{mon}^{\mu\nu} + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}^{mon}, \tag{4.2}$$

where the monopole field satisfies

$$\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma}^{mon} = m^\mu \tag{4.3}$$

with $m^\mu(x) = \sum_b 2\pi \int d\tau \frac{ds_b^\mu}{d\tau}\delta^4(x - s_b)$ and the string current $K_{mon}^{\mu\nu}$ would be given by Eq.(2.6) with the end point of the internal parameter σ_0 would describe the monopole, $q_a^\mu(\tau, \sigma_0) = s_a^\mu$. This vortex current $K_{mon}^{\mu\nu}$ is no longer conserved. By applying ∂_ν to Eq.(4.2), we get $\partial_\nu K^{\mu\nu}_{mon} = -m^\mu$. Here we consider monopoles of the minimum charge allowed by the Dirac quantization. For each magnetic monopole, there will be a attached string. The strings in the configuration space can be closed, half-open with a monopole attached at one side, or open with monopole and antimonopole attached at both ends.

After some steps similar to those in Sec.2, we get a lagrangian

$$\begin{aligned}\mathcal{L}_1 = &\frac{1}{2}(\partial_\mu f)^2 - U(f) - \frac{1}{4e^2}F^2_{\mu\nu} + \frac{\lambda}{8}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} + \mathcal{A}_\mu J^\mu_{ext} \\ &+ \frac{1}{12f^2}H^2_{\mu\nu\rho} + \frac{1}{2}(K^{\mu\nu} + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F^{mon}_{\rho\sigma})B_{\mu\nu} + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\mu\mathcal{A}_\nu B_{\rho\sigma},\end{aligned} \tag{4.4}$$

where the string currents describes both open and closed strings and we dropped the interaction term between monopoles and external charge. In order to integrate over the singlevalued gauge field $\mathcal{A}_\mu$, we introduce an antisymmetric tensor field $N_{\mu\nu}$ so that

$$\begin{aligned}&\delta(\ F_{\mu\nu} - (\partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu) - F^{mon}_{\mu\nu}) \\ &\quad = \int [dN_{\mu\nu}] \exp\{i\int d^4x \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}N_{\mu\nu}[F_{\mu\nu} - (\partial_\mu A_\nu - \partial_\nu A_\mu) - F^{mon}_{\mu\nu}]\}\end{aligned} \tag{4.5}$$

There is no nontrivial Jacobian factor.

Integration over $\mathcal{A}_\mu$ leads to

$$\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\nu(N_{\rho\sigma} - B_{\rho\sigma}) - J^\mu_{ext} = 0, \tag{4.6}$$

which is consistent only if $\partial_\mu J^\mu_{ext} = 0$. If J^μ has a dynamical origin so that it is not conserved identically, we cannot integrate over A_μ without getting multivalued $N_{\rho\sigma}$. With $\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\rho V^{ext}_{\rho\sigma} = J^\mu_{ext}$, we can express

$$N_{\mu\nu} = B_{\mu\nu} + V_{\mu\nu} + V^{ext}_{\mu\nu}, \tag{4.7}$$

where $V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$. A point external electric charge appears as a magnetic monopole in the $V^{ext}_{\mu\nu}$ field. When there is a unform background charge density, we can choose the gauge so that $V_{yz}^{ext} = 2xJ_{ext}^0$ or $V_{\theta\varphi}^{ext} = 2rJ_{ext}^0$. We change variables from $N_{\mu\nu}$ to V_μ, and then $[dN_{\mu\nu}] = [dV_\mu]$.

Now we can integrate over $F_{\mu\nu}$ and get the dual lagrangian,

$$\begin{aligned}\mathcal{L}_{MD} = &\frac{1}{2}(\partial_\mu f)^2 - U(f) + \frac{1}{12f^2}H_{\mu\nu\rho}^2 - \frac{e^2}{4(1+\lambda^2 e^4)}\tilde{B}_{\mu\nu}^2 \\ &- \frac{\lambda e^4}{8(1+\lambda^2 e^4)}\epsilon^{\mu\nu\rho\sigma}\tilde{B}_{\mu\nu}\tilde{B}_{\rho\sigma} + \frac{1}{2}B_{\mu\nu}(K^{\mu\nu} + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}^{mon}),\end{aligned} \tag{4.8}$$

where $\tilde{B}_{\mu\nu} \equiv B_{\mu\nu} + V_{\mu\nu} + V_{\mu\nu}^{ext}$. Note that the dual lagrangian is invariant under the gauge transformation, $\delta B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$ and $\delta V_\mu = -\Lambda_\mu$ because of Eq. (4.2). One can see the above derivation is not affected even when the couplings e, λ are depending on spacetime, for example, describing axionic domain walls. When there is no Higgs field, we can just drop $f, B_{\mu\nu}$ from Eq.(4.8) and get a dual formulation of the Maxwell theory, where magnetic monopoles and electric charges have exchanged their role. The generating functional is now

$$Z = \int [f^{-3}df][dB_{\mu\nu}][dq_a^\mu][dV_\mu]\overline{\Psi}_F \exp\left\{ i\int d^4x \mathcal{L}_{MD}\right\}\Psi_I. \tag{4.9}$$

The massive vector boson is now described by $B_{\mu\nu}$. We could include a kinetic term for magnetic monopoles.

Gauss's law from the variation of B_{0i} is given by

$$-\partial_j(\frac{1}{f^2}H_{0ij}) + \frac{e^2}{1+\lambda^2 e^4}\tilde{B}_{0i} - \frac{\lambda e^4}{2(1+\lambda^2 e^4)}\epsilon_{ijk}\tilde{B}_{jk} + \frac{1}{2}\epsilon_{ijk}F_{jk}^{mon} + K^{0i} = 0. \tag{4.10}$$

Gauss's law from the variation of V_0 leads to

$$\frac{e^2}{1+\lambda^2 e^4}\partial_i\tilde{B}_{0i} - \frac{\lambda e^4}{2(1+\lambda^2 e^4)}\epsilon_{ijk}\partial_i\tilde{B}_{jk} = 0, \tag{4.11}$$

which is a consequence of Eq.(4.10). The constraint (4.10) should be satisfied by the field configuration around strings and monopoles. The classical relation between the original variables and dual variables are given by

$$
\begin{aligned}
f^2(\partial^\mu\theta + A^\mu) &= \frac{1}{6}\epsilon^{\mu\nu\rho\sigma}H_{\nu\rho\sigma}, \\
F_{\mu\nu} + F^{mon}_{\mu\nu} &= \frac{e^2}{2(1+\lambda^2e^4)}\epsilon_{\mu\nu\rho\sigma}\tilde{B}^{\rho\sigma} - \frac{\lambda e^4}{1+\lambda^2e^4}\tilde{B}_{\mu\nu}.
\end{aligned}
\tag{4.12}
$$

When there is a nonzero external background charge J^0_{ext}, the lowest energy configuration would be such that this external charge is shielded completely by the Higgs fields. In terms of the dual fields, there will be nonzero $H_{123} = f^2(\dot{\theta} + A_0 + A_0^{ext}) = -J^0_{ext}$.

There are two mass scales m_f, m_A when there is no background charge. When $m_f \gg m_A$, we expect an effective action for strings and the massive vector bosons, which would be given trivially by a simple generalization of Eq. (2.19). When there are magnetic monopoles, we have to think about the effective action for open strings with massive end points. When there are nonzero charge density, we have to think about the Magnus force and lowenergy modes. Again, there are various questions discussed before.

5. Conclusion

We found the path integral derivation of the dual formulation for various theories of abelian symmetry with strings. Goldstone bosons, axions and massive vector bosons are described by antisymmetric tensor fields of rank two, while cosmic strings appear as the sources for these tensor fields. While the dual formulation has been extensively used before, our derivation puts the dual formulation in a better footing and we hope this leads to a better understanding of the string or vortex dynamics. In addition, we should note that it is trivial to generalize our dual formulation in curved spacetime and euclidean time. In euclidean time, there is a subtlety related to the boundary condition which fixes the conserved charge. (See the second paper of Ref.[8])

However, there are many open questions arising as we have more precise formulation to start. What is an effective action which describes strings and low energy modes of a given theory? When there are nonzero unform background charge density, it is well known that there is a Magnus force on strings and the particle spectrum changes. In this case, what is the low energy effective action for strings, low energy modes and Magnus force in this case? Clearly this question is related to the vortex dynamics in superfluid. Finally, we have the dual formulation in the path integral and could study the quantum dynamics of vortices.

REFERENCES

1. A. Savit, Rev. Mod. Phys. **52**, 453 (1980).
2. M. Kalb and P. Ramond, Phys. Rev. **D9**, 2273 (1974).
3. F. Lund and T. Regge, Phys. Rev. **D14**, 1524 (1976).
4. E. Witten Phys. Lett. **153B**, 243 (1985); A. Vilenskin and T. Vachaspati, Phys. Rev. **D35**, 1138 (1987).
5. R.L. Davis and E.P.S. Shellard, Phys. Lett. **214B**, 219 (1988)
6. R.L. Davis and E.P.S. Shellard, Phys. Rev. Lett. **63**, 2021 (1989) ; R.L. Davis, Phys. Rev. **D40**, 4033 (1989) and R.L. Davis, Mod. Phys. Lett. **A5**, 955 (1990).
7. B. Gradwohl, G. Kalberman, T. Piran and E. Berschtinger, Nucl. Phys. **B338**, 371 (1990); U. Ben-Ya'acov, Nucl. Phys. **B382**, 597 (1992) and Nucl. Phys. **B382**, 616 (1992).
8. E.S. Fradkin and A.A. Tseytlin, Ann. of Phys. (New York) **162**, 32 (1985); S. Coleman and K. Lee, Nucl. Phys. **B329**, 387 (1990).
9. Y. Kim and K. Lee, 'Vortex dynamics in Self-Dual Chern-Simons Higgs Systems,' CU-TP-574, CERN-TH.6701/92 and hep-th 9211035 (1992).
10. C.G. Callan and J.A. Harvey, Nucl. Phys. **B250**, 427 (1985); S.G. Naculich, Nucl. Phys. **B296**, 837 (1988).
11. H.B. Nielsen and P. Olsen, Nucl. Phys. **B61**, 45 (1973).
12. A. Sugamoto, Phys. Rev. **D19**, 1820 (1979).
13. T.J. Allen, M.J. Bowick and A. Lahir, Phys. Lett. **237B**, 47 (1990) and Mod. Phys. Lett. **A 6**, 559 (1991).