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The Possible Role of Event-horizons in Quantum Gravity

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In this paper we extend to the de Sitter universe Bekenstein's result for the minimum variation of the black hole event-horizon area due to the absorption of an extended (classical) particle. Based on these equations we argue that at macroscopic scales the classical and quantum results should be in correspondence with each other (correspondence principle) and conclude that the event-horizon area is quantized in units of Planck's length squared. Consequences are discussed.

1. INTRODUCTION

Since the advent of Quantum Theory and General Relativity, formulating a quantum theory for self-gravitating systems has remained one of the most challenging problems in theoretical physics owing to their apparent incompatibility. Fortunately, some light on the connection between these theories was shed by developments in black hole thermodynamics, due mainly to Bekenstein [1] and Hawking [2,3]. By now we know that both black holes and the de Sitter universe [4] have associated a gravitational entropy proportional to their total event-horizon area—intrinsically a quantum effect. Quantum field theory in curved space-time [5] is a formalism that allows us to address some issues related to these systems without running into the complications arising from the difficulty of quantizing the gravitational field itself (semi-classical gravity). One of the consequences of this formalism is that a black hole is unstable under the emission of (ther-

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mal) particles and, furthermore, this evaporation process is explosive at its final stages (naked singularity?). This instability and the cosmological initial singularity remind us of the hydrogen atom classical instability and lead us to wonder whether the correspondence principle, which was so successful in dealing with the hydrogen atom, cannot be tailored to tackle self-gravitating systems.

According to the correspondence principle, at macroscopic scales, the predictions of quantum theory and classical physics should be put in correspondence with each other. In the specific situation of the hydrogen atom, Bohr studied the classical trajectories of the orbiting electron while assuming the existence of energy levels. Quantum theory enters through the back door when the energy difference between two levels is equated to the energy of an emitted quantum. In this paper we shall exactly retrace Bohr's steps for both the Kerr black hole and the de Sitter space-time. Our classical equations will consist of the first law of thermodynamics combined with the equations of motion for classical point-like test particles as were considered by Christodoulou [6]. Since we wish to put our (classical) result in correspondence to the quantum one, and quantum particles are not point-like, we should consider their generalization for extended particles [1]. We further assume that these self-gravitating systems have some (discrete) quantum spectrum. This is suggested by statistical physics, which tells us that the number of quantum states of these systems ought to be given by the exponential of one quarter of their event-horizon area. Then, after putting into correspondence the change of black hole and de Sitter energies (classical) when they undergo a transition between contiguous levels to the energy of the absorbed/emitted quantum, we shall conclude that the both black hole and the de Sitter event-horizon areas are quantized in units Planck's length squared.

This paper is organized as follows. In Section 2 we briefly review some known results for the Kerr geometry, namely, the first law and formula of the area variation owing to the passage of an extended particle through the horizon. Similar results are derived in connection with the de Sitter geometry in Section 3. In the cosmological situation, instead of dealing with perturbations of the de Sitter geometry, following Davies [7], we consider the Kerr–de Sitter metric in order to incorporate into the space-time a device whose purpose is to aim at the cosmological horizon quanta with definite energy and angular momentum l. This complication comes from the fact that one cannot place the observer (and the device) in an asymptotically flat region where back-reactions could be neglected. Bearing in mind these results, and approaching these systems with the correspondence principle, we shall show in Section 4 that the event-horizon area ought to be quantized in both cases.

2. KERR BLACK HOLE

The Kerr black hole event-horizon area is given by

$$A^{bh} = 4\pi(r_+^2 + a^2), \tag{1}$$

where $r_+ > r_-$ are the roots of

$$r^2 + a^2 - 2Mr = 0 \tag{2}$$

and M is the black hole mass while $J = Ma$ is its angular momentum, both measured by an observer at infinity. After solving eq. (2) for r_+ and inserting the result into eq. (1) we obtain $A = A(M, J)$. Then, after inverting this relation [$M = M(A, J)$] and taking its variation, we obtain the first law of thermodynamics for black holes (bh):

$$\delta M = \frac{\kappa}{4\pi}\,\delta A^{\rm bh} + \Omega\delta J, \tag{3}$$

where κ and Ω are the black hole surface gravity and angular velocity, respectively [1]:

$$\kappa = \frac{(r_+ - r_-)\Omega}{4a} \tag{4}$$

$$\Omega = \frac{a}{r_+^2 + a^2}\,. \tag{5}$$

Let us now see what happens when a point-like particle is swallowed by the black hole. The answer is obtained by writing down the equations of motion at the capture point at the horizon and solving them for the particle's proper energy E [6],

$$E = \Omega l + \frac{\Omega\rho_+^2}{a}\,p_+^r, \tag{6}$$

where l is the particle's proper azimuthal angular momentum, p_+^r the corresponding radial momentum at the event-horizon and $\rho_+^2 = r_+^2 + a^2\cos^2\theta$. If the space-time is asymptotically flat we may identify the variations $\delta M = E$ and $\delta J = l$. Whenever a particle strikes the horizon tangentially, i.e. $p_+^r = 0$, the event-horizon area remains unchanged [6], as may be inferred from inspecting eqs. (3) and (6). This process corresponds to an adiabatic transformation and, accordingly, A should be regarded as a mechanical adiabatic invariant.

As already mentioned, for the purposes we have in mind, this equation must be adapted to deal with extended particles. If μ and R are the particle's proper energy and length and if the horizon is once again approached tangentially (the centre of mass radial momentum vanishes), then, up to the first order in $R \ll r_+ - r_-$ (in quantum jargon this would correspond to large quantum numbers) it was shown by Bekenstein that [1]

$$E \geq \Omega l + 2\kappa\mu R, \tag{7}$$

the equality being attained as $R \to 0$.

This result, combined with the first law [eq. (3)], tells us that after the capture of an extended particle, even under the optimal circumstance when this occurs with no radial momentum, the event horizon area is no longer invariant: the process is always followed by a minimum change in the event-horizon area,

$$\delta A^{\rm bh}_{\rm min} = 8\pi G\mu R. \tag{8}$$

This equation will play a central role in latter discussions and we now proceed to derive its cosmological counterpart.

3. THE DE SITTER UNIVERSE

In contrast with black holes, in the cosmological context, the observer and the particle firing device lie within the event-horizon and their back-reaction to the geometry cannot be neglected. Indeed, they play a pivotal role: if we were to neglect them, the event-horizon area would remain strictly constant. The relevant metric outside the device is not exactly de Sitter but rather Kerr–de Sitter [9]

$$\begin{aligned} ds^2 = {} & \rho^{-2}\Xi^{-2}\Delta_\theta(a\,dt - (r^2+a^2)d\phi]^2\sin^2\theta \\ & - \Delta_r\Xi^{-2}\rho^{-2}(dt - a\sin^2\theta d\phi)^2 + \rho^2(\Delta_r^{-1}dr^2 + \Delta_\theta^{-1}d\theta^2) \end{aligned} \tag{9}$$

where

$$\Delta_r = (a^2+r^2)(1-\Lambda r^2) - 2Mr, \tag{10}$$

$$\Delta_\theta = 1 + \Lambda a^2\cos^2\theta, \tag{11}$$

$$\rho^2 = r^2 + a^2\cos^2\theta, \tag{12}$$

and

$$\Xi = 1 + \Lambda a^2. \tag{13}$$

For convenience, the cosmological constant was replaced, $\Lambda \to 3\Lambda$. Here it will be assumed that the device is tiny in comparison with cosmological scales in the sense that M and $a \ll \Lambda^{-1/2}$.

The cosmological horizon is given by the largest root of $\Delta_r = 0$, a quartic equation. The other three roots do not have any physical bearing: two lie inside the device (r_+, r_-) which is assumed not to be a collapsed system, and the fourth is negative (r_{--}).

The cosmological event-horizon area is

$$A_c = \int_{r_{++}} (g_{\phi\phi}g_{\theta\theta})^{1/2}d\theta d\phi, \tag{14}$$

which after a trivial integration reads

$$A = 4\pi\Xi^{-1}(r_{++}^2 + a^2). \tag{15}$$

In the cosmological context, writing down the first law is more involved than for black holes [eq. (3)] since the lack of an asymptotically flat region precludes regarding either M as the device mass or δM as its corresponding change. This difficulty can be overcome through a Smarr-type formula [10] and its variation. First we recall that for any Killing vector the following identity holds:

$$K^{a;b}_{;b} = -R^a_b K^b. \tag{16}$$

At the event-horizon, some important quantities can be defined. For instance, the Killing vector which coincides with the generators of the horizon $\mathcal{H}$ [8] is

$$\chi^a = \xi^a + \Omega\psi^a \tag{17}$$

where χ^a and ψ^a are, respectively, the time-like and azimuthal Killing vectors and Ω is the event-horizon angular velocity:

$$\Omega = -\frac{g_{\phi t}}{g_{\phi\phi}} = \frac{a}{r_{++}^2 + a^2}. \tag{18}$$

Still at the horizon, the surface gravity κ is defined by

$$\chi^{a;b}\chi_b = \kappa\chi^a, \tag{19}$$

and has been evaluated by Mellor and Moss [11],

$$\kappa = \frac{\Lambda\Omega}{2a\Xi}(r_{++} - r_+)(r_{++} - r_-)(r_{++} - r_{--}). \tag{20}$$

eq. (16), combined with Einstein's equations, we have the identity

$$-\frac{1}{4\pi}\int \xi^{a;b}_{;b} d\Sigma_a = 2\int \left(T^a_b - \frac{1}{2}\delta^a_b T\right)\xi^b d\Sigma_a - \frac{\Lambda}{4\pi}\int \xi^a d\Sigma_a. \qquad (21)$$

The right-hand side of this equation corresponds to the (positive) matter and (negative) cosmological constant energy contributions [4]. Thus, with the aid of Stokes' theorem, we can define the total mass within the cosmological horizon through

$$M_c = -\frac{1}{4\pi}\int_{\mathcal{H}} \xi^{a;b} d\Sigma_{ab}. \qquad (22)$$

It is worthwhile emphasizing that $M_c \neq M$. Rephrasing this equation in terms of the vectors ξ^a and ψ^a [see eq. (17)] yields

$$M_c = -\frac{1}{4\pi}\int_{\mathcal{H}} \chi^{a;b} d\Sigma_{ab} + 2\Omega J_c, \qquad (23)$$

where J_c is the cosmological horizon angular momentum,

$$J_c = \frac{1}{8\pi}\int_{\mathcal{H}} \psi^{a;b} d\Sigma_{a;b} = -\int T^a_b \psi^b d\Sigma_a. \qquad (24)$$

(In order to obtain this identity, one has to recall both Einstein's equations and Stokes' theorem.) Next, one expresses the area element $d\Sigma_{ab} = |dA|\, N_{[a}\chi_{b]}$, where N_a is an outgoing null vector orthogonal to the surface with normalization such that $N_a\chi^a = 1$. In complete analogy with the black hole case [12], it can be shown that the surface gravity is constant over the cosmological event-horizon. This fact, combined with eq. (19), yields

$$\int_{\mathcal{H}} \chi^{a;b} d\Sigma_{ab} = \kappa A. \qquad (25)$$

Therefore

$$M_c = -\kappa A + 2\Omega J_c. \qquad (26)$$

The variation of this formula is much more involved, but it follows the steps taken by Bardeen et al. [12] for black holes, with the minor difference that $\kappa_{bh} \to -\kappa_c$ since their normalization of N^a is opposite to ours (an outgoing rather than an ingoing null generator). The final result is the ***first law of thermodynamics for the de Sitter universe***:

$$\delta M_c = -\frac{\kappa}{4\pi}\delta A + \Omega\delta J_c \qquad (27)$$

where

$$\delta M_c \equiv \int \delta T^a_b \xi^a d\Sigma_b. \qquad (28)$$

and

$$\delta J_c \equiv \int \delta T^a_b \psi^a d\Sigma_b. \qquad (29)$$

Next we focus our attention on the trajectories of test particles. The equations of motion have been derived in the appendix in the following way. Two of them (ϕ and t-directions) are obtained by varying the metric [see eq. (9)], and the other two through the Hamilton–Jacobi principle (which incidentally also leads to the evaluation of the the fourth constant of motion, Carter's invariant [13] for this geometry). Then, still in the appendix, we solve these equations for the particle energy $E = -p^a\xi_a$. In what follows $l = p^a\psi_a$ stands for the azimuthal angular momentum and μ for the proper mass. The final result is

$$E = \frac{\beta + \sqrt{\beta^2 - \alpha\gamma}}{\alpha}, \qquad (30)$$

where

$$\alpha = \Delta_\theta(r^2+a^2)^2 - a^2\sin^2\theta\Delta_r, \qquad (31)$$

$$\beta = al\Delta_\theta(r^2+a^2)^2 - al\sin^2\theta\Delta_r \qquad (32)$$

and

$$\gamma = -\Xi^{-2}\rho^4\left(\Delta_\theta(p^r)^2 + \Delta_r(p^\theta)^2\right) - \Xi^{-2}\mu^2\rho^2\Delta_r\Delta_\theta - l^2\Delta_r\sin^{-2}\theta + a^2l^2\Delta_\theta. \qquad (33)$$

It is very convenient to express the energy in terms of dynamical parameters at the event-horizon where $\Delta_r = 0$,

$$E_0 = \Omega l + \frac{\Omega\rho^2_{++}}{a\Xi}\left|p^r_{++}\right|. \qquad (34)$$

Comparison of this equation with the first law [see eq. (27)] tells us that, as for black holes, the cosmological event-horizon remains unchanged whenever it is struck tangentially ($p^r = 0$) by a point-like particle.

We proceed now to generalize our results for extended particles. Let us assume that the particle touches the horizon with no radial centre of mass momentum and let ϵ be the particle's 'coordinate radius', i.e., $r_{cm} = r_{++} - \epsilon$. For later reference, we display the useful identity

$$\frac{d\Delta_r}{dr}|_{r_{++}} = -2\kappa a \Xi \Omega^{-1}. \tag{35}$$

The relation between ϵ and the particle's proper radius R is

$$R = \int_{r_{++}-\epsilon}^{r_{++}} (g_{rr})^{1/2} dr. \tag{36}$$

Near the horizon

$$g_{rr} = \frac{\rho^2}{\Delta_r} \approx \frac{\rho_{++}^2 \Omega}{2a\kappa \Xi x}, \tag{37}$$

where $x = r_{++} - r$ and eq. (35) was used. Thus

$$R = \left(\frac{\rho_{++}^2 \Omega}{2\kappa_c a \Xi}\right)^{1/2} \int_0^\epsilon x^{-1/2} dx = \left(\frac{2\rho_{++}^2 \Omega \epsilon}{\kappa_c a \Xi}\right)^{1/2}. \tag{38}$$

In order to evaluate the (extended) particle energy, when the horizon is approached tangentially, we must expand eqs. (30)–(33) to lowest order in ϵ. Owing to the fact that the zeroth-order $\sqrt{(\beta^2 - \alpha\gamma)}$ contribution vanishes for Christodoulou's path ($p^r = 0$), the lowest order correction will be $\propto \epsilon^{1/2}$, i.e., of order R; linear corrections in ϵ are already of order R^2 and will be neglected. Thus the (linear) correction in R to the energy E is

$$E_1 = \frac{\sqrt{\alpha\gamma' + \alpha'\gamma - 2\beta\beta'}\, \epsilon^{1/2}}{\alpha} \tag{39}$$

where the primes stand for derivatives with respect to r and this expression is meant to be evaluated at $r = r_{++}$. Albeit tedious, the calculation of E_1 is straightforward:

$$E_1 = \mu\rho_{++} \sqrt{\frac{\kappa\Omega\epsilon}{2a\Xi}}. \tag{40}$$

Combining eqs. (38) and (40) we obtain

$$E_1 = 2\mu R\kappa. \tag{41}$$

Notwithstanding the lack of an asymptotically flat region in the de Sitter geometry, it is still possible to relate variations of spacetime parameters to the particle energy and angular momentum by identifying [14]

$$\delta T_b^a \Leftrightarrow -\frac{p^a v^b}{p^c \xi_c} e(x), \tag{42}$$

where $e(x)$ stands for the particle proper-energy density, a smearing function accounting for the particle finite size (for point particles it is proportional to a delta function). Now, inserting eq. (42) into (28) yields

$$\delta M_c = -\int (e(x) v^a) \xi_a |d\Sigma|. \tag{43}$$

Expanding the left-hand side of these equations in terms of the particle's proper length, we obtain the zeroth-order correction E_0, corresponding to the limit $e(x) \to \mu\delta(x)$, and so on. In other words, as expected,

$$\delta M_c \approx -(E_0 + E_1) + \dots \tag{44}$$

Putting all these pieces together [eqs. (27), (34), (41) and (44)] we conclude that, as for black holes, the passage of an extended particle through the cosmological event-horizon is always followed by the minimum increase in its area

$$\delta A_{\min}^{c} = 8\pi G\mu R \tag{45}$$

4. THE CORRESPONDENCE PRINCIPLE

The connection with quantum theory emerges when we assume that the systems we are considering have some (discrete) spectrum, $A = A(n)$, where n is some 'principal quantum number'. For large n, where classical and quantum results should be put in correspondence,

$$\delta A_{\min} \approx A'(n)\delta n. \tag{46}$$

It is evident that the minimum change in the event-horizon area occurs for transitions between contiguous levels,

$$\delta A_{\min} \approx A'(n). \tag{47}$$

In the quantum limit, the particle's (quanta) proper radius is not arbitrary; it should be of the order of Compton's length. Therefore we set $R = (\eta/8\pi)\hbar/\mu$, where η is presumably a constant of order one. Thus, in the light of eqs. (8) and (45),

$$\delta A_{\min}^{bh,c} = \eta\, l_p^2, \tag{48}$$

where $l_p = \sqrt{G\hbar/c^3}$ stands for Planck's length. Identifying (46) to (47) we establish a connection between gravitation and quantum theory,

$$A(n) = \eta\, l_p^2 (n + \zeta), \tag{49}$$

where ζ is an integration constant, also presumably of order one.

5. REMARKS AND SPECULATIONS

Christodoulou's work [6] on the trajectories of test particles moving on black hole geometries tells us that, from the mechanical point of view, the event-horizon area is to be regarded as an adiabatic invariant. Therefore eq. (48) is nothing but the semi-classical Bohr–Sommerfeld rule applied to self-gravitating systems with event-horizons, which strongly suggests that $\zeta = 1/2$. As a matter of fact, this was the point of view adopted by Bekenstein in an earlier conjecture of event-horizon quantization [1]. We would like to emphazise that Bekenstein's conjecture of event horizon area quantization of black holes, based on the adiabatic nature of the black hole event horizon, and the present one, based on the correspondence principle, must be regarded as two *distinct* pieces of evidence since they are based on different physical premisses.

The present result removes the classical instability occurring both in final stages of black hole evaporation and in cosmology since these systems cannot evolve beyond their ground state. For a black hole, this means that the relic of an evaporated black hole ought to be a microscopic black hole whose area is $A(0) = (\eta/2)l_p^2$, rather than a naked singularity. In the cosmological setting, comparing the de Sitter event-horizon area $A = 4\pi\Lambda^{-1}$ with eq. (48), we conclude that the cosmological constant should be quantized according to

$$\Lambda = \frac{4\pi}{\eta l_p^2 (n + 1/2)} \,. \tag{50}$$

It is widely believed that the universe started its evolution in a de Sitter phase. If we assume that it started its evolution in the ground state, $n = 0$, then, at a very early time, the cosmological constant was huge

$$\Lambda_0 = 8\pi\eta^{-1} l_p^{-2} \,. \tag{51}$$

In a theory where the cosmological constant arises dynamically as in field theory with a symmetry-breaking mechanism, we speculate that as the universe evolves, interactions with matter fields drive the gravitational field to higher-lying levels (gravitational excitations), causing the effective cosmological constant to decrease to its present value where $n \approx 10^{120}$.

Quantum mechanics has taught us over the years that any spectrum arises as the solution of some eigenvalue problem which, in Schrödinger's formalism, is represented through a differential equation. Thus, we expect the event-horizon area quantization to be reproduced by a Schrödinger-like equation. For the time being, our best bet is the Wheeler–DeWitt equation. This issue is currently under investigation. We would also like to emphasize that by taking the microcanonical entropy of a black hole, $S = \ln n$ yields $S \propto \ln A$, the wrong result. Therefore, an exponentially growing (with n) degeneracy must be associated to each level to account for the correct entropy. Accordingly, when the event-horizon area is small there are very few available states, meaning that the canonical formalism is not appropriate to deal with self-gravitating systems whose event-horizon area is small and, in this limit, the concept of temperature is meaningless. In summary, the success of the canonical description of black holes is connected to the high degeneracy of states for large n, where it can be regarded as a macroscopic system. For smaller n's this is no longer true, and one cannot rely upon the canonical description of black holes [see Ref. 17].

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APPENDIX A. TRAJECTORIES OF TEST PARTICLES IN THE KERR-DE SITTER GEOMETRY

The equations of motion of test particles may be obtained through the variations of the 'Lagrangian'

$$\mathcal{L} = \tfrac{1}{2}\left(g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2 + g_{tt}\dot{t}^2 + g_{\phi\phi}\dot{\phi}^2 + 2g_{\phi t}\dot{t}\dot{\phi}\right) \tag{52}$$

where the dots stand for derivatives with respect to an affine parameter defined over the trajectory. Note that t and ϕ are cyclic coordinates and their corresponding conjugate momenta must be conserved:

$$E = -\frac{\partial \mathcal{L}}{\partial \dot{t}} = g_{tt}\dot{t} + g_{\phi t}\dot{\phi} \tag{53}$$

and

$$l = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = g_{\phi\phi}\dot{\phi} + g_{\phi t}\dot{t} . \tag{54}$$

Solving these equations for $\dot{\phi}$ and $\dot{t}$ is a trivial task:

$$\dot{t} = \frac{g_{\phi\phi}E + g_{\phi t}l}{g_{\phi t}^2 - g_{tt}g_{\phi\phi}} \,, \tag{55}$$

$$\dot{\phi} = -\frac{g_{\phi t}E + g_{tt}l}{g_{\phi t}^2 - g_{tt}g_{\phi\phi}} \,. \tag{56}$$

the (first integral) equations of motion in directions t and φ.

$$\Xi^{-2}\rho^2\frac{dt}{d\tau} = (\Delta_r^{-1}(r^2+a^2)^2 - a^2\Delta_\theta^{-1}\sin^2\theta)E + a(\Delta_\theta^{-1} - \Delta_r^{-1}(r^2+a^2))l \qquad (57)$$

and

$$\Xi^{-2}\rho^2\frac{d\phi}{d\tau} = a(\Delta_r^{-1}(r^2+a^2) - \Delta_\theta^{-1})E + (\Delta_\theta^{-1}\sin^{-2}\theta - a^2\Delta_r^{-1})l\,. \qquad (58)$$

The other two equations could be obtained similarly. However, since the other coordinates are not cyclic, following Carter [13] we prefer to search for the fourth constant of motion (the third is the proper energy). If S is the particle's action, the Hamilton-Jacobi equation reads

$$\frac{1}{2}g^{ab}\frac{\partial S}{\partial x^a}\frac{\partial S}{\partial x^b} = -\frac{\partial S}{\partial \tau}\,. \qquad (59)$$

Since this does not depend on t, ϕ, or τ, the action must be of the form [16]

$$S = -\tfrac{1}{2}\mu^2\tau - Et + l\phi + S_r(r) + S_\theta(\theta). \qquad (60)$$

Inserting the inverse matrix elements g^{ab} as well as the above expression into eq. (58) and recalling that $p_r = (\partial S/\partial r)$ and $p_\theta = (\partial S/\partial\theta)$, yields after some trivial but tedious algebra,

$$\begin{aligned}\mathcal{C} &\equiv -\Delta_r p_r^2 - \mu^2 r^2 + \Xi^2\Delta_r^{-1}(al - (r^2+a^2))E) \\ &= +\mu^2 a^2\cos^2\theta + \Delta_\theta p_\theta^2 + \Xi^2\Delta_\theta^{-1}\sin^{-2}\theta(l - aE\sin^2\theta)^2 \end{aligned} \qquad (61)$$

Owing to the fact that left and right-hand sides of these equation are, respectively, functions of r and θ alone, the equations of motion in these directions are separable: the separation constant $\mathcal{C}$ is Carter's [13] fourth constant of motion for this geometry. It is also useful to define the quantities [16]

$$\sqrt{\mathcal{R}} \equiv \Delta_\theta p_\theta \qquad (62)$$

and

$$\sqrt{\Theta} \equiv \Delta_r p_r \qquad (63)$$

which can be expressed by means of eq. (60) as

$$\Theta \equiv (\mathcal{C} - \mu^2 a^2\cos^2\theta)\Delta_\theta - \Xi^2\sin^{-2}\theta(l - aE\sin^2\theta)^2 \qquad (64)$$

and

$$\mathcal{R} = \Xi^2(al - E(a^2+r^2))^2 - (\mathcal{C} + \mu^2 r^2)\Delta_r\,. \qquad (65)$$

As our last step towards obtaining the energy eq. (29) we take the linear combination $\Theta\Delta_r + \mathcal{R}\Delta_\theta$, which after some (tedious) algebra gives

$$\begin{aligned}&[\Delta_\theta(r^2+a^2)^2 - a^2\sin^2\theta\Delta_r]\,E^2 - 2\,[\Delta_\theta(r^2+a^2) - \Delta_r]\,alE \\ &+ [a^2l^2\Delta_\theta - l^2\sin^{-2}\theta\Delta_r - \mu^2\rho^2\Xi^{-2}\Delta_r\Delta_\theta \\ &-\rho^4\Xi^{-2}(\Delta_\theta(p^r)^2 + \Delta_r(p^\theta)^2)] = 0\end{aligned} \qquad (66)$$

which yields the coefficients α, β and γ displayed in (30)–(32).

REFERENCES

1. Bekenstein, J. D. (1973). *Phys. Rev.* *D7*, 2333.
2. Hawking, S. W. (1974). *Nature* **248**, 30.
3. Hawking, S. W. (1975). *Commun. Math. Phys.* **43**, 199.
4. Gibbons, G. W., and Hawking, S. W. (1977). *Phys. Rev.* D**15**, 2738.
5. Birrel, N. D., and Davies, P. C. (1982). *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge).
6. Christodoulou, D. (1970). *Phys. Rev. Lett.* **25**, 1596.
7. Davies, P. C. (1984). *Phys. Rev.* D30, 737.
8. Wald, R. M. (1984). *General Relativity* (University of Chicago Press, Chicago).
9. Carter, B. (1973). In *Black Holes*, C. DeWitt and B. S. DeWitt, eds. (Gordon & Breach, New York).
10. Smarr, L. (1973). *Phys. Rev. Lett.* **30**, 71.
11. Mellor, F. A. and Moss, I. G. (1989). *Class. Quant. Grav.* **6**, 1379.
12. Bardeen, J. M., Carter, B., and Hawking, S. W. (1973). *Commun. Math. Phys.* **31**, 161.
13. Carter, B. (1968). *Commun. Math. Phys.* **10**, 280.
14. Landau, L. D., and Lifshitz, E. M. (1971). *The Classical Theory of Fields* (Pergamon Press, Oxford).
15. Bekenstein, J. D. (1975). *Lett. Nuovo Cimento* **11**, 467.
16. Misner, C. W., Thorne, K. S., and Wheeler, J. A. (1973). *Gravitation* (W. H. Freeman, San Francisco).
17. Holzhey, C. F. E., and Wilczek, F. (1991). "Black Holes as Elementary Particles", preprint IASSNS-HEP-91/71.