Exotic $QQ\overline{q}\overline{q}$ States in QCD

Aneesh V. Manohar

*CERN TH-Division, CH-1211 Geneva 23, Switzerland**

Mark B. Wise

California Institute of Technology, Pasadena, CA 91125

Abstract

We show that QCD contains stable four-quark $QQ\overline{q}\overline{q}$ hadronic states in the limit where the heavy quark mass goes to infinity. (Here Q denotes a heavy quark, $\overline{q}$ a light antiquark and the stability refers only to the strong interactions.) The long range binding potential is due to one pion exchange between ground state $Q\overline{q}$ mesons, and is computed using chiral perturbation theory. For the $Q = b$, this long range potential may be sufficiently attractive to produce a weakly bound two meson state.

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* On leave from the University of California at San Diego.

and baryons (with qqq flavor quantum numbers) in its spectrum. In addition there are

1. Introduction

QCD with three light quark flavors contains mesons (with $\overline{q}q$ flavor quantum numbers)

"nuclei" which are weakly bound states of the baryons. At the present time there is no evidence for other types of states that are stable with respect to the strong interactions. It was originally suggested by Jaffe [1] that there should be hadronic resonances with $qq\overline{qq}$ flavor quantum numbers and there is evidence that some of the observed hadronic resonances should be interpreted in this way. In this paper we study the possibility of *stable* exotic $QQ\overline{qq}$ hadrons, where Q is a heavy quark (i.e., $m_Q \gg \Lambda_{\rm QCD}$) and the stability refers only to the strong interactions. It is easy to see that such states exist in the $m_Q \to \infty$ limit. For very heavy quarks, the quark pair QQ can form a small color antitriplet object of size $(\alpha_s(m_Q)m_Q)^{-1}$ with a binding energy of order $\alpha_s^2(m_Q)m_Q$. The two heavy quarks act as an almost point-like heavy color antitriplet source with a mass of about $2m_Q$ for the two light antiquarks in the $QQ\overline{qq}$ hadron. This results in bound states that have the light degrees of freedom in configurations similar to those in the observed $\overline{\Lambda}_b$ and $\overline{\Lambda}_c$ states, with the QQ pair playing the role of the heavy antiquark. In the heavy quark limit, the binding energy of the QQ pair tends to infinity, whereas the energy of the light degrees of freedom is of order $\Lambda_{\rm QCD}$, so that the $QQ\overline{qq}$ state has a lower energy than two separate $Q\overline{q}$ mesons and is stable with respect to the strong interactions.

This argument for the existence of exotic $QQ\overline{qq}$ states in the spectrum of QCD is based on the short range color Coulomb attraction in the channel $3 \otimes 3 \to \overline{3}$. For the case where Q is the top quark, this description is likely to be quantitatively correct and establishes the existence of exotic $tt\overline{qq}$ states that are stable with respect to the strong interactions. The charm and bottom quarks are not heavy enough for their short range color Coulomb attraction to play an important role in the formation of a $QQ\overline{qq}$ state, since $\alpha_s^2(m_Q)m_Q$ is not large compared with $\Lambda_{\rm QCD}$. If $QQ\overline{qq}$ states exist for $Q = c$ or b, they may be weakly bound two meson systems and the formation of such a bound state depends on the potential between the lowest lying $Q\overline{q}$ mesons. At long distances this potential is determined by one pion exchange and is calculable in chiral perturbation theory. For the remainder of this paper we examine the picture of $QQ\overline{qq}$ hadrons as two weakly bound $Q\overline{q}$ mesons, and apply it to the c and b quark systems.

Section 2 contains a derivation of the long range potential using chiral perturbation theory. In Section 3 the eigenstates of the potential operator are classified and $\Lambda_{\rm QCD}/m_Q$ corrections to the Hamiltonian are discussed. In Section 4 we present a variational calculation that suggests there might be a weakly bound state involving B and B^* mesons. Some concluding remarks are also given.

is a good quantum number. Mesons containing a single heavy quark come in degenerate

2. The Long Range Potential Between Heavy Mesons

In the limit $m_Q \to \infty$, the angular momentum of the light degrees of freedom, s_ℓ,

doublets with total spins $s_\pm = s_\ell \pm 1/2$. The ground state multiplets with $Q\bar{q}_a$ flavor quantum numbers ($q_1 = u, q_2 = d, q_3 = s$) have $s_\ell = 1/2$ and negative parity, giving a doublet of pseudoscalar and vector mesons that we denote by $P_a^{(Q)}$ and $P_a^{*(Q)}$ respectively. For $Q = c$ these are the (D^0, D^+, D_s) and (D^{*0}, D^{*+}, D_s^*) mesons and for $Q = b$ they are the $(B^-, \overline{B}^0, B_s)$ and $(B^{*-}, \overline{B}^{*0}, B_s^*)$ mesons.

The interactions of these heavy mesons with the π, K and η is determined by the chiral and heavy quark symmetries of the strong interactions. The effective Lagrangian that describes the low momentum strong interactions of the pseudo-Goldstone bosons with the $P_a^{(Q)}$ and $P_a^{*(Q)}$ mesons is [2]

$$
\begin{aligned}
\mathcal{L} = &-i\,\mathrm{Tr}\,\overline{H}^{(Q)} v_\mu \partial^\mu H^{(Q)} + \frac{i}{2}\,\mathrm{Tr}\,\overline{H}^{(Q)} H^{(Q)} v^\mu [\xi^\dagger \partial_\mu \xi + \xi \partial_\mu \xi^\dagger] \\
&+ \frac{ig}{2}\,\mathrm{Tr}\,\overline{H}^{(Q)} H^{(Q)} \gamma_\nu \gamma_5 [\xi^\dagger \partial^\nu \xi - \xi \partial^\nu \xi^\dagger] \\
&+ \lambda_1\,\mathrm{Tr}\,\overline{H}^{(Q)} H^{(Q)} [\xi m_q \xi + \xi^\dagger m_q \xi^\dagger] + \lambda_1'\,\mathrm{Tr}\,\overline{H}^{(Q)} H^{(Q)}\,\mathrm{Tr}[m_q \Sigma + m_q \Sigma^\dagger] \\
&+ \frac{\lambda_2}{m_Q}\,\mathrm{Tr}\,\overline{H}^{(Q)} \sigma^{\mu\nu} H^{(Q)} \sigma_{\mu\nu} + \cdots \,,
\end{aligned}
\tag{2.1}
$$

where the ellipsis denotes terms with more derivatives, more factors of the light quark mass matrix

$$
m_q = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} \,,
\tag{2.2}
$$

or more factors of $1/m_Q$ associated with violation of heavy quark spin symmetry. The traces in eq. (2.1) are over flavor and spinor indices. The pseudoscalar and vector heavy meson fields $P_a^{(Q)}$, $P_{a\mu}^{*(Q)}$ are combined to form the 4×4 Lorentz bispinor matrix

$$
H_a^{(Q)} = \frac{(1+\slashed{v})}{2}[P_{a\mu}^{*(Q)}\gamma^\mu - P_a^{(Q)}\gamma_5] \,.
\tag{2.3}
$$

The field $H^{(Q)}$ destroys $P^{(Q)}$ and $P^{*(Q)}$ mesons with four-velocity v^μ. The subscript v on $H^{(Q)}$, $P^{(Q)}$ and $P^{*(Q)}$ has been omitted to simplify the notation. The conjugate "barred" field is defined by

$$
\overline{H}^{(Q)a} = \gamma^0 H_a^{(Q)\dagger} \gamma^0 \,.
\tag{2.4}
$$

The pseudo-Goldstone bosons appear in the Lagrangian through

$$
\xi = \exp\left(\frac{iM}{f}\right) \,,
\tag{2.5}
$$

In eq. (2.5), f is the pion decay constant, $f \simeq 132$ MeV.

$$
\Sigma = \xi^2
\tag{2.7}
$$

$$
M = \begin{bmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \overline{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{bmatrix} \,,
\tag{2.6}
$$

Under $SU(3)_L \times SU(3)_R$ chiral symmetry transformations

$$\Sigma \to L\Sigma R^\dagger , \tag{2.8}$$

where $L \in SU(3)_L$ and $R \in SU(3)_R$. The transformation law of ξ is then

$$\xi \to L\xi U^\dagger = U\xi R^\dagger , \tag{2.9}$$

where U is a complicated function of L, R and the mesons M. In general U depends on space-time, but for transformations in the unbroken $SU(3)_V$ subgroup $V = L = R$, $\xi \to V\xi V^\dagger$, and U is the constant matrix V. Under $SU(2)_v$ heavy quark spin symmetry and $SU(3)_L \times SU(3)_R$ chiral symmetry, the heavy meson fields transform as

$$H^{(Q)} \to S\, H^{(Q)} U^\dagger , \tag{2.10}$$

where $S \in SU(2)_v$ is the heavy quark spin transformation.

The long range potential between heavy mesons is determined by the one pion exchange Feynman diagram in fig. 1, with the virtual pion having momentum transfer $q^\mu = (0, \vec{q})$. At low momentum, the coupling of the $P^{(Q)}$-$P^{*(Q)}$ and $P^{*(Q)}$-$P^{*(Q)}$ to the Goldstone bosons is obtained by expanding the Lagrangian eq. (2.1) up to first order in the pion fields. The only term which contributes to the one-pion coupling at low momentum transfer is the term proportional to g in eq. (2.1). In the $m_Q \to \infty$ limit the $P^{(Q)}$ and $P^{*(Q)}$ are degenerate in mass and can be treated as a single "H particle." This gives the H-pion interaction

$$\mathcal{L}_{\rm int} = -\frac{g}{f} \, {\rm Tr}\, \overline{H}^{(Q)} H^{(Q)} \gamma_\nu \gamma_5 \, \partial^\nu \pi. \tag{2.11}$$

The interaction can be rexpressed in terms of the spin of the light degrees of freedom $\vec{S}_\ell$ and the isospin I^A of the H field as

$$\mathcal{L}_{\rm int} = \frac{2\sqrt{2}g}{f} \left(\vec{S}_\ell \cdot \vec{\partial}\pi^A \right) I^A. \tag{2.12}$$

The Fourier transform of the interaction potential obtained from eq. (2.12) between two H particles is

$$V_\pi(\vec{q}) = -\frac{8g^2}{f^2} \, \vec{I}_1 \cdot \vec{I}_2 \, \frac{\left(\vec{S}_{\ell 1} \cdot \vec{q} \right) \left(\vec{S}_{\ell 2} \cdot \vec{q} \right)}{\vec{q}\,^2 + m_\pi^2} . \tag{2.13}$$

$$V_\pi(\vec{x}) = 4\vec{I}_1 \cdot \vec{I}_2 \left[\left(S_{\ell 1} \cdot \hat{x}\, S_{\ell 2} \cdot \hat{x} \right) W_2(r) + \vec{S}_{\ell 1} \cdot \vec{S}_{\ell 2} W_0(r) \right], \tag{2.14}$$

TABLE 1

I	K	k	V_π
0	0	0	$9W_0/4$
0	1	0	$3X_0$
0	1	± 1	$-3X_1$
1	0	0	$-3W_0/4$
1	1	0	$-X_0$
1	1	± 1	X_1

with the measured branching ratios for $D^{*+} \to D^+\pi^0$ and $D^{*+} \to D^0\pi^+$ leads to the limit $g^2 \lesssim 0.5$. A measurement of the branching ratio for $D^{*+} \to D^+\gamma$ could also lead to valuable information on g [4]. The axial current obtained from the Lagrange density eq. (2.1) is

$$\bar{q}\, T^A\, \gamma_\nu \gamma_5\, q = -g\,\, \mathrm{Tr}\, \overline{H}\, H\, \gamma_\nu \gamma_5\, T^A + \dots \,, \tag{2.18}$$

where the ellipsis denotes terms containing one or more Goldstone boson fields and T^A is a flavor $SU(3)$ generator. Treating the quark fields in eq. (2.18) as constituent quarks and using the nonrelativistic constituent quark model to estimate the D^* matrix element of the l.h.s. of eq. (2.18) gives $g = 1$. (A similar estimate of the pion nucleon coupling constant gives $g_A = 5/3$.) In the chiral quark model there is a constituent quark pion coupling [5]. Using the measured pion-nucleon coupling to determine the constituent quark pion coupling gives $g^2 \simeq 0.6$. Thus, various constituent quark model calculations lead to the expectation that g is near unity.

Equation (2.14) is the leading contribution to the long distance part of the heavy meson interaction potential. Loops and higher derivative operators give contributions that are suppressed by factors of $(4\pi f r)^{-1}$. It seems reasonable that eqs. (2.14)–(2.16) dominate the potential at distances greater than $r_{\min} \equiv (1/2m_\pi) \simeq 0.7$ fm. In the nuclear potential the corrections to one pion exchange are important even at this distance [6]. We believe this is (at least partly) due to integrating out the Δ resonance which has a large coupling to $N\pi$. In the heavy meson case both the $P^{(Q)}$ and $P^{*(Q)}$ mesons are kept in the lagrangian (2.1). The lightest heavy mesons that are integrated out of the theory do not couple very strongly to $P^{(Q)}\pi$ and $P^{*(Q)}\pi$ since in the constituent quark model they correspond to P-wave orbital excitations. At $r = (2m_\pi)^{-1}$, $W_2(r_{\min}) = 518g^2$ MeV and $W_0(r_{\min}) = 9g^2$ MeV, where we have used the neutral pion mass in computing the numerical values.

The eigenvalues of the potential operator eq. (2.14) are easily determined. The position space part of the state vector can be taken to be $|\hat{z}\rangle$, corresponding to the spatial wavefunction $\delta^3(\vec{x} - r\hat{z})$, since the potential is rotationally invariant. Then eigenstates of $V_\pi(\vec{x})$ are $|I\,I^3\rangle|\hat{z}\rangle|K\,k\rangle$ where

$$\vec{K} = \vec{S}_{\ell 1} + \vec{S}_{\ell 2} \,, \tag{2.19}$$

is the total spin of the light degrees of freedom (the spin quantization axis is also taken to be the z-axis) and

$$\vec{I} = \vec{I}_1 + \vec{I}_2 \,, \tag{2.20}$$

is the total isospin. Acting on these eigenstates the potential energy eq. (2.14) can be evaluated. The states with $k \neq \pm 1$ and is given for states with $k = 0$, so we obtain the

$$V_\pi = \left(I^2 - I_1^2 - I_2^2\right)\left[\left(2S_{\ell 1}^z\, S_{\ell 2}^z\, W_2(r) - \frac{1}{3}\, \vec{S}_{\ell 1}\cdot\vec{S}_{\ell 2}\right) W_2(r)\right.$$
$$\left. + \left(K^2 - S_{\ell 1}^2 - S_{\ell 2}^2\right)\left(W_0(r) - \frac{1}{3}W_2(r)\right)\right]. \tag{2.21}$$

where X_0 and X_1 are defined by

$$X_0 = \frac{W_2}{3} - \frac{W_0}{4} \; , \tag{2.22}$$

and

$$X_1 = \frac{W_2}{6} + \frac{W_0}{4} \; . \tag{2.23}$$

Since W_0, X_0 and X_1 are positive, the attractive channels have $(I, K, |k|) = (0, 1, 1)$, $(1, 0, 0)$ and $(1, 1, 0)$. At $r_{\min}$ the potential energies for these states are about $-266g^2$ MeV, $-7g^2$ MeV and $-170g^2$ MeV respectively.

3. Classification of Eigenstates

In the previous section we found spatial $\otimes$ spin parts of the eigenstates of the potential V that were of the form $|\hat{z}\rangle|K\,k\rangle$. By rotational invariance $\hat{R}\,(g)\,[\;|\hat{z}\rangle|K\,k\rangle\;]$ is an eigenstate of V with the same energy for any rotation g. It is convenient to combine these states into ones with a definite "angular momentum of the light degrees of freedom" $\vec{F}$ using[1] [7]

$$|F\,f; K\,k\rangle = \sqrt{2F+1} \int_{SU(2)} dg \; D^{(F)*}_{fk}(g) \; \hat{R}(g)\,[\;|\hat{z}\rangle|K\,k\rangle\;]\;, \tag{3.1}$$

where $D^{(F)}_{fk}(g)$ is the rotation matrix in representation F and the measure is chosen so that

$$\int_{SU(2)} dg = 1. \tag{3.2}$$

Alternatively we can combine states with definite orbital angular momentum

$$|\ell\,m\rangle = \sqrt{2\ell+1} \int dg \; D^{(\ell)*}_{m0}(g) \; \hat{R}(g)|\hat{z}\rangle\;, \tag{3.3}$$

with the spin of the light degrees of freedom to get states

$$|F\,f; \ell\,S\rangle = \sum_{r,s} (\ell\,r; S\,s|F\,f)\;|\ell\,r\rangle|S\,s\rangle\;. \tag{3.4}$$

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[1] F is the total angular momentum minus the spin of the heavy quarks. It is not the true angular momentum of the light degrees of freedom because it contains the orbital angular momentum of the heavy quarks.

Using This allows (3.4) this to write a decomposition of the eigenstates of the potential. Consider for example states with $\ell = 1$. Then $F = 1$ and so we can form states with orbital angular

$$\langle F'\,f'; \ell\,S|F\,f; K\,k\rangle = \delta_{F'F}\,\delta_{f'f}\,\delta_{KS}\,\sqrt{\frac{2\ell+1}{2F+1}}\;(F\,f|\ell\,0; K\,k)$$

$$= \delta_{F'F}\,\delta_{f'f}\,\delta_{KS}\,(-1)^{K+k}\,(\ell\,0|K-k; F\,k)\;. \tag{3.5}$$

momentum $\ell = F-1, F, F+1$. The other restriction is that $F \geq |k|$, so that the D matrix in eq. (3.1) exists. For definiteness, consider the case $F = 1$. Then for $k = 0$ we have according to eq. (3.5) the partial wave decomposition

$$|k = 0\rangle = \sqrt{\frac{1}{3}} \, |\ell = 0\rangle - \sqrt{\frac{2}{3}} \, |\ell = 2\rangle \, . \tag{3.6}$$

For $k = \pm 1$ it is convenient to form the linear combinations

$$|\pm\rangle = \frac{|k = 1\rangle \pm |k = -1\rangle}{\sqrt{2}} \, , \tag{3.7}$$

which decompose as

$$|+\rangle = \sqrt{\frac{2}{3}} \, |\ell = 0\rangle + \sqrt{\frac{1}{3}} \, |\ell = 2\rangle \, ,$$
$$|-\rangle = |\ell = 1\rangle \, . \tag{3.8}$$

Table 2 gives the eigenstates of the potential and their eigenvalues up to $F = 2$.

TABLE 2

F	$K = S$	$\mid k \mid$	$\pm$	ℓ	V_π for $I = 1$	V_π for $I = 0$
0	0	0		0	$-3W_0/4$	$9W_0/4$
1	0	0		1	$-3W_0/4$	$9W_0/4$
2	0	0		2	$-3W_0/4$	$9W_0/4$
0	1	0		1	$-X_0$	$3X_0$
1	1	0		0,2	$-X_0$	$3X_0$
1	1	1+		0,2	X_1	$-3X_1$
1	1	1−		1	X_1	$-3X_1$
2	1	0		1,3	$-X_0$	$3X_0$
2	1	1+		1,3	X_1	$-3X_1$
2	1	1−		2	X_1	$-3X_1$

The "angular momentum of the light degrees of freedom" F must be combined with the spin of the heavy quark pair $S_Q = 0, 1$ to get the total angular momentum J of the bound state. In addition, if the two heavy quarks are of the same flavor, then only states which are completely symmetric are allowed. This gives the additional restriction that

$I + \ell + S_Q$ is even. Since W_0, X_0 and X_1 are all positive it is easy to identify the channels which have an attractive potential. In the $m_Q \to \infty$ limit, the kinetic energy of the heavy mesons can be omitted. In this case, $QQ\bar{q}\bar{q}$ bound states exist if and only if there exists some channel in which the potential is attractive. It is obvious from Table 2 that there exist attractive channels, so we have another demonstration that there exist exotic states in the heavy quark limit of QCD.

There are two Λ_{QCD}/m_Q corrections to the Hamiltonian that are important for the case of heavy but finite quark masses, such as for the b or c quark. The kinetic energies of the heavy mesons

$$\mathcal{H}_{\text{kin}} = \frac{\vec{p}^{\,2}}{2\mu} \ , \tag{3.9}$$

where

$$\frac{1}{\mu} = \frac{1}{m_{Q_1}} + \frac{1}{m_{Q_2}} \ , \tag{3.10}$$

is the reduced mass, should be included. In addition, at order Λ_{QCD}/m_Q the $P^{(Q)} - P^{*(Q)}$ mass difference $\Delta^{(Q)} = m_{P^{*(Q)}} - m_{P^{(Q)}}$ should be taken into account. Experimentally $\Delta^{(c)} \simeq 141$ MeV and $\Delta^{(b)} \simeq 46$ MeV so for $Q = c$ and b this effect is quite significant. We define the mass splitting so that it is zero on $P^{(Q)}$ states and $\Delta^{(Q)}$ on $P^{*(Q)}$ states. This adds a term to the Hamiltonian, $\hat{\Delta}$, that is diagonal on the $P^{(Q)}P^{(Q)}$, $P^{(Q)}P^{*(Q)}$, $P^{*(Q)}P^{*(Q)}$ basis. The relationship between this "meson-type" basis and the $|K\,k\rangle|S_Q\,s_Q\rangle$ basis is straightforward to determine. The meson basis is obtained by first combining the spins of the heavy and light degrees of freedom in each H particle, whereas the K, S_Q basis is obtained by first combining the two heavy spins and the two light spins. A straightforward computation of this change of basis gives:

$$|1,1\rangle|0,0\rangle = \frac{1}{2}\Big\{|P^{*(Q_1)},1\rangle|P^{(Q_2)}\rangle - |P^{(Q_1)}\rangle|P^{*(Q_2)},1\rangle$$
$$+ |P^{*(Q_1)},1\rangle|P^{*(Q_2)},0\rangle - |P^{*(Q_1)},0\rangle|P^{*(Q_2)},1\rangle\Big\} \ ,$$

$$|1,0\rangle|0,0\rangle = \frac{1}{2}\Big\{|P^{*(Q_1)},0\rangle|P^{(Q_2)}\rangle - |P^{(Q_1)}\rangle|P^{*(Q_2)},0\rangle$$
$$+ |P^{*(Q_1)},1\rangle|P^{*(Q_2)},-1\rangle - |P^{*(Q_1)},-1\rangle|P^{*(Q_2)},1\rangle\Big\} \ , \tag{3.11}$$

$$|1,-1\rangle|0,0\rangle = \frac{1}{2}\Big\{|P^{*(Q_1)},-1\rangle|P^{(Q_2)}\rangle - |P^{(Q_1)}\rangle|P^{*(Q_2)},-1\rangle$$
$$- |P^{*(Q_1)},-1\rangle|P^{*(Q_2)},0\rangle + |P^{*(Q_1)},0\rangle|P^{*(Q_2)},-1\rangle\Big\} \ .$$

The mass splitting term is not diagonal in the K, S_Q basis used in Table 2. The general form of the mass splitting in this basis is complicated, and is discussed in the Appendix.

However, it is simple to compute the expectation value of the mass splitting in a given K state. For example,

$$\langle K = 1, k = \pm 1 | \hat{\Delta} | K = 1, k = \pm 1 \rangle = \frac{1}{4} \left(\Delta^{(Q_1)} + \Delta^{(Q_2)} + 2\Delta^{(Q_1)} + 2\Delta^{(Q_2)} \right) . \qquad (3.12)$$

We need to obtain the energy of $B^{(*)}, D^{(*)}$ "meson molecules" including both the kinetic energy and the mass splitting, to determine whether they are bound. A particularly promising entry in Table 2 is the $I = 0$, $S_Q = 0$ state on the sixth line. This has the most attractive potential allowed in the table, and has a small orbital angular momentum barrier. It is also a state which is allowed when the two heavy quarks are identical. In the next section we examine the energy of this state in the case when both heavy quarks are b-quarks using a variational calculation. This state is not an eigenstate of the Hamiltonian, since the mass splitting and the kinetic energy are not diagonal in the K basis. The mass splitting will produce mixing to other $|K\,k\rangle |S_Q\,s_Q\rangle$ states, which will only lower the energy further, and make the state more strongly bound. This $F = 1$ state has total angular momentum one and even parity so it cannot decay (strongly) to BB. As long as the expectation value of the Hamiltonian $\mathcal{H} = \mathcal{H}_{\text{kin}} + V + \hat{\Delta}$ in this state is less than $\Delta^{(b)}$ (the mass of a widely separated B and B^*) it is stable with respect to the strong interactions. The state can decay electromagnetically to $BB\gamma$ if the expectation value of the Hamiltonian is positive. Otherwise, it can only decay by the weak interactions.

There are Λ_{QCD}/m_Q effects we have neglected. For example at this order the heavy quark symmetry relation between the $P^{*(Q)}P^{*(Q)}\pi$ and $P^{(Q)}P^{*(Q)}\pi$ couplings is altered. This effect however is less important than those we have included.

4. Variational Calculation of the Binding Energy for $Q = b$

In this section we consider the case where both heavy quarks are b-quarks and focus on the state in line 6 of Table 2 with $I = 0$ and $S_Q = 0$. The energy E (above $2m_B$) of this trial state is

$$E[\phi] = \int_0^\infty r^2 dr \, \phi^*(r) \, \mathcal{H}\phi(r) \ , \qquad (3.13)$$

where

$$\mathcal{H} = -\frac{1}{m_B} \left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} \right] + \frac{2}{m_B r^2} + V(r) + \frac{3}{2}\left(m_B^* - m_B\right) \ . \qquad (3.14)$$

$\phi(r)$ is a trial radial wavefunction normalized to unity,

$$\int_0^\infty r^2 dr \, \phi^*(r) \, \phi(r) = 1 \ . \qquad (3.15)$$

The orbital angular momentum barrier follows from probabilities 2/3 and 1/3, so the effective

value of L^2 is $(0)(0+1)(2/3) + (2)(2+1)(1/3) = 2$. Thus the angular momentum barrier is the same as for an $\ell = 1$ state.

At large distances the potential $V(r)$ is given by the one pion exchange potential

$$V_\pi(r) = -3X_1(r) \ . \tag{3.16}$$

We expect $V(r)$ to be given by eq. (3.16) for $r \geq r_{\min} = (1/2m_\pi)$. However, some information on the short range part of the potential is needed to make further progress. In the case of nuclear forces there is a short range repulsive core that is often attributed to vector meson exchange [6]. The situation in the heavy meson case is quite different. Suppose the nonet of $\overline{q}q$ vector mesons $V_a^{\mu b}$ is coupled to the heavy mesons via the term

$$\mathcal{L} = g_V \ \mathrm{Tr} \ \overline{H}^a H_b \, v_\mu \, V_a^{\mu b} \ . \tag{3.17}$$

This is the type of coupling the constituent quark model suggests is appropriate. The coupling in eq. (3.17) gives rise to the contribution

$$V_{\rho,\omega}(\vec{q}) = \frac{g_V^2}{\vec{q}^{\,2} + m_V^2} \left\{ \frac{1}{2} \left(\vec{I}^{\,2} - 3/2 \right) + \frac{1}{4} \right\} \ . \tag{3.18}$$

to the potential, where we have taken $m_\rho = m_\omega = m_V$, and defined $\vec{I}$ to be the total isospin in the two particle channel. The first term in the braces comes from ρ exchange and the second comes from ω exchange. Note that ω exchange is repulsive as in the case of the nuclear potential. However, while in the nucleon case the ω-exchange piece is much larger than the rho exchange piece, for the heavy meson potential rho exchange dominates in the $I = 0$ channel giving an *attractive* potential from vector meson exchange. There is no repulsive hard core in the heavy meson potential. Multiple pion exchange contributions to the heavy meson potential are also less important than for nucleons. The pion-nucleon coupling constant is $g^2 = 1.56$, whereas the heavy-meson nucleon coupling is $g^2 \lesssim 0.5$. This, and the fact that the analog of the Δ resonance is not integrated out, implies that multiple pion graphs are relevant in the heavy meson case only for much smaller values of r than in the nucleon case.

Given that vector meson exchange as modeled by eq. (3.18) is attractive in the channel we are considering, a conservative approach is to use in our variational calculation the potential

$$V(r) = \begin{cases} V_\pi(r_{\min}) & r \leq r_{\min}, \\ V_\pi(r) & r > r_{\min}, \end{cases} \tag{3.19}$$

drawn from it should not be taken too seriously. There are physical effects that increase the potential energy which we have neglected. For example, η exchange gives the contribution

$$V_\eta\left(\vec{q}\right) = -\frac{2g^2}{3f^2}\frac{\left(\vec{S}_{\ell 1}\cdot\vec{q}\right)\left(\vec{S}_{\ell 2}\cdot\vec{q}\right)}{\vec{q}^{\,2} + m_\eta^2}\ , \tag{3.20}$$

to the Fourier transform of the potential. In the $I = 0$ channel it has the opposite sign from pion exchange. Its effects, however, are quite small since in the $I = 0$ channel since it is suppressed numerically by a factor of $1/9$. In position space, there is an additional suppression factor because the potential falls off exponentially in a distance m_η^{-1} rather than m_π^{-1}. There are also contributions from derivative vector meson couplings to the heavy mesons. These are less important than eq. (3.17) at large distances but may be of comparable importance at $r \sim 1/m_\rho$. In the case of the nuclear potential their contribution to the tensor force is thought to be very important even at distances as large as 1 fm.

The variational calculation using the energy function eq. (3.13) with Hamiltonian eq. (3.14) and potential eq. (3.19) is straightforward. We have chosen to do the computation with g^2 equal to the present experimental bound of $g^2 = 0.5$. A simple trial wavefunction can be chosen of the form

$$\phi(r) = N\ e^{-ar}r^b\left(1 + cr\right), \tag{3.21}$$

where N is a normalization constant chosen so that ϕ satisfies eq. (3.15). The minimum of the energy is at $a = 6.23\ m_\pi$, $b = 2.26$ and $c = -0.16\ m_\pi$. For these values of the parameters the wavefunction $\phi(r)$ is peaked near $r = r_{\min}$. The state is bound, with a binding energy of 8.3 MeV relative to the BB^* energy. (Recall that the state we are considering cannot decay into BB.) The average radial kinetic energy is 25.5 MeV, the average angular momentum barrier energy is 31.3 MeV, and the average potential energy is -88.2 MeV. The mass splitting $\Delta^{(b)}$ contributes an additional 23 MeV of energy, since the state is 50% BB^* and 50% B^*B^*.

The binding energy is sensitive to the precise value of g^2. For example, with $g^2 = 0.6$, it is 26.9 MeV, whereas for $g^2 = 0.4$, the state is not bound by about 8.2 MeV. The value of the binding energy is also sensitive to the value of the potential below $r_{\min}$. We have chosen to use a flat potential below $r_{\min}$ as a conservative extrapolation, and used the neutral pion mass in the numerical computations (which gives a weaker potential). The state could be much more strongly bound if the potential is more negative than our estimate. We have also investiagated the possibility of DB and DD bound states. With the approximations

we have made, these states are not bound. The reduced mass of these states is small enough that the kinetic energy overwhelms the attraction due to the potential. However, it is possible that these states are bound if the interaction potential is more attractive than our estimate.

It is interesting that there is a limit of QCD in which one can show that there must exist states with exotic quantum numbers. These $QQ\overline{qq}$ states are very difficult to produce or detect experimentally. It is much easier to produce a meson-antimeson bound state. The one-pion exchange potential for meson-antimeson bound states is the negative of the potential for meson-meson bound states that we have studied in this paper. The meson-antimeson spectrum can be investigated by similar methods to those used here. There is one important difference—the meson-antimeson sector has annihilation channels which do not exist in the meson-meson sector, so there will be no stable bound states. However, there might exist resonances.

Appendix A. Transformation of Basis

The transformation between the $|K\,k\rangle\,|S_Q\,s_Q\rangle$ basis (which gives eigenstates of the potential) and the usual angular momentum-meson type basis is computed explicitly in this appendix. The states $|K\,k\rangle$ are obtained by combining the spins of the light degrees of freedom in the two meson, and the states $|Q\,q\rangle$ are obtained by combining the spins of the heavy quarks in the two mesons. (We will denote the states $|S_Q\,s_Q\rangle$ by $|Q\,q\rangle$ from now on, to avoid multiple subscripts in the formulæ.) It is useful to define the states

$$|P\,p; K\,Q\rangle = \sum_{k,q} (K\,k; Q\,q|P\,p)\ |K\,k\rangle\,|Q\,q\rangle\ , \tag{A.1}$$

where the angular momentum P is the sum of K and Q. It is convenient to treat the spin zero meson $P^{(Q)}$ and the three possible polarizations of the spin one meson $P^{*(Q)}$ using a unified notation. For this reason, let $|W_1\,w_1\rangle$ denote the first meson, where $|0\,0\rangle$ is the spin zero $P^{(Q)}$, and $|1\,w\rangle$ denotes the three possible polarization states of the vector $P^{*(Q)}$

meson. The two meson state is then denoted by $|W_1\, w_1\rangle\, |W_2\, w_2\rangle$. Define the state $|S\, s\rangle$ to be the state obtained by combining the total spins of the two mesons,

$$|S\, s; W_1\, W_2\rangle = \sum_{w_1, w_2} (W_1\, w_1; W_2\, w_2 | S\, s)\, |W_1\, w_1\rangle\, |W_2\, w_2\rangle\ . \qquad (A.2)$$

The transformation formulæ will involve the overlap of the two states $|P\, p; K\, Q\rangle$ and $|S\, s; W_1\, W_2\rangle$. Now $|P\, p; K\, Q\rangle$ is obtained by first combining the light spins of the mesons into K and the heavy spins of the mesons into Q, and the resultant into P, whereas $|S\, s; W_1\, W_2\rangle$ is obtained by first combining the light and heavy spins of the first meson into W_1, and of the second meson into W_2, and the resultant into S. The overlap is therefore

$$\langle P\, p; K\, Q\, | S\, s; W_1\, W_2\rangle = \sqrt{(2W_1 + 1)\,(2W_2 + 1)\,(2K + 1)\,(2Q + 1)}$$
$$\times \left\{ \begin{array}{ccc} 1/2 & 1/2 & W_1 \\ 1/2 & 1/2 & W_2 \\ K & Q & S \end{array} \right\} \delta_{PS}\delta_{ps}, \qquad (A.3)$$

using the definition of the 9-j symbol.

The eigenstates of the potential with total angular momentum j are

$$|j\, m; K\, k\; Q\, q\rangle = \sqrt{2j + 1} \int_{SU(2)} dg\; D^{(j)*}_{m\; k+q}(g)\; \hat{R}(g)\, [\, |\hat{z}\rangle\, |K\, k\rangle\, |Q\, q\rangle\,]\,. \qquad (A.4)$$

These states are not identical to the ones defined in eq. (3.1) because we have also included the heavy quark spin $|Q, q\rangle$ along with $|K, k\rangle$ in the definition of the state. The conventional states are obtained by taking the spatial wavefunctions of definite orbital angular momentum eq. (3.3) and combining them with the spin state of the mesons given by eq. (A.2),

$$|j\, m; \ell\, S\, W_1\, W_2\rangle = \sqrt{2l + 1} \sum_{r,s} (\ell\, r; S\, s | j\, m)$$
$$\times \int_{SU(2)} dg\; D^{(l)*}_{r0}(g)\; \left[\, \hat{R}(g)|\hat{z}\rangle\, \right] |S\, s; W_1\, W_2\rangle\ . \qquad (A.5)$$

The transformation matrix is then

$$\langle j\, m; \ell\, S\, W_1\, W_2\, | j\, m; K\, k\, Q\, q\rangle = \sqrt{(2\ell + 1)\,(2j + 1)} \sum_{r,s} (\ell\, r; S\, s | j\, m)$$
$$\times \int_{SU(2)} dg\; D^{(j)*}_{m\; k+q}(g)\; D^{(l)}_{r0}(g)\; \langle S\, s; W_1\, W_2 | \hat{R}(g)\, |K\, k\rangle\, |Q\, q\rangle\,. \qquad (A.6)$$

$$\frac{\langle j\, m; \ell\, S\, W_1\, W_2 | j\, m; K\, k\, Q\, q\rangle}{\sqrt{(2\ell + 1)\,(2j + 1)}} = \sqrt{\frac{(2\ell + 1)}{(2j + 1)}} \sum_{r,s,p,p',P} (\ell\, r; S\, s | j\, m)(K\, k; Q\, q | P\, p) \qquad (A.7)$$

$$\langle j\, m; \ell\, S\, W_1\, W_2 \| \hat{R}(g)\| K\, k\, Q\, q\rangle = D^{(P)}_{p'p}(g)\, \delta_{PP}\delta_{KK}\delta_{QQ} \left\{ \begin{array}{ccc} 1/2 & 1/2 & W_1 \\ 1/2 & 1/2 & W_2 \\ K & Q & S \end{array} \right\} \qquad (A.9)$$

$$\times \int dg\; D^{(j)*}_{m\; k+q}(g)\, D^{(l)}_{r0}(g)\, D^{(P)}_{p'p}(g)\, \langle S\, s; W_1\, W_2 | P\, p'; K\, Q\rangle\,. \qquad (A.8)$$

by $(-1)^\Sigma$ under odd permutations of rows or columns, where Σ is the sum of all nine parameters. Thus the independent 9-j symbols that we need for our problem are

$$\begin{Bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{Bmatrix} = \frac{1}{2}, \qquad \begin{Bmatrix} 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & 1 \\ 0 & 1 & 1 \end{Bmatrix} = \frac{1}{\sqrt{54}},$$

$$\begin{Bmatrix} 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & 1 \\ 0 & 0 & 0 \end{Bmatrix} = \frac{1}{2\sqrt{3}}, \qquad \begin{Bmatrix} 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & 1 \\ 1 & 1 & 0 \end{Bmatrix} = -\frac{1}{18}, \qquad \text{(A.10)}$$

$$\begin{Bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 1 \\ 0 & 1 & 1 \end{Bmatrix} = \frac{1}{6}, \qquad \begin{Bmatrix} 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & 1 \\ 1 & 1 & 1 \end{Bmatrix} = 0,$$

$$\begin{Bmatrix} 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & 1 \\ 1 & 1 & 2 \end{Bmatrix} = \frac{1}{9}.$$

Using these values we can compute the decomposition of the various $|j\,m; K\,k\,Q\,q\rangle$ states, e.g.

$$|1\,m; 1\,1\,0\,0\rangle = \tfrac{1}{2\sqrt{3}}\,|1\,m; 0\,1\,1\,0\rangle - \tfrac{1}{2\sqrt{3}}\,|1\,m; 0\,1\,0\,1\rangle - \tfrac{1}{\sqrt{6}}\,|1\,m; 0\,1\,1\,1\rangle$$
$$- \tfrac{1}{2\sqrt{2}}\,|1\,m; 1\,1\,1\,0\rangle + \tfrac{1}{2\sqrt{2}}\,|1\,m; 1\,1\,0\,1\rangle + \tfrac{1}{2}\,|1\,m; 1\,1\,1\,1\rangle \qquad \text{(A.11)}$$
$$+ \tfrac{1}{\sqrt{24}}\,|1\,m; 2\,1\,1\,0\rangle - \tfrac{1}{\sqrt{24}}\,|1\,m; 2\,1\,0\,1\rangle - \tfrac{1}{2\sqrt{3}}\,|1\,m; 2\,1\,1\,1\rangle$$

where the states on the right hand side of the equation are $|j\,m; \ell\,S\,W_1\,W_2\rangle$ states. The decomposition of the $|j\,m\rangle$ state with $K=1$, $k=-1$, $Q=q=0$ can be obtained from eq. (A.11) by changing the sign of all the $\ell = 1$ terms. Thus the state $K=1$, $k=+$ which is $1/\sqrt{2}$ times the sum of the $k=1$ and $k=-1$ states can be decomposed as

$$|1\,m; 1\,+\,0\,0\rangle = \tfrac{1}{\sqrt{6}}\,|1\,m; 0\,1\,1\,0\rangle - \tfrac{1}{\sqrt{6}}\,|1\,m; 0\,1\,0\,1\rangle - \tfrac{1}{\sqrt{3}}\,|1\,m; 0\,1\,1\,1\rangle$$
$$+ \tfrac{1}{\sqrt{12}}\,|1\,m; 2\,1\,1\,0\rangle - \tfrac{1}{\sqrt{12}}\,|1\,m; 2\,1\,0\,1\rangle - \tfrac{1}{\sqrt{6}}\,|1\,m; 2\,1\,1\,1\rangle. \qquad \text{(A.12)}$$

Rewriting the W_1 and W_2 labels using the more familiar $P^{(Q)}$ and $P^{*(Q)}$ labels gives

$$|1\,m; 1\,+\,0\,0\rangle = \tfrac{1}{\sqrt{6}}\,\left|1\,m; 0\,1\,P^{*(Q_1)}\,P^{(Q_2)}\right\rangle - \tfrac{1}{\sqrt{6}}\,\left|1\,m; 0\,1\,P^{(Q_1)}\,P^{*(Q_2)}\right\rangle$$
$$- \tfrac{1}{\sqrt{3}}\,\left|1\,m; 0\,1\,P^{*(Q_1)}\,P^{*(Q_2)}\right\rangle + \tfrac{1}{\sqrt{12}}\,\left|1\,m; 2\,1\,P^{*(Q_1)}\,P^{(Q_2)}\right\rangle \qquad \text{(A.13)}$$
$$- \tfrac{1}{\sqrt{12}}\,\left|1\,m; 2\,1\,P^{(Q_1)}\,P^{*(Q_2)}\right\rangle - \tfrac{1}{\sqrt{6}}\,\left|1\,m; 2\,1\,P^{*(Q_1)}\,P^{*(Q_2)}\right\rangle.$$

From this decomposition, it is easy to see that the state on the left hand side is 25% $P^{(Q_1)}P^{*(Q_2)}$, 25% $P^{*(Q_1)}P^{(Q_2)}$ and 50% $P^{*(Q_1)}P^{*(Q_2)}$, and is 67% $\ell = 0$ and 33% $\ell = 2$.

The left hand side of eq. (A.13) was expressed in terms of angular momentum eigenstates in eq. (3.8), and in terms of meson states in eq. (3.11), which are special cases of the simultaneous decomposition in eq. (A.13).

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lines are either the $P^{(Q)}$ or $P^{*(Q)}$, and the dashed line is the pion.

Figure Captions

Fig. 1. The one-pion exchange contribution to the meson-meson potential. The solid