Vortex dynamics in self-dual Chern-Simons-Higgs systems

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We consider vortex dynamics in self-dual Chern-Simons-Higgs systems. We show that the naive Aharonov-Bohm phase is the inverse of the statistical phase expected from the vortex spin, and that the self-dual configurations of vortices are degenerate in energy but not in angular momentum. We also use the path integral formalism to derive the dual formulation of Chern-Simons-Higgs systems in which vortices appear as charged particles. We argue that in addition to the electromagnetic interaction, there is an additional interaction between vortices, the so-called Magnus force, and that these forces can be put together into a single "dual electromagnetic" interaction. This dual electromagnetic interaction leads to the right statistical phase. We also derive and study the effective action for slowly moving vortices, which contains terms both linear and quadratic in the vortex velocity. We show that vortices can be bounded to each other by the Magnus force.

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I. INTRODUCTION

Recently, several studies of self-dual Abelian Chern-Simons-Higgs systems in $2+1$ dimensions have appeared [1,2]. These self-dual models have a specific sixth-order potential which has degenerate symmetric and asymmetric vacua. In these systems there is a Bogomol'nyi-type bound on the energy functional, which is saturated by configurations satisfying certain first-order equations. These self-dual configurations consist of topologically stable vortices in the asymmetric phase, and nontopological solitons in the symmetric phase. These solitons carry both electric charge and magnetic flux, resulting in nontrivial spin, and can be regarded as anyons, or particles with fractional spin and statistics [3].

While attention has been paid to the statistics of vortices in the asymmetric phase [4,6], there are many aspects of vortices which still need to be understood clearly. One question is about the spin-statistics theorem of vortices. Another is related to the dynamics of slowly moving vortices in self-dual systems. In this paper, we study various questions related to vortices in Chern-Simons-Higgs systems.

Let us start first with considering the angular momentum of nontopological solitons and vortices. One striking fact is that for a given charge or magnetic flux, the angular momentum of nontopological solitons without any vorticity has the opposite sign compared with that of topological vortices [2]. The spin-statistics theorem implies that the spin of a particle is directly related to the statistics of that particle. The statistics of elementary charged particles [5] and nontopological solitons are determined by the Aharonov-Bohm phase due to electric charge and magnetic flux. Since vortices could have the same charge and magnetic flux but the opposite spin compared with nontopological solitons, the statistics of vortices cannot be explained by the naive Aharonov-Bohm phase. This is the first puzzle we will consider.

A self-dual configuration of n vortices appears to be completely specified by the vortex positions, that is, by $2n$ real parameters [1,2]. As we will see, all configurations of a given number of vortices are degenerate in energy but not in angular momentum. For a system of two vortices, total angular momentum decreases from four times the vortex spin to twice the vortex spin as their separation increases from zero to infinity. The influence of this change on the motion of slowly moving vortices is the second puzzle we shall consider.

In order to understand the statistics of vortices, we reformulate the original theory in a way that makes the interaction between vortices manifest. This reformulation is called the dual formulation, where the massive vector boson in the asymmetric phase is described by the Maxwell-Chern-Simons rather than Chern-Simons-Higgs terms, and where vortices appear as charged particles. The dual formulation has been derived many times in the past using the equations of motion or a lattice model [6]. We present here a clearer derivation using the path integral formalism.

Some physical implications of the dual formulation of various three-dimensional field theories have been studied previously [7]. In the theory of a complex scalar field with a global Abelian symmetry, a vortex in a uniform charged background feels the so-called Magnus force, which has more or less the same origin as the force responsible for the curved flight of a spinning ball. The Magnus force on a curve ball is proportional to its speed and is perpendicular to its direction, very much like the Lorentz force. In the dual formulation, vortices are

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charged particles and the background charge density becomes a magnetic field. The Magnus force on vortices becomes a Lorentz force. The concept of the Magnus force is important for vortex dynamics in superfluids [8]. One can also see the Magnus force for vortices in Maxwell Higgs theories when there is a background electric charge density screened by the Higgs field. While it is possible to see the Magnus force in the original formulation, it appears more transparently in the dual formulation.

In Chern-Simons-Higgs systems, vortices carry both magnetic flux and electric charge. Because vortices feel the charge of other vortices, vortices also feel the Magnus force in the absence of a background charge. In the dual formulation both the electromagnetic and Magnus forces are combined into a single "dual electromagnetic" force. The Aharonov-Bohm phase in the dual formalism determines the statistics of vortices and yields exactly what one expects from the given vortex spin. The key point is that the Magnus force can lead to a nontrivial phase between vortices as the Lorentz force can lead to the Aharonov-Bohm phase. The physical mechanism behind the vortex statistics is now very obvious.

The total angular momentum of many overlapping vortices is equal to the vortex spin times the square of the total vorticity. When vortices at rest are separated from each other by a distance much larger than the vortex core size, one finds that the total angular momentum is just the sum of individual vortex spins. The physical reason is that any gauge-invariant local field falls off exponentially to its vacuum configuration as one moves away from any vortex core. For self-dual vortex configurations characterized by the positions of the vortices, the total energy is just the sum of individual vortex masses but the total angular momentum is a function of the vortex positions. We will express the total angular momentum of vortices in self-dual models as a sum of spins and the orbital angular momentum in a clear way.

The behavior of the total angular momentum could be understood as follows. Consider two noninteracting point anyons of spin s in nonrelativistic quantum mechanics. Two separated anyons at rest have zero classical orbital angular momentum. Quantum mechanically, the orbital angular momentum is given by $2s+2\hbar\times$integer, which, in turn, implies that the total angular momentum is $4s+2\hbar\times$integer. Our vortices are extended objects and can overlap each other. If we quantize the self-dual configurations of two vortices, there would be many states whose orbital angular momentum varies from $2s$ to $2s+2\hbar\times\text{integer}\sim 0$ with the same energy. The average separation of two vortices in these states will increase as the orbital angular momentum decreases.

We are also interested in how the position dependence of the total angular momentum affects the classical dynamics of slowly moving vortices in self-dual Chern-Simons-Higgs systems. To understand the dynamics of vortices, in general, we derive the effective Lagrangian for slowly moving vortices. We follow Manton's approach [9] which means for our case that for a given number of vortices, the field configurations of slowly moving vortices are very close to the field configurations of vortices at rest and the effective action for slowly moving vortices is determined by the self-dual configurations of vortices.

We approach the problem from the Lagrangian point of view. We imagine that the motion of slowly moving vortices is a generalization of the nonrelativistic limit of the Lorentz transformation. This means that the field configurations of vortices in motion satisfy the field equations to first order in vortex velocities. We evaluate the field-theoretic Lagrangian to get a low-energy effective action as a functional of vortex positions and velocities. We show that the orbital angular momentum for vortices at rest calculated from the effective action is identical to that calculated from the field theory.

The contents of this paper are as follows. In Sec. II, we briefly review vortex configurations in self-dual Chern-Simons-Higgs systems. We show that the total angular momentum of vortices at rest can be expressed as a sum of spin and orbital angular momenta. We then present a numerical analysis for two vortices at finite separation. In Sec. III, we present the dual transformation of Chern-Simons-Higgs theories in the path integral formalism. Here we include external currents and fields in the transformation. In Sec. IV, we study various aspects of the dual formulation. We relate the statistics of vortices to the Magnus force. We also discuss the effect of external currents and fields in the dual formulation. In Sec. V, we derive and study the effective Lagrangian of slowly moving vortices. We show that vortices can be bounded to each other by the Magnus force. In Sec. VI, we conclude with some remarks. In Appendix A, we present the dual formulation of the theory of a complex scalar field with a global Abelian symmetry and discuss the Magnus force. In Appendix B, we present the dual formulation of Maxwell-Higgs theories. In Appendix C, we derive the effective Lagrangian for slowly moving vortices of self-dual Maxwell-Higgs systems using the dual formulation of Appendix B. This effective Lagrangian has been studied in detail both numerically and analytically by various authors [10].

II. MODEL

We consider the theory of a complex scalar field $\phi=fe^{i\theta}/\sqrt{2}$ interacting with a gauge field A_μ whose kinetic term is the Chern-Simons term. The Lagrangian for the theory is given by

$$\mathcal{L}=\frac{\kappa}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho+\tfrac{1}{2}(\partial_\mu f)^2+\tfrac{1}{2}f^2(\partial_\mu\theta+A_\mu)^2-U(f)\ . \tag{2.1}$$

Gauss's law constraint obtained from the variation of A_0 is

$$\kappa F_{12}+f^2(\dot\theta+A_0)=0\ , \tag{2.2}$$

where a dot denotes the time derivative. Gauss's law implies that the total magnetic flux $\Psi=\int d^2r F_{12}$ and the total electric charge $Q=\int d^2r f^2(\dot\theta+A_0)$ are related by

$$\kappa\Psi=-Q\ . \tag{2.3}$$

Here we concentrate on the self-dual model where the potential is chosen to be

$$U(f)=\frac{1}{8\kappa^2}f^2(f^2-v^2)^2 . \tag{2.4}$$

For this self-dual model we use Eq. (2.2) to express the energy functional as [1]

$$E=\int d^2r\left\{\frac{1}{2}\dot{f}^2+\tfrac{1}{2}[\partial_i f\mp\epsilon_{ij}f(\partial_j\theta+A_j)]^2+\tfrac{1}{2}f^2\left[(\dot\theta+A_0)\pm\frac{1}{2\kappa}(f^2-v^2)\right]^2\right\}\pm m_pQ , \tag{2.5}$$

where $m_p=v^2/2\kappa$ is the mass of charged particles in the symmetric phase. As the integral in Eq. (2.5) is positive, there is a bound on the energy functional:

$$E\ge m_p|Q| . \tag{2.6}$$

This bound is saturated by the configurations satisfying

$$\begin{aligned}&\dot f=0 ,\\ &\partial_i f\mp\epsilon_{ij}f(\partial_j\theta+A_j)=0 ,\\ &\dot\theta+A_0\pm\frac{1}{2\kappa}(f^2-v^2)=0 ,\end{aligned} \tag{2.7}$$

and Gauss's law (2.2). In the remainder of this section, we will consider only the positively charged configurations.

If there are vortices of unit vorticity at points $\mathbf{q}_a, a=1,\dots,n$, the phase variable can be chosen to be

$$\theta=\sum_{a=1}^{n}\mathrm{Arg}(\mathbf{r}-\mathbf{q}_a) , \tag{2.8}$$

which satisfies $\epsilon_{ij}\partial_i\partial_j\theta=2\pi\sum_a\delta(\mathbf{r}-\mathbf{q}_a)$. Equation (2.7) implies that the total magnetic flux is given by $\Psi=-2\pi n$ for this configuration. Equations (2.2), (2.7), and (2.8) imply that the f field satisfies

$$\partial_i^2\ln f^2-\frac{1}{\kappa^2}f^2(f^2-v^2)=4\pi\sum_a\delta(\mathbf{r}-\mathbf{q}_a) . \tag{2.9}$$

Assuming that vortices are not overlapping, we can analyze the behavior of the f field near $\mathbf{q}_a$. In the complex coordinate of positions, Eq. (2.9) implies

$$\begin{aligned}\ln fs=&\ln|z-q_a|^2+c+b_1(z-q_a)+b_2(z-q_a)^2+b_3(z-q_a)^3+b_4(z-q_a)^4+b_1^*(z^*-q_a^*)\\&+b_2^*(z^*-q_a^*)^2+b_3^*(z^*-q_a^*)^3+b_4^*(z^*-q_a^*)^4-\frac{v^2}{16\kappa^2}e^c|z-q_a|^4+O((z-q_a)^5) ,\end{aligned} \tag{2.10}$$

where real $c[\mathbf{q}_a]$ and complex $b_i[\mathbf{q}_a]$ are defined with respect to $\mathbf{q}_a$ and functions of the positions of other vortices.

As the system is invariant under spatial rotation, in addition to the magnetic flux and the energy there is the angular momentum which characterizes a given configuration. The angular momentum functional $J=\int d^2x\,\epsilon_{ij}r^iT^{0j}$ is given by

$$\begin{aligned}J&=-\int d^2r\epsilon_{ij}r^i[\dot f\partial_j f+f^2(\dot\theta+A_0)(\partial_j\theta+A_j)]\\&=-\int d^2r\epsilon_{ij}r^i[\dot f\partial_j f-\kappa F_{12}(\partial_j\theta+A_j)]\end{aligned} \tag{2.11}$$

with Gauss's law (2.2).

As studied in detail in Refs. [1,2], there are vortices in the asymmetric phase and nontopological solitons in the symmetric phase. The rotationally symmetric configurations of these solitons of a given vorticity n are described by the ansatz, $f(r),\theta=n\varphi$, and $A_i=\epsilon_{ij}r^j[a(r)-n]/r^2$. In the symmetric phase where $f(\infty)=0$, the solution becomes a nontopological soliton with vorticity n. In this case it is shown in Ref. [2] that $a(\infty)=-\alpha$, where $\alpha>n+2$. In the asymmetric phase where $f(\infty)=v$, $a(\infty)=0$ and the solution becomes overlapped vortices. The total magnetic flux of this ansatz is

$$\Psi=-2\pi(n+\alpha) . \tag{2.12}$$

The angular momentum of the solution can be calculated from Eq. (2.11) [2] leading to

$$J=\pi\kappa(\alpha^2-n^2) . \tag{2.13}$$

Since we have used just Gauss's law, Eq. (2.13) is applicable to theories with a more general potential than the self-dual one (2.3). Nontopological solitons in the symmetric phase have the energy per charge identical to that of elementary charged particles, implying that they are at the verge of instability. Vortices in the asymmetric phase are, however, stable for topological reasons.

In the symmetric phase elementary particles have spin $s_p=1/4\pi\kappa$ and nontopological solitons have spin s_pQ^2-Qn. Since total charge would be quantized in integers, the Qn part would be an integer. The statistics of elementary charged particles and nontopological solitons is given by the phase change of the wave function for two identical objects when they are rotated by 180° counterclockwise around the center of mass $e^{2\pi i s_pQ^2}$, and is identical to half of the Aharonov-Bohm phase $e^{i\Psi Q/2}$. On the other hand, vortices of unit vorticity would carry spin $s_v=-\pi\kappa$ and the correct statistics would be $e^{2\pi i s_v}=e^{-2\pi i s_pQ^2}$, which cannot be the naive Aharonov-Bohm phase.

The self-dual configurations of vortices seem to be parametrized only by positions $\mathbf{q}_a$. The energy is independent of vortex positions and so derivatives of fields with respect to vortex positions would become $2n$ zero modes of self-dual equations. For self-dual configurations, the total angular momentum (2.11) becomes

$$J=\kappa\int d^2r\,\epsilon_{ij}r^i(\partial_j\theta+A_j)F_{12}\ . \tag{2.14}$$

It would be interesting for the total angular momentum to be expressed explicitly in terms of the field configuration rather than as an integration over a space of its density. To find such an expression, we note first from Eq. (2.2) that F_{12} vanishes at points $\mathbf{q}_a$'s because f^2 vanishes there. Without changing the value of J, we can then subtract these points from the integration in Eq. (2.14). In the subtracted region, $F_{12}=\partial_1\bar{A}_2-\partial_2\bar{A}_1$ with $\bar{A}_i=A_i+\partial_i\theta$. Noting that

$$\epsilon_{ij}r^i\bar{A}_j\epsilon_{kl}\partial_k\bar{A}_l=\partial_i[\tfrac{1}{2}r^i(\bar{A}_j)^2-\bar{A}_i(r^j\bar{A}_j)]+(\partial_i\bar{A}_i)(r^j\bar{A}_j)\ , \tag{2.15}$$

and $\bar{A}_i$ being transverse from Eq. (2.7), we can write the angular momentum as a boundary integration [2]:

$$J=-\kappa\left[\sum_a\oint dl_a^i-\oint_\infty dl^i\right]\epsilon_{ij}\{\tfrac{1}{2}r^j(\bar{A}_k)^2-\bar{A}_j(r^k\bar{A}_k)\}\ , \tag{2.16}$$

where the sum is over the positions of vortices and the line integral is around a small counterclockwise circle around $\mathbf{q}_a$. There is no spatial infinity term in the asymmetric phase. (For nontopological solitons the boundary at spatial infinity contributes and this contribution does not depend on the position or shape of solitons.)

So far we have reviewed what is known in Ref. [2]. Let us pursue further along this direction. Let us evaluate the integral in Eq. (2.16) at each vortex position $\mathbf{q}_a$. Near $\mathbf{q}_a$, we can put $r^i=q_a^i+\epsilon_{ij}l_a^j$ and Eq. (2.16) becomes

$$J=-\kappa\sum\oint dl_a^i\epsilon_{ij}\{\tfrac{1}{2}q_a^j(\bar{A}_k)^2-\bar{A}_j(q_a^k\bar{A}_k)\}-\kappa\sum\oint dl_a^i\epsilon_{ij}\{\tfrac{1}{2}\epsilon_{jk}l_a^k(\bar{A}_l)^2-\bar{A}_j(\epsilon_{kl}l_a^l\bar{A}_k)\}\ . \tag{2.17}$$

We use Eqs. (2.7) and (2.10) to expand $\bar{A}_i$ near $\mathbf{q}_a$:

$$\bar{A}_i=-\epsilon_{ij}\partial_j\ln f=-\epsilon_{ij}\left\{\frac{l_a^j}{|l_a|^2}+b_1^j+O(l_a)\right\}\ , \tag{2.18}$$

where $b_1=b_1^1=ib_1^2$. We perform the integration in Eq. (2.17) with Eq. (2.18) to get the total angular momentum as

$$J=-2\pi\kappa\sum_a\mathbf{q}_a\cdot\mathbf{b}_1[\mathbf{q}_a]-\pi\kappa|n| \tag{2.19}$$

with the total vorticity n. The first term of the right-hand side of Eq. (2.19) represents the orbital part and the second term represents the spin part.

There is only one length scale v^2/κ in the problem. As the distance between vortices goes to infinity, the field around each vortex position would approach to the rotationally symmetric ansatz exponentially, which with Eq. (2.10) means that $b_i[q_a]$'s vanish exponentially with the mutual distance. Thus, the total angular momentum (2.19) would change from s_vn^2 to s_vn as vortices get separated from each other. The self-dual vortex configurations are degenerate in the energy but not in the angular momentum. The orbital part of the angular momentum is explicitly expressed in terms of self-dual configurations in Eq. (2.19).

Let us now consider the system of two vortices located at points $\mathbf{q}_1=\mathbf{q}/2$ and $\mathbf{q}_2=-\mathbf{q}/2$. The symmetry of the configuration tells us that

$$f(\mathbf{r};\mathbf{q})=f(\mathbf{r};-\mathbf{q})=f(-\mathbf{r};-\mathbf{q})\ ,$$

which, in turn, implies $\mathbf{b}_1(\mathbf{q})=-\mathbf{b}_1(-\mathbf{q})=-\mathbf{b}_2(\mathbf{q})$. In addition, Eq. (2.9) is parity invariant and so the f configuration does not change under the reflection which exchanges two vortices, implying that

$$\mathbf{b}_1=\mathbf{q}\mathcal{B}(q)\ , \tag{2.20}$$

where $q=|\mathbf{q}|$. The total angular momentum (2.19) becomes

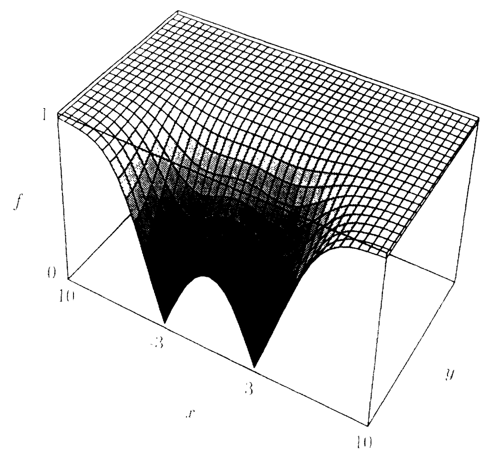

FIG. 1. Plot of the f field in units of v on the x-y plane for two vortices of mutual distance $d=6$ with spatial distance unit v^2/κ.

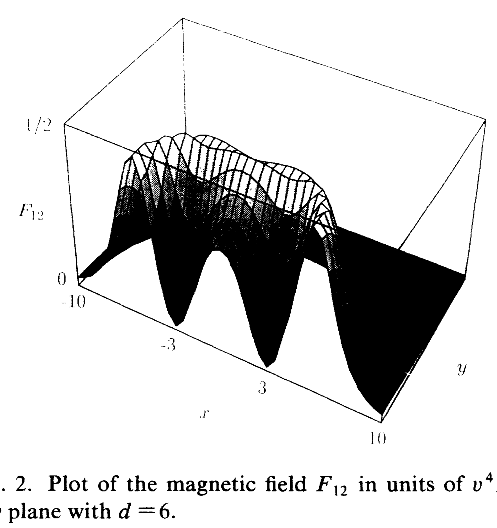

FIG. 2. Plot of the magnetic field F_{12} in units of $v^4/4\kappa^2$ on the x-y plane with $d=6$.

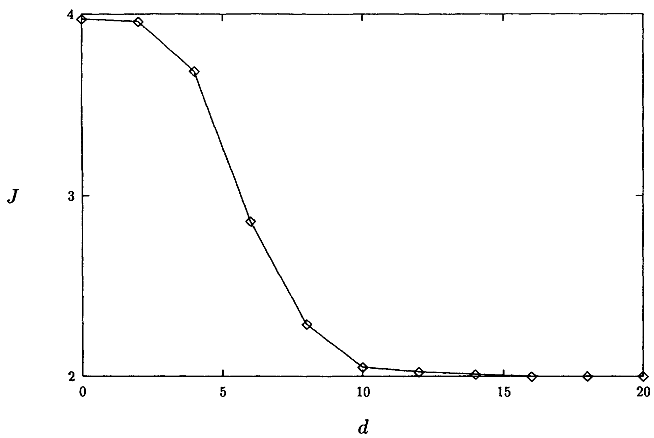

FIG. 3. Plot of the total angular momentum in units of $-\pi\kappa$ as a function of mutual distance d.

$$J=-2\pi\kappa q^2\mathcal{B}(q)-2\pi\kappa\ . \tag{2.21}$$

This angular momentum should approach that of the two overlapped vortices when $q\to 0$, which implies

$$\mathcal{B}=\frac{1}{q^2}+\text{const} \tag{2.22}$$

near $q=0$.

We have studied the configurations of two vortices by a numerical analysis. Although the existence of multivortex solutions is proved [11], no exact solutions are found. Figures 1 and 2 show the magnitude of the scalar field and the magnetic field at $d=6\kappa/v^2$. In Fig. 3, we show the total angular momentum as a function of the separation distance. The total angular momentum decreases from $-4\pi\kappa$ to $-2\pi\kappa$, supporting the argument in the previous paragraphs.

III. DUAL FORMULATION

To understand the interaction between vortices, let us consider transition amplitudes of a Chern-Simons-Higgs system in the path integral formalism. For a generality we include an external gauge field A_μ^{ext} and an external current J^μ. The Lagrangian is then

$$\mathcal{L}=\frac{\kappa}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho+\tfrac{1}{2}(\partial_\mu f)^2+\tfrac{1}{2}f^2(\partial_\mu\theta+A_\mu+A_\mu^{\text{ext}})^2-U(f)+A_\mu J^\mu\ . \tag{3.1}$$

The generating functional is

$$Z=\langle F|e^{-iHT}|I\rangle=\int[df][d\theta][dA_\mu]\prod_x f(x)\Psi_F\exp\left[i\int d^3x\mathcal{L}\right]\Psi_I\ , \tag{3.2}$$

where there is a nontrivial Jacobian factor because we use the radial coordinate for the scalar field. The initial and final wave functions $\Psi_{F,I}$ give necessary boundary conditions.

A given field configuration in the path integral could contain vortices and antivortices and the θ field could be multivalued. We can, in principle, split the θ field into two parts,

$$\theta(\mathbf{r},t)=\bar{\theta}(\mathbf{r},t)+\eta(\mathbf{x},t)\ , \tag{3.3}$$

where the first term describes a configuration of vortices,

$$\bar{\theta}(\mathbf{r},t)=\sum_a(-1)^a\text{Arg}[\mathbf{r}-\mathbf{q}_a(t)]\ , \tag{3.4}$$

with vorticities $(-1)^a$ and locations $\mathbf{q}_a(t)$, and the second term η represents single-valued fluctuations around a given configuration of vortices. From the multivalued $\bar{\theta}$, we can construct the vortex current

$$K^\mu(x)\equiv\frac{1}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu\partial_\rho\bar{\theta}=\sum_a(-1)^a\left[1,\frac{d\mathbf{q}_a}{dt}\right]\delta^2(\mathbf{r}-\mathbf{q}_a(t))=\sum_a(-1)^a\int d\tau\frac{dq_a^\mu}{d\tau}\delta^3(x^\nu-q_a^\nu(\tau))\ , \tag{3.5}$$

which satisfies the conservation law, $\partial_\mu K^\mu=0$. Integration over the θ variable becomes

$$[d\theta]=[d\bar{\theta}][d\eta]=[dq_a^\mu][d\eta]\ , \tag{3.6}$$

which means that we sum over single-valued fluctuations around a given configuration of vortices and then sum over all possible configurations of vortices, including annihilation and creation of vortex pairs.

Let us now linearize the third term of the Lagrangian (3.1) by introducing an auxiliary vector field C^μ:

$$\prod_x f(x)^3\exp\left[i\int d^3x[\tfrac{1}{2}f^2(\partial_\mu\theta+A_\mu+A_\mu^{\text{ext}})^2]\right]=\int[dC^\mu]\exp\left[i\int d^3x\left\{-\frac{1}{2f^2}(C^\mu)^2+C^\mu(\partial_\mu\bar{\theta}+\partial_\mu\eta+A_\mu+A_\mu^{\text{ext}})\right\}\right]\ , \tag{3.7}$$

where the nontrivial Jacobian is essential. As η is single valued, one can integrate over η in the standard way, leading to

$$\int[d\eta]\exp\left[i\int d^3x\ C^\mu\partial_\mu\eta\right]=\delta(\partial_\mu C^\mu)\ . \tag{3.8}$$

Now we introduce the dual gauge field H_μ to satisfy

$$\int[dC^\mu]\delta(\partial_\mu C^\mu)\cdots=\int[dC^\mu][dH_\mu]\delta\left[C^\mu-\frac{1}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu H_\rho\right]\cdots \tag{3.9}$$

where the ellipses denote the integrand. There would be an infinite gauge volume which can be taken care of later, but there is no nontrivial Jacobian factor as the change of variables is linear. By using the fact that

$$\frac{1}{2\pi}\epsilon^{\mu\nu\rho}(\partial_\mu\bar{\theta})\partial_\nu H_\rho = K^\mu H_\mu \tag{3.10}$$

up to a total derivative, we can integrate over C^μ, resulting in the Lagrangian, which is

$$\mathcal{L}' = -\frac{1}{16\pi^2 f^2}H_{\mu\nu}^2 + H_\mu K^\mu + \tfrac{1}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu\left[\kappa A_\rho + \frac{1}{\pi}H_\rho\right] + \frac{1}{2\pi}\epsilon^{\mu\nu\rho}H_\mu\partial_\nu A_\rho^{\text{ext}} + A_\mu J^\mu + \cdots , \tag{3.11}$$

where $H_{\mu\nu}=\partial_\mu H_\nu - \partial_\nu H_\mu$ and the ellipses indicates f-dependent terms.

The exponent is quadratic in A_μ and so the integral over A_μ is easy. The equation of motion for A_μ is

$$\kappa\epsilon^{\mu\nu\rho}\partial_\nu A_\rho = -\frac{1}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu H_\rho - J^\mu , \tag{3.12}$$

whose solution is formally

$$A_\mu = -\frac{1}{2\pi\kappa}H_\mu + V_\mu , \tag{3.13}$$

where $\epsilon^{\mu\nu\rho}\partial_\nu V_\rho = -J^\mu/\kappa$. Rather than integrating over A_μ, we substitute A_μ in the path integral by Eq. (3.13) and then the integration over A_μ becomes the integration over V_μ.

The resulting path integral becomes

$$Z = \int [f^{-2}df][dq_a^\mu][dH_\mu][dV_\mu]\exp\left[i\int d^3x\, \mathcal{L}_D\right] , \tag{3.14}$$

where the dual transformed Lagrangian is

$$\mathcal{L}_D = \tfrac{1}{2}(\partial_\mu f)^2 - U(f) - \frac{1}{16\pi^2 f^2}H_{\mu\nu}^2 - \frac{1}{8\pi^2\kappa}\epsilon^{\mu\nu\rho}H_\mu\partial_\nu H_\rho + H_\mu K^\mu - \frac{1}{2\pi\kappa}H_\mu J^\mu + \frac{1}{2\pi}\epsilon^{\mu\nu\rho}H_\mu\partial_\nu A_\rho^{\text{ext}} + \frac{\kappa}{2}\epsilon^{\mu\nu\rho}V_\mu\partial_\nu V_\rho + V_\mu J^\mu . \tag{3.15}$$

There is no Jacobian factor in the measure. One can introduce the gauge-fixing terms for H_μ and V_μ. The sign difference between the Chern-Simons terms of the original and dual transformed theories will be crucial in understanding the statistics of vortices. The original gauge field is separated into two pieces: H_μ and V_μ. The vortex current K^μ becomes an electric current for the dual field H_μ. The external current is, however, coupled to both the dual gauge field H_μ and reduced gauge field V_μ.

Now the vortex positions appear in the Lagrangian explicitly, however, without any kinetic term. The mass of vortices arises solely from the f, H_μ fields. The vortex position cannot be chosen independently from the field configurations. The variation of H_0 implies Gauss's law constraint

$$\partial_i\left[\frac{1}{f^2}H_{0i}\right] - \frac{1}{\kappa}H_{12} + 4\pi^2 K^0 + 2\pi F_{12}^{\text{ext}} - \frac{2\pi}{\kappa}J^0 = 0 , \tag{3.16}$$

The relation between the original fields and the dual fields in the classical level can be seen from the equations one would get from the Lagrangians at various steps. They are related to each other by

$$f^2(\partial_\mu\theta + A_\mu + A_\mu^{\text{ext}}) = C_\mu = \frac{1}{2\pi}\epsilon_{\mu\nu\rho}\partial^\nu H^\rho . \tag{3.17}$$

Another relation between the original and dual fields is given by Eq. (3.13). The original U(1) charge is then given by

$$\begin{aligned} Q &= \int d^2r\{f^2(\dot{\theta} + A_0 + A_0^{\text{ext}}) + J^0\} \\ &= \int d^2r\left\{\frac{1}{2\pi}H_{12} + J^0\right\} \\ &= \int d^2r\left\{2\pi\kappa K^0 + \frac{\kappa}{2\pi}\partial_i\left[\frac{1}{f^2}H_{0i}\right] + \kappa F_{12}^{\text{ext}}\right\} , \end{aligned} \tag{3.18}$$

where the last equality comes from Gauss's law (3.16). The second-to-last term gives a nonzero contribution to the charge for the configuration of nontopological solitons in the symmetric phase where the f field vanishes only on the isolated points and the spatial infinity. Note that the charge conservation in dual formulation is satisfied by the topology of the field configuration, not by the field equations.

IV. PHYSICAL CONSEQUENCES

We have obtained the dual formulation of Chern-Simons-Higgs systems, which could be useful in understanding the various physical aspects of the asymmetric phase. In the dual formalism the interaction between vortices is more direct because they appear as charged particles rather than topological objects. In addition, we can see the interaction between vortices and external currents and fields more directly. The dual transformation in general changes a weak-coupling theory into a strong-coupling theory and vice versa. The dual formulation has been widely used to understand the phase structure of a given theory. (See Ref. [12] for a review.) If we try to quantize vortices by the semiclassical method, the coupling between elementary particles should be very small, or $\kappa \gg 1$ for the method to be a good approximation. In this case, vortices interact with each other strongly as one can see from the dual formulation. However this aspect of the dual formulation will not be explored in this paper. Let us now make a few observations, about the dual formulation.

A. Massive vector bosons

There are two ways to describe a massive vector boson of spin one in three dimensions: the Maxwell-Chern-Simons terms or the Chern-Simons-Higgs terms. This observation led to the original derivation of the dual transformation [6]. From Eqs. (3.1) and (3.15) with $f=v$ we have two equivalent Lagrangians:

$$\mathcal{L}_1=\frac{\kappa}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho+\tfrac{1}{2}v^2A_\mu^2\ ,$$
$$\mathcal{L}_2=-\frac{1}{16\pi^2v^2}H_{\mu\nu}^2-\frac{1}{8\pi^2\kappa}\epsilon^{\mu\nu\rho}H_\mu\partial_\nu H_\rho\ . \tag{4.1}$$

Both describe a particle of mass $m=v^2/|\kappa|$ and spin $-\kappa/|\kappa|$

B. Quantum Magnus phase

We know how a spinning baseball curves. Let us consider a two-dimensional version. When a ball is moving to the negative x direction with clockwise rotation in a fluid, the wind velocity (at the ball's rest frame) on the positive y part is faster than that on the negative y part, resulting in the pressure difference. The net force on the ball is then pointing the positive y direction. The magnitude of the force is proportional to the ball velocity and so this force is somewhat similar to an effective Lorentz force due to a constant magnetic field. When the moving object is a vortex, it is called the Magnus force [8].

The simplest example in field theories is the theory of a complex scalar field with a global Abelian symmetry. When global vortices in this theory move in a uniform charge background, it feels this Magnus force. As one can see in Appendix A, the charge density appears as a uniform magnetic field and vortices as charged particles in the dual formulation. The Magnus force is given exactly as a Lorentz force

Let us consider now vortices in the asymmetric phase of a Maxwell-Higgs theory. A vortex with nonzero speed moves in a straight line because it is just Lorentz boosted. If we introduce a uniform external electric charge, a vortex moves differently. The background charge will be shielded by the counter charge carried by the Higgs field. There is also a rest frame of charge. As shown in Appendix B, vortices now feel the Magnus force due to the shielding charge carried by the Higgs field, quite similarly as global vortices feel the force due to the global charge carried by the complex scalar field. In Appendix B, we show that in the dual formulation vortices again appear as charge particles and the shielding charge appears as the uniform magnetic field. The Lorentz force due to this effective magnetic field is again the Magnus force in disguise.

Vortices in a Chern-Simons-Higgs theory carry both magnetic flux and charge around their core. When vortices are close to each other, they would feel the electromagnetic force, which leads to an Aharonov-Bohm phase at a large distance. Because vortices carry charge, there would be also the Magnus force between vortices. The Magnus force is like a Lorentz force in nature and would lead to an additional Aharonov-Bohm phase. The total Aharonov-Bohm phase would then be a sum of those from these two forces. As we have seen in the previous section, the dual formulation of Chern-Simons-Higgs systems has the dual gauge interaction between vortices. Thus, one would say the original electromagnetic force and the Magnus force come together as a single dual gauge force. Vortices are charged particles in the dual formulation and so their statistics should be given by the Aharonov-Bohm phase coming from the dual Lagrangian (3.15). The Aharonov-Bohm phase is a long-distance effect and determined only by the Chern-Simons term. The sign of the Chern-Simons term in the dual Lagrangian is opposite that of the original Lagrangian, which makes the real Aharonov-Bohm phase between vortices exactly the inverse of the naive Aharonov-Bohm phase. This implies that the Magnus force is two times larger than and has opposite sign to the original electromagnetic force. The statistics from the dual Aharonov-Bohm phase is consistent with what we expect from the vortex spin. Although the dual formulation has been discovered many times [6], this aspect of the vortex interaction has not been noticed before. Of course, one should be able to derive the statistics between vortices from the original formulation. This is done in the next section.

C. External current and field

We have derived the dual Lagrangian which is valid even with external currents and fields. Let us see first the effect of an external point charge. In the asymmetric phase of a Maxwell-Higgs theory, any charge will be completely screened and the net total charge is zero. In the asymmetric phase of a Chern-Simons-Higgs theory, the charge of vortices cannot, however, be screened because Gauss's law (2.3) or (3.16) implies that the total charge is nonzero when there is a nonzero magnetic flux. If there is no external field and vortex, there cannot be any net magnetic flux for any finite energy configuration and so the total charge is zero, implying that external charges are totally screened. For a given external point charge, the screening charge will surround this charge with the length scale given from the Higgs boson mass.

What is the interaction between external currents and vortices? The dual Lagrangian (3.15) leads to the answer. Both of them are charged currents of the dual gauge field and so there would be a nontrivial phase when the external charge goes around a vortex in a full circle. This phase is determined by the dual Lagrangian and is given by $e^{-i2\pi Q_{\rm ext}}$ with an external charge $Q_{\rm ext}$. If the charge is fractional, the phase is nontrivial because the Higgs field carries a unit charge and can screen only integer charges completely. From Gauss's law (3.16) we see that a uniform external charge density is screened by a uniform dual magnetic field. A single vortex moving on this background would feel the Magnus force which appears as a Lorentz force.

What is the interaction between two external point charges? They are interacting through both gauge fields H_μ and V_μ. At a large distance, the Aharonov-Bohm phases due to two gauge interactions would cancel each other, and there is no nontrivial phase between them. Since the screening charge has a finite core size, at a short distance external charges would see the nontrivial statistics.

We can ask whether the external currents and fields could have a dynamical origin. In Sec. III we have not used the conservation of external currents explicitly. The key aspect is that external currents and the gauge field A_μ couple linearly. There seems to be two simple examples where external currents arise dynamically. The matter could be made of fermions, in which case $A_\mu J^\mu$ would be replaced by

$$\mathcal{L}_F = i\bar{\psi}\gamma^\mu(\partial_\mu + ieA_\mu)\psi + \cdots , \tag{4.2}$$

where the ellipses denotes the mass and Yukawa interaction terms. Or, the matter field could be made of a simple gauge field, in which case the additional Lagrangian would be

$$\mathcal{L}_W = e\epsilon^{\mu\nu\rho}A_\mu\partial_\nu W_\rho + \cdots , \tag{4.3}$$

where the ellipses indicates the kinetic terms for the W_μ field. The external field $A_\mu^{\rm ext}$ can be made dynamical by replacing $A_\mu^{\rm ext}$ in Eq. (3.1) by a gauge field W_μ with some kinetic term. It is trivial to see how these dynamical degrees couple to the dual gauge field, which we will not bother to write down.

One may wonder whether there is any dual formulation of Maxwell-Chern-Simons-Higgs theories. One can follow a similar procedure as in Sec. III and Appendix B and will end with a dual formulation with two gauge fields even when there is no external currents and fields. In the case where the f field is fixed as a constant, a dual formulation with a single gauge field was obtained in Ref. [13] with a different approach.

V. LOW-ENERGY EFFECTIVE LAGRANGIAN

We now consider the dynamics of self-dual vortices with the specific potential (2.3). Vortices at rest are described by the configurations satisfying the self-dual equations (2.7). As they are degenerated in energy, there are no attractive or repulsive forces between them even though there may be velocity-dependent forces. We ask then what is the effective action for slowly moving vortices? As there are no massless modes in the system, the slowly moving vortices would not dissipate their energy to other degrees of freedom and would be described by a nonrelativistic action of interacting particles. This action would consist of terms quadratic and linear in vortex velocities. The most general Lagrangian for the n slowly moving vortices with positions $\mathbf{q}_a(t)$ would then be

$$L = \frac{1}{2}\sum_{ab,ij} R_{ab}^{ij}(\mathbf{q}_c)\dot{q}_a^i\dot{q}_b^j + \sum_{a,i}\dot{q}_a^i H_a^i(\mathbf{q}_c) . \tag{5.1}$$

We want to derive this Lagrangian from the field-theoretic consideration. The linear term would represent the "dual magnetic" interaction between vortices. The effective action also should somehow take into account the fact that vortices are not degenerate in angular momentum. At the moment, we are interested in the classical picture. For this classical picture to be consistent, quantum fluctuations should be very small, which means $\kappa \gg 1$ and vortices interact with each other strongly as argued in the previous section.

There are considerable works [10] for the effective action for slowly moving vortices in self-dual Maxwell-Higgs systems. Their approaches are either geometrical or numerical. Here we take a somewhat different tactic which seems to work also in the Maxwell-Higgs case as shown in Appendix C, where we use the dual formulation even though the original formulation would work equally well. For self-dual Chern-Simons-Higgs systems, the dual formulation seems cumbersome for our present purpose and we start from the original Lagrangian (2.1).

Consider n vortices with a uniform velocity $\mathbf{u}$ and the total mass $M = \pi v^2 n$. The field configuration for this case can be obtained from that of vortices at rest by a Lorentz transformation. We are interested in the slow motion or the nonrelativistic limit. The f field would transform trivially and the gauge field as a vector would have a correction linear in $\mathbf{u}$. The gauge fields would satisfy the field equation to first order in $\mathbf{u}$. We can calculate the Lagrangian $L = \int d^2r\mathcal{L}$ with this transformed configuration and get the expected result $L = M\mathbf{u}^2/2 - M$.

For the field configuration of slowly moving vortices of a given trajectory $\mathbf{q}_a(t)$, we imagine a generalization of the nonrelativistic limit of the Lorentz transformation. For consistency, we will assume that there are first-order corrections to both scalar and gauge fields, and require that they satisfy the field equations to the first order in the vortex velocities, $\dot{\mathbf{q}}_a(t)$. The zeroth order f field would be $f(\mathbf{r};\mathbf{q}_a(t))$ which satisfies Eq. (2.9). The zeroth-order θ field would be $\theta = \sum_a \mathrm{Arg}[\mathbf{r} - \mathbf{q}_a(t)]$. For a given zeroth-order scalar field, the gauge field in the same order can be obtained from Eqs. (2.3) and (2.7):

$$\begin{aligned} A_0(\mathbf{r};\mathbf{q}_a(t)) &= -\frac{1}{2\kappa}(f^2 - v^2) , \\ A_i(\mathbf{r};\mathbf{q}_a(t)) &= -\tfrac{1}{2}\epsilon_{ij}\partial_j \ln\frac{f^2}{\prod_a(\mathbf{r}-\mathbf{q}_a(t))^2} . \end{aligned} \tag{5.2}$$

From the Lagrangian (2.1), we get the field equations

$$\begin{aligned} &-\partial_\mu^2 f + (\partial_\mu\theta - A_\mu)^2 f - U'(f) = 0 , \\ &\kappa F_{12} + f^2(\dot{\theta} + A_0) = 0 , \\ &\kappa\epsilon_{ij}F_{0j} + f^2(\partial_i\theta + A_i) = 0 . \end{aligned} \tag{5.3}$$

The field equation for the θ field can be obtained from the Jacobi identity for the gauge field equations. We demand the field equations are satisfied to first order in the vortex velocities by the field corrections Δf, $\Delta\theta$, ΔA_μ. We choose the gauge where $\Delta\theta = 0$. From Eq. (5.3), we get the first-order field equations

$$\begin{aligned} &\partial_i^2\Delta f + [A_0^2 - (\partial_i\theta + A_i)^2]\Delta f + 2fA_0(\dot{\theta} + \Delta A_0) \\ &\qquad - 2f(\partial_i\theta + A_i)\Delta A_i - U''(f)\Delta f = 0 , \\ &\kappa\epsilon_{ij}\partial_i\Delta A_j + f^2(\dot{\theta} + \Delta A_0) + 2fA_0\Delta f = 0 , \\ &\kappa\epsilon_{ij}(\dot{A}_j - \partial_j\Delta A_0) + f^2\Delta A_i + 2f(\partial_i\theta + A_i)\Delta f = 0 , \end{aligned} \tag{5.4}$$

where f, A_μ are given in zeroth order. We do not know at the present moment the solution $\Delta f, \Delta A_\mu$ of Eq. (5.4) in terms of f explicitly. If there is a unique solution of Eq. (5.4) for a given trajectory of vortices, we have definite configurations for slowly moving vortices of the self-dual Chern-Simons-Higgs system. For the vortices of a uniform motion, the solution of Eq. (5.4) is trivially given taking the nonrelativistic limit of the Lorentz transformed fields.

We imagine the field sum in the path integral to be restricted to these configurations for slowly moving vortices. The field-theoretic action for slowly moving vortices then becomes the effective action as a functional of these vortex trajectories. There will be terms linear and quadratic in the vortex velocities, but no terms which just depend on the vortex positions. We do not need to consider the second-order corrections to the fields because their contribution vanishes due to the field equations satisfied by the zeroth-order fields. By using Gauss's law (2.2), let us write the Lagrangian density (2.1) as

$$\mathcal{L}=\tfrac{1}{2}\dot{f}^2-\kappa\dot{\theta}F_{12}-\frac{\kappa}{2}\epsilon_{ij}\dot{A}_iA_j-\tfrac{1}{2}(\partial_if)^2 -\tfrac{1}{2}f^2(\dot{\theta}+A_0)^2-\tfrac{1}{2}f^2(\partial_i\theta+A_i)^2-U(f)\ . \tag{5.5}$$

The zeroth-order term in the effective action can be calculated trivially and becomes negative of the rest mass. In Eq. (5.5) the last four terms can be put into a sum of two squares plus the rest mass term as shown in Eq. (2.5) and so yields only the second-order terms. Thus, the first-order term $\Delta_1\mathcal{L}$ for the self-dual system is given by

$$\Delta_1\mathcal{L}=-\kappa\dot{\theta}F_{12}-\frac{\kappa}{12}\epsilon_{ij}\dot{A}_iA_j\ , \tag{5.6}$$

where the fields are given in zeroth order. From this Lagrangian, it is not clear how to separate the effect of the electromagnetic interaction from that of the Magnus force. With Eq. (5.2), the second part of the right-hand side of Eq. (5.6) is proportional to

$$2\epsilon_{ij}\dot{A}_iA_j=-A_i\partial_i\partial_0\ln\frac{f^2}{\prod} =-\partial_i\left\{A_i\partial_0\ln\frac{f^2}{\prod}\right\}+(\partial_iA_i)\partial_0\ln\frac{f^2}{\prod}\ , \tag{5.7}$$

with the obvious understanding of $\prod$. Since the zeroth-order gauge field is transverse and the boundary terms lead to no contribution to the effective action, Eq. (5.7) does not contribute to the effective action. With $\dot{\theta}=\sum_a\epsilon_{ij}\dot{q}_a^i\partial_j\ln|\mathbf{r}-\mathbf{q}_a|$, the first part of the right-hand side of Eq. (5.6) is proportional to

$$\dot{\theta}F_{12}=\epsilon_{ij}\epsilon_{kl}\dot{q}_a^i(\partial_j\ln|\mathbf{r}-\mathbf{q}_a|)\partial_kA_l =\dot{q}^i\partial_i(A_j\partial_j\ln|\mathbf{r}-\mathbf{q}_a|)-\dot{q}_a^i\partial_j(A_j\partial_i\ln|\mathbf{r}-\mathbf{q}_a|)+\dot{q}_a^i(\partial_jA_j)\partial_i\ln|\mathbf{r}-\mathbf{q}_a|-\dot{q}_a^i\partial_j(A_i\partial_j\ln|\mathbf{r}-\mathbf{q}_a|)+\dot{q}_a^iA_i\partial_j^2\ln|\mathbf{r}-\mathbf{q}_a|\ , \tag{5.8}$$

where $\epsilon_{ij}\epsilon_{kl}=\delta_{ik}\delta_{jl}\delta_{il}\delta_{jk}$ is used and the sum over the index a is assumed. Now we can get the first-order term of the effective Lagrangian from Eqs. (5.6) and (5.8),

$$\Delta_1L=\int d^2r\,\Delta_1\mathcal{L}=-2\pi\kappa\sum_a\dot{q}_a^iA_i(\mathbf{q}_a)\ , \tag{5.9}$$

where $\partial_i^2\ln|\mathbf{r}-\mathbf{q}_a|=2\pi\delta(\mathbf{r}-\mathbf{q}_a)$ is used and the boundary terms are dropped as they make no contributions to the effective action. There is another way to see that Eq. (5.9) is the only contribution from Eq. (5.8) even though many terms in Eq. (5.8) seem singular at vortex positions. Because $\dot{\theta}\sim1/\delta$ and $F_{12}\sim\delta^2$ with $\boldsymbol{\delta}=\mathbf{r}-\mathbf{q}_a$, $\dot{\theta}F_{12}$ vanishes at vortex positions, which allows us to subtract these points from the integration in Eq. (5.9). Then we can calculate the integration with boundary contributions at vortex positions, getting the same result. From Eqs. (2.7) and (2.10), we get

$$A_i(\mathbf{r}=\mathbf{q}_a)=-\epsilon_{ij}\left\{b_1^j(\mathbf{q}_a)+\sum_{b\neq a}\frac{q_a^j-q_b^j}{|\mathbf{q}_a-\mathbf{q}_b|^2}\right\}\ . \tag{5.10}$$

The second-order term in the Lagrangian would be

$$\Delta_2\mathcal{L}=\kappa\Delta A_0\Delta F_{12}+\tfrac{1}{2}\kappa\epsilon_{ij}(\Delta\dot{A}_iA_j+\dot{A}_i\Delta A_j)+\tfrac{1}{2}\dot{f}^2-\tfrac{1}{2}(\partial_i\Delta f)^2-\tfrac{1}{2}U''(f)(\Delta f)^2+\tfrac{1}{2}f^2(\dot{\theta}+\Delta A_0)^2 -\tfrac{1}{2}f^2(\Delta A_i)^2+\tfrac{1}{2}(\Delta f)^2A_0^2-\tfrac{1}{2}(\Delta f)^2(\partial_i\theta+A_i)^2+2f\Delta fA_0(\dot{\theta}+\Delta A_0)-2f\Delta f(\partial_i\theta+A_i)\Delta A_i\ . \tag{5.11}$$

First note that $\epsilon_{ij}\Delta\dot{A}_iA_j=\epsilon_{ij}\dot{A}_i\Delta A_j$ up to a total time derivative, which does not affect the effective action. We use Eq. (5.4) to remove U'' and the Chern-Simons part up to total derivatives. The resulting second-order term is

$$\Delta_2\mathcal{L}=\tfrac{1}{2}\dot{f}^2+\tfrac{1}{2}f^2(\dot{\theta}+\Delta A_0)^2+\tfrac{1}{2}f^2(\Delta A_i)^2+f\Delta fA_0(\dot{\theta}+\Delta A_0)+f\Delta f(\partial_i\theta+A_i)\Delta A_i\ . \tag{5.12}$$

Hence, we have obtained the effective action for slowly moving vortices. From Eqs. (5.9) and (5.11), we can see that the effective Lagrangian for slowly moving vortices is given by

$$L_{\text{eff}}(\mathbf{q}_a,\dot{\mathbf{q}}_a)=\int d^2r\{\tfrac{1}{2}\dot{f}^2+\tfrac{1}{2}f^2[(\dot{\theta}+\Delta A_0)^2+(\Delta A_i)^2]+f\Delta fA_0(\dot{\theta}+\Delta A_0)+f\Delta f(\partial_i\theta+A_i)\Delta A_i\} -2\pi\kappa\sum_a\dot{q}_a^iA_i(\mathbf{q}_a)\ . \tag{5.13}$$

Here f, A_i given by Eqs. (2.9) and (5.2) and functions of $\mathbf{q}_a$'s. Δf and ΔA_μ are given by the first-order field equation (5.4). The effective Lagrangian is made of the usual quadratic terms and the linear terms which describe the magnetic interactions between vortices.

We made some reasonable assumptions to derive the effective action for slowly moving vortices. A configuration for moving vortices is specified by $f+\Delta f$, $A_\mu+\Delta A_\mu$. The energy functional for this configuration consists of the rest mass and terms linear and quadratic in the vortex velocities. From Eq. (5.4), one can easily show that $\Delta_1 E=\int d^2r\, \kappa\epsilon_{ij}\partial_j[A_0\Delta A_i]$, which vanishes. The quadratic terms in the energy functional are not identical to the quadratic part of the effective Lagrangian. The difference is

$$\begin{aligned}\Delta_2 E-\Delta_2 L=\int d^2r\{&f\Delta f A_0(\dot\theta+\Delta A_0)\\&+f\Delta f(\partial_i\theta+A_i)\Delta A_i+\tfrac{1}{2}(\partial_i\Delta f)^2\\&+\tfrac{1}{2}(\Delta f)^2A_0^2\\&+\tfrac{1}{2}(\Delta f)^2(\partial_i\theta+A_i)^2\}\ ,\end{aligned}\tag{5.14}$$

which does not seem to vanish. We believe that the quadratic part of the effective action is given by $\Delta_2 L$ rather than $\Delta_2 E$ because the linear part cannot be obtained from the energy point of view. For uniformly moving vortices, the first-order correction (5.4) of the fields would be given by the nonrelativistic limit of the Lorentz transformed fields and the effective Lagrangian becomes the total kinetic energy of the system.

We can use Eq. (3.13) to express the linear term in terms of the dual gauge field. As there is no external charge, we can choose the gauge where $V_\mu=0$ and $A_\mu=-H_\mu/2\pi\kappa$. The linear term (5.9) becomes then

$$\Delta_1 L=\sum_a \dot q_a^i H_i(\mathbf{q}_a)\ ,\tag{5.15}$$

which is exactly what we get from the dual formulation and need for the statistics of vortices. The linear part (5.6) implies the "dual magnetic" interaction between vortices. In Sec. IV, we argued that dual magnetic interaction originates both ordinary magnetic and Magnus forces. This linear interacting term (5.14) leads to the statistical phase between vortices. In Ref. [14], Eqs. (5.6) and (5.7) have been examined to get the statistics of vortices at a large separation but was not put into a simple form as Eq. (5.9) or (5.14), let alone its physical meaning.

From Eq. (2.2) we can see the original gauge field strength F_{12} vanishes at $\mathbf{r}=\mathbf{q}_a$. This does not mean that the field strength felt by vortices vanishes. The reason is that the gauge field $A_\mu(\mathbf{r};\mathbf{q}_b)$ as a function of $\mathbf{q}_a$ when $\mathbf{r}=\mathbf{q}_a$ is different from that as a function of $\mathbf{r}$. From Eq. (5.10), we can get the field strength felt by the vortex at $\mathbf{q}_a$;

$$\begin{aligned}\mathcal{H}_{12}(\mathbf{q}_a)&=-2\pi\kappa\epsilon_{ij}\frac{\partial A_j}{\partial q_a^i}\\&=-2\pi\kappa\frac{\partial b_1^i[\mathbf{q}_a]}{\partial q_a^i}\ .\end{aligned}\tag{5.16}$$

Similar consideration would apply as well to the cases studied in Appendices A and B.

There is an interesting check of the linear term. Let us consider the total angular momentum of vortices from the low-energy effective Lagrangian (5.12):

$$\begin{aligned}J_{\text{orbit}}&=\sum_a \epsilon_{ij}q_a^i\frac{\partial L}{\partial \dot q_a^j}\\&=-2\pi\kappa\sum_a \epsilon_{ij}q_a^iA_j(\mathbf{q}_a)+O(\dot q_a^i)\ .\end{aligned}\tag{5.17}$$

With Eqs. (2.10) and (5.2), one can see that

$$J_{\text{orbit}}=-2\pi\kappa\sum_a \mathbf{q}_a\cdot\mathbf{b}_1[\mathbf{q}_a]+O(\dot q_a^i)\ ,\tag{5.18}$$

which is identical to the orbital part in Eq. (2.19) for vortices at rest. Our effective Lagrangian for slowly moving vortices is consistent with the field-theory Lagrangian.

Let us briefly study the dynamics of slowly moving two vortices. First consider two overlapped vortices with a small initial kinetic energy. The initial angular momentum would be very close to $4s_v$. If they can escape form each other to spatial infinity, their angular momentum would be the sum of spins and the orbital angular momentum, $2s_v+u_0b$, where u_0 is the asymptotic speed and b is the impact parameter. As we can choose the kinetic energy, or u_0, arbitrarily small, the angular momentum conservation says that the impact parameter becomes arbitrary large, which is impossible because the force is short ranged. Rather, we think that two vortices are bound together by the mutual magnetic field. By turning around this argument, one can also see that two vortices from the spatial infinity with very small kinetic energy cannot make a head-on collision, rather they will always veer off from each other. By a similar argument, one can see that two vortices of a finite separation and a very small kinetic energy would not escape to the spatial infinity nor overlap each other.

More specifically, consider two vortices at rest whose locations are $\mathbf{q}_1=-\mathbf{q}/2$, $\mathbf{q}_2=\mathbf{q}/2$. From Eqs. (2.10) and (2.20), we see the linear part of the effective Lagrangian (5.9) can be written as

$$\Delta_1 L=-2\pi\kappa\epsilon_{ij}\dot q^iq^j\left[\mathcal{B}(q)-\frac{1}{q^2}\right]\ ,\tag{5.19}$$

which is smooth due to Eq. (2.2). The naive Aharonov-Bohm phase from the electromagnetic interaction would come from the Lagrangian $Q_v\dot q^iA_i$ with the vortex charge $Q_v=2\pi\kappa$, which has the opposite sign as that of the above equation. The magnetic field felt by the reduced one body is

$$\begin{aligned}\mathcal{H}_{12}(q)&=-\frac{2\pi\kappa}{q}\frac{d(q\mathcal{B})}{dq}\\&=\frac{1}{q}\frac{dJ(q)}{dq}\ .\end{aligned}\tag{5.20}$$

with J in Eq. (2.21). The magnitude of the angular momentum decreases with the separation between vortices. The sign of the magnetic field is then opposite the sign of the angular momentum, which tells us the direction to which the vortex trajectories bend. When the vortex spin is positive, or, $\kappa<0$, vortices in a two-vortex system turn right. For the negative spin, vortices turn left. From this one can easily obtain a qualitative picture of the dynamics of two vortices which are either bounded closely or starting and ending at the spatial infinity. However, the detailed pictures seem to be complicated and will not be pursued here.

Somewhat similar behavior has been observed numerically in another kind of self-dual system with global charge and topology in three dimensions [15]. Our approach may shed some light on the physical understanding of the interaction between those solitons.

Finally, let us consider the meaning of the effective Lagrangian (5.11). We do not have any geometric derivation of the quadratic term, but we can take the quadratic term as a metric on the moduli space, the self-dual configurations of vortices. The linear term could be interpreted as a magnetic field in the moduli space. Vortices are then moving along geodesics determined by the metric and magnetic field. Vortices carry spin and may feel the spin connection of the metric on the moduli space. Since the spin connection could be also interpreted as a sort of gauge field, the linear term in our effective Lagrangian may be interpretable as the spin connection, making the effective action fully geometric. To see this, we need a better understanding of the quadratic part of the effective action.

VI. CONCLUSION

We understand now various aspects of vortex dynamics in Chern-Simons-Higgs systems. We have the dual formulation in the path-integral formalism, where the interaction between vortices manifests. The statistics of vortices comes from the Aharonov-Bohm phase of the dual gauge interaction, which combines the usual electromagnetic and Magnus forces. In the dual formulation, we included the external field and current, which could be dynamical. In self-dual models we studied the properties of static vortices and presented an effective action for slowly moving vortices.

There seems to be some interesting directions to take from here. One direction is to find further use of the dual formulation. We can ask whether the perturbative expansion is possible in the dual formulation. For vortices moving on a curved surface whose typical length scale is much larger than the size of vortices, there could be a force on vortices via spin connection because vortices carry spin. Maybe our approaches would shed some light on that. It would also be interesting to find whether there is a dual formulation of the nonrelativistic limit of the theory in the symmetric phase. Another is to understand better the effective action for slowly moving vortices and its dynamical consequences. In addition to the statistics, we have not studied the quantum aspects of vortex dynamics. Quantum aspects of vortices in the field theoretic and effective action levels need further investigation.

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APPENDIX A

Here we derive the Magnus force in the simplest example. Consider the theory of a complex scalar field in three dimensions with a global U(1) symmetry. The Lagrangian is given by

$$\mathcal{L}=|\partial_\mu\phi|^2-U(\phi)\ . \tag{A1}$$

The generating functional is

$$Z=\langle F|e^{-iTH}|I\rangle=\int[d\phi][d\phi^*]\exp\left[i\int d^2x\mathcal{L}\right]\ . \tag{A2}$$

With $\phi=fe^{i\theta}/\sqrt{2}$, the Lagrangian becomes

$$\mathcal{L}=\tfrac{1}{2}(\partial_\mu f)^2+\tfrac{1}{2}f^2(\partial_\mu\theta)^2-U(f)\ . \tag{A3}$$

The conserved current for the global Abelian symmetry is $j_\mu=f^2\partial_\mu\theta$. Suppose we are interested in the minimum energy density configuration for a given uniform charge density $j^0=\rho_B$. The phase becomes $\theta=wt$ with a constant w and the f field is fixed by the minimizing the energy density:

$$U_{\text{eff}}(f)=\frac{1}{2f^2}\rho_B^2+U(f)\ . \tag{A4}$$

There could be a global vortex with this background. The ansatz will be $f(r)$ and $\theta=wt+n\varphi$. These vortices carry logarithmically divergent energy and quadratically divergent angular momentum when the charge density is nonzero.

Let us consider the motion of vortices and antivortices with some background charge. For example, one is imaging some bosonic superfluid or Q matter. In the same way as in Sec. III, we introduce an auxiliary field C^μ to linearize the second term of the Lagrangian (A3). Separate the phase θ into a part for vortex configurations and a part for single-valued fluctuations as in Eq. (3.3). Integrate over the fluctuation to get a new gauge field H_μ for C^μ. After some further steps, similar to Sec. III, we arrive at the dual formulation of the generating functional:

$$Z=\int[f^{-2}df][dH_\mu][dq_a^\mu]\exp\left[i\int d^3x\mathcal{L}_D\right]\ , \tag{A5}$$

where

$$\mathcal{L}_D=\tfrac{1}{2}(\partial_\mu f)^2-\frac{1}{16\pi^2f^2}H_{\mu\nu}^2+H_\mu K^\mu \tag{A6}$$

with K^μ given in Eq. (3.5). There is no Jacobian factor in measure as in Sec. III. In the dual formulation the Goldstone boson is described by the massless vector field with the Maxwell kinetic term. Vortices become charged particles and the logarithmically divergent self-energy comes from the divergent Coulomb energy.

In the dual formalism, the conserved current for the global symmetry becomes

$$j^\mu = f^2 \partial^\mu \theta = \epsilon^{\mu\nu\rho}\partial_\nu H_\rho \ . \tag{A7}$$

The uniform charge density background becomes a uniform magnetic field background. Vortices moving on a uniform charge background would feel the Magnus force as a Lorentz force in the dual formulation.

Let us now do a little bit of the fluid dynamic approach to the Magnus force to figure out the direction. For the positive w and n, the momentum density flow

$$T^{0i} = -\int d^2r [\dot{f}\partial_i f + f^2 \dot{\theta}\partial_i \theta]$$

around the vortex is clockwise, resulting in the negative angular momentum density. Let us consider a vortex moving to the negative x axis. This is very similar to the case where the two-dimensional baseball moving in the same direction with the same rotation, feeling the net Magnus force in the positive y direction. This direction of force is exactly that of the Lorentz force one would get from the dual Lagrangian (A6).

APPENDIX B

Here we present the path integral derivation of the dual transformation for a Maxwell-Higgs theory in three dimensions. For some earlier related works see Ref. [16]. The Lagrangian for a complex scalar field $\phi = fe^{i\theta}/\sqrt{2}$ coupled to the gauge field A_μ is

$$\mathcal{L} = -\frac{1}{4e^2}F_{\mu\nu}^2 + \tfrac{1}{2}(\partial_\mu f)^2 + \tfrac{1}{2}f^2(\partial_\mu\theta + A_\mu + A_\mu^{\rm ext})^2 - U(f) + A_\mu J^\mu \ , \tag{B1}$$

where J^μ is the external current and $A_\mu^{\rm ext}$ is the external gauge field. As in Sec. III, we introduce the vortex current K^μ and integrate over the fluctuation part of the θ field, resulting in a dual gauge field H_μ. The effective Lagrangian becomes

$$\mathcal{L}' = \tfrac{1}{2}(\partial_\mu f)^2 - U(f) - \frac{1}{16\pi^2 f^2}H_{\mu\nu}^2 + H_\mu K^\mu - \frac{1}{4e^2}F_{\mu\nu}^2 + \frac{1}{4\pi}\epsilon^{\mu\nu\rho}H_\mu F_{\nu\rho} + \frac{1}{4\pi}\epsilon^{\mu\nu\rho}H_\mu F_{\mu\nu}^{\rm ext} + A_\mu J^\mu \ . \tag{B2}$$

In order to treat $F_{\mu\nu}$ and A_μ to be independent from each other, we introduce a vector field N_μ so that

$$\int [dF_{\mu\nu}][dA_\mu]\delta(F_{\mu\nu} - (\partial_\mu A_\nu - \partial_\nu A_\mu)) \cdots = \int [dF_{\mu\nu}][dA_\mu][dN_\mu] \exp\left[i\int d^3x \frac{1}{4\pi}\epsilon^{\mu\nu\rho}N_\mu[F_{\nu\rho} - (\partial_\nu A_\rho - \partial_\rho A_\nu)] \right] \cdots \ . \tag{B3}$$

The $F_{\mu\nu}$ integration is just a Gaussian integral and so trivial. The A_μ integration leads to a factor

$$\delta\left[\frac{1}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu N_\rho - J^\mu\right] , \tag{B4}$$

which is consistent only if the external current J^μ is conserved explicitly. Thus, unlike in Sec. III, the external current could have a dynamical origin only for the case when $A_\mu J^\mu$ is replaced by

$$\mathcal{L}_W = \epsilon^{\mu\nu\rho} A_\mu \partial_\nu W_\rho + \text{kinetic terms} \ . \tag{B5}$$

The external gauge field can be made dynamical by simply replacing $A_\mu^{\rm ext}$ by, for example, W_μ in Eq. (B5).

Let us consider a single-valued scalar field ζ such that

$$N_\rho = \bar{N}_\mu + \partial_\rho \zeta \ , \tag{B6}$$

where

$$\epsilon^{\mu\nu\rho}\partial_\nu \bar{N}_\rho = 2\pi J^\mu \tag{B7}$$

and $\partial_\rho \bar{N}_\rho = 0$. A uniform external electric charge density corresponds to a uniform magnetic field in the vector potential $\bar{N}_\rho$. If we put $\bar{N}_\rho = \partial_\rho \bar{\zeta}$ for a point current of unit charge, $\bar{\zeta}$ becomes multivalued with shift 2π, which can be absorbed into ζ. This allows an interpretation that point external charges of integer charge are vortices in the ζ variable.

Putting it together, the generating functional after the dual transformation becomes

$$Z = \int [f^{-2}df][dH_\mu][dq_a^\mu][d\zeta]\delta(\epsilon^{\mu\nu\rho}\partial_\nu \bar{N}_\rho - 2\pi J^\mu) \times \exp\left[\int d^3x \mathcal{L}_D\right] , \tag{B8}$$

where

$$\mathcal{L}_D = \tfrac{1}{2}(\partial_\mu f)^2 - U(f) - \frac{1}{16\pi^2 f^2}H_{\mu\nu}^2 + H_\mu K^\mu + \frac{e^2}{8\pi^2}(H_\mu + \bar{N}_\mu + \partial_\mu \zeta)^2 + \frac{1}{4\pi}\epsilon^{\mu\nu\rho}H_\mu F_{\nu\rho}^{\rm ext} \tag{B9}$$

with K^μ given in Eq. (3.5). There is an obvious Abelian gauge symmetry in the dual Lagrangian. The point external currents of integer charge could appear as vortices in the ζ variable. The massive vector bosons of spin ± 1 are described by the Maxwell-Higgs terms in both formalisms.

If there is a uniform electric charge density background, we know that there should be a uniform charge density background of the opposite charge carried by the Higgs field to have a finite Coulomb energy. In the dual formulation, there is a uniform external magnetic field carried by $\bar{N}_\rho$ which should be balanced by the uniform magnetic field of the opposite sign carried by H_μ for a finite energy density as one can see in the dual Lagrangian (B9). In the dual formulation vortices moving on the uniform charged background are equivalent to charged particles moving on a uniform external magnetic field and vortices feel the Magnus force as an effective Lorentz force.

APPENDIX C

Here we study the effective Lagrangian for slowly moving vortices in self-dual Maxwell-Higgs systems in the dual formulation of Appendix B. The self-dual model is fixed by choosing the potential

$$U(f)=\frac{e^2}{8}(f^2-v^2)^2 . \tag{C1}$$

The energy functional of the dual Lagrangian (B9) can be rewritten as

$$E=\int d^2r\left\{\frac{1}{2}\dot{f}^2+\frac{1}{8\pi^2f^2}H_{12}^2+\frac{e^2}{8\pi^2}(H_i+\bar{N}_i+\partial_i\zeta)^2+\frac{1}{2}\left[\partial_if\pm\frac{1}{2\pi f}H_{0i}\right]^2+\frac{e^2}{8\pi^2}[H_0+\bar{N}_0+\partial_0\zeta\mp\pi(f^2-v^2)]^2\right\}\pm\pi v^2n , \tag{C2}$$

where the vorticity $n=\int d^2rK^0$ appears because of Gauss's law:

$$\partial_i\left[\frac{1}{f^2}H_{0i}\right]+e^2(H_0+\partial_0\zeta)+4\pi^2K^0=0 . \tag{C3}$$

The energy is bounded, $E\geq\pi v^2|n|$. As there is no external charge and field, we choose the gauge where $\bar{N}_\mu=0$ and $\zeta=0$. The energy bound is saturated by the configurations satisfying $\dot{f}=0$, $H_i=0$:

$$H_0=\pm\pi(f^2-v^2) ,\qquad H_{0i}=\mp\pi\partial_if^2 , \tag{C4}$$

and Gauss's law (C3). Two equations in (C4) are consistent to each other. Equations (C3) and (C4) can be put together into an equation for f:

$$\partial_i^2\ln f^2-e^2(f^2-v^2)=4\pi\sum_a\delta(\mathbf{r}-\mathbf{q}_a) . \tag{C5}$$

Let us try to derive the low-energy effective Lagrangian in the dual formulation. We know how the fields transform under the nonrelativistic limit of the Lorentz transformation of all vortices. The scalar field will be invariant but there is a nontrivial correction ΔH_μ to the gauge field. The gauge field satisfies the Maxwell equation in the first order of vortex velocity $\dot{\mathbf{q}}_a$:

$$\partial_\nu\left[\frac{1}{f^2}H^{\nu\mu}\right]+e^2H^\mu+4\pi^2K^\mu=0 . \tag{C6}$$

For slow moving vortices with vortex positions $\mathbf{q}_a(t)$, we assume that the fields transform like a complicated version of the Lorentz transformation. The scalar field would be given simply as $f(\mathbf{r};\mathbf{q}_a(t))$. There would be a correction to the gauge field linear in the velocity. We require that the Maxwell equation is again satisfied to first order in velocity. Note that the velocity of vortices would appear explicitly in the Maxwell equation by the current $K^i=\sum_a\dot{q}_a^i\delta(\mathbf{r}-\mathbf{q}_a)$.

In zeroth order, only H_0 is nonzero as one can see from Eq. (C4). The first-order part of Eq. (C6) is

$$\partial_i\left[\frac{1}{f^2}\partial_i\Delta H_0\right]+e^2\Delta H_0=0 ,\qquad -\partial_0\left[\frac{1}{f^2}\partial_iH_0\right]+\epsilon_{ij}\partial_j\left[\frac{1}{f^2}\Delta H_{12}\right]+e^2\Delta H_i=4\pi^2K^i . \tag{C7}$$

The first part of Eq. (C7) implies that $\Delta H_0=0$. For the second part of Eq. (C7), we apply both ∂_i and $\epsilon_{li}\partial_l$, leading to

$$\partial_i\Delta H_i=\pi\partial_0f^2 ,\qquad \partial_i^2\left[\frac{1}{f^2}\Delta H_{12}\right]-e^2\Delta H_{12}=4\pi^2\sum_a\epsilon_{ij}\dot{q}_a^i\partial_j\delta(\mathbf{r}-\mathbf{q}_a) . \tag{C8}$$

As we know the divergence and curl of ΔH_i, in principle, we can find ΔH_i explicitly.

Before we consider the effective action, let us ask whether the f field satisfies its field equation to first order in the vortex velocity. One can be easily convinced that the first-order correction Δf can be put to be zero consistently.

We now discuss the field configuration of slowly moving vortices for a given trajectory. Let us calculate the field-theory action from Eq. (B9) for these configurations. The zeroth-order term is minus the rest mass. There is no first-order term. The second becomes the effective action for slowly moving vortices. The field equation (C8) is essential in this derivation. The effective Lagrangian is

$$L_{\text{eff}}(\mathbf{q}_a,\dot{\mathbf{q}}_a)=\int d^2r\left\{\frac{1}{2}\dot{f}^2+\frac{1}{8\pi^2f^2}(\Delta H_{12})^2+\frac{e^2}{8\pi^2}(\Delta H_i)^2\right\} . \tag{C9}$$

Let us see what happens in the original formulation. The field equations in the intermediate Lagrangians imply

$$\epsilon^{\mu\nu\rho}\partial_\nu H_\rho=2\pi f^2(\partial^\mu\theta+A^\mu) ,\qquad \epsilon_{\mu\nu\rho}H^\rho=2\pi e^2F_{\mu\nu} , \tag{C10}$$

which implies that

$$\frac{e^2}{2\pi}\Delta H_i=\epsilon_{ij}\dot{A}_j=\partial_i\partial_0\ln f-\epsilon_{ij}\partial_j\theta \tag{C11}$$

in the $A_0=0$ gauge. With this identification, our effective Lagrangian (C9) can be shown easily to be identical to that of Samols in Ref. [10].

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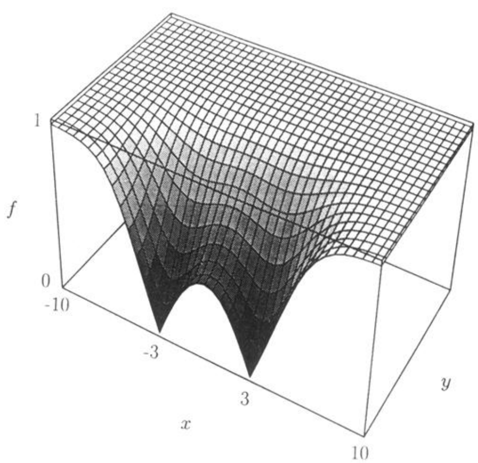

FIG. 1. Plot of the f field in units of v on the x-y plane for two vortices of mutual distance $d=6$ with spatial distance unit v^2/κ.

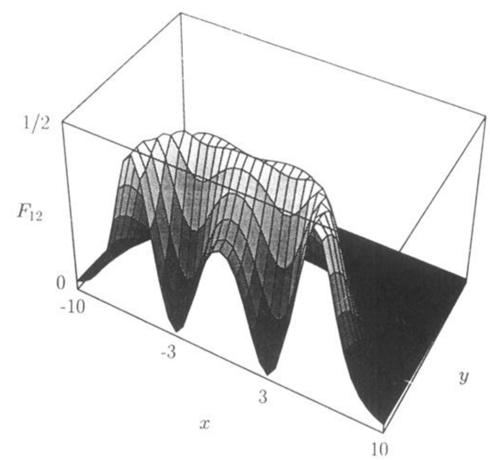

FIG. 2. Plot of the magnetic field F_{12} in units of $v^4/4\kappa^2$ on the x-y plane with $d=6$.