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# High-precision improved-analytic-exponentiation results for multiple-photon effects in low-angle Bhabha scattering at the SLAC Linear Collider and the CERN $e^+e^-$ collider LEP

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Starting from an earlier benchmark analytical calculation of the luminosity process $e^+e^- \to e^+e^- + (\gamma)$ at the SLAC Linear Collider (SLC) and the CERN $e^+e^-$ collider LEP, we use the methods of Yennie, Frautschi, and Suura to develop an analytical improved naive exponentiated formula for this process. The formula is compared to our multiple-photon Monte Carlo event generator BHLUMI (1.13) for the same process. We find agreement on the overall cross-section normalization between the exponentiated formula and BHLUMI below the 0.2% level. In this way, we obtain an important cross-check on the normalization of our higher-order results in BHLUMI and we arrive at formulas which represent the LEP/SLC luminosity process in the below 1% $Z^0$ physics tests of the $SU(2)_L \times U(1)$ theory in complete analogy with the famous high-precision $Z^0$ line-shape formulas for the $e^+e^- \to \mu^+\mu^-$ process discussed by Berends *et al.*, for example.

## I. INTRODUCTION

Currently, high-precision $Z^0$ physics connotes to most physicists the comparison of the cross sections and asymmetries measured at the SLAC Linear Collider (SLC) and CERN $e^+e^-$ collider LEP with standard-model [1] predictions to higher and higher accuracy. Indeed, numerous analysis of effects below the 1% level (and even at the 0.1% level) exist [2]—there is the clear expectation that if LEP (or SLC) collects enough $Z^0$'s, such analyses can be used to determine possible extensions of the standard model if it should fail to conform with expectations in $Z^0$ physics below the 1% regime for example.

Indeed, the 1% regime, from a statistics standpoint, has been surpassed at LEP, where at this time each experiment has $\sim 200\,000$ $Z^0$'s. Hence, current errors on LEP observables are limited by the hardware systematic errors, which we understand [3] will soon be $\sim 0.7\%$, and by the accuracy of the theoretical calculation of the luminosity process $e^+e^- \to e^+e^- + n(\gamma)$, low-angle Bhabha scattering with photon emission: $2m_e/\sqrt{s} \ll \theta_f < 250$ mrad, where $f = e^+, e^-$, where $\theta_f$ is the c.m.-system (c.m.s.) scattering angle of $f$, $\sqrt{s}$ is the c.m.s. total energy, and $m_e$ is the electron rest mass. Hence, the luminosity process is mainly a pure QED effect and it can be calculated in principle to any desired accuracy [4].

Indeed, in Ref. [5], we have used the Yennie-Frautschi-Suura (YFS) theory [6] to introduce the multiple-photon Monte Carlo event generator BHLUMI explicitly for this purpose and we have checked it against known exponential behavior and against [7] available $1\gamma$ Monte Carlo simulations for both its normalization and its $P_\perp$ spectrum, etc., so that, indeed, we have argued that its normalization is accurate to $\sim 0.7\%$. This has been already of some significance, since the difference between BHLUMI and a correct $1\gamma$ Monte Carlo simulation is an estimate of the size of the higher-order corrections to the luminosity process and corresponds to the theoretical uncertainty in any experimental analysis which uses a correct $1\gamma$ Monte Carlo to simulate its detector response to luminosity events and treats the higher-order effects as an attendant uncertainty in that simulation. If the simulation is itself based primarily on BHLUMI, the error on the result due to theory is just the error on BHLUMI's normalization. Hence, in this way Mark II, and later the ALEPH and remaining LEP Collaborations, have been able to quote 1–2 % systematic errors in their luminosities due to theory and, accordingly, upon using this result with an extensive study of the experimental systematic errors, have made the important discovery that the number of massless neutrino generations is 3. 1% luminosity simulations were adequate for this discovery.

LEP and SLC may now search for further restrictions or extensions of the $SU(2)_L \times XU(1)$ theory or for an impressive confirmation of its sufficiency at precision levels below 1%. As a first step in this direction, the theoretical errors on the luminosity simulations should be below or at the level of $\frac{1}{3}$ of the experimental systematic errors: since the latter are soon to be $\sim 0.7\%$, we need now to determine the absolute normalization of BHLUMI to $\sim 0.2\%$ and/or to compute the higher-order corrections to a correct $1\gamma$ calculation to 0.2%. In this paper, we will use the YFS based form of the method [8] of Tsai, Jackson, and Scharre, and of Kuraev and Fadin (the high-precision versions of this method are discussed by Berends *et al.* [8] for the $Z$ line shape) to achieve, using our benchmark analytic work in Ref. [9] on $e^+e^- \to e^+e^-(\gamma)$, an analytic representation of the cross section in BHLUMI for $e^+e^- \to e^+e^- + n(\gamma)$, and, in so doing, present strong evidence that the normalization of BHLUMI is indeed accurate to 0.2%, as required by the current LEP (SLC) data-experimental systematic error scenario. In this way, we apply and introduce high-precision improved naive exponentiation exponential formulas for the SLC/LEP luminosity process in analogy with the formulas discussed by Berends *et al.* in Ref. [8] for $e^+e^- \to \mu^+\mu^-$.

Our work is organized as follows. In Sec. II, we review the relevant aspects of the YFS theory and its realization in BHLUMI. In Sec. III, we review the relevant aspects of our benchmark calculation of $e^+e^- \to e^+e^- + (\gamma)$ in Ref. [9]. In Sec. IV, we combine the YFS theory with our result in Ref. [9] to achieve an analytic representation of the inclusive exponentiated cross section for $e^+e^- \to e^+e^- + n(\gamma)$. Numerical comparison data are illustrated in Sec. V. Section VI contains some summary remarks.

## II. YFS THEORY

In this section, we review the relevant aspects of the methods of Yennie, Frautschi, and Suura [6] as they relate to low-angle Bhabha scattering in the SLC/LEP luminosity regime. We begin with the basic YFS formula.

Specifically, for the process illustrated in Fig. 1, we have the YFS representation

$$d\sigma_{\rm YFS} = \exp[2\alpha({\rm Re}B + \tilde{B})] \int d^4y [1/(2\pi)^4] \exp[iy\cdot(p_1+q_1-q_2-p_2)+D] \left[\bar{\beta}_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \int \prod_{j=1}^{n} \frac{d^3k_j}{k_j} e^{-iy\cdot k_j} \bar{\beta}_n \right] dE_{X'} d^3P_{X'} \tag{1}$$

where $X' = \{\text{outgoing } e^-, \text{ outgoing } e^+\}$ so that $E_{X'} = p_2^0 + q_2^0$ and $\mathbf{P}_{X'} = \mathbf{p}_2 + \mathbf{q}_2$. (The kinematics is summarized in Fig. 1.) Here, we have defined the standard YFS functions

$$B = \frac{-i}{8\pi^3} \int \frac{d^4k}{k^2 - m_\gamma^2 + i\epsilon} \left[ - \left[ \frac{-2q_1+k}{k^2 - 2q_1\cdot k + i\epsilon} + \frac{-2p_1 - k}{k^2 + 2p_1\cdot k + i\epsilon} \right]^2 + \cdots \right], \tag{2}$$

$$D = \int \frac{d^3k}{k} [e^{-iyk} - \theta(K_{\max} - k)] \tilde{S}(k), \tag{3}$$

$$\tilde{S}(k) = -\frac{\alpha}{4\pi^2} \left[ \frac{q_1}{q_1\cdot k} - \frac{p_1}{p_1\cdot k} \right]^2 + \cdots, \tag{4}$$

with

$$2\alpha\tilde{B}(K_{\max}) = \int^{k \le K_{\max}} d^3k\, \tilde{S}(k)/(k^2 + m_\gamma^2)^{1/2}.$$

Thus, $m_\gamma$ is our standard photon mass infrared regulator. It cannot be emphasized too much that (1) is independent of $K_{\max}$. The $\bar{\beta}_n$ are then the usual YFS hard-photon residuals [6] for the SLC/LEP luminosity process in Fig. 1. Equation (1) will be the starting point of our analysis.

More precisely, in Ref. [5], we have realized Eq. (1) via Monte Carlo methods in the FORTRAN program BHLUMI so that we could achieve an event-by-event view of the multiple-photon effects in Fig. 1 in the SLC/LEP luminosity regime in which the physical multiple-photon four-vectors were among the final-particle four-vector list for the respective events for the first time in radiative

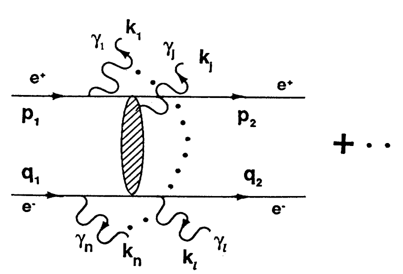


FIG. 1. Low-angle Bhabha scattering with $n$ photon emission: $e^+e^- \to e^+e^- + n(\gamma)$. Here, $(p_1, q_1)$ are the incoming $(e^+, e^-)$ four-momenta, respectively, and $(p_2, q_2)$ are the respective outgoing four-momenta. $k_i$ is the four-momentum of photon $i$, $i = 1, \ldots, n$.

corrections theory in event generators for Bhabha scattering. Indeed, BHLUMI FORTRAN is in use at SLC and LEP. What we wish to do in this paper is to develop methods which will eventually allow us to check just how accurate BHLUMI really is.

Indeed, we should first recall that, in BHLUMI, Eq. (1) is realized for $\bar{\beta}_0$ and $\bar{\beta}_1$: we stop the expansion in the number of residuals of hard photons at one such residual. Some readers [10] may think that we therefore do not generate events with more than one hard photon. This is not true. A quick look at Ref. [6] will show that the exact $O(\alpha^2)$ cross section differs from the cross section in BHLUMI for two photons by just $\bar{\beta}_2$, the two-hard-photon residual: ($dP$ is the differential phase space element)

$$d\sigma^{B2}=[\tilde{S}(k_1)\tilde{S}(k_2)\bar{\beta}_0+\tilde{S}(k_1)\bar{\beta}_1(k_2)+\tilde{S}(k_2)\bar{\beta}_1(k_1)+\bar{\beta}_2(k_1,k_2)]e^{2\alpha \mathrm{Re}B}dP\bigg|_{O(\alpha^2)} \quad (5)$$

is the exact $O(\alpha^2)$ two-photon emission cross section and all of it except $\bar{\beta}_2$ is in BHLUMI. A similar remark holds for $d\sigma^{Bn}$, the exact $[O(\alpha^n)]$ $n$ photon emission cross section is approximated in BHLUMI by

$$d\sigma^{Bn}_{\mathrm{approx}}=\left[\tilde{S}(k_1)\ldots\tilde{S}(k_n)\bar{\beta}_0+\sum_{j=1}^{n}\tilde{S}(k_1)\ldots\tilde{S}(k_{j-1})\tilde{S}(k_{j+1})\ldots\tilde{S}(k_n)\bar{\beta}_1(k_j)\right]dP\bigg|_{O(\alpha^n)} . \quad (6)$$

Thus, any question about the accuracy of BHLUMI can be rigorously addressed by looking into the question of to what part of the cross section the $\bar{\beta}_n$, $n\geq 2$, contribute and, just as importantly, what are the possible numerical procedural errors in the code itself, such as programming errors, etc.

We have developed recently a strategy for achieving such a test of BHLUMI (or any other low-angle Bhabha event generator). Specifically, our idea is to establish, for the luminosity regime, a base line analytic calculation at $O(\alpha)$, which is known to be accurate to some level $\eta \ll 1$. This base line has been established in Ref. [9] to the 0.02% technical precision level. [There is also an uncertainty of $\leq 0.02\%$ for the respective scattering angles below 0.25 rad due to suppressed interference effects. Conservatively, we will quote the combined errors on our base-line formula for the $O(\alpha)$ cross section as 0.03% for such angles]. By looking into the extension of our analytical $O(\alpha)$ result to higher orders by either leading-log methods [11] or by the YFS methods encoded in (1), we can then arrive at analytic expressions which are clearly accurate for the respective luminosity regime at the $\lesssim 0.1\%$ level. Such expressions may then be used to provide an independent check of BHLUMI. This we will now demonstrate in the next two sections. We turn first to a brief review of our base-line result in the next section.

## III. BASE-LINE $O(\alpha)$ LUMINOSITY CALCULATION

In this section we review our $O(\alpha)$ analytic base-line calculation of the SLC/LEP luminosity process $e^+e^-\to e^+e^-+(\gamma)$. We do this with an eye toward developing its improved naive exponentiated form in the next section.

Specifically, in order to realize a 0.03% accurate analytical expression for the process $e^+e^-\to e^+e^-+(\gamma)$ at c.m.s. scattering angles $\theta_e,\theta_{\bar{e}}$ which lie in the regime $2m_e/\sqrt{s} \ll \theta < 0.25$ rad, we may concentrate on the pure $t$-channel exchanges in Fig. 1 and we drop out up-down interferences (interference between radiative effects on the $e^+$(up) line and those on the $e^-$(down) line). We have shown in Ref. [9] that these two approximations do not spoil the 0.03% accuracy of our analytic $O(\alpha)$ result. We want to emphasize that, should we desire to probe the regime of accuracies below 0.03% for our analytic $O(\alpha)$ work, we can reinstate the terms which these two approximations have suppressed in a straightforward way.

With the framework called out by 0.03% accuracy approximations, we get the $O(\alpha)$ cross section for process in Fig. 1 as (here, $k_1\equiv k$)

$$\sigma=\frac{1}{2s}\left[(1+2\beta_t \ln\epsilon+\Delta_{\mathrm{soft}})\int X_0\Theta_{\mathrm{trig}}\,d\Phi_2(p_1+q_1;p_2,q_2)+\int(X_1+X_1')\Theta_{\mathrm{soft}}\Theta_{\mathrm{trig}}\,d\Phi_3(p_1+q_1;p_2,q_2,k)\right] , \quad (7)$$

where

$$
\begin{aligned}
&X_0=2e^4\frac{s^2+u^2}{t^2}, \quad s=2p_1\cdot q_1, \quad t=-2p_1\cdot p_2, \quad u=-2p_1\cdot q_2 ,\\
&\beta_t=\frac{2\alpha}{\pi}\left[\ln\frac{|t|}{m_e^2}-1\right],\\
&\Delta_{\rm soft}=\frac{3}{2}\beta_t+\frac{2\alpha}{\pi}\left[-\frac{1}{2}-\frac{1}{3}\pi^2+{\rm Li}_2(s/|t|)+\ln\frac{|t|}{s}\ln\frac{t}{u}-\frac{1}{2}\ln^2\frac{|t|}{s}\right],\\
&X_1=\frac{e^6}{(k\cdot p_1)(k\cdot p_2)}\left[\frac{s^2+u_1^2}{|t_1|}\left[1-\frac{2m_e^2}{|t_1|}\frac{k\cdot p_1}{k\cdot p_2}\right]+\frac{s_1^2+u^2}{|t_1|}\left[1-\frac{2m_e^2}{|t_1|}\frac{k\cdot p_2}{k\cdot p_1}\right]\right],\\
&X_1'=\frac{e^6}{(k\cdot q_1)(k\cdot q_2)}\left[\frac{s^2+u_1^2}{|t|}\left[1-\frac{2m_e^2}{|t|}\frac{k\cdot q_1}{k\cdot q_2}\right]+\frac{s_1^2+u^2}{|t|}\left[1-\frac{2m_e^2}{|t|}\frac{k\cdot q_2}{k\cdot q_1}\right]\right],\\
&s_1=2p_2\cdot q_2, \quad t_1=-2q_1\cdot q_2, \quad u_1=-2p_2\cdot q_1 ,
\end{aligned}
\tag{8}
$$

and $\Theta_{\rm soft}=\theta(k^0-\epsilon\sqrt{s}/2)$ defines our upper bound on infrared soft photons in the laboratory c.m.s. system $\mathbf{p}_1+\mathbf{q}_1=\mathbf{0}$ where $\epsilon\downarrow 0$. In (7), the idealized SLC/LEP trigger is

$$\Theta_{\rm trig}=\theta(\theta_1-\theta_{\rm min})\theta(\theta_2-\theta_{\rm min})\theta(\theta_{\rm max}-\theta_1)\theta(\theta_{\rm max}-\theta_2) , \tag{9}$$

where $\theta_1\equiv\theta_e$, $\theta_2\equiv\theta_{\bar e}$. Our definition of the invariant phase space in (7) is the conventional one:

$$d\Phi_n(P;p_1,\ldots,p_n)=(2\pi)^4\delta^4\left[P-\sum_{i=1}^{n}p_i\right]\prod_{i=1}^{n}dp_i^3/[(2\pi)^3 2p_i^0] . \tag{10}$$

This then completely defines our calculational framework.

In Ref. [9] we have derived the following explicit analytic representation of the photon emission part of (7): Defining $\xi_i=(1-\cos\theta_i)/2$,

$$\frac{d\sigma}{dk\,d\xi}=\rho_1(k,\xi)+\rho_2(k,\xi) , \tag{11}$$

where $\xi$ is either $\xi_1$ or $\xi_2$ and both are kept within the trigger region and where the $\rho_i$ are given by

$$\rho_1(k,\xi_1)=\frac{4\pi\alpha^2}{s}\left[\frac{\alpha}{\pi}\right]\frac{1}{\xi_1^2}\theta(\xi_1-\xi_{\rm min})\theta(\xi_{\rm max}-\xi_1)[G(k,\tilde a_{\rm up},\xi_1)-G(k,\tilde a_{\rm low},\xi_1)] \tag{12}$$

for

$$\tilde a_{\rm up}=\min\left[\frac{(1-k)\xi_{\rm max}}{\xi_1},1+k/\sqrt{\xi_1}\right], \quad \tilde a_{\rm low}=\max\left[\frac{(1-k)\xi_{\rm min}}{\xi_1},1-k/\sqrt{\xi_1}\right] \tag{13}$$

with

$$
\begin{aligned}
G(k,\tilde a,\xi_1)=\chi(\xi_1,\tilde a)\frac{1+(1-k)^2}{2k}\Bigg[&\theta(\tilde a-\tilde a_1)\left[\ln\frac{s\xi_1}{m_e^2}-\frac{2(1-k)}{1+(1-k)^2}\right]+\theta(\tilde a-\tilde a_1)\ln\frac{1}{(1-\xi_1)(1-k)^2\tilde a_1^2(\tilde a-1)^2}\\
&+\ln_+(\tilde a-\tilde a_1)+\ln_+(\tilde a_1-1)-\theta(1-\tilde a-k)\\
&\times\left[\ln\frac{s\xi_1}{m_e^2}-\frac{2(1-k)}{1+(1-k)^2}+\ln\frac{1}{(1-\xi_1)(\tilde a-1)^2}\right]+\ln_+(\tilde a-1+k)-\ln_+k\Bigg] ,
\end{aligned}
\tag{14}
$$

$\ln_+(x)={\rm sgn}(x)\ln|x|$,

$$\tilde a_1=\frac{1+\nu(\xi_1)}{1+\nu(\xi_1)-k}, \quad \nu(\xi_1)=\frac{(2-k)\xi_1}{1-k-(2-k)\xi_1} , \tag{15}$$

$$\chi(\xi,\tilde a)=\begin{cases}\frac{1}{2}[1+(1-\xi)^2], & \tilde a<1,\\ \frac{1}{2}(1+\{1-\xi/[1-k(1-\xi)]\}^2), & \text{otherwise} ,\end{cases} \tag{16}$$

$\xi_{\max(\min)}=\xi(\theta_{\max(\min)})$ and, similarly,

$$\rho_2(k,\xi_2)=\frac{4\pi\alpha^2}{s}\left[\frac{\alpha}{\pi}\right]\frac{1}{\xi_2^2}\frac{1}{1-k}\theta(\xi_2-\xi_{\min})\theta(\xi_{\max}-\xi_2)[H(k,\tilde{a}_{\rm up},\xi_2)-H(k,\tilde{a}_{\rm low},\xi_2)] \tag{17}$$

for

$$\tilde{a}_{\rm up}=\min\left[\frac{(1-k)\xi_2}{\xi_{\min}},1+k/\sqrt{\xi_2}\right],\quad \tilde{a}_{\rm low}=\max\left[\frac{(1-k)\xi_2}{\xi_{\max}},1-k/\sqrt{\xi_2}\right] \tag{18}$$

with

$$\begin{aligned}H(k,\tilde{a},\xi_2)=\chi^*(\xi_2,\tilde{a})\frac{1+(1-k)^2}{2k}&\left[\frac{\theta(\tilde{a}-\tilde{a}_1^*)}{1-k}\left[\ln\frac{s\xi_2}{m_e^2}-\frac{2(1-k)}{1+(1-k)^2}\right]+\frac{\theta(\tilde{a}-\tilde{a}_1^*)}{1-k}\ln\frac{1}{(1-\xi_2)\tilde{a}_1^{*2}(\tilde{a}_1^*-1)^2}\right.\\&+\theta(\tilde{a}-\tilde{a}_1^*)\ln\frac{(\tilde{a}_1^*-1)^2}{(\tilde{a}-1)^2}+\frac{1}{1-k}[\ln_+(\tilde{a}-\tilde{a}_1^*)+\ln_+(\tilde{a}_1^*-1)]\\&-(1-k)\theta(1-\tilde{a}-k)\left[\ln\frac{s\xi_2}{m_e^2}-\frac{2(1-k)}{1+(1-k)^2}\right]-(1-k)\theta(1-\tilde{a}-k)\ln\frac{1}{(1-\xi_2)k^2}\\&\left.-\theta(1-\tilde{a}-k)\ln\frac{k^2}{(\tilde{a}-1)^2}+(1-k)[\ln_+(\tilde{a}-1+k)-\ln_+k]\right],\end{aligned} \tag{19}$$

$$\tilde{a}_1^*=\frac{1-\xi_2k(2-k)}{1-k}, \tag{20}$$

$$\chi^*(\xi,\tilde{a})=\begin{cases}\frac{1}{2}[1+(1-\xi)^2], & \tilde{a}<1,\\ \frac{1}{2}\{1+[1-\xi(1-k)/(1-\xi k)]^2\}, & \text{otherwise}.\end{cases} \tag{21}$$

We have shown in Ref. [9] that (11)–(21) agree with an exact $O(\alpha)$ Monte Carlo program of the OLDBAB type [12] to 0.03% in our luminosity regime. This $O(\alpha)$ Monte Carlo program is obtained from OLDLAB by modifying it to disentangle up-down interference from the rest and to allow for weighted events and for negative weights (to remove dependence on $\epsilon$). This provides a check on our analytic methods at $O(\alpha)$ and a check on our OLDLAB-type Monte Carlo program.

Our objective now is to develop the improved naive exponentiation of (7) as its photon emission piece is represented by (11). To this we now turn in the next section.

## IV. YFS EXPONENTIATION OF $O(\alpha)$ LOW-ANGLE BHABHA SCATTERING

In this section we wish to develop the analogue for (7) and (11) of the methods of Tsai, Jackson, and Scharre, and of Kuraev and Fadin, which were applied to $e^+e^-\to f\bar{f}+(\gamma)$ and $e^+e^-\to f\bar{f}+(2\gamma)$ when $f\neq e$ by many authors [13]. The methods realized in Ref. [8] by Tsai and Jackson and Scharre are known as naive exponentiation in the literature. The improvements in the naive exponentiation procedure developed by Kuraev and Fadin, initially in response to further work by Tsai [14], and further developed and applied by Berends *et al.*, and others [13] is known in the literature as improved naive exponentiation. However, the sharp cutoffs in (11) have made a naive adaptation of the work in Ref. [13] to (7) and (11) invalid [15]. Here, we return to the basic rigorous YFS formula in (1) as it relates to (7) and (11).

Specifically, our starting point will be the rigorous $v\equiv 1-s'/s$ distribution which follows from (1) via the methods in Ref. [6]. We have from Ref. [6] that (1) implies

$$d\sigma/dv=\left[\bar{\beta}_0\frac{\gamma}{v}F(\gamma)+\frac{\gamma}{v}F(\gamma)\int_0^v dk_1\,k_1 d\Omega_{k_1}\bar{\beta}_1(\mathbf{k}_1)\left[\frac{v}{v-k_1}\right]^{1-\gamma}\right]\exp\{2\alpha[\mathrm{Re}B+\tilde{B}(v)]\} \tag{22}$$

if we truncate at the first hard photon residual $\bar{\beta}_1$. Here, $\gamma=2\beta_t$ and $F(\gamma)$ is the famous YFS function

$$F(\gamma)=\frac{e^{-C\gamma}}{\Gamma(1+\gamma)}, \tag{23}$$

where $C=0.577\,215\,6\ldots=$Euler's constant and $\Gamma(z)$ is the usual gamma function. The respective zero and one hard-photon residuals $\bar{\beta}_0$ and $\bar{\beta}_1$ are already implied by (7) and (11) following their rigorous definitions in Ref. [6]. Thus, the result (22) is indeed a proper starting point for the exponentiation of (7) and (11).

More precisely, on substituting the implied values of $\bar{\beta}_0$ and $\bar{\beta}_1$ from (7) and (11) into (22), we get, with a little rearrangement, the basic formula of this paper:

$$d\sigma/(dv\,d\xi)=\frac{\gamma F(\gamma)}{v}\int_0^v dk\left[\frac{d\sigma}{dk\,d\xi}-\frac{\gamma}{k}\frac{d\sigma_0}{d\xi}+\frac{\gamma}{v}\frac{d\sigma_0}{d\xi}(1+\Delta_{\rm YFS})\right](1-k/v)^{\gamma-1}\exp\{2\alpha[{\rm Re}B+\tilde{B}(v)]\}\ , \tag{24}$$

where $\Delta_{\rm YFS}=\gamma\ln\epsilon+\Delta_{\rm soft}-2\alpha[{\rm Re}B+\tilde{B}(\epsilon)]$, $\epsilon\downarrow 0$. Here, $d\sigma_0$ is the Born cross section in (7), and $d\sigma/dk\,d\xi$ is just the result (11). (The reader should note that due to the amplitude level realization of the YFS theory, the factors of $\gamma d\sigma_0$ have the implied $\xi$ and $k$ dependence as it would follow from $d\sigma/dk\,d\xi$.) In order to complete the analytic discussion we must now discuss the integral over $dk$ in (24).

On substituting $d\sigma_0/d\xi$ and $d\sigma/dk\,d\xi$ from (7) and (11) into (24), we find, to order $(\alpha/\pi)\gamma$, the result

$$\frac{d\sigma}{dv\,d\xi}=\frac{4\pi\alpha^2}{s}\left[\frac{\alpha}{\pi}\right]\frac{1}{\xi^2}\theta(\xi-\xi_{\min})\theta(\xi_{\max}-\xi)[\overline{\rho_1}^{\exp}(v,\xi)+\overline{\rho_2}^{\exp}(v,\xi)]\ , \tag{25}$$

where we define now the fundamental exponentiated distributions

$$\overline{\rho_1}^{\exp}(v,\xi)=G^{\exp}(v,\tilde{a}_{\rm up},\xi)-G^{\exp}(v,\tilde{a}_{\rm low},\xi) \tag{26}$$

with $\{\tilde{a}_{\rm up},\tilde{a}_{\rm low}\}$ defined in (13),

$$\begin{aligned}G^{\exp}(v,\tilde{a},\xi)=&\left[\theta(\tilde{a}-\tilde{a}_1)\left[\ln\frac{s\xi}{m_e^2}CG_1(\xi,v,v',\gamma)-CG_2(\xi,v,v',\gamma)\right]\Big/\chi(\xi,\tilde{a})\right.\\&+\theta(\tilde{a}-\tilde{a}_1)\frac{1+(1-v)^2}{2\gamma}\ln\frac{1}{(1-\xi)(1-v)^2\tilde{a}_1^2(\tilde{a}-1)^2}+\frac{1+(1-v)^2}{2\gamma}[\ln_+(\tilde{a}-\tilde{a}_1)+\ln_+(\tilde{a}_1-1)]\\&-\theta(1-\tilde{a}-v)\left[CG_1(\xi,v,0,\gamma)\ln\frac{s\xi}{m_e^2}-CG_2(\xi,v,0,\gamma)\right]\Big/\chi(\xi,\tilde{a})\\&\left.-\theta(1-\tilde{a}-v)\frac{1+(1-v)^2}{2\gamma}\ln\frac{1}{(1-\xi)(\tilde{a}-1)^2}+\frac{1+(1-v)^2}{2\gamma}[\ln_+(\tilde{a}-1+v)-\ln_+(v)]\right]\\&\times\chi(\xi,\tilde{a})\gamma v^{\gamma-1}F(\gamma)\exp\left[\frac{1}{4}\gamma+\frac{2\alpha}{\pi}(-\pi^2/6-\tfrac{1}{2})\right]\cdot\end{aligned} \tag{27}$$

and the functions $CG_i$ given by

$$\begin{aligned}&CG_1(\xi,v,v',\gamma)=f_1(\xi,v,v',\gamma)-f_2(\xi,v,v',\gamma)+\tfrac{1}{2}f_3(\xi,v,v',\gamma)\\&CG_2(\xi,v,v',\gamma)=f_1(\xi,v,v',\gamma)-f_2(\xi,v,v',\gamma)\end{aligned} \tag{28}$$

for $v'=v(1-\xi)$, where the $f_i$ are recorded in the Appendix, and

$$\overline{\rho_2}^{\exp}(v,\xi)=\frac{1}{1-v}[H^{\exp}(v,\tilde{a}_{\rm up},\xi)-H^{\exp}(v,\tilde{a}_{\rm low},\xi)] \tag{29}$$

with $\{\tilde{a}_{\rm up},\tilde{a}_{\rm low}\}$ defined in (18),

$$\begin{aligned}H^{\exp}(v,\tilde{a},\xi)=&\left[\theta(\tilde{a}-\tilde{a}_1^*)(1-v)\left[\ln\frac{s\xi}{m_e^2}CH_1(\xi,v,v',\gamma)-CH_2(\xi,v,v',\gamma)\right]\Big/\chi^*(\xi,\tilde{a})\right.\\&+\theta(\tilde{a}-\tilde{a}_1^*)\left[\frac{1}{1-v}\ln\frac{1}{(1-\xi)\tilde{a}_1^{*2}(\tilde{a}_1^*-1)^2}+\ln\frac{(\tilde{a}_1^*-1)^2}{(\tilde{a}-1)^2}\right]\frac{1+(1-v)^2}{2\gamma}\\&+\frac{1}{1-v}[\ln_+(\tilde{a}-\tilde{a}_1^*)+\ln_+(\tilde{a}_1^*-1)]\frac{1+(1-v)^2}{2\gamma}\\&-\theta(1-\tilde{a}-v)(1-v)[\ln\frac{s\xi}{m_e^2}CG_1(\xi,v,0,\gamma)-CG_2(\xi,v,0,\gamma)]/\chi^*(\xi,\tilde{a})\\&-\theta(1-\tilde{a}-v)\left[(1-v)\ln\frac{1}{(1-\xi)v^2}+\ln\frac{v^2}{(\tilde{a}-1)^2}\right]\frac{1+(1-v)^2}{2\gamma}\\&\left.+(1-v)(\ln_+(\tilde{a}-1+v)-\ln_+v)\frac{1+(1-v)^2}{2\gamma}\right]\gamma v^{\gamma-1}F(\gamma)\chi^*(\xi,\tilde{a})\exp\left[\frac{\gamma}{4}+\frac{2\alpha}{\pi}(-\pi^2/6-\tfrac{1}{2})\right]\end{aligned} \tag{30}$$

and the functions $CH_i$ defined by

$$CH_1(\xi,v,v',\gamma)=f'_1(\xi,v,v',\gamma)+f'_2(\xi,v,v',\gamma)+f'_3(\xi,v,v',\gamma)$$
$$CH_2(\xi,v,v',\gamma)=2f'_2(\xi,v,v',\gamma)+f'_1(\xi,v,v',\gamma) \tag{31}$$

for $v'=v\xi$, where $f'_i$ are recorded in the Appendix. This completely specifies our YFS-based improved naive exponentiation of the base-line analytic representation of the SLC/LEP luminosity cross section in (7) and (11). Let us now discuss some of its properties relative to the improved naive methods in Ref. [13].

First, our formulas (25) allows us to make an angular cut in $\xi$, whereas the results in Ref. [13] for $e^+e^-\to f\bar{f}$ are already integrated over angles. Second, the result has a sharp structure in $\xi$, $v$ space, just like the base-line $O(\alpha)$ result as we have illustrated in Ref. [9]. Finally, we note that the YFS function $F(\gamma)$ multiplies the entire result. This is very important and has been a point of constant debate. The reason is that the typical value of $\gamma$ for the SLC/LEP luminosity regime is $\sim 0.0965$ so that $F(\gamma)$ is $1-\pi^2\gamma^2/12+\cdots\simeq 1-0.73\times10^{-2}$ and represents a *reduction* in the cross section by $\sim 0.73\%$ due to the competition of multiple soft photons for their respective phase space. $F(\gamma)$ does not exist at $O(\alpha)$. We already have observed this effect in BHLUMI and in YSF2 [16] and we wish to emphasize that no high-precision calculation can leave it out without creating an uncertainty $\sim 1\%$ in the respective result.

The result (25) may now be compared with our YFS Monte Carlo event generator BHLUMI (1.12) to check its normalization and its various distributions. The complete set of comparisons will be taken up elsewhere [17]. Here, we focus on the overall normalization of BHLUMI; this we do in the next section.

## V. COMPARISON WITH BHLUMI

In this section, we discuss the comparison of (25) and BHLUMI. Specifically, we focus on two views of this overall normalization: a wide view and a narrow view. In the wide view, our first check, we consider the symmetric cut scenario of Ref. [18], with $\sqrt{s}=92$ GeV (Here, $x_{\rm MIVIS}$ is the usual minimum energy fraction.):

cut A

$$\min[p_2^0/(\sqrt{s}/2),q_2^0/(\sqrt{s}/2)]\geq x_{\rm MIVIS}=0.5\ ,$$
$$v=1-(p_2+q_2)^2/s\leq v_{\rm max}=0.5\ , \tag{32}$$
$$10\ \text{mrad}\leq\theta\leq 100\ \text{mrad},\quad \theta=\theta_e,\theta_{\bar{e}}\ ,$$

at $\sqrt{s}=92$ GeV whereas in the second check, we look into the relatively small angular interval

cut B

$$x_{\rm MIVIS}=0.5\ ,$$
$$v_{\rm max}=0.5\ , \tag{33}$$
$$1.745\ \text{mrad}\leq\theta\leq 8.727\ \text{mrad},\quad \theta=\theta_e,\theta_{\bar{e}}\ .$$

In this way, we get a narrow and a wide view of the overall normalization of BHLUMI (1.12).

Our results are as follows:

| | cut A | cut B |
|---|---|---|
| $\sigma$, exp. analytic (nb) | 1182.849±0.172 | 37636.10±5.081 |
| $\sigma$, BHLUMI (1.13) (nb) | 1182.368±0.559 | 37652.49±21.06 |

Here, exp. analytic refers to the result (25). We see that our YFS-based exponentiated result agrees with BHLUMI at the $-0.041\%\pm0.050\%$ level in the wide-type scenario of Bardin *et al.* and at the $0.044\%\pm0.058\%$ level in the view provided by cut B. We thus conclude that the overall normalization of BHLUMI is accurate at the $\lesssim 0.2\%$ level when we allow for possible 0.03% uncertainty in the base-line result in (7) and (11).

In order to complete our assessment of the accuracy of BHLUMI, we should estimate the size of the contribution of $\bar{\beta}_2$ to the respective cross sections for cut A and cut B. This we do using crossing from the leading-log formula [16] for $D_2^{(2)}$, the completely differential 2-photon emission formula for initial-state radiation in $e^+e^-\to f\bar{f}+(2\gamma), f\neq e$. Specifically, we may cross to either emission from the electron line or emission from the positron line and we may note that, due our use of crossing, we get from $D_2^{(2)}$ an estimate of the final-state effects as well. This crossed form of $D_2^{(2)}$ may now be used to determine $\bar{\beta}_2$; the latter we introduce into our YFS formula (1) and arrive at the respective contribution to our YFS-based improved naive exponentiation. We get the estimate, for the $e^+$ line,

$$d\sigma/dkd\xi|_{\bar{\beta}_2\ \text{term}}=[\gamma F(\gamma)/v]\exp\{2\alpha[\text{Re}B+\tilde{B}(v)]\}\frac{v^2\gamma^2}{8}\left[3-2v+\left[\frac{7}{2}-\frac{3}{v}-v\right]\ln\frac{1}{1-v}\right]\frac{d\sigma_0}{d\xi}\ . \tag{34}$$

We emphasize that, for consistency, $\bar{\beta}_0$ and $\bar{\beta}_1$ must be computed to $O(\alpha^2)$ also at the leading-log level; the latter results can also be obtained from Ref. [16] via crossing in our approximation of no up-down interference. On computing the respective contributions to the cut A and cut B cross sections, we find

| | cut A | cut B |
|---|---|---|
| $\sigma$, exp. analytic, second order, (nb) | 1181.780±0.175 | 37612.33±5.303 |

Thus, we see that, to second order in $\bar{\beta}_n$, there is a $\sim 0.1\%$ effect that enters the cross section compared with the results to first order in $\bar{\beta}_n$. This still allows us to quote that the normalization of BHLUMI for low-angle $e^+e^- \to e^+e^- + n(\gamma)$ is indeed accurate to $\sim 0.2\%$, conservatively. As a consequence, the limiting factor in the error in the luminosity at LEP is now the experimental systematic error $\sim 0.7-1\ \%$, independent of whether one uses BHLUMI or our recent YFS exponentiated leading-log approach [11] to the SLC/LEP luminosity problem.

## VI. CONCLUSIONS

We have developed in this paper the first, rigorous improved naive exponentiation formula for the luminosity process $e^+e^- \to e^+e^- + n(\gamma)$ at SLC and LEP in the $Z^0$ region. We have used the formula to check the normalization of our YFS Monte Carlo event generator BHLUMI for this process to the $\sim 0.2\%$ level of precision. (Here, the word precision refers to the technical precision relative to such issues as bugs in the code, pseudorandom number effects, rounding errors on the computer, etc., and to the physical precision of the specific process which BHLUMI calculates for the type cuts of Bardin *et al.* discussed in the text. The complete luminosity calculation which deals with what is left out here, i.e., pairs, calorimetric cuts, etc., will appear elsewhere [17].)

The discussion here, in addition to testing BHLUMI, has also provided a new semianalytical formula for the luminosity cross section which is based on the rigorous YFS theory so that it can be systematically improved to arbitrary precision. It is not based on nonrigorous recipes such as those given in Refs. [2] and [10] for Bhabha scattering. This new formula was essential to achieving the type of precise test that we have realized for BHLUMI in the text. We look forward with excitement to the further unravelings that $Z^0$ physics in the attendant below 1% regime may produce.

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## APPENDIX

In this appendix we record the YFS-based improved naive exponentiation functions $f_i$ and $f_i'$ defined in the text.

Specifically, from the results (11) and (24) we get, to order $\gamma^2$ relative $O(\alpha)$,

$$f_1(\xi,v,v',\gamma)=\frac{1}{2\gamma}[1+(1-\xi)^2](1+\Delta_{\rm YFS})+\frac{v'}{\gamma(1-v')}\left\{\left[-\xi+\tfrac{1}{2}\xi^2\left[1+\frac{1}{1-v'}\right]\right]\{1+\gamma\ln(1-v')-\gamma^2{\rm Li}_2[-v'/(1-v')]\}+(\xi^2/2)\left[\frac{-v'\gamma}{1-v'}-\frac{\gamma^2\ln(1-v')}{1-v'}\right]\right\}, \tag{A1}$$

$$f_2(\xi,v,v',\gamma)=\frac{v}{2\gamma}[1+(1-\xi)^2]+\frac{v}{\gamma(1-v')}\{-\xi+\xi^2/[2(1-v')]\}\{v'+\gamma\ln(1-v')-\gamma^2{\rm Li}_2[-v'/(1-v')]\}+\frac{vv'\xi^2}{2\gamma(1-v')}\left[1-\frac{\gamma}{1-v'}-\frac{\gamma^2\ln(1-v')}{v'(1-v')}\right], \tag{A2}$$

$$f_3(\xi,v,v',\gamma)=\frac{v^2}{2\gamma}\left\{\left[1+\left[1-\frac{\xi}{1-v'}\right]^2\right]-2\gamma+2\gamma^2-\frac{2\xi}{(1-v')}\left[\frac{\gamma\ln(1-v')}{v'}-\frac{\gamma^2}{v'}{\rm Li}_2[-v'/(1-v')]\right]+\frac{\xi^2}{(1-v')^2}\left[-\gamma+\gamma\ln(1-v')-\frac{\gamma^2}{v'}\ln(1-v')-\gamma^2{\rm Li}_2[-v'/(1-v')]\right]\right\},$$

$$f_1'(\xi,v,v',\gamma)=\frac{1}{2\gamma}[1+(1-\xi)^2][(1+\Delta_{\rm YFS})/(1-v)^2-v\mathcal{J}_{-1}(v,\gamma)-v\mathcal{J}_{-2}(v,\gamma)]-\frac{v[\xi(\xi-1)]}{\gamma}\mathcal{J}_{-1}(v',\gamma) \tag{A3}$$

$$+\frac{v\xi^2}{2\gamma}[(\xi-1/\xi)\mathcal{J}_{-1}(v',\gamma)+(1-1/\xi)(\xi-1)\mathcal{J}_{-2}(v',\gamma)], \tag{A4}$$

$$f_2'(\xi,v,v',\gamma)=\frac{v}{2\gamma}\mathcal{J}_{-1}(v,\gamma)-\frac{v\xi}{4\gamma}\mathcal{J}_{-1}(v',\gamma)+\frac{v\xi(\xi-1)}{4\gamma}\mathcal{J}_{-2}(v',\gamma), \tag{A5}$$

and

$$f_3'(\xi,v,v',\gamma)=\frac{v}{2\gamma}\mathcal{J}_{-2}(v,\gamma)-\frac{\xi v}{2\gamma(1-\xi)}[\mathcal{J}_{-1}(v,\gamma)-\mathcal{J}_{-1}(v',\gamma)]+\frac{v\xi^2}{4\gamma}\mathcal{J}_{-2}(v',\gamma), \tag{A6}$$

where we have defined the dilogarithm function as usual

$$\mathrm{Li}_2(x)=-\int_0^x \frac{dy}{y}\ln(1-y) \tag{A7}$$

and where, in addition, we have introduced

$$\mathcal{J}_{-1}(v,\gamma)=\frac{1}{1-v}\{1+\gamma\ln(1-v)-\gamma^2\mathrm{Li}_2[-v/(1-v)]\} , \tag{A8}$$

$$\mathcal{J}_{-2}(v,\gamma)=\frac{1}{(1-v)^2}\{1-v\gamma+\gamma\ln(1-v)-\gamma^2\ln(1-v)-\gamma^2\mathrm{Li}_2[-v/(1-v)]\} . \tag{A9}$$

The equations (A1)–(A9) then completely specify $f_i$ and $f_i'$. This completes our Appendix.

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[1] See, for example, F. Dydak, in *Proceedings of the XXVth International Conference on High Energy Physics,* Singapore, 1990, edited by K. K. Phua and Y. Yamaguchi (World Scientific, Singapore, 1991).
[2] See, for example, *Z Physics at LEP 1,* Proceedings of the Workshop, Geneva, Switzerland, 1989, edited by G. Altarelli, R. Kleiss, and C. Verzegnassi (CERN Yellow Report No. 89-08, Geneva, 1989), Vol. 1; D. Kennedy *et al.*, Nucl. Phys. **B231**, 83 (1989), and references therein.
[3] A. Blondel and E. Locci (private communication).
[4] F. J. Dyson, Phys. Rev. **85**, 631 (1952).
[5] S Jadach and B. F. L. Ward, Phys. Rev. D **40**, 3582 (1989).
[6] D. R. Yennie, S. Frautschi, and H. Suura, Ann. Phys. (N.Y.) **13**, 379 (1961); see also K. T. Mahanthappa, Phys. Rev. **126**, 329 (1962).
[7] E. Locci (private communication); G. Bower (private communication); J. Hylen (private communication); B. Harral (private communication); D. Bardin *et al.*, in *Z Physics at LEP 1* [2], Vol. 3. In Ref. [5] we quoted ~1% for the uncertainty in the normalization of BHLUMI; the more recent work noted here has led to the estimate ~0.7% for the uncertainty of the normalization of version 1.12 of BHLUMI.
[8] Y. S. Tsai, SLAC Report No. SLAC-PUB-1515, 1974 (unpubished); J. D. Jackson and D.L. Scharre, Nucl. Instrum. Methods **128**, 13 (1975); E. A. Kuraev and V. S. Fadin, Yad. Fiz. **41**, 733 (1985) [Sov. J. Nucl. Phys. **41**, 466 (1985)]. See also F. Berends *et al.*, in *Z. Physics at LEP 1* [2].
[9] S. Jadach *et al.*, Phys. Lett. B **253**, 469 (1991).
[10] W. Beenakker *et al.*, Nucl. Phys. **B355**, 281 (1991).
[11] S. Jadach *et al.*, Phys. Lett. B **260**, 438 (1991).
[12] F. Berends and R. Kleiss, Nucl. Phys. **B228**, 537 (1983).
[13] See, for example, Berends *et al.*, in Ref. [8], and references therein.
[14] Y. S. Tsai, in *Proceedings of the First Asia-Pacific Conference,* Singapore, 1983, edited by A. Arima *et al.* (World Scientific, Singapore, 1984), Vol. 2.
[15] W. Hollik (private communication).
[16] See, for example, S. Jadach and B. F. L. Ward, Comput. Phys. Commun. **56**, 351 (1990).
[17] S. Jadach *et al.* (unpublished).
[18] See Bardin *et al.*, in *Z Physics at LEP 1* [7].