

Comments on rigid and local supercurrents in $\mathcal{N} = 1$ minimal Supergravity

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Abstract

We discuss local supercurrents as sources of the super-Einstein equations in the superconformal approach in the old and new minimal (auxiliary fields) formulation. Modifications of the Ward identity giving the covariant divergence of the Einstein multiplet are considered in presence of a Fayet–Iliopoulos term. Curvature multiplets can be used as alternative to the gravitino variation in the search for rigid supersymmetric curved backgrounds.

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1 Introduction

In this paper we consider the supercurrent multiplet when coupled to supergravity, using superconformal and superspace techniques. The results presented here are an extension of a previous investigation of the same authors [1]. In rigid supersymmetry the supercurrent multiplet has a certain amount of ambiguity due to the possibility of improvement terms. These redefinitions modify the conservation law of a supercurrent $J_{\alpha\dot{\alpha}}$ expressed through its supercurrent divergence [2–7]

$$\bar{D}^{\dot{\alpha}} J_{\alpha\dot{\alpha}} \approx D_{\alpha} Y + \omega_{\alpha}, \quad (1.1)$$

where Y , ω_{α} are chiral superfields and moreover ω_{α} satisfies a reality condition $D_{\alpha}\omega^{\alpha} = \bar{D}_{\dot{\alpha}}\omega^{\dot{\alpha}}$. In linearized supergravity [3,6] the current couples to the Einstein multiplet $E_{\alpha\dot{\alpha}}$:¹

$$E_{\alpha\dot{\alpha}} \approx -\kappa^2 J_{\alpha\dot{\alpha}}, \quad (1.2)$$

together with a Ward identity, which (without a gauge multiplet) is

$$\begin{aligned} \text{old minimal supergravity:} & \quad \bar{D}^{\dot{\alpha}} E_{\alpha\dot{\alpha}} = D_{\alpha} \mathcal{R}, \\ \text{new minimal supergravity:} & \quad \bar{D}^{\dot{\alpha}} E_{\alpha\dot{\alpha}} = W_{\alpha}^L, \end{aligned} \quad (1.3)$$

where \mathcal{R} is a scalar chiral multiplet and W_{α}^L is a spinor chiral multiplet. The trace equation becomes in the two formulations

$$\mathcal{R} \approx -\kappa^2 Y, \quad W_{\alpha}^L \approx -\kappa^2 \omega_{\alpha}. \quad (1.4)$$

The equations (1.2) and (1.4) are the supergravity extensions of the Einstein equation and its trace

$$\begin{aligned} G_{\mu\nu} & \approx \kappa^2 T_{\mu\nu}, \\ R & \approx -\kappa^2 T_{\mu}^{\mu}, \end{aligned} \quad (1.5)$$

Equations (1.3) also contain the Einstein tensor identity (strong equation =):

$$\nabla^{\mu} G_{\mu\nu} = 0, \quad (1.6)$$

which then implies the ‘weak’ equation (\approx)

$$\nabla^{\mu} T_{\mu\nu} \approx 0, \quad (1.7)$$

The latter is the covariant conservation law of the stress tensor, which holds when the matter field equations are imposed. Note that in the superconformal supergravity context the Planck constant is hidden in the value of the (first component of the) compensating multiplet, which is a chiral superfield in old minimal supergravity and a linear multiplet in new minimal supergravity

$$\text{old minimal supergravity:} \quad \bar{D}_{\dot{\alpha}} X^0 = 0, \quad X^0 \stackrel{\square}{=} \kappa^{-1}$$

¹We will indicate equations valid modulo equations of motion by \approx .

$$\text{new minimal supergravity:} \quad T(L) = \bar{T}(L) = 0, \quad L \stackrel{\square}{=} \kappa^{-2}, \quad (1.8)$$

where we denote by $\stackrel{\square}{=}$ equations due to the (super)conformal gauge fixing. T is the superconformal version of the superspace operator \bar{D}^2 [8, 9], which defines a chiral multiplet from another multiplet, and \bar{T} defines an antichiral multiplet.

1.1 Old minimal Superconformal formulation: results

It has been speculated in the literature that the gravity analogue of the conservation law (1.1) where both Y and ω_α are present corresponds to a new type of supercurrent with $(16+16)$ bosonic + fermionic degrees of freedom, while the two minimal supergravities (1.3) have $(12+12)$ degrees of freedom and the maximal non-minimal supergravity has $(20+20)$ degrees of freedom [4, 6, 7]. In the present paper we will argue that the conservation law when both Y and ω_α are not vanishing is perfectly possible whenever an additional gauge symmetry acts on the chiral compensating multiplet X^0 . This just happens in old minimal supergravity when a Fayet–Iliopoulos (FI) term and a Kähler potential appear.

The basic Lagrangian we consider is

$$\mathcal{L} = [N(X^I, \bar{X}^I)e^{\xi V}]_D + [W^\alpha W_\alpha]_F, \quad \{X^I\} = \{X^0, X^i\}, \quad (1.9)$$

where ξ is the (dimensionless) FI constant. We will assume that the fields X^i appear only in the form

$$N(X^I, \bar{X}^I) = X^0 \bar{X}^0 \Phi(S^i, \bar{S}^i), \quad S^i \equiv X^i/X^0, \quad (1.10)$$

Note that under FI U(1) gauge transformations

$$X^0 \rightarrow X^0 e^{-\xi \Lambda}, \quad V \rightarrow V + \Lambda + \bar{\Lambda}, \quad S^i \text{ inert}, \quad (1.11)$$

while under Kähler transformations

$$X^0 \rightarrow X^0 e^\Xi, \quad \Phi \rightarrow \Phi e^{-\Xi - \bar{\Xi}}, \quad W(S) \rightarrow W(S) e^{-3\Xi}, \quad (1.12)$$

where we mention the transformation of the superpotential for completeness. We do not further consider a superpotential in this paper. However, since in this paper we will confine to an additive Φ such that

$$\Phi = -3 + \Phi^M, \quad (1.13)$$

where the first term corresponds to pure supergravity, the Kähler invariance will be lost. Below, we will see that in the new minimal formulation, unlike the old minimal one, the gravity-matter splitting preserves the Kähler invariance. See below (1.23), where the Kähler invariance is evident.

The super-Einstein equations are obtained from the Lagrangian (1.9) by variation of the auxiliary field A_μ (the gauge field of the U(1) R-symmetry of the superconformal algebra). So the super-Einstein equation becomes²

$$\frac{1}{4} i \gamma_{\alpha\dot{\alpha}}^\mu e^{-1} \frac{\delta \mathcal{L}}{\delta A^\mu} = \mathcal{E}_{\alpha\dot{\alpha}}^{\text{om}, V}(X^0, V) + J_{\alpha\dot{\alpha}}(X^0, V, S^i) + \mathcal{E}_{\alpha\dot{\alpha}}^W \approx 0, \quad (1.14)$$

²We follow the conventions of [10], where the actions, transformations and field equations are given explicitly. Some additional notation is introduced in [9].

where we split the contributions according to (1.9) and further to (1.13), and we have

$$\mathcal{E}_{\alpha\dot{\alpha}}^W = 4W_{\dot{\alpha}}W_{\alpha}, \quad W_{\alpha} \equiv T\mathcal{D}_{\alpha}V = -\frac{1}{2}i\lambda_{\alpha}, \quad (1.15)$$

λ_{α} being the left-projected gaugino in the gauge multiplet.

We will prove a ‘strong identity’ (V, X^0 arbitrary)

$$\bar{\mathcal{D}}^{\dot{\alpha}}\mathcal{E}_{\alpha\dot{\alpha}}^{\text{om},V} = (e^{\xi V}X^0)^3\mathcal{D}_{\alpha}\left[\frac{e^{-3\xi V}T(e^{\xi V}\bar{X}^0)}{(X^0)^2}\right] + 3\xi X^0 e^{\xi V}\bar{X}^0 W_{\alpha}. \quad (1.16)$$

The superspace geometry that encodes the Ward identity in (1.16) was denominated ‘chirally extended supergravity’ in [11]. Note that for $\xi = 0$ this becomes the Ward identity of [1]. Equation (1.16) has both terms as in (1.1) even if we are in old minimal supergravity. The field equations coming from (1.14) have a contribution coming from the V and X^0 field equations (field equations for the auxiliary fields, respectively D and F^0)

$$\begin{aligned} \frac{1}{2}D &= \mathcal{D}_{\alpha}W^{\alpha} = \bar{\mathcal{D}}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}} \approx -\frac{1}{4}\xi N e^{\xi V} = -\frac{1}{4}\xi(N^G + N^M)e^{\xi V}, \\ T(N_0 e^{\xi V}) &\approx 0, \end{aligned} \quad (1.17)$$

where we have introduced

$$N = N^G + N^M, \quad N^G = -3X^0\bar{X}^0, \quad N^M = X^0\bar{X}^0\Phi^M(S^i, \bar{S}^i), \quad (1.18)$$

while

$$N_0 \equiv \frac{\partial N(X, \bar{X})}{\partial X^0} = \bar{X}^0 \left(\Phi - S^i \frac{\partial \Phi}{\partial S^i} \right) = -3\bar{X}^0 + N_0^M, \quad N_0^M = \bar{X}^0 \left(\Phi^M - S^i \frac{\partial \Phi^M}{\partial S^i} \right). \quad (1.19)$$

The second line in (1.17) gives the trace of the Einstein equations in de Sitter space as described in section 4.2. Acting with $\bar{\mathcal{D}}^{\dot{\alpha}}$ on (1.15) using (1.17) we find

$$\bar{\mathcal{D}}^{\dot{\alpha}}\mathcal{E}_{\alpha\dot{\alpha}}^W = -2W_{\alpha}D \approx \xi W_{\alpha}(N^G + N^M)e^{\xi V}. \quad (1.20)$$

From the field equation (1.14) we have

$$\bar{\mathcal{D}}^{\dot{\alpha}}\mathcal{E}_{\alpha\dot{\alpha}}^{\text{om},V} + \bar{\mathcal{D}}^{\dot{\alpha}}J_{\alpha\dot{\alpha}} + \bar{\mathcal{D}}^{\dot{\alpha}}\mathcal{E}_{\alpha\dot{\alpha}}^W \approx 0. \quad (1.21)$$

In this sum, the first term of (1.20) (with N^G) cancels the last term in (1.16) and with the second part of (1.17) we obtain

$$\bar{\mathcal{D}}^{\dot{\alpha}}J_{\alpha\dot{\alpha}} \approx \frac{1}{3}(e^{\xi V}X^0)^3\mathcal{D}_{\alpha}\left[\frac{e^{-3\xi V}T(e^{\xi V}N_0^M)}{(X^0)^2}\right] - \xi W_{\alpha}N^M e^{\xi V}, \quad (1.22)$$

which is the generalization of $R \approx -\kappa^2 T_{\mu}^{\mu}$ of general relativity. We understand the last term in the context of electromagnetism in Appendix C.

Note that for conformally coupled (neutral) chiral multiplets $N_0^M = 0$ and the violation of the superconformal symmetry comes only from the last terms due to the FI term.

1.2 New minimal Superconformal formulation: results

In the new minimal formulation with neutral matter and FI term, the action is

$$\mathcal{L}^{\text{nm}} = \left[3L \ln \frac{L}{X^0 \bar{X}^0} \right]_D + [L K]_D + \xi [L V]_D + [W^\alpha W_\alpha]_F, \quad (1.23)$$

where K is the matter Kähler potential

$$K(S, \bar{S}) = -3 \ln \left(-\frac{1}{3} \Phi(S, \bar{S}) \right). \quad (1.24)$$

The relevant field equation is here the field equation of χ_α , the fermionic component of the linear multiplet, which defines W_α^L , W_α^K and $W_\alpha^{\text{nm,FI}}$ for the three first terms of (1.23)³

$$\begin{aligned} W_\alpha^L + W_\alpha^K + W_\alpha^{\text{nm,FI}} &\approx 0, \\ W_\alpha^L &= 3T\mathcal{D}_\alpha \ln \frac{L}{X^0 \bar{X}^0}, \\ W_\alpha^K &= T\mathcal{D}_\alpha K = -\frac{1}{2}i\lambda_\alpha(K), \\ W_\alpha^{\text{nm,FI}} &= \xi W_\alpha, \end{aligned} \quad (1.25)$$

where $\lambda_\alpha(K)$ is the $P_L\lambda$ component of the real multiplet K .⁴ Clearly, χ does not appear in the last term of (1.23).

The full A_μ field equation is split as

$$\mathcal{E}_a^L + J_a + \mathcal{E}_a^W \approx 0, \quad (1.26)$$

where J_a is related to the part with the Kähler potential in (1.23) (the FI term does not contribute). We will prove below the strong equations

$$\bar{\mathcal{D}}^{\dot{\alpha}} \mathcal{E}_{\alpha\dot{\alpha}}^L = L W_\alpha^L, \quad \bar{\mathcal{D}}^{\dot{\alpha}} \mathcal{E}_{\alpha\dot{\alpha}}^W = -2W_\alpha D. \quad (1.27)$$

The D field equation is now $2D + \xi L \approx 0$, leaving us with

$$\bar{\mathcal{D}}^{\dot{\alpha}} J_{\alpha\dot{\alpha}} \approx L W_\alpha^K, \quad (1.28)$$

which is identical to the rigid equation for the supercurrent of the matter system.

In section 2 we will give a general proof of the strong equations (Ward identities) mentioned above, based on general rules of the transformations of field equations. We obtain the higher θ components of the Einstein multiplet in section 3, and in section 4 we discuss applications related to rigid supersymmetric curved backgrounds. We give an account of the

³e.g. W_α^L is i times the covariant field equation for $P_L\chi$ of the first term of (1.23), see Appendix A.

⁴The superfield W_α^K is also the main ingredient in the construction of super-Kähler invariant actions in [12] and in the new actions for broken supergravity, dubbed ‘Liberated supergravity’ in [13]. In these papers, this superfield is used in the context of old minimal supergravity.

different auxiliary field formulations and related conformal compensators of $\mathcal{N} = 1$ supergravity in section 5. We finish with a conclusion in Section 6. In Appendix A, we give some information on the linear multiplet and the related fermionic chiral superfield. Superconformal Ward identities are given in Appendix B. Appendix C explains an electromagnetic analogue of the conservation law. In Appendix D we discuss the example where supergravity with FI term is coupled to a charged chiral multiplet, and we consider also a nilpotent chiral multiplet so that the model includes a Volkov–Akulov field.

2 Ward identities

2.1 General equation

As mentioned in the introduction, we obtain results by giving first Ward identities that are of the form

$$\overline{\mathcal{D}}^{\dot{\alpha}} \mathcal{E}_{\alpha\dot{\alpha}} = \dots, \quad \mathcal{E}_{\alpha\dot{\alpha}} = \frac{1}{4} i \gamma_{\alpha\dot{\alpha}}^a \mathcal{E}_a, \text{ or } \mathcal{E}_a = 2i \mathcal{E}_{\alpha\dot{\alpha}} (\gamma_a)^{\alpha\dot{\alpha}}, \quad (2.1)$$

where $\mathcal{E}_{\alpha\dot{\alpha}}$ or, equivalently \mathcal{E}_a , is defined by the field equation of A_μ in supergravity actions: in general

$$\mathcal{E}_a \equiv -\frac{4}{3} e^{-1} e_a^\mu \frac{\delta}{\delta A^\mu} S. \quad (2.2)$$

In the notation where $\Theta(\phi^i)$ indicates the covariantized field equation⁵ of the field ϕ^i , the latter equation is written

$$\mathcal{E}_a = -\frac{4}{3} \Theta(A_a). \quad (2.3)$$

The multiplet \mathcal{E}_a has (Weyl, chiral) weights (3,0). The right-hand side of (2.1) should have weights (7/2, 3/2). Other components of \mathcal{E}_a are obtained by successive supersymmetry transformations on this lowest component of the multiplet.

We will consider several actions, e.g. S^{om} and define then $\mathcal{E}_a^{\text{om}}$ as the expression defined by (2.2) when $S = S^{\text{om}}$. But first we find general relations that do not depend on the choice of action S .

The quantity in (2.1) is

$$\overline{\mathcal{D}}^{\dot{\alpha}} \mathcal{E}_{\alpha\dot{\alpha}} = -\frac{1}{4} i (\gamma^a \delta_R \mathcal{E}_a)_\alpha, \quad (2.4)$$

where δ_R is defined by

$$\delta \mathcal{E}_a = \bar{\epsilon} P_L \delta_L \mathcal{E}_a + \bar{\epsilon} P_R \delta_R \mathcal{E}_a. \quad (2.5)$$

To find this, we use a method explained in [1, Appendix D], based upon earlier work in e.g. [14, 15]. The covariant field equations satisfy Ward identities

$$\delta(\epsilon) \phi^j \Theta(\phi)_j = 0. \quad (2.6)$$

⁵ $\Theta(\phi)_i \equiv \frac{\overrightarrow{\delta} S}{\delta \phi^i} + \text{non-covariant terms.}$

We will use the formula that gives the symmetry transformation of covariant field equations

$$\delta(\epsilon)\Theta(\phi^i) = - \left(\frac{\vec{\partial}\delta(\epsilon)\phi^j}{\partial\phi^i} \right)_{\text{cov}} \Theta(\phi)_j, \quad (2.7)$$

where ‘cov’ refers to a covariantization of all spacetime derivatives and omission of terms with undifferentiated gauge fields. The equation depends on which multiplets are taken in the theory and on their transformation laws (see the sum over j in the right-hand side). E.g. since A_μ does not appear in S -supersymmetry transformations of other fields, (2.7) with ϵ replaced by the S -supersymmetry parameter η implies that (2.3) does not transform under S -supersymmetry and therefore $\Theta(A_a)$ defines a superconformal primary.

We will consider this now for superconformal Q -supersymmetry, with⁶

the Weyl multiplet: $\{e_\mu^a, \psi_\mu, b_\mu, A_\mu\}$.

chiral multiplets: $\{X^I, \Omega^I, F^I\}$.

a real vector multiplet: $\{V, \zeta, \mathcal{H}, W_a, \lambda, D\}$, which in Wess–Zumino (WZ) gauge is reduced to $\{W_\mu, \lambda, D\}$.

a real linear multiplet: $\{L, a_{\mu\nu}, \chi\}$.

The supersymmetry transformation of the field equation of A_μ according to (2.7), with ϕ^i replaced by A_μ , depends on the fields that have a term $A_\mu\epsilon$ in their transformation. Such terms are included in covariant derivatives. Considering all the fields mentioned above, we find

$$\begin{aligned} \delta(\epsilon)\Theta(A_a) = \bar{\epsilon}P_R \left[-\frac{3}{2}i\Theta(\psi)_a - \frac{1}{\sqrt{2}}i\gamma_a \left(X^I\Theta(\Omega)_I + \frac{1}{2}\Omega^I\Theta(F)_I \right) \right. \\ \left. -\frac{3}{2}\gamma_a\zeta\Theta(\mathcal{H}) - \frac{3}{4}i\zeta\Theta(W)_a + \frac{3}{4}\gamma_a\lambda\Theta(D) \right] + \text{h.c.} . \end{aligned} \quad (2.8)$$

Combining this with (2.4) and (2.3), we find

$$\begin{aligned} \bar{\mathcal{D}}^{\dot{\alpha}}\mathcal{E}_{\alpha\dot{\alpha}} = \left[\frac{1}{2}\gamma^a\Theta(\psi)_a + \frac{2}{3}\sqrt{2}X^I\Theta(\Omega)_I + \frac{1}{3}\sqrt{2}\Omega^I\Theta(F)_I \right. \\ \left. -2i\zeta\Theta(\mathcal{H}) + \frac{1}{4}\gamma^a\zeta\Theta(W)_a + i\lambda\Theta(D) \right]_{\alpha} . \end{aligned} \quad (2.9)$$

This expression can be rewritten upon use of the Ward identity (2.6) for S -supersymmetry, see (B.9),

$$W(S) \equiv (\delta_S\phi^i)\Theta(\phi)_i = 0. \quad (2.10)$$

Considering the mentioned multiplets, the left projection of this equation is

$$P_L W(S) = P_L \left[\gamma^a\Theta(\psi)_a + \sqrt{2}X^I\Theta(\Omega)_I - 2i\zeta\Theta(\mathcal{H}) + \frac{1}{2}\gamma^a\zeta\Theta(W)_a - 2iL\Theta(\chi) \right]. \quad (2.11)$$

⁶The generalization to several real or linear multiplets is obvious, adding indices that are summed similar to the sum over I for the chiral multiplets.

Combining (2.9) and (2.11), we get

$$\bar{\mathcal{D}}^{\dot{\alpha}} \mathcal{E}_{\alpha\dot{\alpha}} = \left[\frac{1}{6} \sqrt{2} (X^I \Theta(\Omega)_I + 2\Omega^I \Theta(F)_I) - i\zeta \Theta(\mathcal{H}) + i\lambda \Theta(D) + iL \Theta(\chi) \right]_{\alpha}. \quad (2.12)$$

In Appendix A it is shown that the general equation (2.7) also implies that

$$W_{\alpha}^{\chi} = i\Theta(\chi)_{\alpha}, \quad W_{\alpha} = -\frac{1}{2}i\lambda_{\alpha}, \quad (2.13)$$

satisfy the same properties (chiral primaries with real $\mathcal{D}^{\alpha}W_{\alpha}$). Therefore, we still rewrite this equation as

$$\bar{\mathcal{D}}^{\dot{\alpha}} \mathcal{E}_{\alpha\dot{\alpha}} = \frac{1}{6} \sqrt{2} [X^I \Theta(\Omega)_{I\alpha} + 2\Omega_{\alpha}^I \Theta(F)_I] - i\zeta_{\alpha} \Theta(\mathcal{H}) - 2W_{\alpha} \Theta(D) + W_{\alpha}^{\chi} L. \quad (2.14)$$

The upper indication χ for W_{α}^{χ} will later be used to refer to the χ field equation of particular actions, e.g. S^L for the pure supergravity new minimal model: (2.42). The term with $\zeta_{\alpha} \Theta(\mathcal{H})$ is absent in WZ gauge. The last two terms of (2.14) are of the form of the second term in (1.1) when $\Theta(D)_A$ is a constant (FI constant) or L is a constant (conformal gauge choice $L \stackrel{\square}{=} \kappa^{-2}$).

2.2 Old minimal pure supergravity

We first consider pure supergravity with only the compensating multiplet X^0 . Thus we only have the terms with $\Theta(\Omega)_0$ and $\Theta(F)_0$ in the right-hand side of (2.14).

When we apply (2.7) for $\phi^i = F^I$ we get

$$\delta(\epsilon) \Theta(F)_I = -\frac{1}{\sqrt{2}} \bar{\epsilon} P_L \Theta(\Omega)_I. \quad (2.15)$$

Therefore we have

$$\delta(\epsilon) \frac{\Theta(F)_0}{(X^0)^2} = -\frac{1}{\sqrt{2}(X^0)^3} [X^0 \Theta(\Omega)_0 + 2\Omega^0 \Theta(F)_0], \quad (2.16)$$

The quantity in the left-hand side has (Weyl, chiral) weights (0,0). Applying (2.7) for S-supersymmetry for $\phi^i = F^I$ shows that $\Theta(F)_I$ is S-supersymmetry invariant since F^I does not appear in the S-transformations of the fields. Therefore \mathcal{D}_{α} is well defined on this quantity and we can write (2.14) as

$$\bar{\mathcal{D}}^{\dot{\alpha}} \mathcal{E}_{\alpha\dot{\alpha}}^{\text{om}} = -\frac{1}{3} (X^0)^3 \mathcal{D}_{\alpha} \frac{\Theta(F)_0}{(X^0)^2}, \quad (2.17)$$

where ‘om’ stands for pure ‘old minimal’ supergravity. We can apply it to the action

$$S^{\text{om}} = [-3X^0 \bar{X}^0]_D, \quad (2.18)$$

where $\Theta(F)_0 = -3\bar{F}^0 = -3T(\bar{X}^0)$. With the definition

$$\mathcal{R} \equiv \frac{1}{X^0} T(\bar{X}^0), \quad (2.19)$$

we thus get the equation in the form presented in [1]:

$$\bar{\mathcal{D}}^{\dot{\alpha}} \mathcal{E}_{\alpha\dot{\alpha}}^{\text{om}} = (X^0)^3 \mathcal{D}_{\alpha} \frac{\mathcal{R}}{X^0}. \quad (2.20)$$

The value of $\mathcal{E}_a^{\text{om}}$ is in this case

$$\mathcal{E}_a^{\text{om}} = -8X^0 \bar{X}^0 A_a + 4iX^0 \partial_a \bar{X}^{\bar{0}} - 4i\bar{X}^{\bar{0}} \partial_a X^0 + 2i\bar{\Omega}^0 P_L \gamma_a \Omega^{\bar{0}} + 2\sqrt{2}i\bar{\psi}_a \left(P_L \Omega^{\bar{0}} X^{\bar{0}} - P_R \Omega^{\bar{0}} X^0 \right), \quad (2.21)$$

whose flat limit is

$$\mathcal{E}_{\alpha\dot{\alpha}}^{\text{om}}|_{\text{flat}} = -4i\bar{X}^{\bar{0}} \partial_{\alpha\dot{\alpha}} X^0 + 4iX^{\bar{0}} \partial_{\alpha\dot{\alpha}} \bar{X}^{\bar{0}} - 2(D_{\alpha} X^0)(\bar{D}_{\dot{\alpha}} \bar{X}^{\bar{0}}). \quad (2.22)$$

2.3 Old minimum supergravity with a FI term

We now apply the above for the case when X^0 transforms under an Abelian gauge group gauged by a real multiplet V . The supergauge transformations are given in (1.11), where the chiral superfield Λ reduces in the WZ gauge to $\Lambda = -\frac{1}{2}i\theta$ plus contributions to the supersymmetry (see e.g. (2.28) in [16]). The θ transformation is then a phase transformation on the compensating scalar with gauge field W_{μ} .

$$\delta(\theta)X^0 = \frac{1}{2}i\xi\theta X^0, \quad \delta W_{\mu} = \partial_{\mu}\theta. \quad (2.23)$$

The action that provides the FI term is the covariantization of (2.18), and is indicated by ‘om,V’:

$$S^{\text{om},V} = [-3X^0 e^{\xi V} \bar{X}^0]_D. \quad (2.24)$$

This action leads to covariant field equations

$$\Theta(F^0)^V = -3T(e^{\xi V} \bar{X}^0), \quad \Theta(\mathcal{H})^V = \frac{3}{2}\xi X^0 T(e^{\xi V} \bar{X}^0), \quad (2.25)$$

in agreement with the Ward identity for the transformation with parameter the upper complex component of Λ :

$$\frac{1}{2}\Theta(\mathcal{H}) - \xi X^0 \Theta(F_0) = 0. \quad (2.26)$$

Since we still have only one chiral multiplet, the rewriting of the first two terms of (2.14) in (2.17) is still valid. The field equation for the auxiliary field D of the gauge multiplet in WZ gauge gives the moment map, \mathcal{P} , and the supergauge-invariant form is

$$\Theta(D)^V = -\mathcal{P}, \quad \mathcal{P} = \frac{3}{2}\xi X^0 e^{\xi V} \bar{X}^0. \quad (2.27)$$

Therefore we get

$$\bar{\mathcal{D}}^{\dot{\alpha}} \mathcal{E}_{\alpha\dot{\alpha}}^{\text{om},V} = (X^0)^3 \mathcal{D}_\alpha \left[\frac{T(e^{\xi V} \bar{X}^0)}{(X^0)^2} \right] + 2W_\alpha \mathcal{P} - \frac{3}{2} i \xi \zeta_\alpha X^0 T(e^{\xi V} \bar{X}^0). \quad (2.28)$$

Since $e^{3\xi V} \mathcal{D}_\alpha e^{-3\xi V} = -\frac{3}{2} i \xi \zeta_\alpha$, the last term can be absorbed in the first one

$$\bar{\mathcal{D}}^{\dot{\alpha}} \mathcal{E}_{\alpha\dot{\alpha}}^{\text{om},V} = (e^{\xi V} X^0)^3 \mathcal{D}_\alpha \left[\frac{e^{-3\xi V} T(e^{\xi V} \bar{X}^0)}{(X^0)^2} \right] + 2W_\alpha \mathcal{P}. \quad (2.29)$$

When using this in WZ gauge, the exponentials of V do not appear, and the equation simplifies to (we denote the \mathcal{E}_a field now as $\mathcal{E}_a^{\text{om,FI}}$)

$$\bar{\mathcal{D}}^{\dot{\alpha}} \mathcal{E}_{\alpha\dot{\alpha}}^{\text{om,FI}} = (X^0)^3 \mathcal{D}_\alpha \frac{\mathcal{R}}{X^0} + 2W_\alpha \mathcal{P}. \quad (2.30)$$

But one should take into account that the supersymmetry transformations of a chiral multiplet are modified, i.e.

$$\delta(\epsilon) \bar{F}^0 = \dots - \frac{1}{2} i \bar{\epsilon} \xi P_L \lambda \bar{X}^0, \quad (2.31)$$

and covariant spacetime derivatives get a W_μ connection. Thus the meaning of supercovariant derivatives \mathcal{D}_α is different, and (2.31) induces an extra term

$$(X^0)^3 \mathcal{D}_\alpha \frac{\mathcal{R}}{X^0} = (X^0)^3 \mathcal{D}_\alpha \frac{\bar{F}^0}{(X^0)^2} = \dots - \frac{1}{2} i \xi X^0 \bar{X}^0 P_L \lambda = \dots + \frac{2}{3} \mathcal{P} W_\alpha. \quad (2.32)$$

The latter replaces the contribution in (2.29) where the derivatives \mathcal{D}_α and T act on V .

We can write a supergauge-invariant version of (2.22)

$$\begin{aligned} \mathcal{E}_{\alpha\dot{\alpha}}^{\text{om},V} \Big|_{\text{flat}} &= 2e^{\xi V} X^0 D_\alpha e^{-\xi V} \bar{D}_{\dot{\alpha}} e^{\xi V} \bar{X}^0 - 2e^{\xi V} \bar{X}^0 \bar{D}_{\dot{\alpha}} e^{-\xi V} D_\alpha e^{\xi V} X^0 \\ &\quad - 2e^{-\xi V} D_\alpha (e^{\xi V} X^0) \bar{D}_{\dot{\alpha}} (e^{\xi V} \bar{X}^0). \end{aligned} \quad (2.33)$$

This expression makes no sense in conformal language since e.g. \mathcal{D}_α can only be applied on a multiplet with $c = -w$. Thus this expression is only applicable to the rigid supersymmetry limit. Therefore we write D_α and not \mathcal{D}_α , ... But we can check the coefficient of the last term in (2.29) by using this flat limit. Indeed, $\bar{\mathcal{D}}^{\dot{\alpha}}$ on (2.33) gives a term⁷

$$\bar{\mathcal{D}}^{\dot{\alpha}} \mathcal{E}_{\alpha\dot{\alpha}}^{\text{om},V} \Big|_{\text{flat}} = \dots + \frac{8}{3} W_\alpha \mathcal{P}. \quad (2.34)$$

This contains the contributions of the last term of (2.29) and of (2.32).

⁷The conventions in Appendix A of [1] imply $T = \bar{D}^2 = -\bar{D}^{\dot{\alpha}} \bar{D}_{\dot{\alpha}}$ and we neglect all terms that have less than 3 derivatives on V .

In WZ gauge, the field equation of A_μ is the covariantized version of (2.21), which implies that there is an extra term:

$$\begin{aligned}\mathcal{E}_a^{\text{om,FI}} &= -8X^0\bar{X}^0 A_a + 4iX^0\hat{\partial}_a\bar{X}^0 - 4i\bar{X}^0\hat{\partial}_aX^0 + 2i\bar{\Omega}^0 P_L\gamma_a\Omega^{\bar{0}} \\ &\quad + 2\sqrt{2}i\bar{\psi}_a \left(P_L\Omega^0 X^{\bar{0}} - P_R\Omega^{\bar{0}} X^0 \right) \\ &= \mathcal{E}_a^{\text{om}} - \frac{8}{3}W_a\mathcal{P},\end{aligned}\tag{2.35}$$

where $\hat{\partial}_aX^0 = \partial_aX^0 - \frac{1}{2}i\xi W_aX^0$. The transformation of $\mathcal{E}_a^{\text{om,FI}}$ contains then the transformation of W_a

$$\delta_{\text{extra}}(\epsilon)\mathcal{E}_a^{\text{om,FI}} = -\frac{8}{3}\mathcal{P}\delta(\epsilon)W_a = \frac{4}{3}\mathcal{P}\bar{\epsilon}\gamma_a\lambda,\tag{2.36}$$

which leads to the same term (2.34).

2.4 Gauge kinetic terms

Now we consider

$$S^W = -\frac{1}{4}[\bar{\lambda}P_L\lambda]_F = [W^\alpha W_\alpha]_F,\tag{2.37}$$

which added to (2.24) gives the off-shell version of the Freedman model [17, 11]. The A_μ field equation of this action leads to

$$\mathcal{E}_a^W = i\bar{\lambda}\gamma_*\gamma_a\lambda, \quad \mathcal{E}_{\alpha\dot{\alpha}}^W = \lambda_{\dot{\alpha}}\lambda_\alpha = 4W_{\dot{\alpha}}W_\alpha.\tag{2.38}$$

Here there are no chiral multiplets, and the only contribution in the right-hand side of (2.14) comes from

$$\Theta^W(D) = D = i\mathcal{D}^\alpha\lambda_\alpha = -2\mathcal{D}^\alpha W_\alpha.\tag{2.39}$$

Hence the equation is

$$\bar{\mathcal{D}}^{\dot{\alpha}}\mathcal{E}_{\alpha\dot{\alpha}}^W = -2W_\alpha D,\tag{2.40}$$

which can easily be checked from the supersymmetry transformation of λ . For the full model with FI term and kinetic terms, we thus get

$$\bar{\mathcal{D}}^{\dot{\alpha}}(\mathcal{E}^{\text{om,V}} + \mathcal{E}^W)_{\alpha\dot{\alpha}} = (e^{\xi V}X^0)^3\mathcal{D}_\alpha \left[\frac{e^{-3\xi V}T(e^{\xi V}\bar{X}^0)}{(X^0)^2} \right] + 2W_\alpha(\mathcal{P} - D).\tag{2.41}$$

where the last factor is the D -field equation.

2.5 New minimal pure supergravity

Here we have only a real linear multiplet L (and the chiral multiplet X^0 , which, however, does not appear in the action):

$$\begin{aligned}
S^L &= \left[3L \ln \frac{L}{X^0 \bar{X}^0} \right]_D \\
&= \frac{3}{2} \int d^4x e \left[-\square^c L + \frac{1}{2L} (v_a v^a + \mathcal{D}_a L \mathcal{D}^a L + \bar{\chi} \not{\mathcal{D}} \chi) + \frac{1}{4L^3} \bar{\chi} P_L \chi \bar{\chi} P_R \chi \right. \\
&\quad \left. + \frac{1}{2} i \bar{\psi}_\mu \gamma^\mu \gamma_* \left(\not{\mathcal{D}} \chi + \frac{1}{2L} (i \gamma_* \not{\psi} - \not{\mathcal{D}} L) \chi + \frac{1}{2L^2} \chi \bar{\chi} \chi \right) \right] \\
&\quad + \varepsilon^{\mu\nu\rho\sigma} \left(2A_\mu - \frac{1}{L} v_\mu - \frac{1}{4L^2} i \bar{\chi} \gamma_* \gamma_\mu \chi + \frac{1}{2L} \bar{\psi}_\mu \chi \right) \partial_\nu a_{\rho\sigma}, \tag{2.42}
\end{aligned}$$

using fields and notations as in Appendix A. The A_μ field equation gives now

$$\mathcal{E}_a^L = -4v_a + \frac{3}{2L} \bar{\chi} \gamma_* \gamma_a \chi. \tag{2.43}$$

According to (2.14), this satisfies

$$\bar{\mathcal{D}}^{\dot{\alpha}} \mathcal{E}_{\alpha\dot{\alpha}}^L = L W_\alpha^L, \tag{2.44}$$

where now we use for $\Theta(\chi)$ the field equation for the action (2.42):

$$W_\alpha^L = i\Theta(\chi)^L, \quad \Theta(\chi)^L = \frac{3}{2L} P_L \not{\mathcal{D}} \chi + \frac{3}{4L^3} P_L \chi \bar{\chi} P_R \chi - \frac{3}{4L^2} i \not{\psi} P_R \chi. \tag{2.45}$$

In Poincaré gauge, $L = \kappa^{-2}$ and $\chi = 0$, this is

$$W_\alpha^L \stackrel{\square}{=} -\frac{1}{2} P_L \gamma \cdot \widehat{R}'(Q), \quad \widehat{R}'_{\mu\nu}(Q) = 2 \left(\partial_{[\mu} - \frac{3}{2} i A_{[\mu} \gamma_* + \frac{1}{4} \omega_{[\mu}^{ab} \gamma_{ab} + \frac{1}{4} i \kappa^2 \gamma_* \gamma_{[\mu} \not{\psi}] \nu]} \right) \psi_{\nu]}, \tag{2.46}$$

which agrees with what is found in [18, 19].

2.6 The FI term in new minimal supergravity

The FI term in new minimal supergravity is

$$S^{\text{nm,FI}} = \xi [L V]_D. \tag{2.47}$$

A_μ does not appear in this action. Hence (2.14) reduces here to

$$0 = -2W_\alpha \Theta^{\text{nm,FI}}(D) + W_\alpha^{\text{nm,FI}} L. \tag{2.48}$$

Indeed, $\Theta^{\text{nm,FI}}(D) = \frac{1}{2} \xi L$ and $W_\alpha^{\text{nm,FI}} = -\xi W_\alpha$. The FI term in new minimal does therefore not contribute to \mathcal{E}_a and this is consistent with the conservation equations.

We can consider then the total action

$$S^{\text{nm,total}} = S^L + S^{\text{nm,FI}} + S^W. \quad (2.49)$$

The conservation equation is then

$$\overline{\mathcal{D}}^{\dot{\alpha}} (\mathcal{E}^L + \mathcal{E}^W)_{\alpha\dot{\alpha}} = L W_{\alpha}^L - 2D W_{\alpha}, \quad (2.50)$$

Adding the zero result (2.48) and using (2.39) this is

$$\begin{aligned} \overline{\mathcal{D}}^{\dot{\alpha}} (\mathcal{E}^L + \mathcal{E}^W)_{\alpha\dot{\alpha}} &= L W_{\alpha}^{\text{nm,total}} - 2W_{\alpha} \Theta^{\text{nm,total}}(D), \\ W_{\alpha}^{\text{nm,total}} &= W_{\alpha}^L + \xi W_{\alpha}, \quad \Theta^{\text{nm,total}}(D) = D + \frac{1}{2}\xi L, \end{aligned} \quad (2.51)$$

and thus the right-hand side also vanishes by the field equations.

3 Components of the Einstein curvature multiplet

The Einstein current multiplet is defined by the field equation of A_{μ} , the U(1) gauge field in the Weyl multiplet, see (2.2) or (2.3). This is the first component of a real multiplet since it is a superconformal primary (it does not transform under S -supersymmetry). Then all other components are defined by the transformations of this first component. These components are denoted as

$$\{\mathcal{C}_a, \mathcal{Z}_a, \mathcal{H}_a, \mathcal{B}_{ba}, \Lambda_a, D_a\}, \quad (3.1)$$

where the fermions \mathcal{Z}_a and Λ_a are Majorana spinors, while from the bosons only \mathcal{H}_a is complex and the others are real.

We omit now the overall coefficient in (2.3), and start from $\mathcal{C}_a = \Theta(A)_a$. We then use (2.7) to derive the further components. E.g. (2.8) determines the \mathcal{Z}_a component. Sometimes we use Ward identities (2.6) for several symmetries to rewrite expressions. In (6.3) of [1] we obtained this result for the coupling of the Weyl multiplet to chiral multiplets. Now we will add a real multiplet and a linear multiplet as in section 2.1.

The relevant components are then

$$\begin{aligned} \mathcal{C}_a &= \Theta(A)_a, \\ \mathcal{Z}_a &= 3\Theta(\psi)_a - \gamma_a \gamma \cdot \Theta(\psi) + \frac{1}{\sqrt{2}} \gamma_a (\Omega^I \Theta(F)_I + \text{h.c.}) - \frac{3}{2} i \gamma_* \gamma_a \lambda^A \Theta(D)_A - 2i \gamma_* \gamma_a \Theta(\chi) L, \\ \mathcal{H}_a &= 2i \bar{X}^{\bar{I}} \mathcal{D}_a \Theta(F)_{\bar{I}} - 4i \Theta(F)_{\bar{I}} \mathcal{D}_a \bar{X}^{\bar{I}} + 3i \bar{\lambda}^A \gamma_a P_L \Theta(\lambda)_A + 3i \bar{\chi} \gamma_a P_R \Theta(\chi), \\ \mathcal{B}_{ba} &= 3\Theta(e)_{ab} - \eta_{ab} \Theta(e)_c{}^c + \frac{1}{2} \varepsilon_{abcd} \mathcal{D}^c \Theta(A)^d - \frac{1}{2} \eta_{ab} (\bar{\Omega}^I \Theta(\Omega)_I + 2F^I \Theta(F)_I + \text{h.c.}) \\ &\quad + \eta_{ab} D^A \Theta(D)_A + 3i \tilde{F}_{ab}{}^A \Theta(D)_A + \frac{3}{2} \bar{\lambda}^A \gamma_{ab} \Theta(\lambda)_A \\ &\quad - 3L \Theta(a)_{ba} - 2\eta_{ab} L \Theta(L) - \bar{\chi} \Theta(\chi) - \frac{3}{2} \bar{\chi} \gamma_{ab} \Theta(\chi), \\ \Lambda_a &= 2\gamma^b \mathcal{D}_{[a} \mathcal{Z}_{b]} - 3\sqrt{2} (\Theta(\Omega)^I \mathcal{D}_a X_I + \Omega^I \mathcal{D}_a \Theta(F)_I + \text{h.c.}) \\ &\quad + \frac{3}{2} (\frac{1}{2} \gamma \cdot F^A - i \gamma_* D^A) \gamma_a \Theta(\lambda)_A + \frac{3}{2} \gamma_b \gamma_a \lambda^A \Theta(W)_A^b \end{aligned}$$

$$\begin{aligned}
& + \frac{3}{2}i\gamma_*\gamma^b\gamma_a\lambda^A\mathcal{D}_b\Theta(D)_A - 6i\gamma_*\mathcal{D}_a(\lambda^A\Theta(D)_A) \\
& - \frac{3}{2}(\not{\psi} - i\gamma_*\not{\mathcal{D}}L)\gamma_a\Theta(\chi) - \frac{3}{2}i\gamma_*\gamma_a\chi\Theta(L) - \frac{3}{4}i\gamma_*\gamma_{bc}\gamma_a\chi\Theta(a)^{bc} - 6i\gamma_*\mathcal{D}_a(\Theta(\chi)L) , \\
D_a = & - 2\mathcal{D}^b\mathcal{D}_{[b}\Theta(A)_{a]} - 2\mathcal{D}_{[a}\mathcal{D}_{b]}\Theta(A)^b \\
& - \frac{3}{2}i\left(2\mathcal{D}_aX^I\Theta(X)_I - \bar{\Omega}^I\overleftrightarrow{\mathcal{D}}_a\Theta(\Omega)_I - 2F^I\mathcal{D}_a\Theta(F)_I - \text{h.c.}\right) \\
& + \Theta(W)_{aA}D^A + \frac{3}{2}i\bar{\Theta}(\lambda)_A\gamma_*\gamma_a{}^b\overleftrightarrow{\mathcal{D}}_b\lambda^A \\
& + 3v_a\Theta(L) + 3v^b\Theta(a)_{ba} + 3i\mathcal{D}^bL\tilde{\Theta}(a)_{ba} - \frac{3}{2}i\bar{\Theta}(\chi)\gamma_*\gamma_a{}^b\overleftrightarrow{\mathcal{D}}_b\chi .
\end{aligned} \tag{3.2}$$

We will further restrict ourselves to the components up to \mathcal{B}_{ab} arguing that the higher components follow from the constraint that the multiplet is a generalization of a linear multiplet, i.e.

$$\overline{\mathcal{D}}^{\dot{\beta}}\frac{1}{L}\overline{\mathcal{D}}^{\dot{\alpha}}\mathcal{E}_{\alpha\dot{\alpha}} = 0 . \tag{3.3}$$

In the Poincaré limit, L is a constant, and this equation reduces to $\overline{D}^2E_{\alpha\dot{\alpha}} = 0$. The constraint (3.3) applies also to the gauge multiplet, replacing $\frac{1}{L}$ with $\frac{1}{D}$.

3.1 Components in old minimal formulation with FI term

The values of the covariantized field equations depend on the considered action. For the action in section 2.3 in WZ gauge we have

$$\begin{aligned}
\Theta(A)_a & = -3iX^0\mathcal{D}_a\bar{X}^0 + 3i\bar{X}^0\mathcal{D}_aX^0 - \frac{3}{2}i\bar{\Omega}^0P_L\gamma_a\Omega^0 , \\
\Theta(\psi)_a & = \dots + \frac{3}{4}i\xi\gamma_a\left(X^0P_L - \bar{X}^0P_R\right)\lambda , \\
\Theta(e)_{ab} & = \dots + \frac{3}{2}\eta_{ab}\xi\left[X^0\bar{X}^0D + \sqrt{2}i\bar{\lambda}\left(P_R\Omega^0X^0 - P_L\Omega^0X^0\right)\right] , \\
\Theta(F)_0 & = -3\bar{F}^0 , \\
\Theta(\Omega)_0 & = 3P_L\not{\mathcal{D}}\Omega^0 - \frac{3}{2}\sqrt{2}i\xi\bar{X}^0P_L\lambda , \\
\Theta(X)_0 & = -3\Box^c\bar{X}^0 + \frac{3}{2}\xi\left(-\bar{X}^0D + \sqrt{2}i\bar{\lambda}P_R\Omega^0\right) , \\
\Theta(D) & = -\mathcal{P} = -\frac{3}{2}\xi X^0\bar{X}^0 , \\
\Theta(\lambda) & = \frac{3}{2}\sqrt{2}i\xi\left(P_R\Omega^0X^0 - P_L\Omega^0X^0\right) , \\
\Theta(W)_a & = \frac{3}{2}i\xi\left(\bar{X}^0\mathcal{D}_aX^0 - X^0\mathcal{D}_a\bar{X}^0 + \bar{\Omega}^0\gamma_aP_R\Omega^0\right) ,
\end{aligned} \tag{3.4}$$

where the \dots are the expressions in (6.5) of [1], and the covariant derivatives also include the connection for (2.23) with gauge field W_μ .

We consider the conformal gauge

$$\text{Conformal gauge old minimal : } X^0|_{\square} = \kappa^{-1} , \quad \Omega^0|_{\square} = 0 , \quad b_\mu = 0 , \tag{3.5}$$

after which the conformal U(1) symmetries mix with the θ gauge symmetries, $\lambda_T = -\frac{1}{2}\xi\theta$, and as such the fields transform with a different weight under the remaining gauge symmetries, as in (D.4) in Appendix D. Correspondingly, we define

$$A_a^\xi = A_a + \frac{1}{2}\xi W_a, \quad (3.6)$$

which is thus inert under the remaining transformations, and also appears in the S -super-symmetry decomposition law instead of A_a :

$$\begin{aligned} \delta_{\square}(\epsilon) &= \delta_Q(\epsilon) + \delta_S \left(\eta = \frac{1}{2}(\mathrm{i}\gamma_* A^\xi - P_R u - P_L \bar{u})\epsilon \right) + \delta_K \left(\lambda_{Ka} = -\frac{1}{4}\bar{\epsilon}\hat{\phi}_a \right), \\ P_L \hat{\phi}_a &= P_L \phi_a + \frac{1}{2}P_L(\mathrm{i}A^\xi + \bar{u})\psi_a, \quad u \equiv \kappa \bar{F}^0. \end{aligned} \quad (3.7)$$

Then the field equations become

$$\begin{aligned} \kappa^2 \Theta(A)_a &\stackrel{\square}{=} 6A_a^\xi, \\ \kappa^2 \Theta(\psi)_a &\stackrel{\square}{=} -\frac{1}{2}\gamma_{abc}\hat{R}^{bc}(Q) - \frac{3}{4}\mathrm{i}\xi\gamma_*\gamma_a\lambda, \\ \kappa^2 \Theta(e)_{ab} &\stackrel{\square}{=} \hat{G}_{ab} + 6A_a^\xi A_b^\xi + 3\eta_{ab}(u\bar{u} - A_c^\xi A^{\xi c} + \frac{1}{2}D), \\ \kappa \Theta(F^0) &= -3u, \\ \sqrt{2}\kappa \Theta(\Omega^0) &\stackrel{\square}{=} P_L \gamma^{ab}\hat{R}_{ab}(Q) - 3\mathrm{i}\xi P_L \lambda, \\ \kappa \Theta(X^0) &\stackrel{\square}{=} -3\mathrm{i}\hat{\mathcal{D}}^a A_a^\xi + \frac{1}{2}\hat{R} + 3A^{\xi a} A_a^\xi - \frac{3}{2}\xi D, \\ \kappa^2 \Theta(D) &\stackrel{\square}{=} -\frac{3}{2}\xi, \quad \Theta(\lambda) \stackrel{\square}{=} 0, \quad \kappa^2 \Theta(W)_a \stackrel{\square}{=} 3\xi A_a^\xi, \end{aligned} \quad (3.8)$$

where

$$\begin{aligned} \hat{R}_{\mu\nu}(Q) &= 2 \left(\partial_{[\mu} - \frac{3}{2}\mathrm{i}A_{|\mu}\gamma_* + \frac{1}{4}\omega_{[\mu}{}^{ab}(e, \psi)\gamma_{ab} + \frac{1}{2}\gamma_{|\mu}(-\mathrm{i}\gamma_* A^\xi + \bar{u})P_L + uP_R \right) \psi_{\nu]}, \\ \hat{\mathcal{D}}_a A_b^\xi &= \nabla_a A_b^\xi + \frac{1}{2}\mathrm{i}\bar{\psi}_a \gamma_* \hat{\phi}_b, \quad \hat{\phi}_a = \frac{1}{2}\gamma^b \hat{R}_{ba}(Q) + \frac{1}{12}\gamma_a \gamma^{bc} \hat{R}_{bc}(Q) \\ \hat{G}_{ab} &= \hat{R}_{ab} - \frac{1}{2}\eta_{ab}\eta^{cd}\hat{R}_{cd} \quad \hat{R}_{ab} = R_{(ab)} - \frac{1}{2}\bar{\psi}^c \gamma_{(a}\hat{R}_{b)c}(Q) + \frac{1}{2}\bar{\psi}_{(a}\gamma^c \hat{R}_{b)c}(Q), \end{aligned} \quad (3.9)$$

and R_{ab} is the torsionful Ricci tensor.

The first components of the Einstein curvature multiplet (3.2) in the gauge (3.5) are then

$$\begin{aligned} \kappa^2 \mathcal{C}_a &= 6A_a^\xi, \\ \kappa^2 \mathcal{Z}_a &= -\frac{3}{2}\gamma^{\mu\nu\rho}\hat{R}_{\nu\rho}(Q) - \gamma_a \gamma^{\mu\nu}\hat{R}_{\mu\nu}(Q) + 3\mathrm{i}\gamma_*\gamma_a\lambda\xi, \\ \kappa^2 \mathcal{H}_a &= -6\mathrm{i}\hat{\mathcal{D}}_a \bar{u}, \quad \hat{\mathcal{D}}_a \bar{u} \equiv (\partial_a - \frac{3}{2}\mathrm{i}\xi W_a)\bar{u} + \bar{\psi}_a P_R \gamma \cdot \hat{\phi}, \\ \kappa^2 \mathcal{B}_{ba} &= 3\hat{G}_{ab} - \eta_{ab}\hat{G}_c{}^c + 18A_a^\xi A_b^\xi + 3\eta_{ab}(u\bar{u} - A_c^\xi A^{\xi c} - \xi D) \\ &\quad + 3\varepsilon_{abcd}\hat{\mathcal{D}}^c A^{\xi d} - \frac{9}{2}\mathrm{i}\xi \tilde{F}_{ab}(W). \end{aligned} \quad (3.10)$$

3.2 Components in new minimal supergravity

Since A_μ does not appear in the action (2.47), the first component of the Einstein multiplet, proportional to $\Theta(A)_a$, does not get a contribution from the FI term, and the Einstein tensor multiplet is unchanged.⁸

We immediately consider the conformal gauge

$$\text{Conformal gauge new minimal : } L|_{\square} = \kappa^{-2}, \quad \chi|_{\square} = 0, \quad b_\mu|_{\square} = 0, \quad (3.11)$$

which implies that the Poincaré supersymmetry transformations are

$$\begin{aligned} \delta_{\square}(\epsilon) &= \delta_Q(\epsilon) + \delta_S \left(\eta = \frac{1}{4} i \kappa^2 \gamma_* \psi \epsilon \right) + \delta_K \left(\lambda_{Ka} = -\frac{1}{4} \bar{\epsilon} \hat{\phi}_a \right), \\ \hat{\phi}_a &= \phi_a - \frac{1}{4} i \kappa^2 \gamma_* \psi \psi_a. \end{aligned} \quad (3.12)$$

and $\hat{\phi}_a$ is of the same form as in (3.9), but with

$$\begin{aligned} \widehat{R}_{\mu\nu}(Q) &= 2 \left(\partial_{[\mu} - \frac{3}{2} i A_{[\mu} \gamma_* + \frac{1}{4} \omega_{[\mu}{}^{ab} \gamma_{ab} + \frac{1}{4} i \gamma_* \gamma_{[\mu} \not{H}] \right) \psi_{\nu]}, \\ H_a &= \kappa^2 v_a = e^{-1} e_{a\mu} \varepsilon^{\mu\nu\rho\sigma} \left(\kappa^2 \partial_\nu a_{\rho\sigma} - \frac{1}{4} \bar{\psi}_\nu \gamma_\rho \psi_\sigma \right). \end{aligned} \quad (3.13)$$

The action (2.42) in this Poincaré form is

$$\begin{aligned} \left[3L \ln \frac{L}{X^0 \bar{X}^0} \right]_D \Big|_{\square} &= \int d^4x e^{-\frac{1}{2\kappa^2}} \left[R - \frac{1}{2} \bar{\psi}_\mu \gamma^{\mu\nu\rho} \widehat{R}_{\nu\rho}(Q) + \frac{3}{2} H_a H^a \right] \\ &+ \frac{3}{2} \varepsilon^{\mu\nu\rho\sigma} (2A_\mu - H_\mu) \partial_\nu a_{\rho\sigma}. \end{aligned} \quad (3.14)$$

The action (2.42) leads to the following field equations

$$\begin{aligned} \Theta(L) &\stackrel{\square}{=} \frac{1}{2} \eta^{ab} \widehat{R}_{ab} + \frac{3}{4} H_a H^a, \\ \Theta(\chi) &= \frac{3}{2L} \not{D}\chi + \frac{3}{2} P_L \chi \bar{\chi} P_R \chi + \frac{3}{2} P_R \chi \bar{\chi} P_L \chi - \frac{3}{4L^2} i \gamma_* \not{\psi} \chi \\ &\stackrel{\square}{=} 3i \gamma^a \gamma_* \hat{\phi}_a = \frac{1}{2} i \gamma_* \gamma \cdot \widehat{R}(Q), \\ \kappa^2 \Theta(e)_{ab} &\stackrel{\square}{=} \widehat{G}_{ab} - \frac{3}{4} \eta_{ab} H_c H^c + \frac{3}{2} H_a H_b, \\ \Theta(a)_{ab} &\stackrel{\square}{=} 3i \tilde{F}_{ab}(A) - \frac{3}{2} \varepsilon_{abcd} \widehat{D}^c H^d, \\ \kappa^2 \Theta(\psi)_a &\stackrel{\square}{=} -\frac{1}{2} \gamma_a{}^{bc} \widehat{R}_{bc}(Q), \\ \Theta(A)_a &= -\frac{9}{8L} i \bar{\chi} \gamma_* \gamma_a \chi + 3v_a \stackrel{\square}{=} \frac{3}{\kappa^2} H_a. \end{aligned} \quad (3.15)$$

The first components of the Einstein curvature multiplet (3.2) in the gauge (3.11) are then

$$\kappa^2 \mathcal{C}_a = 3H_a,$$

⁸Other field equations, like $\Theta(\psi)$, $\Theta(e)$, do change but these contributions cancel since the first component of the multiplet is the same.

$$\begin{aligned}
\kappa^2 \mathcal{Z}_a &= -\frac{3}{2} \gamma_a{}^{bc} \widehat{R}_{bc}(Q), \\
\mathcal{H}_a &= 0, \\
\kappa^2 \mathcal{B}_{ba} &= 3 \left(\widehat{G}_{ab} - \frac{3}{4} \eta_{ab} H_c H^c + \frac{3}{2} H_a H_b - \varepsilon_{abcd} \widehat{\mathcal{D}}^c H^d + 3i \widetilde{F}_{ab}(A) \right). \tag{3.16}
\end{aligned}$$

4 Curved backgrounds

To obtain supersymmetric backgrounds that preserve four supersymmetries, we consider the three curvature multiplets: Weyl, Einstein and scalar curvatures and impose that the high θ components (in presence of auxiliary fields) are vanishing in the background. For the Weyl multiplet, $W_{\alpha\beta\gamma}$, the vanishing of the second component leads to the vanishing of the Weyl tensor

$$W_{\mu\nu\rho\sigma} = 0, \tag{4.1}$$

which will be valid for all the examples below.

Since the scalar multiplet follows from the transformation of the Einstein multiplet, the other constraints follow already from the latter. In particular we have to consider the transformation of the fermion field \mathcal{Z}_a in the Einstein multiplet:

$$P_L \delta \mathcal{Z}_a = \frac{1}{2} \kappa^2 P_L (i \mathcal{H}_a - \gamma^b \mathcal{B}_{ba} - i \mathcal{D} \mathcal{C}_a) \epsilon + i P_L (-3 \mathcal{C}_a + \gamma_{ab} \mathcal{C}^b) \eta. \tag{4.2}$$

We can write the S supersymmetry parameter of (3.7) and (3.12) in a uniform way using the values of \mathcal{C}_a in (3.10) and (3.16):

$$P_L \eta = \left(\frac{1}{12} i \kappa^2 \mathcal{C} - \frac{1}{2} \bar{u} \right) \epsilon, \tag{4.3}$$

just taking $u = 0$ in case of new minimal. This leads to

$$P_L \delta \mathcal{Z}_a = \frac{1}{2} P_L [i (\mathcal{H}_a + 3 \kappa^2 \mathcal{C}_a \bar{u}) - \gamma^b \kappa^{-2} B_{ba} - i \mathcal{D} \mathcal{C}_a - i \gamma_{ab} \mathcal{C}^b \bar{u}] \epsilon, \tag{4.4}$$

where

$$B_{ba} \equiv \kappa^2 \mathcal{B}_{ba} + \frac{1}{6} \kappa^4 (-4 \mathcal{C}_a \mathcal{C}_b + \eta_{ab} \mathcal{C}_c \mathcal{C}^c). \tag{4.5}$$

The vanishing of the right-hand side of (4.4) constrains supersymmetric backgrounds. Splitting in independent parts we get

$$\mathcal{C}_a u = 0, \quad \mathcal{H}_a = 0, \quad B_{ba} = 0, \quad \mathcal{D}_a \mathcal{C}_b = 0. \tag{4.6}$$

We can then further consider only the bosonic part of these equations.

In the previous section, we did not include here the contributions of the gauge kinetic terms, see section 2.4. Note, however, that (2.38) implies that \mathcal{C}_a does not have bosonic terms from this part, and from (3.2) we see that also \mathcal{H}_a does not get extra bosonic terms. There will be new terms in $\Theta(e)_{ab}$, whose bosonic part is

$$\Theta^W(e)_{ba} = F_{c(a}(W) F_b)^c(W) + \frac{1}{4} \eta_{ab} F_{cd}(W) F^{cd}(W) - \frac{1}{2} \eta_{ab} D^2. \tag{4.7}$$

Taking also into account (2.39), this gives from (3.2) an extra contribution to \mathcal{B}_{ba} :

$$\mathcal{B}_{ba}^W = 3 F_{c(a}(W) F_b)^c(W) + \frac{3}{4} \eta_{ab} F_{cd}(W) F^{cd}(W) + \frac{3}{2} \eta_{ab} D^2 + 3i \widetilde{F}_{ab}(W) D. \tag{4.8}$$

4.1 Curved backgrounds old minimal

The equations in (4.6), apart from $B_{ba} = 0$, imply with the bosonic parts of the values in (3.10):

$$uA_\mu^\xi = 0, \quad (\partial_\mu + \frac{3}{2}i\xi W_\mu) u = 0, \quad \nabla_\mu A_\nu^\xi = 0. \quad (4.9)$$

Then the antisymmetric part of $B_{ba} = 0$, has only the terms with $\tilde{F}_{ab}(W)$, which vanish adding (3.10) to (4.8) upon use of the D field equation

$$D \approx \mathcal{P} = \frac{3}{2}\xi X^0 \bar{X}^0 \stackrel{\square}{=} \frac{3}{2}\xi \kappa^{-2}. \quad (4.10)$$

In this case, we get from (4.5), without taking into account the gauge terms (4.8), that the symmetric part of B_{ab} is

$$B_{(ba)} = 3\widehat{G}_{ab} - \eta_{ab}\widehat{G}_c{}^c - 6A_a^\xi A_b^\xi + 3\eta_{ab}(u\bar{u} + A_c^\xi A^{c\xi} - \xi D). \quad (4.11)$$

Hence, without matter, the vanishing of this expression leads to

$$R_{ab} - 2A_a^\xi A_b^\xi + \eta_{ab}(2A^{c\xi} A_c^\xi + 3u\bar{u} - 3\xi D) = 0. \quad (4.12)$$

The above equations for $\xi = 0$ were also obtained in section 6.1 of [1] and agree with [20–22]. Note that $R + 6A_a A^a + 12u\bar{u} = 0$ and $\nabla^\mu A_\mu = 0$ are the vanishing of the upper component of the multiplet \mathcal{R} . The equations allow supersymmetric solutions⁹

$$\begin{aligned} \text{AdS}_4 : A_a &\approx 0, & R + 12u\bar{u} &\approx 0, & u &\neq 0, \\ S^3 \times L : A_a &\approx (A_0, 0, 0, 0), & u &\approx 0, & B_{ab} &\approx 0, \\ & R_{00} \approx R_{0i} \approx 0, & R_{ij} &\approx 2A_0^2 \delta_{ij}, & (i = 1, 2, 3), & R \approx 6A_0^2, \\ \text{AdS}_3 \times L : A_a &\approx (0, 0, 0, A_3), & u &\approx 0, & B_{ab} &\approx 0, \\ & R_{33} \approx R_{3i} \approx 0, & R_{ij} &\approx -2A_3^2 \eta_{ij}, & (i = 0, 1, 2), & R \approx -6A_3^2, \end{aligned} \quad (4.13)$$

where A_0 or A_3 are constants.

For a de Sitter configuration, an extra nilpotent Volkov–Akulov field X^1 was introduced in [1] and a possible solution is

$$\begin{aligned} \text{dS} : F^0 &\approx \kappa^{-2}\lambda, & F^1 &\approx -\frac{1}{3}\kappa^{-2}\mu, \\ \kappa\mathcal{R}|_{\text{last}} &= -\frac{1}{6}R\kappa - 2\kappa^3|F_0|^2 \approx -\frac{2}{9}\kappa^{-1}\mu^2, & \kappa\mathcal{R}|_{\text{first}} &\approx \lambda. \end{aligned} \quad (4.14)$$

The scalar curvature is then $R \approx 4\kappa^2 V = \frac{4}{3}\kappa^{-2}\mu^2 - 12\kappa^{-2}\lambda^2$ and thus

$$B_a{}^a \approx R + 12u\bar{u} \approx 16\kappa^{-2}\mu^2, \quad (4.15)$$

signals the breaking of supersymmetry.

⁹In [20–22] also Euclidean theories are considered by treating real supergravity fields as complex fields. Such an extension is not included in the investigation in this paper.

4.2 FI term in components

In absence of matter (S^i) the FI model is the Freedman model [17, 11], which has a field equation (2.25)

$$T(e^{\xi V} \bar{X}^0) \approx 0. \quad (4.16)$$

This is the covariantized \mathcal{R} chiral scalar curvature multiplet. Note that the above equation in components and in the WZ gauge reproduces the field equations. The first component is

$$u = \kappa \bar{F}^0 \approx 0, \quad (4.17)$$

while the last component is proportional to $\Theta(X)_0$ in (3.4)

$$(\partial^\mu + iA^{\xi\mu}) (\partial_\mu + iA_\mu^\xi) \bar{X}^0 - \frac{1}{6}R\bar{X}^0 + \frac{1}{2}\xi\bar{X}^0 D \approx \text{fermionic terms} \quad (4.18)$$

With zero fermions and in conformal gauge a de Sitter solution is obtained with

$$A_\mu^\xi \stackrel{\square}{\approx} 0, \quad R \approx 3\xi D \stackrel{\square}{\approx} \frac{9}{2}\kappa^{-2}\xi^2. \quad (4.19)$$

The breaking of supersymmetry is evident from

$$B_a{}^a \approx R - 12\xi D \approx -9\xi D. \quad (4.20)$$

4.3 Curved backgrounds new minimal

In this case, with the values in (3.16), the first two equations of (4.6) are empty. The last equation and the antisymmetric part of B_{ab} imply

$$\nabla_\mu H_\nu = 0, \quad \partial_{[\mu} A_{\nu]} = 0, \quad (4.21)$$

The symmetric part of B_{ab} is in this case

$$B_{(ba)} = 3 \left(\widehat{G}_{ab} - \frac{1}{2}H_a H_b - \frac{1}{4}\eta_{ab} H_c H^c \right). \quad (4.22)$$

and its vanishing implies

$$R_{ab} = \frac{1}{2} (H_a H_b - \eta_{ab} H^c H_c). \quad (4.23)$$

The trace of the latter, $R + \frac{3}{2}H^a H_a = 0$ is also the θ component of the chiral curvature. These equations agree with the constraints in [20]. The solutions $S^3 \times L$ and $AdS_3 \times L$ in (4.13) are also possible here, in this case with A_μ vanishing, and $H_a/2$ taking the value of A_a of the old-minimal solutions. However, there is no room for the AdS_4 solution in this case. These are just examples of solutions of the supersymmetry-preservation conditions.

5 Auxiliary fields and conformal compensators

We give here a summary and discussion of auxiliary field formulations of $\mathcal{N} = 1$, $D = 4$ supergravity. First of all we repeat the argument that auxiliary field formulations that have a consistent Lagrangian and describe pure supergravity can only have $(12 + 12)$ + a multiple of $(8 + 8)$ degrees of freedom. This is so because additional fermions should come in pairs of 4-component spinors in the form $\mathcal{L} = \dots + \bar{\lambda}\chi + \dots$. This is e.g. also mentioned in [23], where they clearly mention that their $(16 + 16)$ set has either to be reduced to one of the $(12 + 12)$ sets, or it describes additional physical degrees of freedom. Similarly, the structure in [11] with $(16 + 16)$ fields is in fact an off-shell description of the Freedman model [17], discussed also in this paper, and thus contains other propagating degrees of freedom.

Still there have been made claims of sets of auxiliary field different from the old minimal [24–26], new minimal [27], or non-minimal set [28, 29]. There have been proposals for new non-minimal sets of auxiliary fields or $(16 + 16)$ sets. The latter suffer certainly from the feature mentioned above that they cannot describe pure supergravity. Still one may wonder whether the theory with extra physical degrees of freedom can be obtained using the minimal sets of auxiliary fields.

The known sets of auxiliary fields can be obtained by coupling the Weyl multiplet to a compensating multiplet, and perform gauge fixings. The old minimal supergravity is obtained by using a chiral compensating multiplet [30, 31], new minimal supergravity is obtained by a real linear compensator [32, 33] and non-minimal supergravity by a complex linear multiplet [32, 33]. The latter has an arbitrary Weyl weight w and for generic Weyl weight the multiplet is irreducible. For Weyl weight $w = 2$ the real part can be separated, and the multiplet can be restricted to a real linear multiplet, and for $w = 0$, a chiral multiplet can be separated (by acting with T on the complex linear multiplet). For these values of the parameter the non-minimal set thus reduces to one of the minimal sets plus a meaningless separate part.

Theories with the same set of multiplets but different gauge fixings are equivalent, i.e. different gauge fixings can always be rephrased as field redefinitions [34]. In [35] it has been checked that the new non-minimal sets are reproduced by a gauge fixing of a reducible set of compensating multiplets, containing a chiral multiplet or a linear multiplet, and another multiplet. Hence, by choosing a gauge fixing in the chiral or linear multiplet, these set of auxiliary fields become reducible, and one can restrict to one of the minimal sets of auxiliary fields. Later the $(16 + 16)$ sets were brought up [36, 37]. However, it was shown again that these are just obtained by a different gauge fixing when using together a chiral and a real linear multiplet [38–40]. These authors reproduced the action and transformation law of the $(16 + 16)$ set by some gauge choice, and showed by another gauge choice that the theory can be reduced to one of the minimal versions coupled to a physical real linear or chiral multiplet. They even exhibited explicitly the redefinitions that are equivalent to these changes of gauge conditions.

This shows that the known $\mathcal{N} = 1$ sets of auxiliary fields can be restricted to the old and new minimal and the non-minimal ones, each described by one irreducible compensating multiplet. In this paper we considered old and new minimal sets. Then the auxiliary fields

are

$$\begin{aligned} \text{old minimal: } & (A_a, u = \kappa \bar{F}^0), \\ \text{new minimal: } & (A_\mu, a_{\mu\nu}), \quad H^\mu \equiv e^{-1} \varepsilon^{\mu\nu\rho\sigma} \left(\kappa^2 \partial_\nu a_{\rho\sigma} - \frac{1}{4} \bar{\psi}_\nu \gamma_\rho \psi_\sigma \right). \end{aligned} \quad (5.1)$$

The actions for pure Poincaré supergravity are

$$\begin{aligned} \kappa^2 e^{-1} \mathcal{L}^{\text{om}} &= \frac{1}{2} R - \frac{1}{2} \bar{\psi}^\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho - \frac{1}{3} u \bar{u} + 3 A^a A_a \\ \kappa^2 e^{-1} \mathcal{L}^{\text{nm}} &= \frac{1}{2} R - \frac{1}{2} \bar{\psi}^\mu \gamma^{\mu\nu\rho} (D_\nu - \frac{3}{2} i \gamma_* A_\nu) \psi_\rho + \frac{3}{4} H_a H^a + \frac{3}{2} \kappa^2 \varepsilon^{\mu\nu\rho\sigma} (2A_\mu - H_\mu) \partial_\nu a_{\rho\sigma}. \end{aligned} \quad (5.2)$$

6 Conclusion

We obtained the conservation laws in old and new minimal supergravity, also in case that there are FI terms. They can be written as equations on conformal superprimary multiplets. They follow from Ward identities that can be derived in general. Such general relations allow us also to obtain the different components of the Einstein supermultiplet. This in turn gives an alternative derivation of the Festuccia–Seiberg relations [20] for supersymmetric backgrounds.

One of the main results of the present paper is the Ward Identity given by (1.16), which shows that (1.1) is perfectly compatible with the old minimal formulation provided one realises that the FI gauging makes the chiral compensator not inert and the mere existence of the FI gauge field introduces eight more (4B+4F) off-shell degrees of freedom. This is the reason why the authors of [11] called it chirally extended Supergravity, although it was originally introduced in the context of the old minimal formulation by Stelle and West [41]. We point out that these results are the local exact nonlinear version of the linearized results of [6] obtained by coupling the rigid S super current multiplet to linearised supergravity. A similar modification is expected to work also for the Kähler invariance. The modification of the super current conservation law occurs since the chiral compensator is not inert under FI or Kähler symmetry. Such modification doesn't occur in new minimal supergravity because the linear multiplet compensator is inert under these gauge symmetries (see (1.25)).

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A Linear multiplet and related spinor chiral superfield

For convenience, we repeat the transformations of the independent fields of the linear multiplet

$$\begin{aligned}\delta L &= \frac{1}{2}i\bar{\epsilon}\gamma_*\chi, \\ \delta\chi &= -\frac{1}{2}\left(\not{\psi} + i\gamma_*\not{D}L\right)\epsilon - 2i\gamma_*\eta L, \\ \delta a_{\mu\nu} &= \frac{1}{4}i\bar{\epsilon}\gamma_{\mu\nu}\gamma_*\chi + \frac{1}{2}\bar{\epsilon}\gamma_{[\mu}\psi_{\nu]}L.\end{aligned}\tag{A.1}$$

where

$$v_a = e^{-1}e_{a\mu}\varepsilon^{\mu\nu\rho\sigma}\left(\partial_\nu a_{\rho\sigma} - \frac{1}{4}\bar{\psi}_\nu\gamma_\rho\psi_\sigma L\right) + \frac{1}{2}\bar{\chi}\gamma_{ab}\psi^b.\tag{A.2}$$

Therefore, (2.7) implies that in any action containing this multiplet

$$\delta(\epsilon)P_L\Theta(\chi) = -\frac{1}{2}iP_L\epsilon\Theta(L) + \frac{1}{4}i\gamma^{\mu\nu}P_L\epsilon\Theta(a)_{\mu\nu}.\tag{A.3}$$

First of all this implies that $P_L\Theta(\chi)$ is a chiral multiplet. It does not transform under S-supersymmetry, and is thus a super-primary field. Since $P_L\chi$ has (Weyl,chiral) weights $(5/2, -3/2)$, the field equation $P_L\Theta(\chi)$ has weights $(3/2, 3/2)$, consistent with the requirements for a chiral multiplet [8, 9]. Furthermore we find

$$\mathcal{D}^\alpha\Theta(\chi)_\alpha = i\Theta(L).\tag{A.4}$$

Therefore it is convenient to define

$$W_\alpha^\chi \equiv i\Theta(\chi)_\alpha,\tag{A.5}$$

such that $D^\alpha W_\alpha^\chi = -\Theta(L)$ is real. The components of this chiral multiplet are

$$W_\alpha^\chi = \left\{ i\Theta(\chi)_\alpha, \frac{1}{\sqrt{2}}(P_L)_{\beta\alpha}\Theta(L) + \frac{1}{2\sqrt{2}}(\gamma^{ab})_{\beta\alpha}\Theta(a)_{ab}, -i(\not{D}\Theta(\chi))_\alpha \right\}.\tag{A.6}$$

B Superconformal Ward identities

To derive the components of the Einstein curvature multiplet, we made use of the Ward identities that are granted by the superconformal symmetries. In our formalism Ward identities are constructed from the action in the following way¹⁰

$$W(T^A) = (\delta(\epsilon^A)\phi^i) \frac{\delta S}{\delta\phi^i} = (\delta(\epsilon^A)\phi^i) \Theta(\phi)_i + \text{curvature terms} = 0.\tag{B.1}$$

¹⁰After partial integrating terms with $\partial_\mu\epsilon^A$, extra curvature terms can appear in (B.1), from (D.12) in [1].

The ϕ^i are the independent fields of the action and T^A the operator of one of the symmetries. The Ward identities transform into Ward identities under the symmetries and are therefore invariant. We will now summarize the Ward identities for the conformal supergravity algebra

$$\begin{aligned} \text{Cov. gct: } W(P)_a &\equiv \mathcal{D}^b \Theta(e)_{ab} + \Theta(A)^b R_{ab}(T) + \frac{3}{4} \bar{\Theta}(\psi)^b R_{ab}(Q) \\ &\quad + [\Theta(X)^I \mathcal{D}_a X_I + \bar{\Theta}(\Omega)^I \mathcal{D}_a \Omega_I + \Theta(F)^I \mathcal{D}_a F_I + \text{h.c.}] \\ &\quad + \Theta(W)_A^b F_{ab}^A + \bar{\Theta}(\lambda)^A \mathcal{D}_a \lambda_a + \Theta(D)^A \mathcal{D}_a D_A \\ &\quad + \Theta(L) \mathcal{D}_a L + \bar{\Theta}(\chi) \mathcal{D}_a \chi + 3\Theta(a)^{bc} \mathcal{D}_{[a} a_{bc]}, \end{aligned} \quad (\text{B.2})$$

$$\text{Lorentz: } W(M)_{ba} \equiv \Theta(e)_{[ba]} + \frac{1}{4} [\bar{\Omega}^I \gamma_{ba} \Theta(\Omega)_I + \text{h.c.}] + \frac{1}{4} \bar{\lambda}^A \gamma_{ba} \Theta(\lambda)_A + \frac{1}{4} \bar{\chi} \gamma_{ba} \Theta(\chi), \quad (\text{B.3})$$

$$\begin{aligned} \text{Dilatations: } W(D) &\equiv \Theta(e)_a^a \\ &\quad + [w_I X^I \Theta(X)_I + (w_I + \frac{1}{2}) \Omega^I \Theta(\Omega)_I + (w_I + 1) \Theta(F)^I F_I + \text{h.c.}] \\ &\quad + \frac{3}{2} \bar{\Theta}(\lambda)^A \lambda_A + 2\Theta(D)^A D_A + 2L\Theta(L) + \frac{5}{2} \bar{\chi} \Theta(\chi), \end{aligned} \quad (\text{B.4})$$

$$\text{spec.conf.: } W(K)_a \equiv \Theta(b)_a, \quad (\text{B.5})$$

$$\begin{aligned} \text{T-symmetry: } W(T) &\equiv -\mathcal{D}^a \Theta(A)_a + i \left[X^I \Theta(X)_I - \frac{1}{2} \bar{\Omega}^I \Theta(\Omega)_I - 2F^I \Theta(F)_I - \text{h.c.} \right] \\ &\quad + \frac{3}{2} i \bar{\Theta}(\lambda)^A \gamma_* \lambda_A - \frac{3}{2} i \bar{\chi} \gamma_* \Theta(\chi), \end{aligned} \quad (\text{B.6})$$

$$\text{Q-susy: } W(Q) \equiv -\mathcal{D}^a \Theta(\psi)_a \quad (\text{B.7})$$

$$\begin{aligned} &\quad + \frac{1}{\sqrt{2}} [\Omega^I \Theta(X)_I + (-\not{D} X^I + F^I) \Theta(\Omega)_I + \not{D} \Omega^I \Theta(F)_I + \text{h.c.}] \\ &\quad - \frac{1}{2} \gamma_a \lambda^A \Theta(W)_A^a + \frac{1}{2} (-\frac{1}{2} \gamma \cdot F^A + i \gamma_* D^A) \Theta(\lambda)_A \\ &\quad + \frac{1}{2} i \gamma_* \not{D} \lambda^A \Theta(D)_A + \frac{1}{2} i \gamma_* \chi \Theta(L) + \frac{1}{2} (\psi - i \gamma_* \not{D} L) \Theta(\chi) \\ &\quad + \frac{1}{2} (\frac{1}{2} i \gamma_{\mu\nu} \gamma_* \chi + \gamma_{[\mu} \psi_{\nu]} L) \Theta(a)^{\mu\nu}, \end{aligned} \quad (\text{B.8})$$

$$\text{S-susy: } W(S) \equiv \gamma_a \Theta(\psi)^a + \sqrt{2} [X^I \Theta(\Omega)_I + \text{h.c.}] - 2i \gamma_* \Theta(\chi) L. \quad (\text{B.9})$$

C Current conservation with gauge symmetry: electro-dynamics analogue

In this section, we understand the presence of the W_α contribution in (1.22) in the context of electro-dynamics. After all this contribution is not so surprising because in electro-dynamics the conservation of the gauge-invariant matter stress tensor gets a contribution

$$\partial^\mu T_{\mu\nu}^{\text{M}} \approx -F_{\nu\rho} J^{\rho\text{M}}, \quad (\text{C.1})$$

which just cancels the $\partial^\mu T_{\mu\nu}^{\text{e.m.}}$, which satisfies

$$\partial^\mu T_{\mu\nu}^{\text{e.m.}} = 2F_{\nu\rho} \partial^\lambda F_\lambda{}^\rho. \quad (\text{C.2})$$

This corresponds to higher θ terms in (1.22). The term $N^{\text{M}} W_\alpha \propto \Phi^{\text{M}} W_\alpha$ contains at the θ^3 level the right-hand side of (C.1) since $F_{\nu\rho}$ is at the θ level of W_α and at the θ^2 level of Φ^{M}

is the component

$$B_\mu(\Phi) = i \left(\frac{\partial\Phi}{\partial S^i} \partial_\mu S^i - \frac{\partial\Phi}{\partial \bar{S}^i} \partial_\mu \bar{S}^i \right) = J_\mu^M. \quad (\text{C.3})$$

D Coupling chiral supergravity to a charged chiral multiplet

In this Appendix we give the Lagrangian when the supergravity with FI term is coupled to a charged chiral multiplet. The charge under the U(1) of the gauge multiplet is fixed by the requirement that a superpotential term be invariant. This analysis extends to an arbitrary number of charged fields [16] but here we will confine the example to a single chiral multiplet S with a superpotential

$$(X^0)^3 W(S) = \ell (X^0)^3 S, \quad (\text{D.1})$$

linear in S , so that if $S^2 = 0$ is imposed we have the Volkov–Akulov (VA) theory. The gauge-invariant supergravity action that includes (D.1) is

$$\mathcal{L} = [-3X^0 \bar{X}^0 e^{\xi V}]_D + [X^0 \bar{X}^0 S \bar{S} e^{-2\xi V}]_D + \ell [(X^0)^3 S]_F + [W^\alpha W_\alpha]_F, \quad S^2 = 0, \quad (\text{D.2})$$

and the (FI) gauge transformations (1.11) contain then also a transformation for S :

$$X^0 \rightarrow X^0 e^{-\xi\Lambda}, \quad S \rightarrow S e^{3\xi\Lambda}, \quad V \rightarrow V + \Lambda + \bar{\Lambda}. \quad (\text{D.3})$$

In WZ gauge, this leads to transformations as in (2.23). Furthermore there are the conformal U(1) symmetries, e.g. $\delta X^0 = i\lambda_T X^0$. After the conformal gauge fixing that makes X real, these transformations mix such that X^0 does not transform, i.e. $\lambda_T = -\frac{1}{2}\xi\theta$. The fields that are charged under these symmetries, denoting the components of the multiplet S as $\{S, P_L\chi, f\}$, are

field	$i\lambda_T$	$i\xi\theta$	resulting $i\xi\theta$
$P_L\psi_\mu$	$\frac{3}{2}$	0	$-\frac{3}{4}$
$P_L\lambda$	$\frac{3}{2}$	0	$-\frac{3}{4}$
X^0	1	$\frac{1}{2}$	0
$P_L\Omega^0$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$
F^0	-2	$\frac{1}{2}$	$\frac{3}{2}$
S	0	$-\frac{3}{2}$	$-\frac{3}{2}$
$P_L\chi$	$-\frac{3}{2}$	$-\frac{3}{2}$	$-\frac{3}{4}$
f	-3	$-\frac{3}{2}$	0
D	0	0	0

(D.4)

and opposite for the complex or charged conjugates.

The spectrum of this theory contains gravity coupled to Maxwell in de Sitter space with a charged scalar, a spin 1/2 field and a spin 3/2 massive gravitino. The scalar disappears if

S is nilpotent. This can be obtained by adding a chiral term $[\sigma S^2(X^0)^3]_F$, with σ a chiral multiplet with zero Weyl weight and transforming under (D.3) as $\sigma \rightarrow \sigma \exp(-3\xi\Lambda)$. Thus the FI model can be written in a Volkov–Akulov form, as suggested in a final comment in [11]. Deleting ξ we have the Volkov–Akulov model coupled to supergravity [42, 43].

The Goldstino is then a linear combination of the gaugino λ and the VA fermion χ , since both F and D have a non-zero value. The cosmological constant is

$$\kappa^4 V = \frac{9}{8}\xi^2 + |\ell|^2, \quad (\text{D.5})$$

with an additional charged scalar contribution if S is not nilpotent. By deleting S we have the Freedman model, while by deleting V (or setting $\xi = 0$) we have pure supergravity [24] or Maxwell-Einstein supergravity with auxiliary fields [16].

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