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# $\mathcal{N} = 2^*$ (non-)Abelian theory in the $\Omega$ background from string theory

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We present a D-brane realisation of the Abelian and non-Abelian $N = 2^*$ theory both in five and four dimensions. We compute topological amplitudes in string theory for $\Omega$ deformed spacetime first with one and then with two parameters. In the field theory limit we recover the perturbative partition function of the deformed $N = 2^*$ theory in agreement with the existing literature.

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PoS(EPS-HEP2017)546

## 1. Introduction

$\mathcal{N}=2$ supersymmetric theories' dynamics is determined by a single holomorphic function called prepotential, which only receives one-loop and instantonic corrections. The explicit computation of the non-perturbative contributions via localisation techniques in the deformed spacetime, called $\Omega$ background allowed the indirect reconstruction of the $k$-instanton contribution to the prepotential [1]. In [2] it was shown that the perturbative free energy of the $\mathcal{N}=2$ gauge theory on the $\Omega$ background with a single parameter turned on corresponds to the all-genus partition function of the topological string on a conical singularity [3]. In turn, it is connected to higher-derivative F-terms, $F_g W^{2g}$ in the low-energy action, involving powers of the chiral Weyl superfield W, whose lowest component is the graviphoton field strength. These terms are computed in string theory by the topological amplitudes [4]. A special case on the field theory side is the case of Abelian gauge group, which is actually non-trivial even at the perturbative level in the presence of a massive neutral hypermultiplet called $\mathcal{N}=2^*$ [2, 5].

We present a string realisation of the Abelian and non-Abelian 5d and 4d $\mathcal{N}=2^*$ theory and compute the couplings $F_{g,n}$ of a double series of higher-dimensional F-terms. This amounts to compute a series of amplitudes involving four gravitini, $2g-2$ anti-self-dual graviphotons and $2n$ self-dual gauge fields belonging to the multiplet of the D5-brane coupling modulus. In the field theory limit, the result reproduces the Nekrasov partition function in the two-parameter $\Omega$ background, in agreement with the proposal [6]. This paper is based on [7].

## 2. $\mathcal{N}=2^*$ gauge theory construction

$\mathcal{N}=2$ gauge theory becomes $\mathcal{N}=2^*$ when a massive hypermultiplet of mass $m$ in the adjoint representation of the gauge group is present. When the mass is zero $\mathcal{N}=2$ vector and $\mathcal{N}=2$ hypermultiplet combine forming $\mathcal{N}=4$ vector multiplet. Then non-trivial cancellations occur and the theory becomes conformal at the quantum level. For generic values of mass parameter, the adjoint hypermultiplet participates non-trivially to the dynamics of gauge theory. An easy way to realise $\mathcal{N}=2^*$ in string theory is in terms of D-branes. To give mass to the hypermultiplet we use the Scherk-Schwarz deformation as in field theory [8], more precisely we use the connection between Scherk-Schwarz reductions and freely acting orbifolds [9, 10, 11], where the analogue of the non-trivial boundary condition of field theory is a simultaneous action of the rotation on the $T^4$ coordinates and a shift along the $S^1(x_4)$ circle.

For a gauge group $U(1)$ we take one $D5$ brane and put it on the $\mathbb{C}^2$ origin of the $\mathcal{M}_{1,3}\times S^1_m\times S^1_R\times\mathbb{C}^2/\mathbb{Z}_N$ space, where $S^1_m$ is the Scherk-Schwarz circle with radius $R_{\rm SS}=m^{-1}$, while $S^1_R$ is a spectator circle. We replaced compact $T^4$ with non-compact $\mathbb{C}^2$, because a $T^4$ with the presence of supersymmetry allowes symmetries that are the discrete rotations $\mathbb{Z}_N$, with $N=2,3,4,6$. With this replacement we can take the $N$ of $\mathbb{Z}_N$ arbitrarily large, and thus decouple the Kaluza-Klein scale and the Scherk-Schwarz scale in a consistent way. This choice also leads automatically to the decoupling of the gravitational sector, since the four-dimensional Newton constant vanishes and yields the desired gauge theory in five and four dimensions. The non-compactness of the space $\mathbb{C}^2/\mathbb{Z}_N$ makes the minimal D-brane construction consistent, and we should not care about tadpole conditions for the (untwisted) Ramond-Ramond forms [12, 13].



PoS(EPS-HEP2017)546

Twisted tadpole conditions, which should be imposed independently of the compactness or not of the transverse space, are also absent in this construction, because of the free action of the Scherk-Schwarz deformation, only massive states with odd winding number do propagate in the transverse channel. The partition function associated to a single D5 brane then reads

$$\mathscr{A} = \frac{1}{N}\left[\sum_{\ell=0}^{N-1} \rho\left[{0 \atop \ell}\right] \sum_{r\in\mathbb{Z}} e^{2i\pi r\ell/N} P_r(1/m)\right] \sum_{s\in\mathbb{Z}} P_s(R)\,, \tag{2.1}$$

where, $P_n(\rho) = q^{\frac{1}{2}(n/\rho)^2} = e^{-\pi t (n/\rho)^2}$ is the generic contribution of the quantised momenta. $\rho\left[{0 \atop \ell}\right]$ encodes the contribution of the world-sheet bosons and fermions, where

$$\rho\left[{0 \atop 0}\right] = \tfrac{1}{2}\sum_{a,b=0,1} (-1)^{a+b+ab}\,\frac{\theta^4\left[{a/2 \atop b/2}\right]}{\eta^{12}}\,, \tag{2.2}$$
$$\rho\left[{0 \atop \ell}\right] = \tfrac{1}{2}\sum_{a,b=0,1} (-1)^{a+b+ab}\,\frac{\theta^2\left[{a/2 \atop b/2}\right]}{\eta^{6}}\left(2\sin(\pi\ell/N)\,\frac{\theta\left[{a/2 \atop b/2+\ell/N}\right]}{\theta\left[{1/2 \atop 1/2+\ell/N}\right]}\right)\left(2\sin(-\pi\ell/N)\,\frac{\theta\left[{a/2 \atop b/2-\ell/N}\right]}{\theta\left[{1/2 \atop 1/2-\ell/N}\right]}\right),$$

for $\ell \neq 0$. We can separate $\rho\left[{0 \atop \ell}\right]$ into space-time bosons and space-time fermions $\rho\left[{0 \atop \ell}\right] = \beta\left[{0 \atop \ell}\right] - \varphi\left[{0 \atop \ell}\right]$, where $\beta\left[{0 \atop \ell}\right]$ with $a=0$ gives space-time bosons, the $\varphi\left[{0 \atop \ell}\right]$ with $a=1$ gives space-time fermions. For bosons we can write (fermions follow by supersymmetry)

$$\beta\left[{0 \atop 0}\right] = V_4\,\frac{O_4}{\eta^{10}} + O_4\,\frac{V_4}{\eta^{10}} \quad \text{and} \quad \beta\left[{0 \atop \ell}\right] = V_4\,\frac{O_4(\ell/N)}{\eta^4} + O_4\,\frac{V_4(\ell/N)}{\eta^4}\,, \tag{2.3}$$

where $O_4$ and $V_4$ are the SO(4) level-one characters with non-trivial argument. Combining their $q$-Taylor expansion with their space-time counterpart and taking into account that fermions have similar expansions, one arrives at the following expansion for the light states of the annulus partition function

$$\mathscr{A}_0 \simeq (V_4 - 2S_4)\left[\frac{1}{N}\sum_{\ell=0}^{N-1} e^{2i\pi r\ell/N} q^{\frac{1}{4}(rm)^2}\right] + (4\,O_4 - 2C_4)\left[\frac{1}{N}\sum_{\ell=0}^{N-1} \cos(2\pi\ell/N)\, e^{2i\pi r\ell/N}\, q^{\frac{1}{4}(rm)^2}\right], \tag{2.4}$$

from which one finds that the vector multiplet has the following KK masses along the shifted direction

$$M_V^2 = \tfrac{1}{2}(Nkm)^2\,, \qquad k = 0, \pm 1, \pm 2, \ldots \tag{2.5}$$

whereas there are two hypermultiplets with masses

$$M_{H_1}^2 = \tfrac{1}{2}(1+kN)^2 m^2\,, \quad k = 0,1,2,\ldots\,, \qquad \text{and} \qquad M_{H_2}^2 = \tfrac{1}{2}(1-kN)^2 m^2\,, \quad k = 1,2,\ldots\,. \tag{2.6}$$

In the field theory limit $N \to \infty$ the second set of hypermultiplets decouples, and one is left with an Abelian massless vector multiplet and a neutral hypermultiplet with mass $\frac{1}{2}m^2$.

## 3. Topological amplitude for two parameters deformation

In the previous section we constructed $\mathcal{N}=2^*$ Abelian theory and derived its partition function in flat Minkowski space. To realise Nekrasov $\Omega$ background with two deformation parameters in string theory one should compute a topological amplitude involving two gravitons, anti-self-dual graviphotons and self-dual backgrounds for the partners of gauge coupling on the D5 branes [6].

$$\mathcal{A}_{g,n}=\left\langle (V^{+}_{\rm grav})^2\,(V^{-}_{\rm grav})^2\,V^{2g-2}_{\rm gph}V^{2n}_{S'_+}\right\rangle . \tag{3.1}$$

The latter insertions in the amplitude are absent for one parameter deformation. This amplitude we should compute in the vacuum described in the previous section to realise $\Omega$ deformed Abelian $N=2^*$. Since we are interested in the one-loop amplitude with open-string fields running in the loop, it suffices to restrict our attention to the Riemann surface with the topology of an annulus. Choosing suitable kinematics for vertex operators, after tedious computations following [6] one arrives at the following result [7]

$$\begin{aligned}\mathcal{A}_{g,n}\left[{}^{0}_{\ell}\right] = &-4\sin^2\left(\tfrac{\pi\ell}{N}\right)\int_0^\infty \tfrac{dt}{t}\sum_{r,s\in\mathbb{Z}}\left(\cos^2(\pi\hat\epsilon_+)-\cot\left(\tfrac{\pi\ell}{N}\right)\sin^2(\pi\hat\epsilon_+)\right)P_r(1/m)\,P_s(R)\,\mathbb{Z}_{K3}\left[{}^{0}_{\ell}\right]\\ &\times\frac{\pi^2(\epsilon_--\epsilon_+)(\epsilon_-+\epsilon_+)}{\sin\pi(\hat\epsilon_--\hat\epsilon_+)\,\sin\pi(\hat\epsilon_-+\hat\epsilon_+)}\left[H_1\left(\tfrac{\hat\epsilon_-}{2};0;\tfrac{t}{2}\right)\right]^2\frac{H_1\left(\frac{\hat\epsilon_+}{2};\frac{\ell}{N};\frac{t}{2}\right)H_1\left(\frac{\hat\epsilon_+}{2};-\frac{\ell}{N};\frac{t}{2}\right)}{H_1\left(\frac{\hat\epsilon_--\hat\epsilon_+}{2};0;\frac{t}{2}\right)H_1\left(\frac{\hat\epsilon_-+\hat\epsilon_+}{2};0;\frac{t}{2}\right)},\end{aligned} \tag{3.2}$$

where $\mathbb{Z}_{K3}\left[{}^{0}_{\ell}\right]$ is the standard contribution of the (non-compact) K3 bosons in the $\ell$ projected sector, $\hat\epsilon_\pm=\epsilon_\pm t(rm-is/R)$. In the field theory limit both the string oscillators and the Kaluza-Klein modes along the Scherk-Schwarz direction should decouple. This is achieved by taking $t\to\infty$ in the theta and $H_1$ function and $N\to\infty$ in the momentum sum. One thus finds

$$\begin{aligned}\mathcal{F}(\epsilon_+,\epsilon_-) = &\lim_{t,N\to\infty}\frac{1}{N}\sum_{\ell=0}^{N-1}\mathcal{A}_{g,n}\left[{}^{0}_{\ell}\right]\\ = &\tfrac{1}{4}\pi^2(\epsilon_--\epsilon_+)(\epsilon_-+\epsilon_+)\left[-2F^V(\epsilon_+,\epsilon_-;0)+F^H(\epsilon_+,\epsilon_-;m)+F^H(\epsilon_+,\epsilon_-;-m)\right],\end{aligned} \tag{3.3}$$

with

$$F^V(\epsilon_+,\epsilon_-;0)=\int_0^\infty\frac{dt}{t}\sum_{s\in\mathbb{Z}}\frac{1}{\sin(\pi(\hat\epsilon_--\hat\epsilon_+))}\,\frac{1}{\sin(\pi(\hat\epsilon_-+\hat\epsilon_+))}\cos(2\pi\hat\epsilon_+)\,e^{-\pi t(s/R)^2}, \tag{3.4}$$

$$F^H(\epsilon_+,\epsilon_-;\pm m)=\int_0^\infty\frac{dt}{t}\sum_{s\in\mathbb{Z}}\frac{1}{\sin(\pi(\hat\epsilon_--\hat\epsilon_+))}\,\frac{1}{\sin(\pi(\hat\epsilon_-+\hat\epsilon_+))}\,e^{-\pi t(m^2+(s/R)^2)}. \tag{3.5}$$

where $\epsilon_\pm=\epsilon_1\pm\epsilon_2$ and $\epsilon_+=0$ corresponds to one parameter deformation. These integrals are divergent in the UV ($t\to 0$) and hence need a proper regularisation. After performing change of variable, summing over Kaluza-Klein momenta $s$, one can deform the integration domain into the Hankel contour and get [7]

$$\begin{aligned}F^H_{g_1,g_2}(m)= &\sum_{g_1,g_2=0}^{\infty}4\,\frac{B_{g_1}B_{g_2}}{g_1!\,g_2!}(g_1+g_2-3)!\,(4R)^{g_1-1}\,(4R)^{g_2-1}\epsilon_1^{g_1-1}\epsilon_2^{g_2-1}\\ &\times\left[\zeta(g_1+g_2-2;-2\epsilon_+R-imR)+(-1)^{g_1+g_2}\zeta(g_1+g_2-2;1+2\epsilon_+R+imR)\right],\end{aligned} \tag{3.6}$$

for the massive hypermultiplet, and

$$F^V_{g_1,g_2}(0)=\sum_{g_1,g_2=0}^{\infty}\lim_{\mu\to 0}\; 2\,\frac{B_{g_1}B_{g_2}}{g_1!\,g_2!}(g_1+g_2-3)!\left(1+(-1)^{g_1+g_2}\right)(4R)^{g_1-1}(4R)^{g_2-1}\epsilon_1^{g_1-1}\epsilon_2^{g_2-1} \tag{3.7}$$

$$\times\left[\zeta(g_1+g_2-2;-i\mu R)+\zeta(g_1+g_2-2;1+i\mu R)\right], \tag{3.8}$$

where we introduced a mass parameter $\mu$ for the vector multiplet, and tend it to zero at the end of computations. By simply putting $\epsilon_+=0$ one recovers the one parameter deformation [7]. After taking care about a non-regular terms $g_1+g_2\le 0$, one can compare with the existing literature [1, 2] requiring a change $2\pi R\to R$ and $4i\epsilon\to\epsilon$. One finds

$$\left.\frac{\mathcal{F}(\epsilon_+,\epsilon_-)}{-\pi^2\epsilon_1\epsilon_2}\right|_{\substack{2\pi R\to R\\ 4i\epsilon_i\to\epsilon_i}} = -\tfrac{1}{3}\left(\frac{\epsilon_1}{\epsilon_2}+3+\frac{\epsilon_2}{\epsilon_1}\right)\left[\log\left(2\sinh\left(\tfrac{\epsilon_+R}{4}-\tfrac{mR}{2}\right)\right)+\log\left(2\sinh\left(\tfrac{\epsilon_+R}{4}+\tfrac{mR}{2}\right)\right)-2\log(\Lambda R)\right]$$
$$-4\sum_{\substack{g_1,g_2=0\\ g_1+g_2>2}}^{\infty}\frac{B_{g_1}}{g_1!}\frac{B_{g_2}}{g_2!}(R\epsilon_1)^{g_1-1}(R\epsilon_2)^{g_2-1}\Big[(1+(-1)^{g_1+g_2})\zeta(3-g_1-g_2)$$
$$-\mathrm{Li}_{3-g_1-g_2}\left(e^{-mR+\frac{1}{2}R\epsilon_+}\right)-\mathrm{Li}_{3-g_1-g_2}\left(e^{mR+\frac{1}{2}R\epsilon_+}\right)\Big]. \tag{3.9}$$

The four dimensional limit corresponds to $R\to 0$. One can also write down a genus expansion of this perturbative free energy, see [7] for details. The four dimensional limit is obtained straightforwardly taking $R\to 0$.

### 3.1 The non-Abelian extension

The results of the Abelian $\mathcal{N}=2^*$ can be straightforwardly generalised to the non-Abelian $\mathrm{U}(M)$ $\mathcal{N}=2^*$ gauge theory. Instead of one D-brane we should simply consider $M$ D-branes on the $\mathbb{C}^2$ origin . Since the freely-acting orbifold does not act on the Chan-Paton factors, the annulus amplitude for the non-Abelian case is obtained from eq. (2.1) by simply multiplying it by $M^2$. Hence, the free energies with one or two parameter deformations of the previous section are also multiplied by $M^2$. One can also find the Coulomb branch by introducing suitable Wilson lines along the compact directions either along the Scherk-Schwarz circle $S^1_m$, or along the spectator circle $S^1_R$, or both.

## 4. Conclusions

We presented a realisation of an Abelian and non-Abelian $\mathcal{N}=2^*$ theory and its radius deformation in five dimensions in a D-brane set-up, based on [7]. For $\Omega$ deformed $\mathcal{N}=2^*$ theory we briefly described the computation of the topological amplitudes involving two gravitons (four gravitini), $2g-2$ anti-self-dual graviphotons and $n$ self-dual gauge fields belonging to the multiplet of the D5-brane coupling modulus. We took an appropriate field theory limit and showed that the results reproduce the perturbative part of the Nekrasov partition function in the $\Omega$ background.

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## References

[1] N. A. Nekrasov, Adv. Theor. Math. Phys. **7** (2004) 831 [hep-th/0206161].

[2] N. Nekrasov and A. Okounkov, Prog. Math. **244** (2006) 525 [hep-th/0306238].

[3] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, Commun. Math. Phys. **165** (1994) 311 [hep-th/9309140].

[4] I. Antoniadis, E. Gava, K. S. Narain and T. R. Taylor, Nucl. Phys. B **455** (1995) no.1-2, 109 [hep-th/9507115].

[5] R. Poghossian and M. Samsonyan, J. Phys. A **42** (2009) 304024 [arXiv:0804.3564 [hep-th]].

[6] I. Antoniadis, I. Florakis, S. Hohenegger, K. S. Narain and A. Zein Assi, Nucl. Phys. B **875** (2013) 101 [arXiv:1302.6993 [hep-th]].

[7] C. Angelantonj, I. Antoniadis and M. Samsonyan, Nucl. Phys. B **923** (2017) 32 [arXiv:1702.04998 [hep-th]].

[8] J. Scherk and J. H. Schwarz, Nucl. Phys. B **153** (1979) 61.

[9] R. Rohm, Nucl. Phys. B **237** (1984) 553.

[10] C. Kounnas and M. Porrati, Nucl. Phys. B **310** (1988) 355.

[11] C. Kounnas and B. Rostand, Nucl. Phys. B **341** (1990) 641.

[12] E. Dudas, Class. Quant. Grav. **17** (2000) R41 [hep-ph/0006190].

[13] C. Angelantonj and A. Sagnotti, Phys. Rept. **371** (2002) 1 Erratum: [Phys. Rept. **376** (2003) no.6, 407] [hep-th/0204089].