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All rational one-loop Einstein-Yang-Mills amplitudes at four points

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Abstract

All four-point mixed gluon-graviton amplitudes in pure Einstein-Yang-Mills theory with at most one state of negative helicity are computed at one-loop order and maximal powers of the gauge coupling, using D-dimensional generalized unitarity. The resulting purely rational expressions take very compact forms. We comment on the color-kinematics duality and a relation to collinear limits of pure gluon amplitudes.

Contents

1 Introduction and conclusions

It is a classic result in the field of scattering amplitudes that supersymmetric Ward identities force gluon and graviton tree-level amplitudes to vanish if all particles carry the same helicities or at most one state of opposite helicity [1],

$$A_n(\pm,+,+,\ldots,+) = M_n(\pm,+,+,\ldots,+) = 0\,. \tag{1.1}$$

While this result holds at tree level in *any* quantum field theory, in the presence of supersymmetry the vanishing persists to all loops. In non-supersymmetric field theories, in particular in the “pure” Yang-Mills and gravity theories, the above amplitudes are very interesting as they receive their leading contributions at one loop and are remarkably simple – resembling tree-level expressions, although with more subtle factorization properties [2]. Their unitarity cuts vanish in four dimensions since the helicity configuration of any two-particle cut of the one-loop expressions in (1.1) implies that there is at least one vanishing tree-level piece. Hence, these one-loop amplitudes are finite rational functions of the momentum invariants.

In the case of pure Yang-Mills theory they were efficiently constructed through their analytic properties and even the all-multiplicity expression has been established in the all-plus case [3], resulting in a remarkably compact formula

$$A_n^{\text{1-loop}}(1^+,\ldots,n^+) = \frac{iN_p}{96\pi^2}\sum_{1\le k_1<k_2<k_3<k_4\le n}\frac{\langle k_1k_2\rangle[k_2k_3]\langle k_3k_4\rangle[k_4k_1]}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}\,, \tag{1.2}$$

using spinor helicity variables[1][2]. These one-loop amplitudes are also generated by the self-dual Yang-Mills theory and represent their only non-vanishing amplitudes [5]. The single-minus gluon amplitudes at one loop are also known for all multiplicities and have been constructed using Berends-Giele type [6], as well as BCFW-type recursion relations [2]. Their form is considerably more involved.

All-plus and single-minus helicity amplitudes have also been constructed in pure gravity. A conjecture for the all-plus graviton amplitude at any multiplicity exists [7] and agrees with explicit constructions at $n\le 7$ points. Again, this infinite series of graviton amplitudes is identical to one-loop self-dual gravity. For the single-minus amplitudes, an explicit, yet not very compact expression has been recently derived [8] using a spin-off of the BCFW method known as augmented recursion [9], following earlier work in [10]. As is often the case, the analytic structure, in particular consistency of soft and collinear limits, helped to constrain the ansatz.

In this work we focus on explicit S-matrix elements for mixed graviton and gluon scattering in Einstein gravity minimally coupled to Yang-Mills theory, or EYM for short. In the 1990s EYM amplitudes in four dimensions for the maximally-helicity violating (MHV) case, i.e. two negative-helicity states, were given at tree level in [11]. Only rather recently modern approaches

[1] N_p is the color weighted number of bosonic minus fermionic states circling in the loop.
[2] See [4] for comprehensive reviews.

to scattering amplitudes based on the scattering equation formalism of CHY [12], or the color-kinematic duality relations [13], were applied to the realm of EYM amplitudes, leading to a number of explicit results. Double-copy constructions for gluon-graviton scattering in supergravity theories were given in [14]. However, the most efficient way of establishing EYM amplitudes is by expanding them in a basis of pure gluon amplitudes multiplied by kinematic numerators to be determined (also featuring in color-kinematic duality):

$$A_{\text{EYM}}^{\text{tree}}(1,2,\ldots,n;h_1,\ldots,h_m) = \sum_{\beta\in\text{Perm}(2,\ldots,n-1;h_1,\ldots,h_m)} n(1,\{\beta\},n)\, A_{\text{YM}}^{\text{tree}}(1,\{\beta\},n)\,. \tag{1.3}$$

This form was initially presented by a string-based construction for one graviton and n-gluon scattering in [15], the field theory proof followed shortly thereafter [16, 17] and was further lifted to the sector of three gravitons in [16] employing the CHY formalism. A color-kinematic duality based construction extended this to amplitudes involving up to five gravitons [18]. The complete recursive solution for the numerators $n(1,\{\beta\},n)$ has recently been constructed in the single-trace sector in [19] and for multi-traces in [20]. This, together with the existing result for all tree-level color-ordered gluon amplitudes [21], constitutes the complete solution for the EYM S-matrix at tree level.

This state of affairs sets the stage for the investigation of the present paper. Here we compute the remaining rational amplitudes of the EYM theory at the leading one-loop level at multiplicity four. These are the three all-plus helicity amplitudes involving one, two or three gravitons, as well as the six single-minus amplitudes involving one, two or three gravitons. An elegant way to determine such amplitudes consists in employing two-particle unitarity cuts in $D = 4-2\epsilon$ dimensions [22] (see also [23] for the first uses of D-dimensional generalized unitarity). The main idea is that a rational term in four dimensions, $\mathcal{R}$, will in D dimensions acquire a discontinuity, but to a higher order in the dimensional regularization parameter ϵ. Schematically,

$$\mathcal{R} \;\to\; \mathcal{R}(-s)^{-\epsilon} \;=\; \mathcal{R}\left[1-\epsilon\log(-s)\right] + \cdots\;. \tag{1.4}$$

Technically, the calculation is greatly simplified by using the general supersymmetric Ward identity of (1.1) at the one-loop order, which implies that the contribution of an arbitrary state in the loop is proportional to that from a scalar circulating in the loop,

$$A_{n+m}^{\text{any state in loop}}(1,2,\ldots,n;h_1,\ldots,h_m) = N_p\, A_{n+m}^{\text{scalar in loop}}(1,2,\ldots,n;h_1,\ldots,h_m)\,. \tag{1.5}$$

It is important to realize that "any state in loop" refers to a "pure" contribution of a definite quantum field excitation (e.g. graviton or gluon) propagating in the loop. This relation may therefore be straightforwardly applied to the EYM situation of a gluon circulating inside the loop of a mixed gluon-graviton amplitude, see Figure 1 for a four-point example: A one-loop single-graviton three-gluon amplitude will have one-loop contributions of order κg^3 and $\kappa^3 g$. A generic one-loop m-graviton and n-gluon amplitude will have g-leading contributions of order $g^n\kappa^m$ representing only gluons in the loop, whereas the g-subleading contributions $g^{n-2k}\kappa^{m+2k}$ reflect contributions where $2k$ gluon propagators are turned into graviton propagators. Note that there is no single-gluon l-graviton vertex.

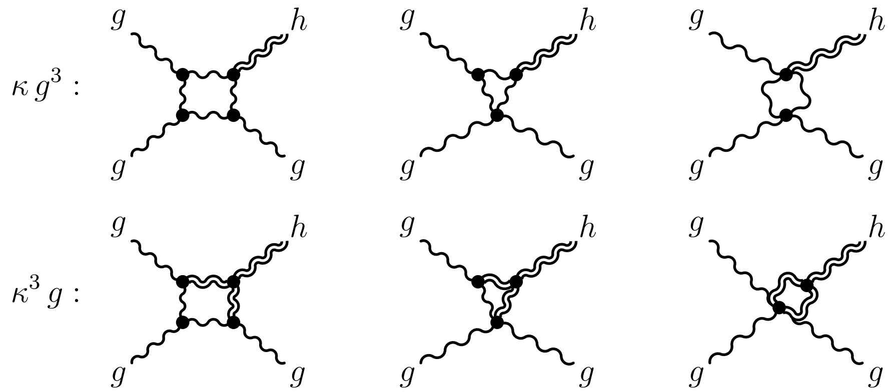


Figure 1: *Contributions to the one-loop EYM amplitude $A_{3+1}(1,2,3;h)$ at different orders in κ and g. Only the top line of the $\mathcal{O}(\kappa\, g^3)$ contribution of the amplitude is constructible by replacing the gluon by a (massive) scalar from (1.5) in the rational case.*

For the contributions to the amplitude maximizing the powers of the gauge coupling constant, i.e. the contributions to $A_{n+m}(1,2,\ldots,n;h_1,\ldots,h_m)$ at order $g^n\kappa^m$, we only have gluons running in the loop, and the relation (1.5) applies with $N_p=1$, i.e. this contribution may be computed upon replacing the gluon inside the loop by a scalar. The cuts are performed in D dimensions, where a generic loop momentum L satisfies $L^2=0=l_{(-2\epsilon)}^2-l_{(4)}^2=0$, where $l_{(-2\epsilon)}$ and $l_{(4)}$ represent the (-2ϵ)- and four-dimensional part of L. Because the external kinematics is four-dimensional, at one loop there is just one $l_{(-2\epsilon)}$. Setting $l_{(-2\epsilon)}^2:=\mu^2$, one then has $l_{(4)}^2=\mu^2$, i.e. all internal D-dimensional scalar can effectively be treated as four-dimensional massive scalar with uniform mass μ^2, over which one integrates at the end [22].

The "non-pure" contributions of order $g^{n-2k}\kappa^{m+2k}$, however, have a mixture of gluons and gravitons running inside the loop. Here the situation is less clear, as (1.5) does not hold. A simple dimensional analysis also reveals that the mixed graviton-gluon contributions in the loop are not represented by (1.5).

Hence in this work we only aim at finding the maximal g contributions to the one-loop rational amplitudes in EYM theory. Here we find intriguingly simple results, to wit[3]

$$\begin{aligned}
A^{(1)}(1^+,2^+,3^+;4^{++})\Big|_{\kappa g^3} &= 0\,,\\
A^{(1)}(1^+,2^+,3^+;4^{--})\Big|_{\kappa g^3} &= -\frac{i}{(4\pi)^2}\,\frac{[12][34]}{\langle 12\rangle\langle 34\rangle}\,(\langle 42\rangle[23]\langle 34\rangle)^3\,\frac{s^2+t^2+u^2}{6\,s^2\,t^2\,u^2}\,,\\
A^{(1)}(1^-,2^+,3^+;4^{++})\Big|_{\kappa g^3} &= \frac{i}{(4\pi)^2}\,\frac{[24][34]}{\langle 24\rangle\langle 34\rangle}\,\frac{1}{\langle 23\rangle[21][31]}\,\frac{1}{6}\,(s^2+u^2)\,,\\
A^{(1)}(1^+,2^+;3^{++},4^{++})\Big|_{\kappa^2 g^2} &= \frac{i}{(4\pi)^2}\,\frac{[12]}{\langle 12\rangle}\,\frac{[34]^2}{\langle 34\rangle^2}\,\frac{s}{6}\,,
\end{aligned}\tag{1.6}$$

[3]In our conventions we have $s=\langle 12\rangle[21]$, $t=\langle 23\rangle[32]$ and $u=\langle 13\rangle[31]$.

$$
\begin{aligned}
A^{(1)}(1^-,2^+;3^{++},4^{++})\Big|_{\kappa^2 g^2} &= \frac{i}{(4\pi)^2}\,\frac{[24]^2[34]^2\langle 14\rangle^2}{\langle 34\rangle^2}\,\frac{s}{6\,t\,u}\,,\\
A^{(1)}(1^+,2^+;3^{++},4^{--})\Big|_{\kappa^2 g^2} &= \frac{i}{(4\pi)^2}\,\frac{[1\,2][1\,3]^4\langle 1\,4\rangle^4}{\langle 1\,2\rangle}\,\frac{t^2+u^2}{6\,s\,t^2\,u^2}\,,\\
A^{(1)}(1^\pm;2^{++},3^{++},4^{++})\Big|_{\kappa^3 g} &= 0\,,\\
A^{(1)}(1^+;2^{++},3^{++},4^{--})\Big|_{\kappa^3 g} &= 0\,.
\end{aligned}
$$

It would be interesting to also construct the missing "non-pure" pieces at higher orders in κ as well, even though they will be numerically subleading at energies well below the Planck mass. This should be possible using the double-copy techniques initiated in [18].

The rest of our paper is organized as follows. In the next section we collect all relevant tree-level amplitudes involving gluons, gravitons and massive scalars entering the cuts needed to compute the rational amplitudes we are interested in. Sections 3–5 are devoted to the calculation of all one-loop amplitudes with one graviton and three gluons. A particularly interesting case is that of Section 3, where we find that the all-plus amplitude $\langle 1^+2^+3^+4^{++}\rangle$, although non-vanishing in D dimensions, actually vanishes in the four-dimensional limit. Sections 6–8 discuss the derivation of the amplitudes with two gravitons and two gluons, while Sections 9–10 contain the (vanishing) amplitudes with three gravitons and one gluon. Finally in Section 11 we rederive the curiously vanishing single-graviton all-plus amplitude from a double-copy construction. Two appendices complete the paper. In Appendix A we list the D-dimensional expressions of the relevant integrals and the appropriate limits contributing to the amplitudes of interest, while in Appendix B we derive all the four-point tree-level amplitudes with two massive scalars and gluons/gravitons using recursion relations.

2 Relevant tree-level amplitudes

In this section we collect all the tree-level amplitudes entering our calculation. The basic building blocks are the three-point amplitudes involving a gluon or graviton and two massive scalars. The color-ordered gluon-scalar-scalar amplitudes are [24]

$$
A(1^+,2_\phi,3_{\bar\phi}) \;=\; i\,\frac{\langle q|3|1]}{\langle q1\rangle}\,, \qquad A(1^-,2_\phi,3_{\bar\phi}) \;=\; i\,\frac{\langle 1|3|q]}{[1q]}\,, \tag{2.1}
$$

where $p_2^2=p_3^2=\mu^2$, and μ is the mass of the scalar particles. In these formulae, λ_q and $\widetilde{\lambda}_q$ are reference spinors, and the amplitudes themselves are independent of their choice. The amplitudes involving a graviton are similarly given by the square of the previous amplitudes[4]

$$
A(1^{++};2_\phi,3_{\bar\phi}) \;=\; i\left[A(1^+,2_\phi,3_{\bar\phi})\right]^2\,, \qquad A(1^{--};2_\phi,3_{\bar\phi}) \;=\; i\left[A(1^-,2_\phi,3_{\bar\phi})\right]^2\,. \tag{2.2}
$$

[4] We have confirmed this calculation also from Feynman rules, for which a good source is [25].

We will also need four-point amplitudes involving two gluons/gravitons and two scalars. The amplitudes involving gluons have been derived in [24] using BCFW recursion relations [26] applied to massive scalars, and the relevant amplitudes with gravitons can be obtained similarly (see Appendix B for details). We quote here the expression of the relevant Yang-Mills amplitudes with two gluons and two scalars:

$$A(1^+, 2^+, 3_\phi, 4_{\bar{\phi}}) = \mu^2 \frac{[12]}{\langle 12\rangle} \frac{i}{(p_4+p_1)^2-\mu^2}, \tag{2.3}$$

$$A(1^-, 2^+, 3_\phi, 4_{\bar{\phi}}) = \frac{\langle 1|4|2]^2}{s_{12}} \frac{i}{\left[(p_4+p_1)^2-\mu^2\right]}, \tag{2.4}$$

while for the amplitudes involving a graviton, a gluon and two scalars we have:[5]

$$A(1^+, 2_\phi, 3_{\bar{\phi}}; 4^{++}) = -\mu^2 \frac{[14]}{\langle 14\rangle^2} \langle 1|3|4] \left[\frac{i}{(p_3+p_4)^2-\mu^2} + \frac{i}{(p_2+p_4)^2-\mu^2}\right], \tag{2.5}$$

$$A(1^-, 2_\phi, 3_{\bar{\phi}}; 4^{++}) = -\frac{\langle 1|3|4]^3}{s_{14}\langle 14\rangle} \left[\frac{i}{(p_3+p_4)^2-\mu^2} + \frac{i}{(p_2+p_4)^2-\mu^2}\right], \tag{2.6}$$

$$A(1^+, 2_\phi, 3_{\bar{\phi}}; 4^{--}) = -\frac{\langle 4|3|1]^3}{s_{14}[41]} \left[\frac{i}{(p_3+p_4)^2-\mu^2} + \frac{i}{(p_2+p_4)^2-\mu^2}\right]. \tag{2.7}$$

We have also double-checked these amplitudes through a direct Feynman diagrammatic calculation. The two-graviton/two-scalar amplitudes in turn read

$$A(2_\phi, 3_{\bar{\phi}}; 4^{++}, 1^{++}) = -\mu^4 \frac{[41]^2}{\langle 41\rangle^2} \left[\frac{i}{(p_3+p_4)^2-\mu^2} + \frac{i}{(p_2+p_4)^2-\mu^2}\right], \tag{2.8}$$

$$A(2_\phi, 3_{\bar{\phi}}; 4^{++}, 1^{--}) = -\frac{\langle 1|3|4]^4}{s_{14}^2} \left[\frac{i}{(p_3+p_4)^2-\mu^2} + \frac{i}{(p_2+p_4)^2-\mu^2}\right]. \tag{2.9}$$

Note that (2.3), (2.5) and (2.8) manifestly vanish if the scalars are massless.

For later convenience we shall split up (2.5)–(2.9) into a sum of two partial amplitudes which treat the single graviton effectively as if it were color ordered, in the sense that

$$A(1^\pm, 2_\phi, 3_{\bar{\phi}}; 4^{++}) := \mathcal{A}(4^{++}, 1^\pm, 2_\phi, 3_{\bar{\phi}}) + \mathcal{A}(1^\pm, 4^{++}, 2_\phi, 3_{\bar{\phi}}), \tag{2.10}$$

with

$$\mathcal{A}(4^{++}, 1^+, 2_\phi, 3_{\bar{\phi}}) = \mu^2 \frac{[41]}{\langle 41\rangle^2} \langle 1|3|4] \frac{i}{(p_3+p_4)^2-\mu^2}, \tag{2.11}$$

$$\mathcal{A}(1^+, 4^{++}, 2_\phi, 3_{\bar{\phi}}) = \mu^2 \frac{[41]}{\langle 41\rangle^2} \langle 1|3|4] \frac{i}{(p_3+p_1)^2-\mu^2}, \tag{2.12}$$

$$\mathcal{A}(4^{++}, 1^-, 2_\phi, 3_{\bar{\phi}}) = \frac{\langle 1|2|4]^3}{\langle 14\rangle\, s_{14}} \frac{i}{(p_3+p_4)^2-\mu^2}, \tag{2.13}$$

[5] The derivation of (2.5), (2.6) and (2.8) is presented in Appendix B.

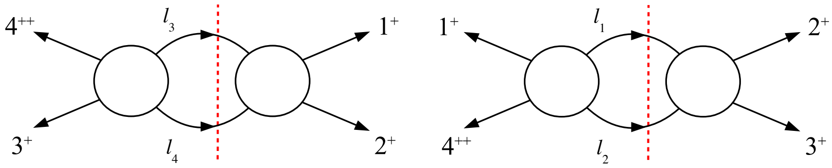


Figure 2: *The s- and t-channel cuts of the all-plus single-graviton amplitude. Cyclic permutations of the labels* $(1, 2, 3)$ *should also be added.*

$$\mathcal{A}(1^-, 4^{++}, 2_\phi, 3_{\bar{\phi}}) = \frac{\langle 1|2|4]^3}{\langle 14\rangle\, s_{14}}\, \frac{i}{(p_3 + p_1)^2 - \mu^2}\,, \tag{2.14}$$

and similarly for the other amplitudes. In the unitarity-based construction of the one-loop amplitudes to be discussed, we then symmetrize explicitly in the graviton leg(s) attached.

3 The $\langle 1^+\, 2^+\, 3^+\, 4^{++}\rangle$ amplitude

We begin our investigation with the four-point same-helicity amplitude with one graviton and three gluons. We will derive the integrand of this amplitude, as well as its four-dimensional limit. We anticipate the interesting outcome of this computation, namely that this amplitude is zero in the four-dimensional limit – a result that we will also confirm from the double-copy perspective in Section 11.[6]

To organize the computation efficiently, we employ the effective "color"-ordered graviton partial amplitudes introduced in the previous section. The diagrams to be considered are shown in Figure 2. As all gluons carry the same helicity, we need only to evaluate the first diagram in Figure 2; the final result will then be obtained by adding the terms obtained by cycling $(1, 2, 3)$ in the partial result.

For the configuration (1234) of Figure 2 there are two two-particle cuts, in the $s_{12} = s$ and $s_{23} = t$ channels. We start with the t-channel cut which is given by the product of the two partial amplitudes:

$$\begin{aligned} A^{(1)}(4^{++}; 1^+, 2^+, 3^+)\big|_{(1234),t} &= \mathcal{A}(4^{++}, 1^+, l_{1,\phi}, l_{2,\bar{\phi}})A(2^+, 3^+, -l_{2,\phi}, -l_{1,\bar{\phi}}) \\ &= 2\mu^4 \frac{[41][23]}{\langle 41\rangle\langle 23\rangle} \frac{\langle 1|l_2|4]}{\langle 14\rangle} \frac{i}{(l_2 + p_4)^2 - \mu^2} \frac{i}{(l_1 - p_2)^2 - \mu^2}\,, \end{aligned} \tag{3.1}$$

where the explicit expressions of the tree-level amplitudes entering the cut are given in (2.3) and (2.5), and the factor of two arises from summing of the possible assignment (ϕ, $\bar{\phi}$ and $\bar{\phi}$, ϕ) for the internal scalar particles.

[6]We thank Henrik Johansson and Radu Roiban for confirming the vanishing of this amplitude in four dimensions from the double-copy approach implemented in [18].

For the s-channel cut of the (1234)-configuration, one similarly arrives at an integrand

$$\begin{aligned} A^{(1)}(4^{++};1^+,2^+,3^+)\big|_{(1234),s} &= \mathcal{A}(3^+,4^{++},l_{3,\phi},l_{4,\bar{\phi}})A(1^+,2^+,-l_{4,\phi},-l_{3,\bar{\phi}}) \\ &= 2\mu^4 \frac{[43][12]}{\langle 43\rangle\langle 12\rangle}\frac{\langle 3|l_4|4]}{\langle 12\rangle}\frac{i}{(l_4+p_3)^2-\mu^2}\frac{i}{(l_3-p_1)^2-\mu^2}\,. \end{aligned} \tag{3.2}$$

The strategy to find the integrand is now to rewrite the t-channel expression in such a way as to reproduce the s-channel expression modulo terms that vanish on the s-cut. For this we first introduce a uniform parametrization of the (1234) box diagram in terms of a single loop momentum l:

$$l_1 = l - p_1\,, \qquad l_2 = -l - p_4\,, \qquad l_3 = l\,, \qquad l_4 = p_1 + p_2 - l\,, \tag{3.3}$$

with

$$D_i = (l-q_i)^2 - \mu^2\,, \qquad q_0 = 0, \quad q_1 = p_1\,, \quad q_2 = p_1 + p_2\,, \quad q_3 = -p_4\,. \tag{3.4}$$

Using these the, s- and t-channel cuts take the compact forms

$$A^{(1)}(4^{++};1^+,2^+,3^+)\Big|_{(1234),t} = 2i\mu^4 \frac{[12][34]}{\langle 12\rangle\langle 34\rangle}\frac{\langle 1|l|4]}{\langle 14\rangle}\frac{\big[(2\pi)\delta(D_1)\big]\big[(2\pi)\delta(D_3)\big]}{D_0\,D_2}\,, \tag{3.5}$$

$$A^{(1)}(4^{++};1^+,2^+,3^+)\Big|_{(1234),s} = 2i\mu^4 \frac{[12][34]}{\langle 12\rangle\langle 34\rangle}\frac{\langle 3|l|4]}{\langle 34\rangle}\frac{\big[(2\pi)\delta(D_0)\big]\big[(2\pi)\delta(D_2)\big]}{D_1\,D_3}\,, \tag{3.6}$$

where we have explicitly indicated the cut propagators. From this it is obvious that we need to relate $\langle 1|l|4]$ to $\langle 3|l|4]$. The trick to do this is to exploit the identity

$$\langle 3|l|4] = \frac{[12]}{[23]}\langle 1|l|4] + \frac{[24]}{[32]}s_{l4}\,, \tag{3.7}$$

where $s_{l4} = \langle 4|l|4] = 2\,(l\cdot p_4)$, which in turn may be written as

$$s_{l4} = (l+p_4)^2 - \mu^2 - (l^2-\mu^2) = D_3 - D_0 \mathrel{\hat{=}} D_3\Big|_{\text{on } s\text{-cut}}\,. \tag{3.8}$$

We also note the identity

$$\frac{\langle 3|l|4]}{\langle 34\rangle} = \frac{\langle 1|l|4]}{\langle 14\rangle} + \frac{[24]}{[32]\langle 34\rangle}(D_3 - D_0)\,. \tag{3.9}$$

Inserting this into the s-cut expression (3.6) and dropping the D_0 term gives us an integrand which may be lifted off the cuts (with the usual replacement $(2\pi)\delta(D) \to i/D$ for the cut propagators):[7]

$$A^{(1)}(4^{++};1^+,2^+,3^+)\big|_{1234} = -2i\frac{[12][34]}{\langle 12\rangle\langle 34\rangle}\int\frac{d^4l}{(2\pi)^4}\frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}}\frac{\mu^4}{D_0D_1D_2D_3}\left[\frac{\langle 1|l|4]}{\langle 14\rangle} + \frac{[24]}{[32]\langle 34\rangle}D_3\right]. \tag{3.10}$$

[7]The factor of -1 in the following expression arises from reinstating two (uncut) propagators.

The partial one-loop amplitude is thus given by a linear box integral and a scalar triangle.

The final step is to now reduce the linear box integral. Here we use the `Mathematica` package `FeynCalc` [27], which efficiently implements the Passarino-Veltman reduction algorithm [28]. Doing this we arrive at the final result[8]

$$A^{(1)}(1^+,2^+,3^+;4^{++}) = \frac{2}{(4\pi)^{2-\epsilon}}\frac{[41][42][43]}{\langle 41\rangle\langle 23\rangle}\frac{t}{u}\left[\frac{1}{2}I_4[\mu^4;s,t]+\frac{1}{t}I_3[\mu^4;t]+\frac{1}{s}I_3[\mu^4;s]\right] \;+\; \text{perms}\,, \tag{3.11}$$

where by "perms" we indicate the two permutations (2314) and (3124) of (1234), which interchange the Mandelstam invariants as $(s,t,u) \to (t,u,s)$ and $(s,t,u) \to (u,s,t)$, respectively. However, we need not do this explicitly as taking the four-dimensional limit using the relations in (A.7) we get a vanishing result:

$$A^{(1)}(1^+,2^+,3^+;4^{++}) = 0\,. \tag{3.12}$$

It would be desirable to understand the deeper reason for this curious vanishing.

We also quote an alternative expression of the D-dimensional amplitude, which is given by:

$$A^{(1)}(1^+,2^+,3^+;4^{++}) = \frac{2}{(4\pi)^{2-\epsilon}}\frac{[12]\,[34]}{\langle 12\rangle\,\langle 34\rangle}\frac{1}{\langle 41\rangle[12]\langle 42\rangle}\left[\frac{s\,t}{2}I_4[\mu^4;s,t] \;-\; sI_3[\mu^4;s] \;+\; \text{perms}\right]\,, \tag{3.13}$$

where the two permutations are the same as in (3.11). The vanishing of (3.13) is of course obtained again upon using the formulae of Appendix A. We also comment that this integrand is manifestly odd under the exchange of any two same-helicity gluons. In color space this means that this amplitude is proportional to $f^{a_1a_2a_3}$, with no $d^{a_1a_2a_3}$ contribution. We will see that the same property is shared by all amplitudes involving three gluons computed in this paper – they only come with an $f^{a_1a_2a_3}$ color factor.

4 The $\langle 1^-\,2^+\,3^+\,4^{++}\rangle$ amplitude

Constructing this amplitude is a slightly harder task, hence as an introduction we will first re-derive the four-point gluon amplitude with a single negative-helicity gluon of [22] and then apply a similar procedure to the more complicated EYM case. The form of the four-gluon integrand is also of use for a double-copy based construction of the EYM amplitudes.

Warmup. As for the case of the all-plus amplitude derived in the previous section, we work with two-particle cuts. Because only gluons are involved, color ordering leaves us with only two channels to consider, see Figure 3. For the s-channel we have

$$A_4^{(1)}(1^-,2^+,3^+,4^+)\Big|_s = A(3^+,4^+,l_{1,\phi},l_{3,\bar{\phi}})\,A(1^-,2^+,-l_{3,\phi},-l_{1,\bar{\phi}})$$

[8]The integral functions appearing in (3.11) and in the rest of the paper are defined in Appendix A, following the conventions of [22] up to a minus sign for the I_3 integrals.

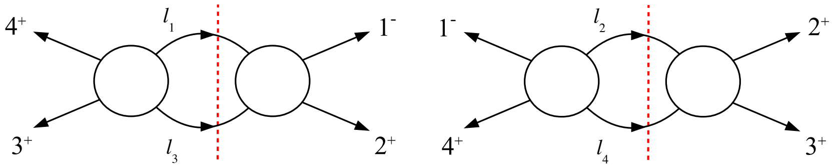


Figure 3: *The s- and t-channel cuts of the* $A^{(1)}(1^-, 2^+, 3^+, 4^+)$ *amplitude in pure Yang-Mills.*

$$= \mu^2 \frac{[34]}{\langle 34\rangle\,(l_4^2-\mu^2)}\;\frac{-\langle 1|l_1|2]^2}{\langle 12\rangle[21]\,(l_2^2-\mu^2)}\,, \tag{4.1}$$

whereas the t-channel cut reads

$$\begin{aligned} A_4^{(1)}(1^-,2^+,3^+,4^+)\Big|_t &= A(4^+,1^-,l_{2,\phi},l_{4,\bar\phi})\,A(2^+,3^+,-l_{4,\phi},-l_{2,\bar\phi}) \\ &= -\frac{\langle 1|l_4|4]^2}{\langle 41\rangle[14]\,(l_1^2-\mu^2)}\,\mu^2\frac{[23]}{\langle 23\rangle\,(l_3^2-\mu^2)}\,. \end{aligned} \tag{4.2}$$

The strategy to find the integrand is now to rewrite the t-channel expression in such a way to reproduce the s-channel one modulo terms that vanish on the s-cut. For this, we will make use of the following identity to rewrite the numerator in (4.2):

$$\langle 1|l_1|4] = \frac{1}{\langle 34\rangle}\Big[\langle 13\rangle\, s_{l_1 1} + \langle 1|l_1|2]\,\langle 23\rangle\Big]\,, \tag{4.3}$$

where $s_{l_1 1} = \langle 1|l_1|1] = 2\, l_1\cdot p_1$, which in turn may be written as

$$s_{l1} = (l_1^2-\mu^2) - (l_2^2-\mu^2) \mathrel{\hat=} (l_1^2-\mu^2)\Big|_{\text{on } t\text{-cut}}\,. \tag{4.4}$$

This last expression holds on the t-channel cut. Inserting the expression (4.3) for $\langle 1|l_1|4]$ into the t-channel cut amplitude $A_4|_t$ of (4.2) then yields an expression which may straightforwardly be lifted off the cut. Thus we get an integrand[9]

$$\begin{aligned} A^{(1)}(1^-,2^+,3^+,4^+) = -\int \frac{d^4 l}{(2\pi)^4}\,\frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}}\left(-\frac{\mu^2}{\langle 34\rangle^2}\right)\Big[\langle 1|l|2]^2 \\ + 2\frac{\langle 13\rangle}{\langle 23\rangle}\,D_0\,\langle 1|l|2] + \frac{\langle 13\rangle^2}{\langle 23\rangle^2}\,D_0\, s_{l1}\Big]\frac{1}{D_0 D_1 D_2 D_3}\,, \end{aligned} \tag{4.5}$$

where we have chosen the loop momentum parametrization as $l = l_1$, and

$$D_0 = l^2-\mu^2\,,\quad D_1 = (l-p_1)^2-\mu^2\,,\quad D_2 = (l-p_1-p_2)^2-\mu^2\,,\quad D_3 = (l+p_4)^2-\mu^2\,. \tag{4.6}$$

[9]Again, the minus sign in front of the following expression arises from two cut propagators.

Note that there is an ambiguity in treating the last term in (4.5). By the logic laid out above we could have also replaced $s_{l_1 1}$ by D_0 as the resulting expression would agree with (4.2) and (4.1) on the respective cuts. However, only the choice quoted above does reproduce the result in the literature.[10] The final step is to now reduce the tensor integrals appearing in (4.5), which we do again using the `Mathematica` package `FeynCalc` [27]. Doing this we find

$$A^{(1)}(1^-, 2^+, 3^+, 4^+) = \frac{2i}{(4\pi)^{2-\epsilon}} \frac{\langle 1|4|2]^2}{\langle 34\rangle^2} \frac{s}{tu} \Big[I_4[\mu^4; s, t] + \frac{st}{2u} I_4[\mu^2; s, t] + \frac{s(u-t)}{tu} I_3[\mu^2; t] \\ + \frac{t(s-u)}{su} I_3[\mu^2; s] + \frac{u-s}{t^2} I_2[\mu^2; t] + \frac{u-t}{s^2} I_2[\mu^2; s] \Big] . \tag{4.7}$$

This result agrees with the result in the literature [22].[11]

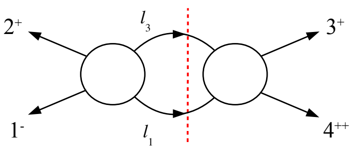
Single graviton amplitude. After this warmup let us now consider the EYM amplitude for a single graviton and three gluons with one negative-helicity state. Again we shall construct the integrand from two-particle cuts. Now, due to the presence of the graviton 4^{++} which we here include with the effectively colored ordered tree-amplitudes $\mathcal{A}$ of (2.10), we will have to consider three distinct type of two-particle cut diagrams. These follow from the particle configurations (1234), (1243) and (1423) pushing the graviton leg 4^{++} through the color-ordered gluons. The full amplitude is then divided into three parts,

$$A^{(1)}(1^-, 2^+, 3^+; 4^{++}) = A_{(1234)} + A_{(1243)} + A_{(1423)} , \tag{4.8}$$

which we now construct in turn from two-particle cuts.

Diagram (1234). Here we encounter an s-channel and a t-channel cut. For the s-channel of the (1234)-configuration we find

$$A_{(1234)}|_s =$$



$$= A(1^-, 2^+, \phi_{l_3}, \bar{\phi}_{l_1})\, \mathcal{A}(3^+, 4^{++}, \phi_{-l_1}, \bar{\phi}_{-l_3}) \tag{4.9}$$

$$= 2\mu^2 i^2 \frac{[12][34]}{\langle 12\rangle\langle 34\rangle} \frac{\langle 3|l_1|4]\, \langle 1|l_1|2]^2}{[12]^2\, \langle 34\rangle} \frac{\big[(2\pi)\delta(D_0)\big]\, \big[(2\pi)\delta(D_2)\big]}{D_1\, D_3} ,$$

where for the diagram (1234) we use the following loop momentum assignments:

$$D_0 = l_1^2 - \mu^2 =: l^2 - \mu^2 , \qquad D_1 = l_2^2 - \mu^2 = (l - p_1)^2 - \mu^2 ,$$

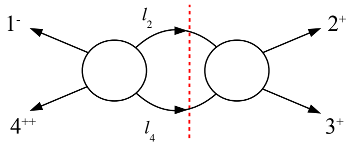
[10] It would be valuable to understand this seeming ambiguity better. Such an ambiguity does not appear in the procedure of merging cuts employed in later sections, which we have used to confirm all calculations of this paper. In the latter procedure, vanishing integrals are omitted, which may obscure a double-copy interpretation of the results.

[11] Had we taken D_0^2 instead of $D_0\, s_{l_1}$ in the last term of (4.5) we would on top find a term proportional to $[u/(st)]\, I_3[\mu^4]$ in the above, in disagreement with [22].

$$D_2 = l_3^2 - \mu^2 = (l - p_1 - p_2)^2 - \mu^2\,, \qquad\qquad D_3 = l_3^2 - \mu^2 = (l + p_4)^2 - \mu^2\,. \tag{4.10}$$

Note that we have set $l_1 = -l$. The t-channel cut of the (1234)-configuration on the other side takes the form

$$A_{(1234)}|_t =$$



$$= \mathcal{A}(4^{++}, 1^-, \phi_{l_2}, \bar{\phi}_{l_4})\, A(2^+, 3^+, \phi_{-l_4}, \bar{\phi}_{-l_2})$$

$$= -2\mu^2 i^2\, \frac{[12][34]}{\langle 12\rangle\langle 34\rangle}\, \frac{\langle 1|l|4]^3}{[41]^2\,\langle 14\rangle}\, \frac{\big[(2\pi)\delta(D_1)\big]\,\big[(2\pi)\delta(D_3)\big]}{D_0\, D_2}\,. \tag{4.11}$$

We now lift the two expressions (4.9) and (4.11) off the cuts by the same strategy that was applied previously. We rewrite the two l-dependent spinorial expressions in $A_{(1234)}|_s$ as

$$\langle 3|l|4] = \frac{[12]}{[23]}\,\langle 1|l|4] + \frac{[42]}{[23]}\, s_{l4}\,, \qquad \langle 1|l|2] = \frac{\langle 34\rangle}{\langle 23\rangle}\,\langle 1|l|4] + \frac{\langle 31\rangle}{\langle 23\rangle}\, s_{l1}\,. \tag{4.12}$$

Using these relations, we observe the identity

$$\begin{aligned}\langle 3|l|4]\,\langle 1|l|2]^2 &= \frac{[12]\langle 34\rangle^2}{s_{23}\,\langle 32\rangle}\,\langle 1|l|4]^3 + \frac{[24]\langle 23\rangle}{s_{23}} s_{l4}\,\langle 1|l|4]^2 \\ &\quad + \frac{[12]\langle 31\rangle}{s_{23}\langle 32\rangle} s_{l1}\,\langle 1|l|4]\,\Big(\langle 34\rangle\,\langle 1|l|4] + \langle 23\rangle\,\langle 1|l|2]\Big)\,. \end{aligned} \tag{4.13}$$

Inserting this into the s-cut amplitude (4.9), and rewriting the Mandelstam invariants $s_{li} = 2(l{\cdot}p_i)$ as

$$s_{l4} = D_3 - D_0 \mathrel{\hat{=}} D_3\Big|_{\text{on } s\text{-cut}}\,, \qquad s_{l1} = D_0 - D_1 \mathrel{\hat{=}} -D_1\Big|_{\text{on } s\text{-cut}}\,, \tag{4.14}$$

leads us to an expression for the $A_{(1234)}$ integrand manifestly agreeing with both cuts (4.9) and (4.11),

$$\begin{aligned} A_{(1234)} = 2i^2 \frac{[12][34]}{\langle 12\rangle\langle 34\rangle} \int \frac{d^4 l}{(2\pi)^4}\, \frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}} \Big\{ &\frac{\langle 1|l|4]^3}{[41]^2\,\langle 14\rangle} - \frac{[42]}{\langle 14\rangle[12]^3}\, D_3\,\langle 1|l|2]^2 \\ &- \frac{\langle 31\rangle\,\langle 1|l|4]}{\langle 14\rangle[41]^2\langle 34\rangle^2}\, D_1\, \Big(\langle 34\rangle\,\langle 1|l|4] + \langle 23\rangle\,\langle 1|l|2]\Big) \Big\}\, \frac{\mu^2}{D_0 D_1 D_2 D_3}\,. \end{aligned} \tag{4.15}$$

This expression may be straightforwardly reduced to scalar integrals using e.g. `FeynCalc`. As a matter of fact, one quickly sees that the second term in the above vanishes upon integration.

An alternative representation for $A_{(1234)}$ is obtained if one rewrites the t-cut expression (4.11) in terms of the s-cut one plus D_0 terms, arriving at

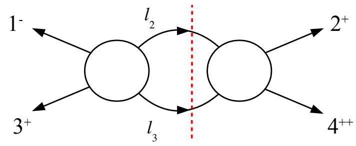
$$A'_{(1234)} = 2i^2 \frac{[12][34]}{\langle 12\rangle\langle 34\rangle} \int \frac{d^4 l}{(2\pi)^4}\, \frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}} \Big\{ \frac{\langle 3|l|4]\langle 1|l|2]^2}{[12]^2\,\langle 34\rangle} + \frac{[24]}{[14][12]s_{23}}\, D_0\,\langle 1|l|4]^2$$

$$+\frac{\langle 13\rangle\,[23]\,\langle 3|l|4]}{[12][41]\langle 34\rangle^2\, s_{23}}\, D_0\Big(\langle 34\rangle\,\langle 1|l|4] + \langle 23\rangle\,\langle 1|l|2]\Big)\Big\}\,\frac{\mu^2}{D_0D_1D_2D_3}\,, \tag{4.16}$$

which upon Passarino-Veltman reduction indeed matches $A_{(1234)}$ of (4.15). The result after reduction reads:

$$\begin{aligned} A_{(1234)} =& \frac{2i}{(4\pi)^{2-\epsilon}}\frac{[24][34]}{\langle 24\rangle\langle 34\rangle}\frac{1}{[12]\langle 23\rangle[31]}\Big[-\frac{3}{2}stI_4[\mu^4;s,t] - \frac{s^2t^2}{2u}I_4[\mu^2;s,t] \\ & - \frac{s^2(s+3u)}{t^2}I_3[\mu^4;t] - \frac{s^2\,(s^2+3st+3t^2)}{tu}I_3[\mu^2;t] + \frac{t(u-s)}{s}I_3[\mu^4;s] + \frac{s^2t}{u}I_3[\mu^2;s] \\ & + \frac{s^2(2t-u)+u^3}{2st}I_2[\mu^2;s] - \frac{s(2s-u)(s+3u)}{2t^2}I_2[\mu^2;t]\Big]\,. \end{aligned} \tag{4.17}$$

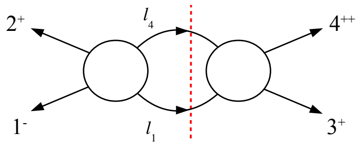
Diagram (1243). For the (1243)-contribution we have a u-channel and a s-channel cut, which read

$A_{(1243)}|_u =$ 

$= A(3^+,1^-,\phi_{l_2},\bar\phi_{l_3})\,\mathcal{A}(2^+,4^{++},\phi_{-l_3},\bar\phi_{-l_2})$

$$= 2\mu^2 i^2\frac{[12][34]}{\langle 12\rangle\langle 34\rangle}\,\frac{\langle 2|l|4]\,\langle 1|l|3]^2}{\langle 24\rangle\,[31]^2}\,\frac{\big[(2\pi)\delta(D_0)\big]\,\big[(2\pi)\delta(D_2)\big]}{D_1D_3}\,, \tag{4.18}$$

and

$A_{(1243)}|_s =$ 

$= A(1^-,2^+,\phi_{l_4},\bar\phi_{l_1})\,\mathcal{A}(4^{++},3^+,\phi_{-l_1},\bar\phi_{-l_4})$

$$= -2\mu^2 i^2\frac{[12][34]}{\langle 12\rangle\langle 34\rangle}\,\frac{\langle 3|l|4]\,\langle 1|l-p_3|2]^2}{\langle 34\rangle\,[21]^2}\,\frac{\big[(2\pi)\delta(D_1)\big]\,\big[(2\pi)\delta(D_3)\big]}{D_0D_2}\,, \tag{4.19}$$

where we have introduced the loop parametrization $l := -l_3$ along with

$$\begin{aligned} D_0 &= l_3^2-\mu^2 =: l^2-\mu^2\,, & D_1 &= l_1^2-\mu^2 = (l-p_3)^2-\mu^2\,, \\ D_2 &= l_2^2-\mu^2 = (l-p_1-p_3)^2-\mu^2\,, & D_3 &= l_4^2-\mu^2 = (l+p_4)^2-\mu^2\,. \end{aligned} \tag{4.20}$$

The s-cut expression may now be lifted off the cut by using the identities

$$[31]\,\langle 3|l|4] = [12]\,\langle 2|l|4] + [14]\,s_{l4}\,,\qquad \langle 42\rangle\,\langle 1|l-p_3|2] = \langle 34\rangle\,\langle 1|l|3] + \langle 14\rangle\,2(l-p_3)\cdot p_1\,. \tag{4.21}$$

On the s-cut (where $D_1 = D_3 = 0$) we may replace $s_{l4} = D_3 - D_0 \hat{=} -D_0$ as well as $2(l-p_3)\cdot p_1 = D_2 - D_1 \hat{=} D_2$. Using this we arrive at the integrand for the (1243)-type contribution,

$$A_{(1243)} = 2i^2 \frac{[12][34]}{\langle 12\rangle\langle 34\rangle} \int \frac{d^4l}{(2\pi)^4} \frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}} \Big\{ \frac{\langle 2|l|4]\,\langle 1|l|3]^2}{\langle 24\rangle[31]^2} - \frac{[14]}{[12]\langle 24\rangle[31]^2} D_0\,\langle 1|l|3]^2 - \frac{\langle 14\rangle}{[12]^2\langle 24\rangle^2\langle 34\rangle} D_2\,\langle 3|l|4]\Big(\langle 34\rangle\,\langle 1|l|3] + \langle 42\rangle\,\langle 1|l-p_3|2]\Big)\Big\} \frac{\mu^2}{D_0D_1D_2D_3}\,. \tag{4.22}$$

Again we have an expression in terms of box and triangle tensor integrals amenable to standard integral reduction techniques. An alternative and more compact expression may derived if one rewrites the u-cut in terms of the s-cut followed by a shift in the integration variable $l \to l + p_3$. One then finds

$$A'_{(1243)} = 2i^2 \frac{[12][34]}{\langle 12\rangle\langle 34\rangle} \int \frac{d^4l}{(2\pi)^4} \frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}} \Big\{ \frac{\langle 3|l|4]\,\langle 1|l|2]^2}{\langle 34\rangle[12]^2} - \frac{\langle 14\rangle}{[12]^2\langle 24\rangle^2\langle 34\rangle} D_1\,\langle 3|l|4]\Big(\langle 34\rangle\,\langle 1|l|3] + \langle 42\rangle\,\langle 1|l|2]\Big)\Big\} \frac{\mu^2}{D_0D_1D_2D_3}\,, \tag{4.23}$$

where now

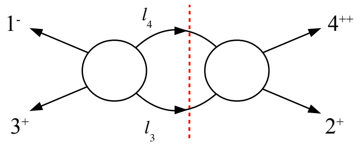
$$D_0 = (l+p_3)^2-\mu^2\,, \qquad D_1 = l^2-\mu^2\,, \qquad D_0 = (l-p_1)^2-\mu^2\,, \qquad D_0 = (l+p_3+p_4)^2-\mu^2\,. \tag{4.24}$$

Passarino-Veltman reducing (4.22) or (4.23), one arrives at

$$A_{(1243)} = -\frac{2i}{(4\pi)^{2-\epsilon}} \frac{[24][34]}{\langle 24\rangle\langle 34\rangle} \frac{1}{[12]\langle 23\rangle[31]} \Big[-\frac{us}{2} I_4[\mu^4; u, s] + \frac{s^2}{u} I_3[\mu^4; u] + \frac{u^2}{s} I_3[\mu^4; s] - \frac{tu}{2s} I_2[\mu^2; s] - \frac{st}{2u} I_2[\mu^2; u]\Big]\,. \tag{4.25}$$

Diagram (1423). The remaining (1423)-contribution carries a u-channel and a t-channel cut. These read

$$A_{(1423)}|_t =$$

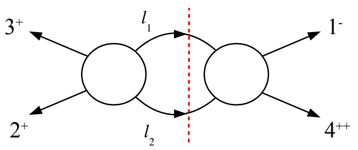


$$= A(3^+, 1^-, \phi_{l_4}, \bar\phi_{l_3})\,\mathcal{A}(4^{++}, 2^+, \phi_{-l_3}, \bar\phi_{-l_4})$$

$$= -2i^2\mu^2 \frac{\langle 1|l|4]^3}{\langle 14\rangle\langle 23\rangle^2} \frac{\big[(2\pi)\delta(D_0)\big]\,\big[(2\pi)\delta(D_2)\big]}{D_1\,D_3}\,, \tag{4.26}$$

and

$$A_{(1423)}|_u =$$



$$= A(2^+, 3^+, \phi_{l_1}, \bar\phi_{l_2})\,\mathcal{A}(1^-, 4^{++}, \phi_{-l_2}, \bar\phi_{-l_1})$$

$$= -2i^2\mu^2 \, \frac{\langle 2|l|4] \, \langle 1|l-p_2|3]^2}{\langle 24\rangle^3} \, \frac{\big[(2\pi)\delta(D_1)\big]\,\big[(2\pi)\delta(D_3)\big]}{D_0\,D_2}\,, \tag{4.27}$$

where we identified the loop momentum as $l := -l_2$ and used the inverse propagators suitable for diagram (1423),

$$\begin{aligned} D_0 &= l_2^2 - \mu^2 =: l^2 - \mu^2\,, & D_1 &= l_3^2 - \mu^2 = (l-p_2)^2 - \mu^2\,,\\ D_2 &= l_1^2 - \mu^2 = (l-p_2-p_3)^2 - \mu^2\,, & D_3 &= l_4^2 - \mu^2 = (l+p_4)^2 - \mu^2\,. \end{aligned} \tag{4.28}$$

However, by inspection we see that $A_{(1423)}$ may be obtained from the (1234)-configuration by simply swapping $2 \leftrightarrow 3$ (or $s \leftrightarrow u$). Hence we conclude that

$$\begin{aligned} A_{(1423)} &= A_{(1234)}\Big|_{2\leftrightarrow 3}\\ &= -\frac{2i}{(4\pi)^{2-\epsilon}}\frac{[24][34]}{\langle 24\rangle\langle 34\rangle}\frac{1}{[12]\langle 23\rangle[31]}\Big[\frac{3}{2}\,u\,tI_4[\mu^4;u,t] + \frac{u^2t^2}{2s}I_4[\mu^2;u,t]\\ &+ \frac{u^2(u+3s)}{t^2}I_3[\mu^4;t] + \frac{u^2\,(u^2+3su+3s^2)}{ts}I_3[\mu^2;t] + \frac{t(u-s)}{u}I_3[\mu^4;u] - \frac{u^2t}{s}I_3[\mu^2;u]\\ &- \frac{u^2(2t-s)+s^3}{2ut}I_2[\mu^2;u] + \frac{u(2u^2+5us-3s^2)}{2t^2}I_2[\mu^2;t]\Big]\,. \end{aligned} \tag{4.29}$$

Final result. Adding all the three contributions $A_{(1234)} + A_{(1243)} + A_{(1423)}$ leads to the final D-dimensional result:

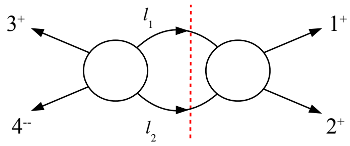
$$\begin{aligned} A^{(1)}(1^-,2^+,3^+;4^{++}) =& -\frac{2i}{(4\pi)^{2-\epsilon}}\frac{[24][34]}{\langle 24\rangle\langle 34\rangle}\frac{1}{[12]\langle 23\rangle[31]}\Big\{-\frac{3}{2}stI_4[\mu^4;s,t] - \frac{s^2t^2}{2u}I_4[\mu^2;s,t]\\ &+ \frac{1}{2}suI_4[\mu^4;u,s] - \frac{3}{2}tuI_4[\mu^4;u,t] - \frac{t^2u^2}{2s}I_4[\mu^2;u,t] - \frac{(-t^2u-2tu^2+u^3)}{su}I_3[\mu^4;s]\\ &- \frac{(t^4+3t^3u+3t^2u^2+tu^3)}{su}I_3[\mu^2;s] - \frac{(t^2u+tu^2)}{su}I_3[\mu^4;t]\\ &- \frac{(-t^4-2t^3u-t^2u^2+2tu^3+u^4)}{su}I_3[\mu^2;t] - \frac{(2t^2+4tu+u^2)}{u}I_3[\mu^4;u] + \frac{tu^2}{s}I_3[\mu^2;u]\\ &\frac{t(t+2u)}{s}I_2[\mu^2;s] + \frac{(t^2+2tu+2u^2)}{t}I_2[\mu^2;t] - \frac{t(t+2u)}{u}I_2[\mu^2;u]\Big\}\,. \end{aligned} \tag{4.30}$$

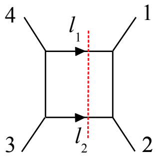
Taking the four-dimensional limit yields the compact final expression

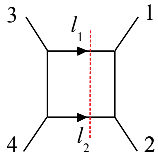
$$A^{(1)}(1^-,2^+,3^+;4^{++}) = \frac{i}{(4\pi)^2}\,\frac{[24][34]}{\langle 24\rangle\langle 34\rangle}\,\frac{1}{\langle 23\rangle[21][31]}\,\frac{1}{6}\,(s^2+u^2)\,. \tag{4.31}$$

5 The $\langle 1^+\,2^+\,3^+\,4^{--}\rangle$ amplitude

We now consider the rational one-loop amplitude with a single negative-helicity graviton and three positive-helicity gluons $A^{(1)}(1^+,2^+,3^+;4^{--})$. For amplitudes containing progressively more negative helicities, the procedure described in previous sections to construct the integrand becomes tedious. Hence, from now on, rather than constructing the integrand, we will use the standard approach of [29, 30] where we directly merge all two-particle cuts into a single function. The case at hand is particularly simple given the very symmetric helicity configuration chosen. Using the tree-level amplitudes in Section 2, we find that the s-cut of the amplitude is given by

s-cut: 

$= -i^2\mu^2 \frac{[12]}{\langle 12\rangle}\frac{\langle 4|l_1|3]^3}{s[34]}\Big[$ 

$+$ 

$\Big]\ .$ (5.1)

This amplitude also has t- and u-cuts which are obtained by simply cycling the labels (312) $\to$ (123) and (312) $\to$ (231), respectively. As in the previous sections, we use `FeynCalc` [27] to perform efficiently all relevant Passarino-Veltman reductions of the three-tensor box in (5.1) (and its permutations). We work first in the s-cut, and focus on the tensor box with particle ordering (1234). We lift the integral off the cut, and perform a Passarino-Veltman reduction. This will generate scalar boxes with particle ordering (1234) (and powers of the (-2ϵ)-momentum μ in the numerator), whose coefficient(s) we will then confirm from the t-cut. It will also generate one-mass triangles and bubbles in the s-channel (again with powers of μ in the numerator), which we keep, as well as spurious one-mass triangles and bubbles with a t-channel discontinuity, which we drop. We then repeat the same operation for the two other box topologies with particle orderings (1243) and (1324). Merging all contributions thus obtained, we arrive at our final result:

$$A^{(1)}(1^+,2^+,3^+;4^{--}) \;=\; 2i^2\frac{[12][34]}{\langle 12\rangle\langle 34\rangle}(\langle 42\rangle[23]\langle 34\rangle)^3\Big[f(s,t,u)\;+\text{perms}\Big]\ , \tag{5.2}$$

where

$$\begin{aligned} f(s,t,u) = \;& \frac{i}{(4\pi)^{2-\epsilon}}\frac{1}{stu^2}\Big[\frac{3}{2}I_4[\mu^4;s,t] - \frac{t(s-2u)}{s^3}I_3[\mu^4;s] - \frac{s(t-2u)}{t^3}I_3[\mu^4;t] \\ & + \frac{st}{2u}I_4[\mu^2;s,t] + \frac{s\,(s^2-3tu)}{t^2u}I_3[\mu^2;t] + \frac{t\,(t^2-3su)}{s^2u}I_3[\mu^2;t] \\ & + \frac{(s-2u)(u-2t)}{2s^3}I_2[\mu^2;s] + \frac{(u-2s)(t-2u)}{2t^3}I_2[\mu^2;t]\Big]\ . \end{aligned} \tag{5.3}$$

As in the case of the $\langle 4^{++}1^+2^+3^+\rangle$ amplitude computed in Section 3, by "perms" we denote the two permutations 2314 and 3124 of 1234, with the the Mandelstam invariants interchanged as $(s\to t, t\to u, u\to s)$ and $(s\to u, t\to s, u\to t)$. Performing the four-dimensional limit using the results of Appendix A, we find:

$$f(s,t,u) \;\to\; -\frac{i}{(4\pi)^2}\frac{3t^2+3tu+2u^2}{24\,s^3t^3}\ . \tag{5.4}$$

Adding the permutations, we arrive at a very compact final result:

$$A^{(1)}(1^+,2^+,3^+;4^{--}) \;=\; -\frac{i}{(4\pi)^2}\,\frac{[12][34]}{\langle 12\rangle\langle 34\rangle}\,(\langle 42\rangle[23]\langle 34\rangle)^3\,\frac{s^2+t^2+u^2}{6\,s^2\,t^2\,u^2}\;. \tag{5.5}$$

Note that the kinematic function in (5.5) is an odd function under any exchange of two gluons, and hence the complete amplitude is even under such an exchange (including a minus sign from the colour factor f^{abc}), as it should.

6 The $\langle 1^+\, 2^+\, 3^{++}\, 4^{++}\rangle$ amplitude

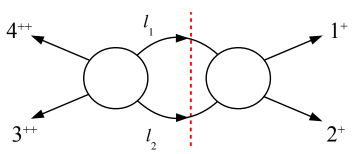
In this section we move on to amplitudes which contain two gravitons and two gluons. The simplest case to consider occurs when all particles have the same helicity – a particularly symmetric configuration.

We briefly describe the outline of the derivation, similarly with previous calculations. As usual there are three cut diagrams to consider, in the s-, t- and u-channels. These cuts will give rise to tensor boxes with particle ordering (1234), (1243) and (1324). These are given by:

$$\begin{aligned}
s\text{-cut:}\quad & A\big(3^{++},4^{++},l_{1,\bar\phi},l_{2,\phi}\big)\,\big[A\big(1^+,2^+,-l_{2,\bar\phi},-l_{1,\phi}\big)\;+\;1\leftrightarrow 2\big]\,,\\
t\text{-cut:}\quad & A\big(4^{++},1^+,l_{1,\bar\phi},l_{2,\phi}\big)\;\;A\big(2^+,3^{++},-l_{2,\bar\phi},-l_{1,\phi}\big)\,,\\
u\text{-cut:}\quad & A\big(3^{++},1^+,l_{1,\bar\phi},l_{2,\phi}\big)\;\;A\big(2^+,4^{++},-l_{2,\bar\phi},-l_{1,\phi}\big)\,.
\end{aligned} \tag{6.1}$$

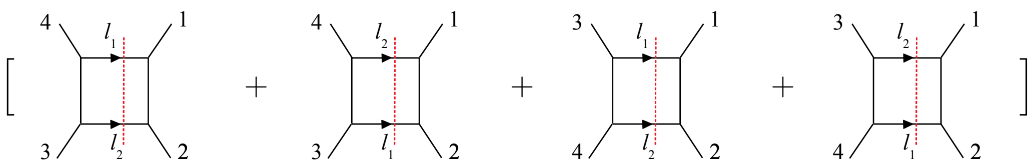
Note that on the right-hand side of the the s-cut in (6.1) we have to include the sum of two color-ordered amplitudes, $A\big(1^+,2^+,-l_{2,\bar\phi},-l_{1,\phi}\big)$ and $A\big(2^+,1^+,-l_{2,\bar\phi},-l_{1,\phi}\big)$. Indeed, since the left-hand side of the cut is an amplitude with a colorless (two-graviton) external state, both terms contribute to the same color ordering. This will be a recurrent feature of all cuts where one side of the cut is colorless. Moreover, there will be an additional contribution from the cut obtained by swapping ϕ with $\bar\phi$, which will double up the result of the previous cuts, as usual.

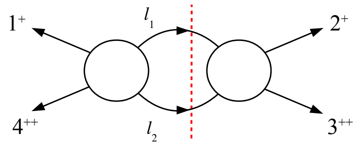
Using the tree-level amplitudes given in Section 2, we work out the expressions of these cuts, which give rise to three tensor boxes with the different particle orderings (1234), (1243) and (1324). Inspecting all cuts we can reconstruct the amplitude. We find the following results:

s-cut: 

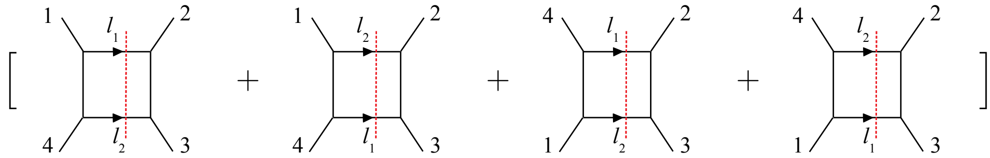
$$= \; 2\mu^6\,\frac{[34]^2}{\langle 34\rangle^2}\,\frac{[12]}{\langle 12\rangle}$$

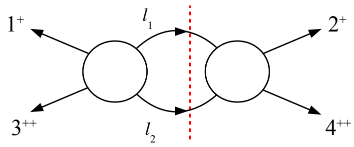
(6.2)



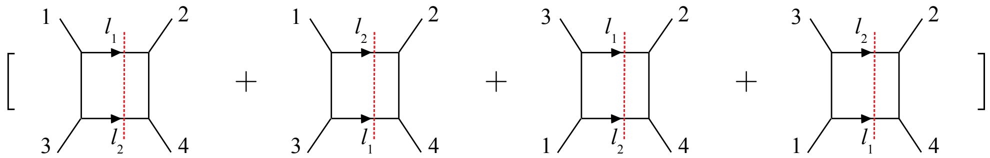
t-cut: 

$$= 2\mu^4 \frac{[41]}{\langle 41\rangle^2}\frac{[32]}{\langle 32\rangle^2}\langle 1|l_1|4]\langle 2|l_2|3] \tag{6.3}$$

$\cdot$  ,

u-cut: 

$$= 2\mu^4 \frac{[31]}{\langle 31\rangle^2}\frac{[42]}{\langle 42\rangle^2}\langle 1|l_1|3]\langle 2|l_2|4] \tag{6.4}$$

$\cdot$  .

Note that our cut integrand contains tensor boxes with cut momenta l_1 and l_2 as well as the same contribution but with l_1 and l_2 flipped. At the level of the integral, this will be taken into account by doubling up the contribution of a single copy.

The next step consists in combining all cuts, which we will do for each box topology separately. Doing so, we arrive at the following result for the topology (1234):

$$(1234):\qquad -\frac{i}{(4\pi)^{2-\epsilon}}\frac{[12]}{\langle 12\rangle}\frac{[34]^2}{\langle 34\rangle^2}\cdot 4\Big(I_4[\mu^6;s,t]-\frac{1}{t}I_2[\mu^4;t]\Big)\ , \tag{6.5}$$

which is obtained from combining the relevant terms in the s-cut given in (6.2) and the t-cut of (6.3). The topology (1243) is simply obtained by swapping $3 \leftrightarrow 4$, or $s \to s, t \to u, u \to s$ in the previous result:

$$(1243):\qquad -\frac{i}{(4\pi)^{2-\epsilon}}\frac{[12]}{\langle 12\rangle}\frac{[34]^2}{\langle 34\rangle^2}\cdot 4\Big(I_4[\mu^6;s,u]-\frac{1}{u}I_2[\mu^4;u]\Big)\ . \tag{6.6}$$

The last topology to consider is (1324), which is obtained from combining the relevant terms from the s- and u-cuts, given in (6.2) and (6.4). Doing so we get:

$$(1324): \qquad -\frac{i}{(4\pi)^{2-\epsilon}}\frac{[12]}{\langle 12\rangle}\frac{[34]^2}{\langle 34\rangle^2}\cdot \\ 4\Big(I_4[\mu^6;u,t]+\frac{ut}{2s}I_4[\mu^4;u,t]-\frac{t}{s}I_3[\mu^4;t]-\frac{u}{s}I_3[\mu^4;u]+\frac{I_2[\mu^4;t]}{t}+\frac{I_2[\mu^4;u]}{u}\Big)\ . \tag{6.7}$$

Finally we take the four-dimensional limit:

$$4\Big(I_4[\mu^6;s,t]-\frac{1}{t}I_2[\mu^4;t]\Big)\to -\frac{s}{15}\ , \tag{6.8}$$

while

$$4\Big(I_4[\mu^6;u,t]+\frac{ut}{2s}I_4[\mu^4;u,t]-\frac{t}{s}I_3[\mu^4;t]-\frac{u}{s}I_3[\mu^4;u]+\frac{I_2[\mu^4;t]}{t}+\frac{I_2[\mu^4;u]}{u}\Big)\to -\frac{s}{30}\ . \tag{6.9}$$

Combining all terms we arrive at a remarkably simple final result:

$$A^{(1)}(1^+,2^+;3^{++},4^{++}) \;=\; \frac{i}{(4\pi)^2}\frac{[12]}{\langle 12\rangle}\frac{[34]^2}{\langle 34\rangle^2}\frac{s}{6}\ . \tag{6.10}$$

We note that (6.10) is symmetric under the exchange of the two gluons. This is consistent with the colour factor δ^{ab} of this amplitude – indeed, the complete, color-dressed result should be symmetric under a swapping of the two gluons.

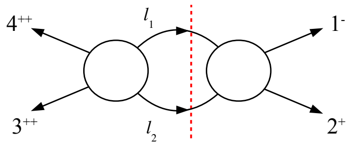
We also quote the compact expression of the D-dimensional result:

$$A^{(1)}(1^+,2^+;3^{++},4^{++}) = -\frac{4i}{(4\pi)^{2-\epsilon}}\frac{[12]}{\langle 12\rangle}\frac{[34]^2}{\langle 34\rangle^2}\Big\{I_4[\mu^6;s,t]+I_4[\mu^6;s,u]+I_4[\mu^6;u,t] \\ +\frac{tu}{2s}I_4[\mu^4;u,t]-\frac{t}{s}I_3[\mu^4;t]-\frac{u}{s}I_3[\mu^4;u]\Big\}\ . \tag{6.11}$$

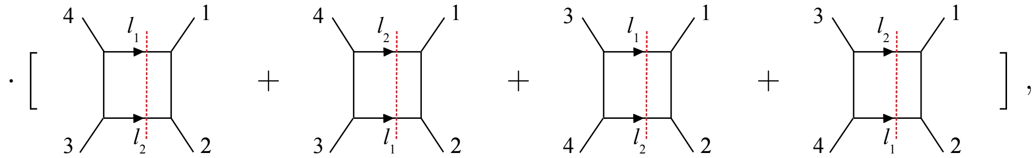
7 The $\langle 1^-\,2^+\,3^{++}\,4^{++}\rangle$ amplitude

Here we follow the same strategy as in the previous section, and derive the complete amplitude from merging two-particle cuts. As we will see, this procedure will now give rise to three tensor boxes with different particle orderings as before, with numerators that are up to quartic order in the loop momenta. These will then be Passarino-Veltman reduced as usual.

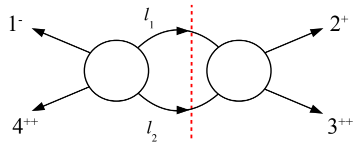
We now compute the three possible two-particle cuts of the amplitude. We also include the usual factor of two from swapping ϕ and $\bar{\phi}$ in the loop. The s-cut is given by

s-cut: 

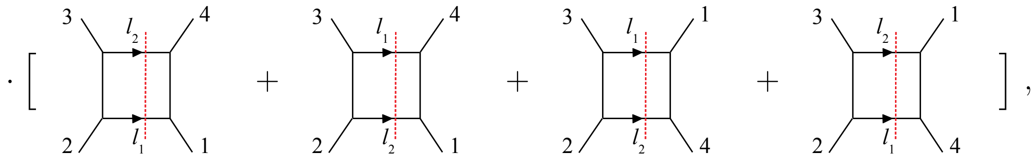
$$= 2\mu^4 \frac{[34]^2}{\langle 34\rangle^2} \frac{\langle 1|l_1|2]^2}{s} \tag{7.1}$$



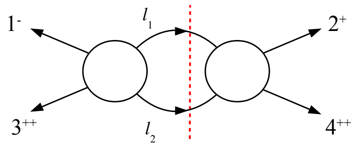
arising from $A(3^{++}, 4^{++}, l_{1,\phi}, l_{2,\bar{\phi}})\big[A(1^-, 2^+, -l_{2,\phi}, -l_{1,\bar{\phi}}) + A(2^+, 1^-, -l_{2,\phi}, -l_{1,\bar{\phi}})\big]$. Again, the appearance of two terms on the right-hand side of the cut, with two different gluon orderings, is due to the fact that the amplitude on the left-hand side of the cut contains a colorless external state. The next cut to look at is:

t-cut: 

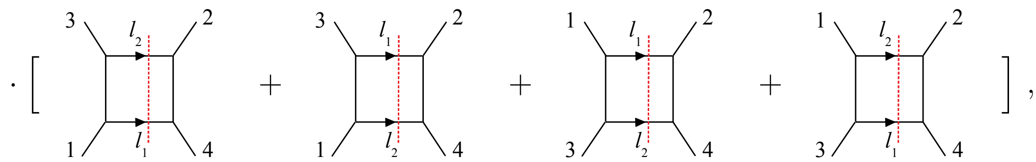
$$= 2\mu^2 \frac{[32]}{\langle 32\rangle^2\langle 14\rangle} \frac{\langle 1|l_1|4]^3\langle 2|l_1|3]}{t} \tag{7.2}$$



obtained from $A(4^{++}, 1^-, l_{1,\phi}, l_{2,\bar{\phi}})A(2^+, 3^{++}, -l_{2,\phi}, -l_{1,\bar{\phi}})$. Finally,

u-cut: 

$$= 2\mu^2 \frac{[42]}{\langle 42\rangle^2\langle 13\rangle} \frac{\langle 1|l_1|3]^3\langle 2|l_1|4]}{u} \tag{7.3}$$



from $A(3^{++}, 1^-, l_{1,\phi}, l_{2,\bar{\phi}})A(2^+, 4^{++}, -l_{2,\phi}, -l_{1,\bar{\phi}})$. We also define a convenient spinor prefactor which has the correct spinor weights for the given amplitude:

$$\mathcal{J} = \frac{[2\,4]^2[3\,4]^2\langle 1\,4\rangle^2}{\langle 3\,4\rangle^2} \ . \tag{7.4}$$

We are now ready to merge the different cuts. From the topology (1234) we get:

$$\begin{aligned} (1234):\quad & \frac{4i}{(4\pi)^{2-\epsilon}}\,\mathcal{J}\,\Bigg\{-I_4[\mu^6;s,t]\Big(\frac{1}{ut}\Big) - I_4[\mu^4;s,t]\Big(\frac{s}{2u^2}\Big) \\ & + I_3[\mu^4;t]\Big(\frac{s^2(2s+3t)}{u^2t^3}\Big) + I_3[\mu^4;s]\Big(\frac{-(2s+t)}{su^2}\Big) \\ & + I_2[\mu^4;t]\Big(\frac{(t-2s)(4s+3t)}{3ut^4}\Big) + I_2[\mu^4;s]\Big(\frac{(s+2t)}{uts^2}\Big) + I_2[\mu^2;t]\Big(\frac{s}{3t^3}\Big)\Bigg\}. \end{aligned} \tag{7.5}$$

The box topology (1243) is simply obtained from the topology (1234) in (7.5) by swapping $3 \leftrightarrow 4$, or $(s,t,u) \to (s,u,t)$. Note that $\mathcal{J}$ is invariant under this swap, hence the result for the (1243) topology is immediately found to be:

$$\begin{aligned} (1243):\ & \frac{4i}{(4\pi)^{2-\epsilon}}\,\mathcal{J}\,\Bigg\{-I_4[\mu^6;s,u]\Big(\frac{1}{ut}\Big) - I_4[\mu^4;s,u]\Big(\frac{s}{2t^2}\Big) \\ & + I_3[\mu^4;u]\Big(\frac{s^2(2s+3u)}{t^2u^3}\Big) + I_3[\mu^4;s]\Big(\frac{-(2s+u)}{st^2}\Big) \\ & + I_2[\mu^4;u]\Big(\frac{(u-2s)(4s+3u)}{3tu^4}\Big) + I_2[\mu^4;s]\Big(\frac{(s+2u)}{uts^2}\Big) + I_2[\mu^2;u]\Big(\frac{s}{3u^3}\Big)\Bigg\}. \end{aligned} \tag{7.6}$$

Note that in (7.5) and (7.6) the $I_2[\mu^2]$ functions only appear in the u- and t-channel.

The last topology is (1324), for which we obtain

$$\begin{aligned} (1324):\ & \frac{4i}{(4\pi)^{2-\epsilon}}\,\mathcal{J}\,\Bigg\{-I_4[\mu^6;u,t]\Big(\frac{1}{ut}\Big) - I_4[\mu^4;u,t]\Big(\frac{2}{s}\Big) \\ & + I_3[\mu^4;t]\Big(\frac{-2(3t^2+3ut+u^2)}{st^3}\Big) + I_3[\mu^4;u]\Big(\frac{-2(3u^2+3ut+t^2)}{su^3}\Big) \\ & + I_2[\mu^4;t]\Big(\frac{(t+4u)(3t+2u)}{3ut^4}\Big) + I_2[\mu^4;u]\Big(\frac{(4t+u)(2t+3u)}{3tu^4}\Big) \\ & - I_4[\mu^2;u,t]\Big(\frac{ut}{2s^2}\Big) + I_3[\mu^2;u]\Big(\frac{u}{s^2}\Big) + I_3[\mu^2;t]\Big(\frac{t}{s^2}\Big) - I_2[\mu^2;t]\Big(\frac{11t^2+7ut+2u^2}{6st^3}\Big) \\ & - I_2[\mu^2;u]\Big(\frac{2t^2+7ut+11u^2}{6su^3}\Big)\Bigg\}. \end{aligned} \tag{7.7}$$

The expression (7.7) is symmetric in $u \leftrightarrow t$.

Finally we take the four-dimensional limit of (7.5), (7.6) and (7.7) using (A.7), thus getting

$$\frac{i}{(4\pi)^2}\,\mathcal{J}\,\frac{(t+2u)}{30\,t^2}\,,\qquad \frac{i}{(4\pi)^2}\,\mathcal{J}\,\frac{(u+2t)}{30\,u^2}\,,\qquad \frac{i}{(4\pi)^2}\,\mathcal{J}\,\frac{s^3}{15\,u^2\,t^2}\,, \tag{7.8}$$

respectively. Thus, we arrive at the final result for the four-dimensional limit of the amplitude (using the expression of $\mathcal{J}$ in (7.4)):

$$A^{(1)}(1^-,2^+;3^{++},4^{++}) \;=\; \frac{i}{(4\pi)^2}\,\frac{[2\,4]^2[3\,4]^2\langle 1\,4\rangle^2}{\langle 3\,4\rangle^2}\,\frac{s}{6\,t\,u}\;. \tag{7.9}$$

The D-dimensional answer is easily obtained by adding (7.5), (7.6) and (7.7).

8 The $\langle 1^+\,2^+\,3^{++}\,4^{--}\rangle$ amplitude

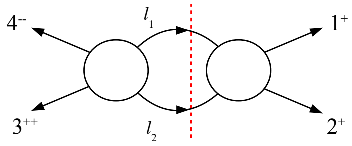
We proceed similarly to the previous sections and study all two-particle cuts of this amplitude. As in earlier examples, we find three box topologies with tensor numerators. In this case, an appropriate spinor prefactor which has the correct spinor weights for the given amplitude is

$$\mathcal{J} = \frac{[1\,2][1\,3]^4\langle 1\,4\rangle^4}{\langle 1\,2\rangle}\;. \tag{8.1}$$

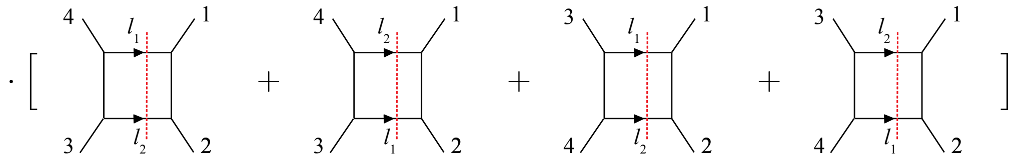
We construct the two-particle cuts of this amplitude using the tree-level expressions in Section 2. The corresponding cuts will again give rise to three tensor boxes with different particle orderings and numerators which are now quartic in the loop momenta. The expression of the relevant cut diagrams are:

$$\begin{aligned} s\text{-cut}: &\quad A(3^{++},4^{--},l_{1,\phi},l_{2,\bar\phi})\,\left[A(1^+,2^+,-l_{2,\phi},-l_{1,\bar\phi})\,+\,1\leftrightarrow 2\right],\\ t\text{-cut}: &\quad A(4^{--},1^+,l_{1,\phi},l_{2,\bar\phi})\;A(2^+,3^{++},-l_{2,\phi},-l_{1,\bar\phi})\,,\\ u\text{-cut}: &\quad A(3^{++},1^+,l_{1,\phi},l_{2,\bar\phi})\;A(2^+,4^{--},-l_{2,\phi},-l_{1,\bar\phi})\;. \end{aligned} \tag{8.2}$$

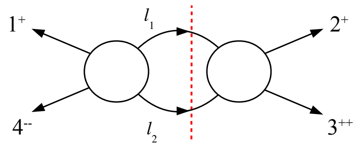
As in the cases studied in Sections 6 and 7, the s-cut integrand includes the sum of two color-ordered tree amplitudes on the right-hand side of the cut, which contribute to the same color-ordered amplitude, given that the external state on the left-hand side of the cut is colorless. Using the expressions of the relevant tree-level amplitudes and including a factor of two from the two possible assignments from the internal scalar fields, we obtain the following expressions for the cuts:

s-cut: 

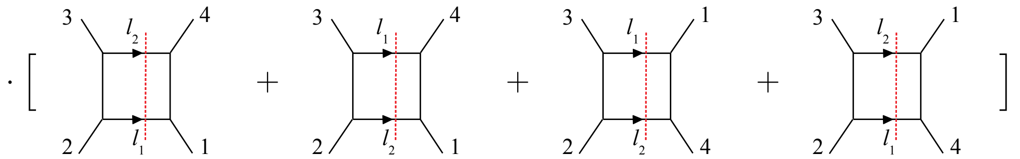
$$= 2\mu^4 \frac{[12]}{\langle 12\rangle} \frac{\langle 4|l_1|3]^4}{s^2} \tag{8.3}$$



,

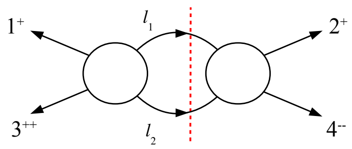
t-cut: 

$$= 2\mu^2 \frac{[32]}{\langle 32\rangle^2[41]} \frac{\langle 4|l_1|1]^3\langle 2|l_1|3]}{t} \tag{8.4}$$

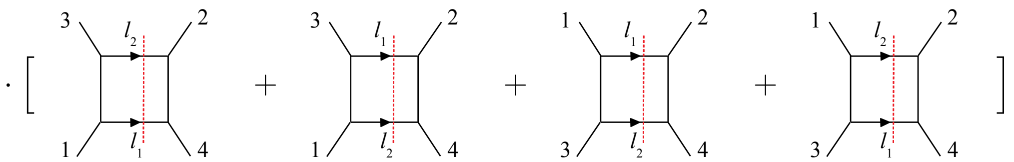


,

and finally,

u-cut: 

$$= 2\mu^2 \frac{[31]}{\langle 31\rangle^2[42]} \frac{\langle 4|l_1|2]^3\langle 1|l_1|3]}{u} \tag{8.5}$$



.

As usual, we now merge the cuts focusing separately on the three different box integrals. Merging the s- and t-cut for the topology (1234) we get:

$$
\begin{aligned}
(1234):\ & \frac{4i}{(4\pi)^{2-\epsilon}}\,\mathcal{J}\Bigg\{-I_4[\mu^6;s,t]\Big(\frac{1}{u^2t^2}\Big)-I_4[\mu^4;s,t]\Big(\frac{2s}{tu^3}\Big)\\
&+I_3[\mu^4;t]\Big(\frac{2(t^3+u^3)}{t^4u^3}\Big)+I_3[\mu^4;s]\Big(\frac{2(6s^2+8st+3t^2)}{s^3u^3}\Big)\\
&+I_2[\mu^4;t]\Big(\frac{(2u-t)(4u+3t)}{3t^5u^2}\Big)+I_2[\mu^4;s]\Big(\frac{(s+2t)(3s^2-8st-8t^2)}{3s^4t^2u^2}\Big)\\
&-I_4[\mu^2;s,t]\Big(\frac{s^2}{2u^4}\Big)\\
&+I_3[\mu^2;t]\Big(\frac{s}{u^4}\Big)-I_3[\mu^2;s]\Big(\frac{(2s+t)(2s^2+2st+t^2)}{s^2u^4}\Big)\\
&-I_2[\mu^2;t]\Big(\frac{s(6t^2-3tu+2u^2)}{6t^4u^3}\Big)+I_2[\mu^2;s]\Big(\frac{11s^3+59s^2t+64st^2+22t^3}{6s^3tu^3}\Big)\Bigg\}.
\end{aligned}
\tag{8.6}
$$

The topology (1243) can be obtained by swapping $3\leftrightarrow 4$ in (8.6), or $(s,t,u)\to(s,u,t)$. Noting that $\mathcal{J}$ is invariant under this swap we get:

$$
\begin{aligned}
(1243):\ & \frac{4i}{(4\pi)^{2-\epsilon}}\,\mathcal{J}\Bigg\{-I_4[\mu^6;s,u]\Big(\frac{1}{u^2t^2}\Big)-I_4[\mu^4;s,u]\Big(\frac{2s}{ut^3}\Big)\\
&+I_3[\mu^4;u]\Big(\frac{2(t^3+u^3)}{u^4t^3}\Big)+I_3[\mu^4;s]\Big(\frac{2(6s^2+8su+3u^2)}{s^3t^3}\Big)\\
&+I_2[\mu^4;u]\Big(\frac{(2t-u)(4t+3u)}{3u^5t^2}\Big)+I_2[\mu^4;s]\Big(\frac{(s+2u)(3s^2-8su-8u^2)}{3s^4t^2u^2}\Big)\\
&-I_4[\mu^2;s,u]\Big(\frac{s^2}{2t^4}\Big)+I_3[\mu^2;u]\Big(\frac{s}{t^4}\Big)-I_3[\mu^2;s]\Big(\frac{(2s+u)(2s^2+2su+u^2)}{s^2t^4}\Big)\\
&-I_2[\mu^2;u]\Big(\frac{s(6u^2-3tu+2t^2)}{6u^4t^3}\Big)\ +\ I_2[\mu^2;s]\Big(\frac{11s^3+59s^2u+64su^2+22u^3}{6s^3ut^3}\Big)\Bigg\}.
\end{aligned}
\tag{8.7}
$$

Next, we merge the u- and t-cuts for the topology (1324):

$$
\begin{aligned}
(1324):\ & \frac{4i}{(4\pi)^{2-\epsilon}}\,\mathcal{J}\Bigg\{-I_4[\mu^6;u,t]\Big(\frac{1}{u^2t^2}\Big)-I_4[\mu^4;u,t]\Big(\frac{1}{2sut}\Big)+I_3[\mu^4;t]\Big(\frac{2u+3t}{st^4}\Big)\\
&+I_3[\mu^4;u]\Big(\frac{2t+3u}{su^4}\Big)+I_2[\mu^4;t]\Big(\frac{(t-2u)(3t+4u)}{3u^2t^5}\Big)+I_2[\mu^4;u]\Big(\frac{(u-2t)(4t+3u)}{3t^2u^5}\Big)\\
&+I_2[\mu^2;u]\Big(\frac{s}{3u^4t}\Big)+I_2[\mu^2;t]\Big(\frac{s}{3ut^4}\Big)\Bigg\}.
\end{aligned}
\tag{8.8}
$$

As expected, the expression (8.8) is symmetric in $u\leftrightarrow t$.

Finally we take the four-dimensional limit of (8.6), (8.7) and (8.8). These are given by

$$-\frac{i}{(4\pi)^2}\,\mathcal{J}\,\frac{(3t^2+ut+u^2)}{15\,s^2\,t^3}\,, \qquad -\frac{i}{(4\pi)^2}\,\mathcal{J}\,\frac{(3u^2+ut+t^2)}{15\,s^2\,u^3}\,, \qquad -\frac{i}{(4\pi)^2}\,\mathcal{J}\,\frac{s(2t^2+ut+2u^2)}{30\,u^3\,t^3}\,, \tag{8.9}$$

respectively. Thus, we arrive at the final result for the four-dimensional limit of the amplitude, using the expression for $\mathcal{J}$ in (8.1),

$$A^{(1)}(1^+,2^+;3^{++},4^{--}) \;=\; \frac{i}{(4\pi)^2}\,\frac{[1\,2][1\,3]^4\langle 1\,4\rangle^4}{\langle 1\,2\rangle}\,\frac{t^2+u^2}{6\,s\,t^2\,u^2}\;. \tag{8.10}$$

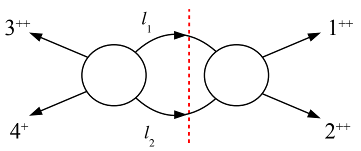
The D-dimensional result is obtained by adding (8.6), (8.7) and (8.8).

9 The $\langle 1^{++}\,2^{++}\,3^{++}\,4^{\pm}\rangle$ amplitudes

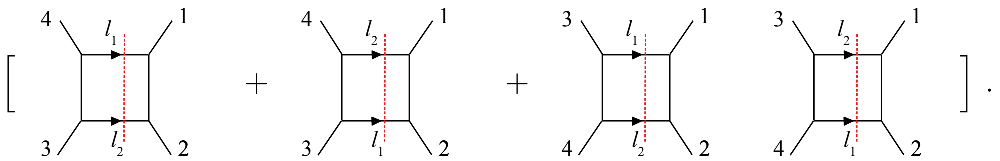
We now move on to consider the one-loop amplitudes with three gravitons and a gluon, beginning with the amplitude with three same-helicity gravitons and one gluon. It is easy to show that this amplitude vanishes upon integration. Consider for instance the s-cut diagram of the $\langle 1^{++}\,2^{++}\,3^{++}\,4^{+}\rangle$ amplitude. Its expression is

$$\mu^2\frac{[34]}{\langle 34\rangle^2}\langle 4|l_2|3]\,\left[\frac{i}{(l_2+p_3)^2-\mu^2}+l_1\leftrightarrow l_2\right]\,\mu^4\frac{[12]^2}{\langle 12\rangle^2}\left[\frac{-i}{(l_1-p_1)^2-\mu^2}+p_1\leftrightarrow p_2\right], \tag{9.1}$$

or

s-cut: 

$$= \; -\mu^2\frac{[34]}{\langle 34\rangle^2}\langle 4|l_2|3]\;\mu^4\frac{[12]^2}{\langle 12\rangle^2}\;. \tag{9.2}$$



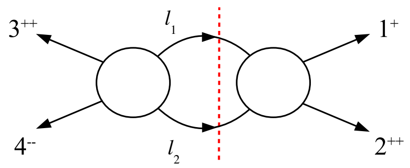
Although the integrand does not vanish, the integrated expression does because it is an odd function under the exchange of $l_1 \leftrightarrow l_2$. The t- and u-cut are simply given by permutations of the s-cut and hence combining the three cuts one obtains a vanishing integrated expression. Finally, using (2.10) it is immediate to see that also the $\langle 1^{++}\,2^{++}\,3^{++}\,4^{-}\rangle$ amplitude vanishes for the same reason. In conclusion,

$$A^{(1)}(1^{++},2^{++},3^{++};4^{\pm}) \;=\; 0\;. \tag{9.3}$$

10 The $\langle 1^+ \, 2^{++} \, 3^{++} \, 4^{--} \rangle$ amplitude

Similarly to the previous section, we can easily show that the amplitude $\langle 1^+ \, 2^{++} \, 3^{++} \, 4^{--} \rangle$ vanishes upon integration. Consider for instance its s-channel cut. This is given by

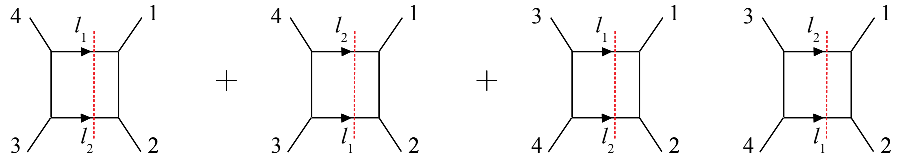
s-cut:



$$= A(3^{++}, 4^{--}, l_{1,\phi}, l_{2,\bar{\phi}}) A(1^+, 2^{++}, -l_{2,\phi}, -l_{1,\bar{\phi}})$$

$$= \left[\frac{-\langle 4|l_2|3]^4}{s^2} \left[\frac{i}{(l_2+p_3)^2 - \mu^2} + l_1 \leftrightarrow l_2 \right] \right] \left[\mu^2 \frac{[21]}{\langle 21 \rangle^2} \langle 1| - l_1 |2] \left[\frac{i}{(l_1 - p_2)^2 - \mu^2} + l_1 \leftrightarrow l_2 \right] \right]$$

$$= \mu^2 \frac{[21]}{\langle 21 \rangle^2} \frac{\langle 4|l_2|3]^4 \langle 1|l_2|2]}{s^2} \, .$$



$$\Big[\quad \cdots \quad \Big] \, . \tag{10.1}$$

Again, the integrated expression is an odd function under $l_1 \leftrightarrow l_2$ and hence it vanishes. The same holds true for the t- and u- channel cuts. In summary, we get

$$A^{(1)}(1^+; 2^{++}, 3^{++}, 4^{--}) = 0 \, . \tag{10.2}$$

11 The $\langle 1^+ \, 2^+ \, 3^+ \, 4^{++} \rangle$ amplitude from the double copy

The color-kinematic duality or double copy [13] was extended in the works [14, 18] also to the domain of mixed graviton-gluon amplitudes in the Einstein-Yang-Mills theory. In particular [18] exposed explicitly how to construct an Einstein-Yang-Mills amplitude through a double copy from Yang-Mills and Yang-Mills + ϕ^3 theory:

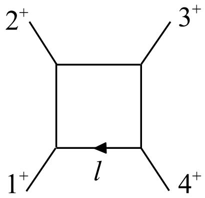
$$A_{\text{EYM}} = A_{\text{YM}} \otimes A_{\text{YM}+\phi^3} \, . \tag{11.1}$$

The latter Yang-Mills-Scalar theory contains biadjoint scalars $\phi^{A\hat{a}}$ next to the gluons $A_\mu^{\hat{a}}$ and is defined through the Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{YM}+\phi^3} = & -\frac{1}{4} F^{\hat{a}}_{\mu\nu} F^{\mu\nu\,\hat{a}} + \frac{1}{2} (D_\mu \phi^A)^{\hat{a}} (D^\mu \phi^A)^{\hat{a}} + \frac{1}{3!} \lambda \, g \, F^{ABC} f^{\hat{a}\hat{b}\hat{c}} \phi^{A\hat{a}} \phi^{B\hat{b}} \phi^{C\hat{c}} \\ & - \frac{g^2}{4} f^{\hat{a}\hat{b}\hat{d}} f^{\hat{d}\hat{c}\hat{d}} \phi^{A\hat{a}} \phi^{B\hat{b}} \phi^{A\hat{c}} \phi^{B\hat{d}} \, . \end{aligned} \tag{11.2}$$

As a one-loop application of (11.1), we wish to derive the vanishing of the $\langle 1^+\,2^+\,3^+\,4^{++}\rangle$ amplitude, which we observed with a direct computation in Section 3. Thus we need to construct integrands for the two amplitudes $A^{(1)}(1^+,2^+,3^+,4^+)$ and $A^{(1)}(1_\phi^{A_1},2_\phi^{A_2},3_\phi^{A_3},4^+)$ where color ordering is performed in both cases with respect to the hatted gauge group index. The first one, the all-plus helicity four-gluon amplitude, is well-known and takes the form

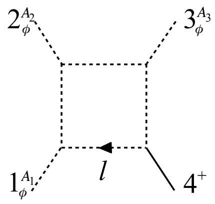
$$A^{(1)}(1^+,2^+,3^+,4^+) =$$



$$= \frac{[12][34]}{\langle 12\rangle\langle 34\rangle}\int \frac{d^4l}{(2\pi)^4}\frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}}\,\frac{\mu^4}{D_0D_1D_2D_3}\,. \tag{11.3}$$

As this is a pure box-integral, in the construction of the one-loop YM + ϕ^3 amplitude integrand we only need to construct the box-contribution to the $A^{(1)}(1_\phi^A,2_\phi^B,3_\phi^C,4^+)$ amplitude as well:

$$A^{(1)}(1_\phi^{A_1},2_\phi^{A_2},3_\phi^{A_3},4^+)\Big|_{\text{boxes}} =$$



$$= i\int \frac{d^4l}{(2\pi)^4}\frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}}\,\frac{f^{A_1A_2A_3}}{D_0D_1D_2D_3}\frac{\langle q|l|4]}{\langle q4\rangle}$$

$$+\ \text{cycl(1,2,3)}\,. \tag{11.4}$$

Here we have simply inserted the scalar-scalar-on-shell-gluon vertex of (2.1) in the south-east corner with a reference spinor λ_q. The numerator emerging from this integrand respects color-kinematics duality as it is built entirely from three-valent graphs. Employing the double-copy prescription [18] of (11.1) we are therefore led to the following representation of the all-plus single-gluon EYM-amplitude

$$A^{(1)}(1^+_{A_1},2^+_{A_2},3^+_{A_3},4^{++}) = if^{A_1A_2A_3}\,\frac{[12][34]}{\langle 12\rangle\langle 34\rangle}\int\frac{d^4l}{(2\pi)^4}\frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}}\frac{\mu^4}{D_0D_1D_2D_3}\,\frac{\langle q|l|4]}{\langle q4\rangle}+\text{cycl(1,2,3)}\,. \tag{11.5}$$

Passarino-Veltman reducing the integral one arrives at the D-dimensional expression

$$\begin{aligned}A^{(1)}(1^+_{A_1},2^+_{A_2},3^+_{A_3},4^{++}) = if^{A_1A_2A_3}\,\frac{[12][34]}{\langle 12\rangle\langle 34\rangle}\,\frac{1}{\langle q4\rangle}\,\frac{1}{u}\Big\{&\tfrac{1}{2}(t\langle q|3|4]-s\langle q|1|4])I_4[\mu^4;s,t]\\ &+\langle q|2|4]\,(I_3[\mu^4;s]-I_3[\mu^4;t])\Big\}+\text{cycl(1,2,3)}\,.\end{aligned} \tag{11.6}$$

Going to four dimensions simplifies this result considerably, and one arrives at

$$A^{(1)}(1^+_{A_1},2^+_{A_2},3^+_{A_3},4^{++}) = if^{A_1A_2A_3}\,\frac{[12][34]}{\langle 12\rangle\langle 34\rangle}\,\frac{1}{\langle q4\rangle}\,\frac{1}{12}\Big\{\langle q|3|4]+\tfrac{1}{2}\langle q|2|4]\Big\}+\text{cycl(1,2,3)} = 0\,. \tag{11.7}$$

The expression above vanishes as the prefactor is invariant under cyclic shifts in $(1,2,3)$ and obviously the bracketed terms sum to zero, as $\langle q|3|4] + \text{cycl}(1,2,3) = \langle q|\, p_1+p_2+p_3\, |4] = 0$. Hence, we have reproduced the vanishing result of Section 3.

Finally, we comment on the question whether the amplitude relations of Stieberger and Taylor [15, 31] relating pairs of collinear gluons to gravitons extend to the one-loop level for the one-loop rational amplitudes we have considered in this paper.

We will test this for the simplest case of the all-plus amplitude with one graviton. For such a relation to be true, the vanishing four-dimensional result must follow from the specific collinear limit proposed by Stieberger and Taylor on the five-point all-plus rational amplitude in pure Yang-Mills. In analogy to the tree-level relation, in four dimensions we expect to have:

$$A^{(1)}_{\text{EYM;ST}}(1^+,2^+,3^+,P^{++}) \overset{?}{=} \frac{\kappa}{g^2}\,\mathcal{G}(x) \lim_{p_4\|p_5} s_{24} A^{(1)}_{\text{YM}}(1^+,5^+,2^+,4^+,3^+) \;+\; \text{cycl}(1,2,3)\,, \tag{11.8}$$

where the equality would hold in the collinear limit $\{p_4 \to xP,\, p_5 \to (1-x)P\}$ on the right-hand side of (11.8), and $\mathcal{G}(x)$ is an undetermined function of the momentum splitting fraction x which is expected to be independent of the helicities of the particles. Note that $\mathcal{G}(x)$ has been determined for tree amplitudes in [31]. We have also added cyclic permutations of the three gluons to secure cyclic symmetry in these particles. Using the well-known expression for the all-plus five-point rational amplitude in Yang-Mills [32], we see that the right-hand side of (11.8) contains the factor

$$s_{24}A^{(1)}_{\text{YM}}(1^+,5^+,2^+,4^+,3^+) = s_{24}\frac{i}{48\pi^2}\frac{-s_{15}s_{52} - s_{13}s_{43} + \langle 5\,2\rangle\langle 4\,3\rangle[2\,4][3\,5]}{\langle 1\,5\rangle\langle 5\,2\rangle\langle 2\,4\rangle\langle 4\,3\rangle\langle 3\,1\rangle}\,. \tag{11.9}$$

Performing the above-mentioned collinear limit on (11.9), followed by a cyclic permutation of the three gluon legs in order to reflect the anticipated color structure, and relabelling $P \to p_4$ (with p_4 being the momentum of the graviton leg), we arrive at

$$\lim_{p_4\|p_5} s_{24}\, A^{(1)}_{\text{YM}}(1^+,5^+,2^+,4^+,3^+) \;+\; \text{cycl}(1,2,3) \;=\; \left[\frac{1}{(1-x)} - 2x\right]\; \frac{1}{2}(st+ut+su)\,\mathcal{A}_0\,, \tag{11.10}$$

with

$$\mathcal{A}_0 \;:=\; \frac{i}{48\pi^2}\frac{\langle 2|1|4]}{\langle 2\,4\rangle}\frac{1}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle}\,. \tag{11.11}$$

Clearly this does not vanish and hence invalidates the conjecture (11.8). However we note the following rather intriguing similarity: Consider again our D-dimensional result for this amplitude as obtained in (3.11), and focus only on the pure box contribution; evaluated in the $D\to 4$ limit, it gives

$$6\mathcal{A}_0\left[\frac{st}{2}I_4[\mu^4;s,t]\right]\Bigg|_{D=4} \;+\; \text{perms} \;=\; -\frac{1}{2}(st+ut+su)\mathcal{A}_0\,. \tag{11.12}$$

This is curiously proportional to the x-independent part of the right-hand side of (11.10), which was obtained from the Stieberger-Taylor collinear limit. Given the vanishing of our final result in four dimensions, also the triangle contribution in (3.11) can be written in a similar way:

$$6\,\mathcal{A}_0\left[\,sI_3[\mu^4;t]\;+\;tI_3[\mu^4;s]\right]\Big|_{D=4}\;+\;\text{perms}\;=\;\frac{1}{2}(st+ut+su)\,\mathcal{A}_0\,. \tag{11.13}$$

In conclusion, even though the amplitude (3.11) vanishes in four dimensions, we find the similarities between (11.12) (or (11.13)) and (11.10) intriguing, and worth further investigation.

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A Integrals

The integral functions used in this paper are defined as:

$$\begin{aligned}\int\frac{d^{4-2\epsilon}L}{(2\pi)^{4-2\epsilon}}\frac{\mu^m}{L^2\cdots\left[(L-\sum_{i=1}^{n-1}p_i)^2\right]}\;&=\;\int\frac{d^4l}{(2\pi)^4}\frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}}\frac{\mu^m}{(l^2-\mu^2)\cdots\left[(l-\sum_{i=1}^{n-1}p_i)^2-\mu^2\right]}\\&:=\;\frac{i}{(4\pi)^{2-\epsilon}}\,I_n[\mu^m]\;,\end{aligned} \tag{A.1}$$

where $L^2=L^2_{(4)}+L^2_{(-2\epsilon)}:=l^2-\mu^2$.[12] The exact expressions for the bubble, one-mass triangle and zero-mass box integral functions in $4-2\epsilon$ dimensions following from the definition (A.1) are

$$I_2[1;s]\;=\;r_\Gamma\frac{(-s)^{-\epsilon}}{\epsilon(1-2\epsilon)}\;, \tag{A.2}$$

[12]Our definition (A.1) differs from [22] in that we do not include a factor of $(-)^n$ on the right-hand side of this equation. Hence, note the minus sign on the right-hand side of (A.3), in contradistinction with e.g. (I.4) of [30].

$$I_3[1;s] \;=\; -\frac{r_\Gamma}{\epsilon^2}(-s)^{-1-\epsilon}\;, \tag{A.3}$$

for the bubble and one-mass triangle, while for the zero-mass box function one has [33]

$$I_4[1;s,t] \;=\; r_\Gamma \frac{2}{st}\left[\frac{(-s)^{-\epsilon}}{\epsilon^2}\,{}_2F_1\left(1,-\epsilon,1-\epsilon;1+\frac{s}{t}\right)+\frac{(-t)^{-\epsilon}}{\epsilon^2}\,{}_2F_1\left(1,-\epsilon,1-\epsilon;1+\frac{t}{s}\right)\right]\;, \tag{A.4}$$

where

$$r_\Gamma := \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\;. \tag{A.5}$$

The results (A.2), (A.3) and (A.4) are exact to all orders in ϵ, and the expression of the corresponding integral functions in a different number of dimensions can be obtained by simply replacing ϵ to the appropriate value, for instance $\epsilon \to \epsilon - 1$ and $\epsilon \to \epsilon - 2$ for $D = 6 - 2\epsilon$ and $D = 8 - 2\epsilon$, respectively. The dependence on the relevant kinematic invariants is indicated in brackets along with the power of μ. Using [22]

$$I_n^{D=4-2\epsilon}[(\mu^2)^p] \;=\; -\epsilon(1-\epsilon)(2-\epsilon)\cdots(p-1-\epsilon)\,I_n^{D=2p+4-2\epsilon}\;, \tag{A.6}$$

along with the expressions (A.2), (A.3) and (A.4), which are correct in any number of dimensions, one easily arrives at the following result, used widely in this paper:

$$\begin{aligned}
I_2[\mu^2;s] &= -\frac{s}{6}+\mathcal{O}(\epsilon)\;, \qquad I_2[\mu^4;s] = -\frac{s^2}{60}+\mathcal{O}(\epsilon)\;,\\
I_3[\mu^2;s] &= \frac{1}{2}+\mathcal{O}(\epsilon)\;, \qquad I_3[\mu^4;s] \;=\; \frac{s}{24}+\mathcal{O}(\epsilon)\;,\\
I_4[\mu^2;s,t] &= \mathcal{O}(\epsilon)\,, \qquad I_4[\mu^4;s,t] \;=\; -\frac{1}{6}+\mathcal{O}(\epsilon),\\
I_4[\mu^6;s,t] &= -\frac{s+t}{60}+\mathcal{O}(\epsilon)\;,\\
I_4[\mu^8;s,t] &= -\frac{1}{840}\left(2s^2+st+2t^2\right)+\mathcal{O}(\epsilon)\;\;,
\end{aligned} \tag{A.7}$$

in complete agreement with results of [22, 7] (after taking into account the opposite sign in the definition of triangle functions compared to those papers).

B Tree-level amplitudes via recursion relations

In this appendix we derive the relevant tree amplitudes involving gravitons, gluons and massive scalars which enter the one-loop calculations in EYM performed in earlier sections.

The $A(4^{++},1^+,2_\phi,3_{\bar\phi})$ amplitude

We use a BCFW recursion relation with a $\langle 4\,1]$ shift, i.e. we perform a shift

$$\hat\lambda_4 = \lambda_4 + z\lambda_1\,, \qquad \hat{\tilde\lambda}_1 = \tilde\lambda_1 - z\tilde\lambda_4\;. \tag{B.1}$$

There are two recursion diagrams to compute, A and B. The first one is

$$A_A(4^{++}, 1^+, 2_\phi, 3_{\bar{\phi}}) \;=\; A(\hat{4}^{++}, \hat{P}_\phi, 3_{\bar{\phi}}) \, \frac{i}{(p_3+p_4)^2-\mu^2} \, A(\hat{1}^+, 2_\phi, -\hat{P}_{\bar{\phi}}) \; . \tag{B.2}$$

In accordance with (2.1) and (2.2) we have

$$\begin{aligned} A(\hat{4}^{++}, \hat{P}_\phi, 3_{\bar{\phi}}) &= -i \; \frac{\langle q_1|3|4]^2}{\langle q_1 \hat{4}\rangle^2} \; , \\ A(\hat{1}^+, 2_\phi, -\hat{P}_{\bar{\phi}}) &= i \; \frac{\langle q_2| - \hat{P}|\hat{1}]}{\langle q_2 1\rangle} \; , \end{aligned} \tag{B.3}$$

with $\hat{P} = \hat{p}_1 + p_2$. The reference spinors q_1 and q_2 can be conveniently chosen to be $q_2 = \hat{4}$ and $q_1 = 1$. Using

$$\langle 1|3|4]^2 \; \langle \hat{4}| - \hat{P}|\hat{1}] = -\mu^2 \, s_{14} \, \langle 1|3|4] \; , \tag{B.4}$$

one quickly arrives at the result

$$A_A(4^{++}, 1^+, 2_\phi, 3_{\bar{\phi}}) \;=\; -i^2 \, \mu^2 \, \frac{[41]}{\langle 41\rangle^2} \langle 1|3|4] \; \frac{i}{(p_3+p_4)^2-\mu^2} \; . \tag{B.5}$$

The second diagram corresponds to swapping the position of the graviton with the gluon, to account for the fact that the graviton is colour blind. We have

$$A_B(4^{++}, 1^+, 2_\phi, 3_{\bar{\phi}}) \;=\; A(\hat{1}^+, \hat{P}_\phi, 3_{\bar{\phi}}) \frac{i}{(p_2+p_4)^2-\mu^2} \, A(\hat{4}^{++}, 2_\phi, -\hat{P}_{\bar{\phi}}) \; . \tag{B.6}$$

With the same choice of reference spinors, we get

$$A_B(4^{++}, 1^+, 2_\phi, 3_{\bar{\phi}}) \;=\; -i^2 \, \mu^2 \, \frac{[41]}{\langle 41\rangle^2} \langle 1|3|4] \; \frac{i}{(p_2+p_4)^2-\mu^2} \; , \tag{B.7}$$

and hence the result for the complete amplitude is

$$A(4^{++}, 1^+, 2_\phi, 3_{\bar{\phi}}) \;=\; \mu^2 \, \frac{[41]}{\langle 41\rangle^2} \langle 1|3|4] \; \left[\frac{i}{(p_3+p_4)^2-\mu^2} + \frac{i}{(p_2+p_4)^2-\mu^2} \right] . \tag{B.8}$$

Note that this amplitude vanishes for $\mu^2 = 0$.

Soft limits of the $A(4^{++}, 1^+, 2_\phi, 3_{\bar{\phi}})$ amplitude

It is an interesting check to confirm that the amplitude obtained in this way has the correct soft limits. To this end we consider the case with gluon 1^+ becoming soft. We then expect the amplitude to factorize as

$$A(4^{++}, 1^+, 2_\phi, 3_{\bar{\phi}}) \xrightarrow[p_1 \to 0]{} S_1^{(0)} A(4^{++}; 2_\phi, 3_{\bar{\phi}}) \; , \tag{B.9}$$

where the soft function is

$$S_1^{(0)} = \frac{p_2 \cdot \varepsilon(p_1)}{\sqrt{2}(p_2 \cdot p_1)} - \frac{p_3 \cdot \varepsilon(p_1)}{\sqrt{2}(p_3 \cdot p_1)} . \tag{B.10}$$

Using $\varepsilon_\nu^{(+)}(p_1) = \langle\xi|\nu|1]/(\sqrt{2}\langle\xi\, 1\rangle)$, where $|\xi\rangle$ is a reference spinor, and choosing for convenience $\xi = 4$, we get

$$S_1^{(0)} A(4^{++}; 2_\phi, 3_{\bar\phi}) = i\,\frac{\langle 4|3|1]}{\langle 41\rangle}\Big[\frac{1}{2(p_2 \cdot p_1)} + \frac{1}{2(p_3 \cdot p_1)}\Big]\frac{\langle q|3|4]^2}{\langle q\, 4\rangle^2} . \tag{B.11}$$

In the soft limit, one easily finds that

$$\langle 4|3|1]\langle q|3|4] \xrightarrow[p_1 \to 0]{} -\mu^2 \langle q4\rangle[41] , \tag{B.12}$$

and choosing the arbitrary spinor q to be equal to 1, we finally get

$$S_1^{(0)} A(4^{++}; 2_\phi, 3_{\bar\phi}) \xrightarrow[p_1 \to 0]{} i\,\mu^2 \frac{[41]}{\langle 41\rangle^2}\langle 1|3|4]\Big[\frac{1}{2(p_2 \cdot p_1)} + \frac{1}{2(p_3 \cdot p_1)}\Big] , \tag{B.13}$$

which is identical to the result for $A(4^{++}, 1^+, 2_\phi, 3_{\bar\phi})$.

The $A(4^{++}, 1^-, 2_\phi, 3_{\bar\phi})$ amplitude

We will use the same BCFW shift as in (B.1). Again, there are two recursion diagrams to compute, A, and B. The first one is

$$A_A(4^{++}, 1^-, 2_\phi, 3_{\bar\phi}) = A(\hat{4}^{++}, \hat{P}_\phi, 3_{\bar\phi})\,\frac{i}{(p_3 + p_4)^2 - \mu^2}\,A(\hat{1}^-, 2_\phi, -\hat{P}_{\bar\phi}) , \tag{B.14}$$

with

$$\begin{aligned} A(\hat{4}^{++}, \hat{P}_\phi, 3_{\bar\phi}) &= -i\,\frac{\langle q_1|3|4]^2}{\langle q_1 \hat{4}\rangle^2} , \\ A(\hat{1}^-, 2_\phi, -\hat{P}_{\bar\phi}) &= i\,\frac{\langle 1| - \hat{P}|q_2]}{[\hat{1} q_2]} , \end{aligned} \tag{B.15}$$

and with $\hat{P} = \hat{p}_1 + p_2$. A convenient choice for the reference spinors q_1 and q_2 is again $q_2 = \hat{4}$ and $q_1 = 1$, which immediately leads to

$$A_A(4^{++}, 1^-, 2_\phi, 3_{\bar\phi}) = -i^2\,\frac{\langle 1|2|4]^3}{\langle 14\rangle\, s_{14}}\,\frac{i}{(p_3 + p_4)^2 - \mu^2} . \tag{B.16}$$

Similarly

$$A_B(4^{++}, 1^-, 2_\phi, 3_{\bar\phi}) = A(\hat{1}^-, \hat{P}_\phi, 3_{\bar\phi})\,\frac{i}{(p_2 + p_4)^2 - \mu^2}\,A(\hat{4}^{++}, 2_\phi, -\hat{P}_{\bar\phi}) , \tag{B.17}$$

which leads to

$$A_B(4^{++},1^-,2_\phi,3_{\bar\phi}) \;=\; -i^2\,\frac{\langle 1|2|4]^3}{\langle 14\rangle\, s_{14}}\;\frac{i}{(p_2+p_4)^2-\mu^2}\;. \tag{B.18}$$

Adding the two contributions, we get

$$A(4^{++},1^-,2_\phi,3_{\bar\phi}) \;=\; \frac{\langle 1|2|4]^3}{\langle 14\rangle\, s_{14}}\;\Big[\frac{i}{(p_3+p_4)^2-\mu^2}+\frac{i}{(p_2+p_4)^2-\mu^2}\Big]\;. \tag{B.19}$$

Note that this amplitude does not vanish for $\mu^2=0$.

The $A(1^{++},2^{++},3_\phi,4_{\bar\phi})$ amplitude

We now consider the case of two gravitons and two scalars. The simplest case to consider is that of two gravitons of the same helicity, already considered in [7] in the computation of all-plus graviton amplitudes. We will use the shifts

$$\hat\lambda_1=\lambda_1+z\lambda_2\,,\qquad\qquad \hat{\tilde\lambda}_2=\tilde\lambda_2-z\tilde\lambda_1\;. \tag{B.20}$$

As usual, there are two diagrams to consider. The first one is

$$A_A(1^{++},2^{++},3_\phi,4_{\bar\phi}) \;=\; A(\hat 1^{++},\hat P_\phi,4_{\bar\phi})\,\frac{i}{(p_4+p_1)^2-\mu^2}\,A(\hat 2^{++},3_\phi,-\hat P_{\bar\phi})\;, \tag{B.21}$$

while $A_B(1^{++},2^{++},3_\phi,4_{\bar\phi})=\big[A_A(1^{++},2^{++},3_\phi,4_{\bar\phi})\big]_{1\leftrightarrow 2}$. Thus, using (2.1) we get

$$A_A=(-i\,)\frac{\langle q_1|4|1]^2}{\langle q_1\hat 1\rangle^2}\frac{i}{(p_4+p_1)^2-\mu^2}(-i\,)\frac{\langle q_2|-\hat P|\hat 2]^2}{\langle q_2 2\rangle^2}\;. \tag{B.22}$$

Choosing $q_2=\hat 1$, $q_1=2$ and using $\langle q_1|4|1]\langle q_2|-\hat P|\hat 2]=-\mu^2 s_{12}$, we finally arrive at

$$A(1^{++},2^{++},3_\phi,4_{\bar\phi}) \;=\; -\mu^4\frac{[12]^2}{\langle 12\rangle^2}\Big[\frac{i}{(p_4+p_1)^2-\mu^2}+\frac{i}{(p_3+p_4)^2-\mu^2}\Big]\;. \tag{B.23}$$

Note that (B.23) agrees with (4.10) of [7].

References

[1] M. T. Grisaru and H. Pendleton, *"Some Properties of Scattering Amplitudes in Supersymmetric Theories"*, Nucl. Phys. B124, 81 (1977).

[2] Z. Bern, L. J. Dixon and D. A. Kosower, *"On-shell recurrence relations for one-loop QCD amplitudes"*, Phys. Rev. D71, 105013 (2005), hep-th/0501240.

[3] Z. Bern, G. Chalmers, L. J. Dixon and D. A. Kosower, *"One loop N-gluon amplitudes with maximal helicity violation via collinear limits"*, Phys. Rev. Lett. 72, 2134 (1994), hep-ph/9312333. • G. Mahlon, *"One loop multi-photon helicity amplitudes"*, Phys. Rev. D49, 2197 (1994), hep-ph/9311213.

[4] L. J. Dixon, *"Calculating scattering amplitudes efficiently"*, hep-ph/9601359. • H. Elvang and Y.-t. Huang, *"Scattering Amplitudes in Gauge Theory and Gravity"*, Cambridge University Press (2015). • J. M. Henn and J. C. Plefka, *"Scattering Amplitudes in Gauge Theories"*, Lect. Notes Phys. 883, 1 (2014).

[5] W. A. Bardeen, *"Selfdual Yang-Mills theory, integrability and multiparton amplitudes"*, Prog. Theor. Phys. Suppl. 123, 1 (1996), in: *"From the standard model to grand unified theories. Proceedings, 6th Yukawa International Seminar, YKIS'95, Kyoto, Japan, August 21-25, 1995"*, pp. 1-8. • G. Chalmers and W. Siegel, *"The self-dual sector of QCD amplitudes"*, Phys. Rev. D54, 7628 (1996), hep-th/9606061. • D. Cangemi, *"Selfdual Yang-Mills theory and one loop like-helicity QCD multi-gluon amplitudes"*, Nucl. Phys. B484, 521 (1997), hep-th/9605208.

[6] G. Mahlon, *"Multi-gluon helicity amplitudes involving a quark loop"*, Phys. Rev. D49, 4438 (1994), hep-ph/9312276.

[7] Z. Bern, L. J. Dixon, M. Perelstein and J. S. Rozowsky, *"Multileg one loop gravity amplitudes from gauge theory"*, Nucl. Phys. B546, 423 (1999), hep-th/9811140.

[8] S. D. Alston, D. C. Dunbar and W. B. Perkins, *"n-point amplitudes with a single negative-helicity graviton"*, Phys. Rev. D92, 065024 (2015), arxiv:1507.08882.

[9] D. C. Dunbar, J. H. Ettle and W. B. Perkins, *"Augmented Recursion For One-loop Gravity Amplitudes"*, JHEP 1006, 027 (2010), arxiv:1003.3398.

[10] Z. Bern, D. C. Dunbar and T. Shimada, *"String based methods in perturbative gravity"*, Phys. Lett. B312, 277 (1993), hep-th/9307001. • D. C. Dunbar and P. S. Norridge, *"Calculation of graviton scattering amplitudes using string based methods"*, Nucl. Phys. B433, 181 (1995), hep-th/9408014. • A. Brandhuber, S. McNamara, B. Spence and G. Travaglini, *"Recursion relations for one-loop gravity amplitudes"*, JHEP 0703, 029 (2007), hep-th/0701187.

[11] K. G. Selivanov, *"SD perturbiner in Yang-Mills + gravity"*, Phys. Lett. B420, 274 (1998), hep-th/9710197. • Z. Bern, A. De Freitas and H. L. Wong, *"On the coupling of gravitons to matter"*, Phys. Rev. Lett. 84, 3531 (2000), hep-th/9912033.

[12] F. Cachazo, S. He and E. Y. Yuan, *"Scattering of Massless Particles in Arbitrary Dimensions"*, Phys. Rev. Lett. 113, 171601 (2014), arxiv:1307.2199. • F. Cachazo, S. He and E. Y. Yuan, *"Einstein-Yang-Mills Scattering Amplitudes From Scattering Equations"*, JHEP 1501, 121 (2015), arxiv:1409.8256.

[13] Z. Bern, J. Carrasco and H. Johansson, *"New Relations for Gauge-Theory Amplitudes"*, Phys. Rev. D78, 085011 (2008), arxiv:0805.3993. • Z. Bern, J. J. M. Carrasco and H. Johansson, *"Perturbative Quantum Gravity as a Double Copy of Gauge Theory"*, Phys. Rev. Lett. 105, 061602 (2010), arxiv:1004.0476.

[14] M. Chiodaroli, M. Günaydin, H. Johansson and R. Roiban, *"Scattering amplitudes in $\mathcal{N} = 2$ Maxwell-Einstein and Yang-Mills/Einstein supergravity"*, JHEP 1501, 081 (2015), arxiv:1408.0764. • M. Chiodaroli, M. Günaydin, H. Johansson and R. Roiban, *"Spontaneously Broken Yang-Mills-Einstein Supergravities as Double Copies"*, JHEP 1706, 064 (2017), arxiv:1511.01740. • M. Chiodaroli, *"Simplifying amplitudes in Maxwell-Einstein and Yang-Mills-Einstein supergravities"*, arxiv:1607.04129.

[15] S. Stieberger and T. R. Taylor, *"New relations for Einstein-Yang-Mills amplitudes"*, Nucl. Phys. B913, 151 (2016), arxiv:1606.09616.

[16] D. Nandan, J. Plefka, O. Schlotterer and C. Wen, *"Einstein-Yang-Mills from pure Yang-Mills amplitudes"*, JHEP 1610, 070 (2016), arxiv:1607.05701.

[17] L. de la Cruz, A. Kniss and S. Weinzierl, *"Relations for Einstein-Yang-Mills amplitudes from the CHY representation"*, Phys. Lett. B767, 86 (2017), arxiv:1607.06036.

[18] M. Chiodaroli, M. Günaydin, H. Johansson and R. Roiban, *"Explicit Formulae for Yang-Mills-Einstein Amplitudes from the Double Copy"*, JHEP 1707, 002 (2017), arxiv:1703.00421.

[19] F. Teng and B. Feng, *"Expanding Einstein-Yang-Mills by Yang-Mills in CHY frame"*, JHEP 1705, 075 (2017), arxiv:1703.01269.

[20] Y.-J. Du, B. Feng and F. Teng, *"Expansion of All Multitrace Tree Level EYM Amplitudes"*, JHEP 1712, 038 (2017), arxiv:1708.04514.

[21] J. M. Drummond and J. M. Henn, *"All tree-level amplitudes in $\mathcal{N} = 4$ SYM"*, JHEP 0904, 018 (2009), arxiv:0808.2475. • L. J. Dixon, J. M. Henn, J. Plefka and T. Schuster, *"All tree-level amplitudes in massless QCD"*, JHEP 1101, 035 (2011), arxiv:1010.3991. • J. L. Bourjaily, *"Efficient Tree-Amplitudes in N=4: Automatic BCFW Recursion in Mathematica"*, arxiv:1011.2447.

[22] Z. Bern and A. Morgan, *"Massive loop amplitudes from unitarity"*, Nucl. Phys. B467, 479 (1996), hep-ph/9511336.

[23] A. Brandhuber, S. McNamara, B. J. Spence and G. Travaglini, *"Loop amplitudes in pure Yang-Mills from generalised unitarity"*, JHEP 0510, 011 (2005), hep-th/0506068.

[24] S. D. Badger, E. W. N. Glover, V. V. Khoze and P. Svrcek, *"Recursion relations for gauge theory amplitudes with massive particles"*, JHEP 0507, 025 (2005), hep-th/0504159.

[25] T. Schuster, *"Lee-Wick Gauge Theory and Effective Quantum Gravity"*, Diploma thesis, 2008, Humboldt University Berlin, available at http://qft.physik.hu-berlin.de.

[26] R. Britto, F. Cachazo and B. Feng, *"New recursion relations for tree amplitudes of gluons"*, Nucl. Phys. B715, 499 (2005), hep-th/0412308. • R. Britto, F. Cachazo, B. Feng and E. Witten, *"Direct proof of tree-level recursion relation in Yang-Mills theory"*, Phys. Rev. Lett. 94, 181602 (2005), hep-th/0501052.

[27] R. Mertig, M. Bohm and A. Denner, *"FEYN CALC: Computer algebraic calculation of Feynman amplitudes"*, Comput. Phys. Commun. 64, 345 (1991). • V. Shtabovenko, R. Mertig and F. Orellana, *"New Developments in FeynCalc 9.0"*, Comput. Phys. Commun. 207, 432 (2016), arxiv:1601.01167.

[28] G. Passarino and M. Veltman, *"One Loop Corrections for $e^+ e^-$ Annihilation Into $\mu^+ \mu^-$ in the Weinberg Model"*, Nucl. Phys. B160, 151 (1979).

[29] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, *"One loop n-point gauge theory amplitudes, unitarity and collinear limits"*, Nucl. Phys. B425, 217 (1994), hep-ph/9403226.

[30] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, *"Fusing gauge theory tree amplitudes into*

loop amplitudes", Nucl. Phys. B435, 59 (1995), hep-ph/9409265.

[31] S. Stieberger and T. R. Taylor, *"Graviton Amplitudes from Collinear Limits of Gauge Amplitudes"*, Phys. Lett. B744, 160 (2015), arxiv:1502.00655.

[32] Z. Bern, L. J. Dixon and D. A. Kosower, *"One loop corrections to five gluon amplitudes"*, Phys. Rev. Lett. 70, 2677 (1993), hep-ph/9302280.

[33] M. B. Green, J. H. Schwarz and L. Brink, *"$\mathcal{N}=4$ Yang-Mills and $\mathcal{N}=8$ Supergravity as Limits of String Theories"*, Nucl. Phys. B198, 474 (1982). • Z. Bern, L. J. Dixon and D. A. Kosower, *"Dimensionally regulated pentagon integrals"*, Nucl. Phys. B412, 751 (1994), hep-ph/9306240.