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Direct numerical computation and its application to the higher-order radiative corrections

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Abstract. The direct computation method(DCM) is developed to calculate the multi-loop amplitude for general masses and external momenta. The ultraviolet divergence is under control in dimensional regularization. In this paper we report on the progress of DCM to several scalar multi-loop integrals after the presentation in ACAT2016. Also the discussion is given on the application of DCM to physical 2-loop processes including numerator functions.

1. Introduction

For the investigation of physics in the current and future collider experiments, a precise evaluation of higher order corrections in perturbative quantum field theory(QFT) is required. The calculation of the higher order radiative corrections turns out to be a large scale computation since the required number of Feynman diagrams is huge and the integral of each diagram is sometimes very complicated. The role of an automated system for the calculation of perturbative series in QFT is not only to manage such a large scale computation but to avoid possible errors caused by humans.

The starting point of the study in high-energy physics is QFT, i.e., the symbolic representation of the theory and the final output is the predicted numerical values to be compared with experimental results. So the specific feature of the system is depicted at which point one switches from the symbolic treatment to the numerical one. In this sense, our system would be described as the maximally numerical method.

The library of the multi-loop integrals is an important component of the automated system and the libraries are working well in the 1-loop calculations[1, 2, 3, 4, 5, 6, 7]. Beyond 1-loop, we have worked on the development of a computational method for Feynman loop integrals with a fully numerical approach. It is based on numerical integration and extrapolation techniques. In this paper, we describe the status and new developments in our techniques for the numerical computation of Feynman loop integrals.

2. Direct computation method

The multi-loop integral is essential for the higher-order radiative corrections. The integral can have singularities originating from the physics.

Table 1. The progress of DCM at ACAT2017. The number shown as 'x-dim.' stands for the maximum dimension of integrals.

	2-point(self-energy)	3-point(vertex)	4-point(box)
2-loop	massless, massive ref.[13, 22] 4-dim.	massive ref.[14, 15, 20] 5-dim.	massive ref.[16, 17] 6-dim.
3-loop	massless, massive ref.[20, 21] 7-dim.	massless ref.[18, 19] 6-dim.	
4-loop	massless ref.[20] 8-dim.		

- The function in the denominator can vanish, which is normally avoided by the analytic continuation with $m \to m - i0$ in the analytical method.
- The ultraviolet(UV) divergence can appear when the integral is divergent for the large momentum region when it is calculated in 4 space-time dimension.
- The infrared(IR) divergence can appear when a massless particle is included in the integral and the integral is divergent in the soft momentum region.

The singularity in the integrand disappears if we introduce the regularization, e.g., $m^2 \to m^2 - i\rho$, taking spacetime dimension to be $4 - 2\varepsilon$, or the introduction of the fictitious mass λ for a massless particle. With non-zero ρ, ε and λ the integral can be computed numerically and the physical value can be obtained by extrapolation to the limit $\rho \to 0$ and so forth.

It is already shown that DCM can handle these singularities numerically. For the scalar integrals the status is shown in Table.1. After the last ACAT[20], we have filled the box for 3-loop vertex functions. The complexity of numerical integration increases with the dimension of the integral and we have computed integrals up to dimension 8, or a diagram with 9 propagators. For the numerical integration, we use robust integration software in ParInt[8] or the double exponential transformation method[9] and use MPI[10] or other parallel environments for accelerating the computation.

An important feature of DCM is that one does not need to separate terms by hand as is done in the analytic treatment. In a large-scale computation, manual operation is the point where some error can happen. Suppose an integral I has a UV singularity as

$$I = \frac{C_{-K}}{\varepsilon^K} + \cdots + \frac{C_{-1}}{\varepsilon} + C_0 + C_1\varepsilon + C_2\varepsilon^2 + \cdots \tag{1}$$

where we take the spacetime dimension to be $n = 4 - 2\varepsilon$. Then, we compute I as the integral whose integrand includes the numerical value of ε. From a set of numerical values, we can obtain all values of the leading coefficients using a linear solver or Wynn's algorithm[11]. When the most singular term is $1/\varepsilon^K$, in the first case, we solve the linear equation

$$I(\varepsilon_j) = \sum_{k=-K}^{-K+N-1} C_k \varepsilon^k \qquad (j = 1, \ldots, N) \tag{2}$$

using an appropriate linear solver such as `dgefs.f` from the SLATEC Common Mathematical Library[12], and in the second case the following iteration is performed

$$a(j, k+1) = a(j+1, k-1) + \frac{1}{a(j+1, k) - a(j, k)} \qquad a(j, 0) = \varepsilon^{-n} I(\varepsilon_j), a(j, -1) = 0 \tag{3}$$

to obtain approximations to the coefficient C_n(in the columns with k odd).

3. Application

The 2-loop amplitude including the numerator is processed in the following way. As an explicit example, we consider the calculation of the 2-loop electroweak self-energy function, $\Pi(s)$. The function is the sum of a number of diagrams, i.e., $\Pi(s) = \sum \Pi_j(s)$ and later we do not write the index j explicitly unless it is necessary. First, GRACE[1, 23, 24, 25] automatically generates the diagrams. Then for each diagram its numerator and denominator are given by a symbolic code in REDUCE to compute the integral:

$$\Pi(s) = \int [d\ell_1][d\ell_2] \frac{\mathbf{N}(\ell_1, \ell_2)}{P_1 P_2 \cdots P_N} = \int dx_1 dx_2 \cdots dx_N \delta(1 - \sum x_k) G \tag{4}$$

where $\mathbf{N}$ is the numerator and $P_k = p_k^2 - m_k^2 + i\rho$. Here,

$$G = \Gamma(N) \int [d\ell_1][d\ell_2] \frac{\mathbf{N}(\ell_1, \ell_2)}{\Delta^N} \tag{5}$$

where $\Delta = {}^{\mathrm{t}}\vec{\ell} A \vec{\ell} + 2{}^{\mathrm{t}}\vec{\ell}\vec{b} + C$. Here $\vec{\ell} = \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix}$. After a sequence of the variable transformation, we have

$$\Delta = \ell_1^2 + \ell_2^2 - V, \quad \mathbf{N} = f^{00} + f^{10}\ell_1^2 + f^{01}\ell_2^2 + f^{20}(\ell_1^2)^2 + f^{11}\ell_1^2\ell_2^2 + f^{02}(\ell_2^2)^2 , \tag{6}$$

and G is given by

$$G = \sum \frac{(-1)^{N+k+m}}{(4\pi)^n} \frac{\Gamma(N-n-k-m)\Gamma(n/2+k)\Gamma(n/2+m)}{(\Gamma(n/2))^2} \frac{f^{km}}{U^{n/2}V^{N-n-k-m}}. \tag{7}$$

where U and the product of U and V are polynomials of x's,

$$U = \det A, \quad UV = -\det \begin{pmatrix} A & \vec{b} \\ {}^{\mathrm{t}}\vec{b} & C \end{pmatrix} . \tag{8}$$

The derivative function $d\Pi(s)/ds$ is also computed in a similar manner. The system generates the FORTRAN code for G automatically. However, at the present status, one must prepare a part of REDUCE code dependent on the topology of the diagram manually. This part needs more work for the complete automation. This point can be automated since the number of the topologies in 2-loop diagrams is rather limited, so that we can prepare all possible code for these in the system.

There is a variant of the treatment of the numerator used in the calculation of the 2-loop electroweak correction to muon $g-2$ in ref.[26]. In the future, we would also like to implement this method since it is preferable to have another method for the automated computation to confirm the results.

When the integral has UV divergence, we keep ε finite. Also when the denominator in the integrand vanishes in the integration region, we keep ρ finite. For an integral with both singularities, we execute a double extrapolation: First, we fix the value of ε and compute the integral for several values of ρ to estimate the limit as $\rho \to 0$ using either of the methods explained in the previous section. Then, we calculate the C coefficients of series in ε. A detailed discussion on the double extrapolation is found in ref.[27, 28].

As an instance, we consider the Higgs 2-point function of the 2-loop order in the electro-weak theory with the non-linear gauge fixing[25]. Then GRACE generates 3082 diagrams for the function including counter-term diagrams. So the system should perform well to handle such size of computation. Some test computations have shown that this example can be calculated within realistic computer time.

4. Summary

The DCM is well developed as an important tool to calculate the radiative corrections in 2-loop order when it is combined with GRACE system. Since it is based on a fully numerical method, once it is proved to work for the scalar integral, it can compute the physical amplitude without any special extension of the computational method.

As was already discussed, in the large scale calculation for the higher-order correction it is desirable to perform the calculation automatically, so that one can avoid possible human error. From this view point, the DCM still needs two points to be upgraded. One is to control the selection of the series of numerical values of $\{\varepsilon_j\}$ using some iterative test to find the proper values, i.e., a variation of a machine-learning method. Second is to introduce a system to analyze the structure of diagrams to provide the topology-dependent part of the numerator handler. In a future publication we plan to present explicit results for 2-loop processes.

Acknowledgments

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