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Mathematical cosmology

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Abstract. Many topics were covered in the submitted papers, showing much life in this subject at present. They ranged from conventional calculations in specific cosmological models to provocatively speculative work. Space and time restrictions required selecting from them, for summarisation here; the book of Abstracts should be consulted for a full overview.

There is continuing interest in various forms of imperfect fluid solution; for example M. L. Bedran and M. O. Calvao [Feederal Univeristy of Rio de Janeiro] discuss universe models where imperfect fluids evolve reversibly owing to the presence of a conformal Killing vector field, and L. P. Chimento and A. S. Jakubi [University of Buenos Aires] consider stability of solutions with causal viscous fluids, concluding that qualitative asymptotic behaviour in the future is not altered by relaxation processes but that in the past it is significantly changed.

The ongoing study of Mixmaster universe dynamics was represented by two papers based on numerical simulations of its dynamical behaviour. The problem is that the standard indicators of chaotic behaviour, such as Lyaponov exponents, give different results when applied on the one hand to the one-dimensional Return Map, characterising the evolution as a change of parameters in a series of Kasner epochs, and on the other hand to the exact field equations, represented in terms of evolution of parameters in a two-dimensional anisotropy plane. A. Burd and R. Tavakol [Queen Mary College, London] argue that the gauge freedom in general relativity makes all such standard indicators of chaotic behaviour problematical, and that indeed chaos is an inherently gauge-dependent phenomenon. B. Berger [Oakland University, Michigan] however argues that use of Minisuperspace proper time gives a definitive answer, showing that there is chaotic behaviour in the full solutions, in agreement with analyses based on the Return Map.

More general dynamics of homogeneous models is studied in papers by K. Rosquist [Stockholm University], discussing the nature of the symplectic structure needed in order to represent Bianchi Class B dynamics in Hamiltonian form, and by C. Uggla [Syracuse University] and R. Jantzen [Villanova University], indicating a hierarchical structure emerging from the study of invariant manifolds in the space of solutions. Thus "simpler models constitute building blocks for the construction of the dynamical structure of more complicated ones". A similar theme emerges in the paper by C. Hewitt [University of Waterloo], showing how self-similar solutions appear to be asymptotic states at late time for more general (diagonal G2) inhomogeneous cosmologies. These approaches seem to be very helpful in obtaining an overview of the kinds of dynamics possible in cosmological models.

The use of piecewise Friedmann-Tolman models for the expanding universe (generalised "Swiss-Cheese" models) is discussed by A. Chamorro [Bilbao], in a paper representative of studies by a number of authors. Overdense or underdense regions can be imbedded in external expanding universes, provided they are surrounded by compensating intervening Tolman zones. One can thus construct a model of an intermediate scale inhomogeneous universe made up of Friedmann underdense and overdense spherical regions surrounded by compensating thick Tolman shells imbedded in a Friedmann expanding background, in line with current ideas about the cell structure of the universe.

The study of inhomogeneous models will be considerably helped by a survey project reported on by A. Krasinski [Copernicam Astronomical Centre, Warsaw], who emphasizes that while in the old days there was a view that solutions of the Einstein Field Equations are so difficult to come by that any new solution was worth having, now the situation is different. There are so many published solutions (most discovered many times) that the first thing to do when looking for exact solutions is to see if what you are planning has already been done, for there is a good chance it will already be in the literature; and the need is to understand the solutions obtained and their relations to each other, rather than just to find new solutions.

The project assembles and classifies exact inhomogeneous solutions of the Einstein equations that contain the FLRW (Friedmann-Lemaitre-Robertson-Walker) universes as limiting cases, and so can be understood as inhomogeneous cosmological models; results of 247 papers have been included in this compilation so far. The relationships between the models (in particular, specialisations that lead from one to another) have been examined, leading to a broad classification into five main types, and characterising which models are subcases of others; in many cases, multiple discoveries of the same model have been catalogued.

Krasinski points out that many interesting inhomogeneous models were already studied in the 1930's and 1940's, particularly papers by R. C. Tolman [1] and by N. R. Sen [2] contain proposals that are still attractive today. Sen showed that the Lemaitre [3] solution predicts a behaviour of density distribution that today would be called formation of voids, also implying that the Einstein-Strauss "swiss-cheese" model is unstable to velocity perturbations (i.e. to perturbations that allow non-comoving walls). Krasinski also comments that despite all the work done to the present day, no rotating generalisation of the expanding FLRW models are explicitly known; we also lack explicit shearing and accelerating FLRW generalisations.

Given the special nature of exact solutions, perturbation solutions are inevitable; two important issues arise.

One is their linearisation stability, that is, how well the linearised solution represents the behaviour of the exact solutions. An interesting study by J. Frauendiener and B. G. Schmidt [Max Planck Institute for Astrophysics, Garching] looks at this issue in the case of spherically symmetric spacetimes, comparing the linearised and exact solutions with each other. Not surprisingly, the linear and exact solutions deviate from each other more and more as the density contrast grows. Such studies are invaluable in analysing the reliability of perturbation theory.

The second issue is the averaging problem in cosmology, raised in a survey talk on cosmology at the GR10 meeting and now gradually becoming a focus of activity. The point is that practical cosmological models, being patently unable to represent all the structure in the universe down to the finest details, represent the universe averaged over some suitable scale; and different models represent it at different averaging scales (for example some may contain perturbations representing clusters of galaxies but others only a smoothed out cosmological substratum). There are two implications. Firstly, it is clear that cosmological models should state explicitly state the averaging scale envisioned in their application, for this is crucial to their interpretation. The issue then is that averaging does not commute with process of working out the Einstein field equations [4]. Consequently the field equations in cosmology at smoothed out scales should include an effective polarisation term resulting from the averaging process (as in the well-known Isaacson term in the case of gravitational radiation).

Various authors have examined this issue in an interesting manner, for example Bildhauer and Futamase [5] claim that the effect could be large enough to seriously change the relation between the Hubble constant and the age of the universe. Now in a series of papers summarised in a poster presented to the meeting (being unable to give an oral presentation for financial reasons), R. Zalaletdinov [Uzbek Academy of Sciences, Tashkent] gives a systematic way of tackling the problem by use of bitensors that enable averaging of tensor fields over a finite volume. He works out the consequences for averaging covariant derivatives, and so the effect on the field equations, in terms of structural functions and a series of correlation tensors, the latter determining splitting rules for averages. The result is a scheme for averaging out a Riemann space resulting in the appearance of an averaged space with a metric and two equi-affine symmetric connections. He obtains the averaged Einstein equations and contracted Bianchi identities. The result is a very promising scheme for tackling this fundamental problem from the foundations; its implications, and its relation to other proposals such as those of Futamase and Kasai, have still to be determined.

To broaden the scope of the discussion , some studies considered more general issues in cosmology. D. H. King [Vancounver] and C. Klein and H. Pfister [Tubingen] consider Machian properties of rotating universe from different viewpoints, both claiming (in different contexts) that a FLRW universe cannot rotate with respect to its inertial frame, in agreement with previous work by D. J. Raine. In a different spirit, R. Tavakol [Queen Mary College, London] asks the questions "Is general relativity fragile?", examining its stability under various possible changes both in terms of imposing symmetries on models (which are never truly satisfied) and in terms of various ways of generalising General Relativity. Various examples show that in general structural stability will not hold. In the long term this kind of issue will become important in determining the questions we ask and the models we use. This kind of issue underlies some of the other papers presented, for instance that by Chimento and Jakubi mentioned above.

Finally, the most speculative paper of the session, by L. Smolin [Syracuse University], considered the possibility of natural selection in cosmology. The issue here is that, as emphasized by Dawkins and others, Darwinian selection is a powerful mechanism for creating apparently purposeful structure and order where none existed before, and indeed is the only mechanism known that can do so. The issue then is whether this might be introduced as an explanatory principle in cosmology, explaining some of the coincidences that are otherwise inexplicable (except perhaps on an anthropic basis, that many people reject). This becomes a possibility if one conceives of situations where collapse of a black hole gives rise to new expansion phase (a "daughter universe"), with the possibility of the constants of physics being different in the new expansion phase than in the old universe. Thus there is a source of variation of conditions, one of the necessities for evolution. One also needs some mechanism for selection: here the proposal is that it is simply numbers of progeny universes that is the mechanism acting, leading eventually to an overwhelming likelihood of universe models existing with a maximal creation of daughter universes. The issue is to show that the constants realised around us are indeed such as to maximise black hole production and consequent creation of daughter universes.

This is highly speculative, but certainly in the spirit of much modern theoretical cosmology. Smolin presents a detailed argument for his proposal [6]. It can be criticised in detail, as was shown by the workshop discussion, and there is room for development and testing of the proposal; however it provides an exciting prospect of uniting two of the major paradigms of scientific understanding (evolution through natural selection, and the expanding and evolving universe) into a new way of understanding cosmology. Smolin proposes to explain in this way why many of the dimensionless numbers which characterise particle physics and cosmology take unnatural values. This will be regarded with skepticism by many, but is certainly an interesting idea.

Overall there is much interesting activity in this area. It was a pleasure to have Charles Misner, one of the pioneers in much innovative work in theoretical cosmology, join us for some of the discussions; we wish him well on the occasion of his 60th birthday.

References

[1] R. C. Tolman 1934 *Proc Nat Acad Sci* **20** 169

[2] N. R. Sen 1934 *Z Astrophys* **9** 215, 1935 *Z Astrophys* **10** 291

[3] Lemaitre 1933 *Ann Soc Sci Bruxelles* **A53** 51

[4] G. Ellis 1983 *GR10 conference proceedings, Ed. B. Bertotti et al, Plenum Press*

[5] Bildhauer and Futamase 1991 *GRG* **23** 1251

[6] Smolin 1992 *Class. Qu. Grav.* **9** 173