Comments on nonlinear sigma models coupled to supergravity

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A B S T R A C T

$N = 1$, $D = 4$ nonlinear sigma models, parametrized by chiral superfields, usually describe Kählerian geometries, provided that Einstein frame supergravity is used. The sigma model metric is no longer Kähler when local supersymmetry becomes nonlinearly realized through the nilpotency of the supergravity auxiliary fields. In some cases the nonlinear realization eliminates one scalar propagating degree of freedom. This happens when the sigma model conformal-frame metric has co-rank 2. In the geometry of the inflaton, this effect eliminates its scalar superpartner. We show that the sigma model metric remains semidefinite positive in all cases, due the to positivity properties of the conformal-frame sigma model metric.

1. Introduction

In this note we comment on some general properties of the sigma-model metric in $N = 1$ supergravity coupled to chiral multiplets [1,2]. In particular we discuss properties of the metric in the conformal and Einstein frames. These frames are particular cases in the superconformal approach to supergravity [3]. Different frames are suitable for uncovering the physics of different explicit models. Positivity properties of the sigma-model metric are maintained by the metric in different frames, but the sigma model metric in the conformal frame can have lower rank than in the Einstein frame, before elimination of the axial vector auxiliary field. In this note, we show examples where the conformal metric is non-invertible but the final sigma model metric is nevertheless positive definite.

In the case of nonlinear realizations of supersymmetry, the nilpotency of the auxiliary field A_μ may reduce the rank of the full scalar metric. A particularly striking example is the inflaton, where the nilpotency of A_μ removes the pseudoscalar partner of the inflaton (the "sinflaton").

2. Supergravity in the conformal and Einstein frames

Conformal-frame supergravity is the formulation that follows directly from the tensor calculus [3–5] that uses a minimal set of auxiliary fields; namely, a complex scalar u and axial vector A_μ. This corresponds to a particular choice of Jordan frame (see e.g. [6]) in which the frame function, $\phi(z, \bar{z})$, is the first component of a real superfield, $\Phi(Z, \bar{Z})$, whose local D-density is the Lagrangian of supergravity coupled to a nonlinear sigma model [1]

$$\Phi(Z, \bar{Z})\Big|_D = L_{CS} = e\Big[\frac{\phi}{6}R - \phi_{i\bar{j}}\partial_\mu z^i \partial_\nu \bar{z}^{\bar{j}} g^{\mu\nu} - \frac{\phi}{9}A_\mu A_\nu g^{\mu\nu}$$
$$+ \frac{i}{3}A_\mu(\phi_i \partial_\nu z^i - \phi_{\bar{i}}\partial_\nu \bar{z}^{\bar{i}})g^{\mu\nu} + ...\Big]. \qquad (1)$$

Here $e = \det e_\mu^a$, $\phi_i = \partial_i \phi$ etc. and we only wrote explicitly bosonic terms relevant for our discussion. The function $\phi(z, \bar{z})$ is negative with non-negative sigma model metric $\phi_{i\bar{j}}$. We call eq. (1) "conformal frame Lagrangian" because its Einstein equation is sourced by the improved energy–momentum tensor in curved space [7] (see [8] for the supergravity extension). We remark that the conformal-frame Lagrangian is additive in the ϕ function, differently from other Jordan frames. This property was important in formulating the "sequestering" scenario of ref. [9]. The physical properties of a supergravity model may be transparent in one frame but hidden in another. For instance, the example of ref. [9], based on brane constructions, was naturally additive in ϕ. The same is true for conformally coupled scalars. Other models, such as sequential toroidal compactifications of higher-dimensional supergravity, are instead additive in the Kähler potential K.

The Einstein frame action is obtained by performing a Weyl rescaling of the vierbein, $e_\mu^a \to e_\mu^a \exp(\sigma)$, such that the R curva-

ture term coincides with the Einstein–Hilbert action.[1] This rescaling is

$$e^{2\sigma} = -\frac{3}{\phi}, \tag{2}$$

$$\frac{1}{6}e\phi R \rightarrow -\frac{1}{2}eR - \frac{3}{4}eg^{\mu\nu}\partial_\mu \log\phi\partial_\nu \log\phi + \text{ total derivative}. \tag{3}$$

Under this rescaling we have $L_{CS} \rightarrow L_{ES}$ with

$$L_{ES}/e = -\frac{1}{2}R + \frac{3}{\phi}\phi_{i\bar{j}}\partial_\mu z^i\partial_\nu \bar{z}^{\bar{j}}g^{\mu\nu}$$
$$- \frac{3}{4}[(\log\phi)_i\partial_\mu z^i + (\log\phi)_{\bar{i}}\partial_\mu \bar{z}^{\bar{i}}]^2 +$$
$$+ \frac{1}{3}A_\mu A_\nu g^{\mu\nu} - iA_\mu[(\log\phi)_i\partial_\nu z^i - (\log\phi)_{\bar{i}}\partial_\nu \bar{z}^{\bar{i}}]g^{\mu\nu}. \tag{4}$$

Finally, if one integrates out the A_μ field, one gets

$$L_{ES}/e = -\frac{1}{2}R + \frac{3}{\phi}\phi_{i\bar{j}}\partial_\mu z^i\partial_\nu \bar{z}^{\bar{j}}g^{\mu\nu}$$
$$- \frac{3}{4}[(\log\phi)_i\partial_\mu z^i + (\log\phi)_{\bar{i}}\partial_\mu \bar{z}^{\bar{i}}]^2$$
$$+ \frac{3}{4}[(\log\phi)_i\partial_\mu z^i - (\log\phi)_{\bar{i}}\partial_\mu \bar{z}^{\bar{i}}]^2. \tag{5}$$

This is a nonlinear sigma model with Kähler metric ($d \equiv dz^i\partial_i$, $\bar{d} \equiv d\bar{z}^{\bar{i}}\partial_{\bar{i}}$)

$$(d \otimes \bar{d})K = -\frac{3}{\phi}(d \otimes \bar{d})\phi + \frac{3}{4}(d\log\phi + \bar{d}\log\phi)$$
$$\otimes (d\log\phi + \bar{d}\log\phi) +$$
$$- \frac{3}{4}(d\log\phi - \bar{d}\log\phi) \otimes (d\log\phi - \bar{d}\log\phi)$$
$$= -\frac{3}{\phi}(d \otimes \bar{d})\phi + 3d\log\phi \otimes \bar{d}\log\phi = -3d \otimes \bar{d}\log\phi. \tag{6}$$

3. Properties of the sigma-model metric

By inspection of eq. (6), the Einstein-frame Kähler metric is the sum of three $2n \times 2n$ matrices: the matrix $\phi_{i\bar{j}}$ and two positive rank-one matrices, the first coming from the Weyl rescaling and the second from integrating out the A_μ field. The physical requirement is that $\phi_{i\bar{j}}$ is non-negative and of rank $\geq 2n - 2$. For $n = 1$ the rank-zero example is the inflaton metric discussed in the next section. We also observe that the splitting of the Kähler metric in three factors does not respect the Kähler invariance

$$K \rightarrow K + \Lambda + \bar{\Lambda}, \tag{7}$$

where Λ is a holomorphic function of the coordinates. In the ϕ variables, the transformation corresponds to

$$\phi \rightarrow \phi e^{-\frac{1}{3}(\Lambda + \bar{\Lambda})}. \tag{8}$$

Let us now consider a case where the $\phi_{i\bar{j}}$ metric has rank $2n - 2$. This is the model with

$$\phi^3 = -\frac{1}{3}d_{ijk}(z^i + \bar{z}^{\bar{i}})(z^j + \bar{z}^{\bar{j}})(z^k + \bar{z}^{\bar{k}}). \tag{9}$$

This metric appears in the large-volume limit of the Kähler class moduli in Calabi–Yau compactifications of type IIA superstrings [10]. Since ϕ depends only on Re z^i and is homogeneous of degree one, it follows that $\phi_{i\bar{j}}$ Re $z^j = 0$. So $\phi_{i\bar{j}}$ has a null eigenvector. Since the metric for Im z^i is the same as the one for Re z^i, it follows that the sigma-model metric splits into a rank-$2n - 2$ part plus a rank-2 part

$$K_{i\bar{j}} = -3\left(\frac{\phi_{i\bar{j}}}{\phi} - \frac{\phi_i\phi_{\bar{j}}}{\phi^2}\right). \tag{10}$$

Notice that if we only retain the volume modulus, t, then $\phi_{t\bar{t}} = 0$ and the entire metric resides in the last term, which, in this case, is the full metric.

We remark that the conformal frame action is invariant under the Kähler transformation (7), (8), even if the sigma model metric is not. This happens because the transformation also acts nontrivially on the conformal frame metric $g^C_{\mu\nu}$:

$$g^C_{\mu\nu} \rightarrow g^C_{\mu\nu}e^{\frac{1}{3}(\Lambda + \bar{\Lambda})}. \tag{11}$$

4. Inflaton disk geometry

In many supersymmetric models such as the supergravity extension [11,12] of the $R + R^2$ "Starobinsky" model [13,14], the inflaton φ has a Kähler potential

$$K = -3 \log[(\varphi + \bar{\varphi})/3]. \tag{12}$$

The standard inflaton is the real part of φ while the imaginary part is its supersymmetric partner, the "sinflaton." As in the previous example, $\phi_{\varphi,\bar{\varphi}} = 0$. This is due to the fact that $R + R^2$ supergravity is dual to a standard (conformal frame) supergravity with Lagrangian

$$L_{CS} = e\frac{\phi}{6}R + ..., \qquad \phi = -(\varphi + \bar{\varphi}). \tag{13}$$

This formula shows that the Kähler metric of this model is entirely due to curved space effects, since the two degrees of freedom Re φ, Im φ acquire kinetic terms only though the Weyl rescaling and the A_μ field equation. Dropping the A_μ contributions one obtains the $R + R^2$, $N = 0$ theory [15].

5. Nonlinear realizations

Nonlinear realizations of local supersymmetry have been widely discussed in the recent past. Beyond the Volkov–Akulov [16] nilpotent chiral superfield X ($X^2 = 0$) [17], other superfields can undergo nonlinear realizations if they satisfy nilpotency conditions. In supergravity one can impose constraints which have no analog in rigid supersymmetry, since they create nonlinear restrictions on the underlying local superspace geometry. In this case, the only constraints that do not affect the gauge field sector are nilpotency constraints on the auxiliary fields u and/or A_μ. They have the form [18]

$$X\bar{X}(\mathcal{R} - c) = 0, \qquad X\bar{X}G_{\alpha\dot{\alpha}} = 0. \tag{14}$$

Here $\mathcal{R}$ is the scalar curvature chiral superfield and $G_{\alpha\dot{\alpha}}$ is the real Einstein superfield [19]. These constraints imply that u and A_μ are nilpotent. The first constraint will affect the scalar potential and the latter will affect the kinetic terms. Recently these constraints have been used in higher derivative supergravity [20]. In this note we discuss only the properties of the sigma model metric, so we confine our discussion to the latter constraint.

Nilpotency of A_μ implies that its contribution to the kinetic term in eqs. (1), (5) should be deleted. This procedure in some

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[1] In the superconformal approach, different Jordan frames correspond to different gauge choices for the compensating multiplet [3].

cases has the effect of removing the propagating degree of freedom associated to one of the rank-one matrix contributions to the sigma-model metric. In particular, in the inflaton model discussed above, it deletes the "sinflaton" degree of freedom.

In the multi-field case with ϕ potential given by eq. (9), the full metric will have rank $2n - 1$ so that one scalar degree of freedom will always become nondynamical. It is important however to make sure that the remaining degrees of freedom have positive metric. This follows from the positivity conditions on the $\phi_{i\bar{j}}$ metric. We also observe that two ϕ potentials related by coordinate transformations may have a different number of propagating degrees of freedom once the nonlinear constraint on A_μ is imposed. This is due to the fact that when the A_μ field becomes nilpotent, the underlying geometry is no longer Kähler. This is easily seen in the inflaton example. When $\phi = -(\varphi + \bar{\varphi})$ the sinflaton does not propagate. The potential $\phi = \varphi\bar{\varphi} - 1$, which corresponds to a conformally coupled complex scalar, is obtained from the former, up to a Kähler transformation, by the coordinate transformation $\varphi \to (\varphi + i)/(\varphi - i)$. The latter potential is easily seen to give a sigma model where two scalars propagate, since it corresponds to a flat metric $\phi_{\varphi,\bar{\varphi}} = 1$ in the conformal frame.

Acknowledgments

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