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THE TWISTED N=2 SUPERSYMMETRIC SIGMA-MODEL: A FOUR-LOOP CALCULATION OF THE BETA-FUNCTION

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Abstract

The twisted $N = 2$ supersymmetric σ-model in two dimensions is a modification of the standard $N = 2$ σ-model which has a torsion background in addition to the usual metric background. We compute the β-function at four loops and find an elegant expression in terms of the generalized curvature constructed from the connection with torsion. The result is a natural generalization of that for the ordinary $N = 2$ σ-model.

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1. Introduction

The study of the interplay between 2-dimensional non-linear σ-models and string theory has been very fruitful. The conformal invariance conditions for the σ-model are interpreted as equations defining consistent backgrounds for string propagation[1]; these equations are equivalent to those derived from the string effective action which is constructed as a generating function for string interaction amplitudes[2]. These ideas have been tested for a wide variety of different σ-models. One of the most interesting results was obtained in the context of the $N = 2$ supersymmetric σ-model[3]. The target manifold, where the fields take their values, is required to be Kähler for the ordinary $N = 2$ supersymmetric σ-model[4]; it was thought on general grounds that $N = 2$ σ-models defined on Ricci-flat Kähler manifolds would be finite to all orders[5]. This hope was dashed by the discovery in Ref. [3] of a non-vanishing counterterm at the four-loop order. Moreover the resulting conformal invariance conditions were shown[6-8] to follow as equations of motion from the string effective action constructed by Gross and Witten [9] from a study of string amplitudes. The calculation of Ref.[3] has an appealing elegance resulting from the simple form of the action for the $N = 2$ supersymmetric σ-model and the use of $N = 2$ D-algebra to streamline the calculation.

However, the conventional $N = 2$ σ-model contains only a metric background coupling, whereas the most general renormalizable σ-model can contain backgrounds corresponding to the full set of massless excitations of the string, namely the graviton (or metric) g_{ij}, the antisymmetric tensor b_{ij} and the dilaton D. It was shown in Ref. [10] that it is possible to incorporate an antisymmetric tensor background by generalizing to the so-called twisted $N = 2$ σ-model, which differs from the conventional version in having a modified chirality condition for the fields. The target manifold is then no longer Kähler; it is a complex manifold with *two* independent complex structures, which are each covariantly constant with respect to a connection with torsion. There is however still a potential (which we shall call the "twisted Kähler potential") from which the metric and antisymmetric tensor may be obtained by taking derivatives, and as in the usual $N = 2$ case the σ-model action is written as an integral over the potential.

Our intention in this paper is to extend the results of Ref. [3] to the twisted case, again relying heavily on the simple form of the action and the properties of the D-algebra to achieve a concise treatment. The main result of the paper is an extremely simple form for the four-loop β-function, expressed in terms of the generalized Riemann curvature tensor (constructed from the connection with torsion), and is a natural extension of the result for the untwisted case. The layout of the paper is as follows: In Section 2 we describe both the conventional $N = 2$ σ-model and the twisted $N = 2$ σ-model. In Section 3 we discuss the Feynman diagram calculation for the twisted model and formulate rules for constructing diagrams which contribute to the β-function at four loops. Then in Section 4 we postulate an expression for the four-loop correction to the simple pole in the twisted Kähler potential, involving only the generalized Riemann tensor which is constructed from the connection with torsion. We show that the rules for generating a non-zero term from this expression correspond precisely to the rules given in Section 3 and hence prove that the postulated expression does indeed reproduce the results of the diagrammatic calculation. Finally in Section 5 we write down the four-loop β-function, and discuss some unresolved

issues and directions for further investigation. In particular we show that there is a close connection between the form of the twisted $N = 2$ β-function in terms of the generalized curvature, and the vanishing of the twisted $N = 4$ β-function, and we speculate that this connection may persist to all orders.

2. The twisted $N = 2$ sigma model

In this section we shall define the twisted $N = 2$ σ-model and describe its properties. Our discussion in this Section will of necessity follow closely that of Ref. [10] where the twisted σ-model was first introduced. First of all, however, for purposes of comparison and also for future reference, it will be useful to discuss the ordinary untwisted $N = 2$ σ-model.

The 2-dimensional bosonic non-linear σ-model has the action

$$S = -\frac{1}{4}\int d^2x g_{ij}\partial_\mu \phi^i \partial^\mu \phi^j \tag{2.1}$$

where $g_{ij}(\phi)$ is the metric and $\phi^i(x)$, $i = 1 \ldots D$ gives a map from the 2-dimensional worldsheet to the D-dimensional target manifold. For a generic target manifold, this action can always be extended to an $N = 1$ supersymmetric version given, in superspace notation, by the action

$$S = -\frac{1}{4}\int d^2x d^2\theta g_{ij}(\Phi) D^\alpha \Phi^i D_\alpha \Phi^j, \tag{2.2}$$

where $\Phi^i(x, \theta)$ are real scalar superfields. The action of Eq. (2.2) is manifestly $N = 1$ supersymmetric; however, it has a further, non-manifest supersymmetry if and only if the target manifold is Kähler[4]. In fact, postulating an extra supersymmetry of the form

$$\delta_\eta \phi^i = (\eta^\alpha D_\alpha \Phi^j) J_i{}^j \tag{2.3}$$

we find, on requiring these tranformations to generate the supersymmetry algebra,

$$\begin{aligned} J_i{}^j J_j{}^k &= -\delta_i{}^k \\ J_i{}^l J_{[j}{}^k{}_{,l]} - J_j{}^l J_{[i}{}^k{}_{,l]} &= 0, \end{aligned} \tag{2.4}$$

which are the conditions for J to be a complex structure. Then requiring the action to be invariant under Eq. (2.3), we find

$$J_i{}^k g_{kj} = -J_j{}^k g_{ki}, \tag{2.5}$$

which is the condition for the manifold to be hermitian, and we also find

$$\nabla_i J_j{}^k = 0, \tag{2.6}$$

where ∇ is the usual Levi-Civita connection. Eqs. (2.4)-(2.6) are precisely the definitions of a Kähler manifold–a hermitian complex manifold with a covariantly constant complex structure. For Kähler manifolds, one can choose a local complex co-ordinate system $z^p, \bar{z}^{\bar{p}}$ in which the metric has the form

$$g_{p\bar{q}}(z,\bar{z}) = \frac{\partial}{\partial z^p}\frac{\partial}{\partial \bar{z}^{\bar{q}}}K(z,\bar{z})$$
$$g_{pq} = g_{\bar{p}\bar{q}} = 0. \tag{2.7}$$

$K(z,\bar{z})$ is then referred to as the Kähler potential. We may rewrite the action in manifestly $N=2$ supersymmetric form by introducing complex spinor derivatives D_α, $\bar{D}_\alpha$ and complex superfields Φ^p, $\bar{\Phi}^{\bar{q}}$, satisfying the chirality condition

$$\bar{D}_\alpha \Phi^p = D_\alpha \bar{\Phi}^{\bar{q}} = 0 \tag{2.8}$$

and with $\Phi^p|_{\theta=\bar{\theta}=0} = z^p$, $\bar{\Phi}^{\bar{q}}|_{\theta=\bar{\theta}=0} = \bar{z}^{\bar{q}}$. Eq. (2.2) then becomes

$$S = \int d^2x d^2\theta d^2\bar{\theta} K(\Phi^p, \bar{\Phi}^{\bar{q}}). \tag{2.9}$$

The twisted $N=2$ σ-model can be derived in an analogous fashion, by seeking an additional supersymmetry in the $N=1$ supersymmetric σ-model with torsion (i.e. with an antisymmetric tensor field). For a general target manifold we may write the manifestly $N=1$ supersymmetric action

$$S = -\frac{1}{4}\int d^2x d^2\theta [g_{ij}(\Phi) D^\alpha \Phi^i D_\alpha \Phi^j + b_{ij}(\Phi) D^\alpha \Phi^i (\gamma_5 D)_\alpha \Phi^j] \tag{2.10}$$

where $b_{ij}(\Phi) = -b_{ji}(\Phi)$ is an antisymmetric tensor field and Φ^i, $i = 1 \ldots D$ are real scalar superfields. We look for an additional supersymmetry of the form

$$\delta_\eta \Phi^i = (\eta^\alpha D_\alpha \Phi^j) J_j{}^i(\Phi) + (\eta^\alpha (\gamma_5 D)_\alpha \Phi^j) \tilde{J}_j{}^i$$
$$= -i(\eta_+ D_- \Phi^j) J^+{}_j{}^i + i(\eta_- D_+ \Phi^j) J^-{}_j{}^i \tag{2.11}$$

where

$$J^\pm{}_j{}^i = \tilde{J}_j{}^i \pm J_j{}^i, \tag{2.12}$$

which generalizes Eq. (2.3). Requiring that these tranformations generate the supersymmetry algebra, we find that $J^\pm$ both satisfy Eq. (2.4) and hence both are complex structures. Demanding invariance of Eq. (2.10) under the transformation Eq. (2.11), we find that $J^\pm$ each satisfy the hermiticity condition Eq. (2.5), and we also find the condition

$$\mathcal{D}_i^\pm J^\pm{}_j{}^k \equiv J^\pm{}_j{}^k{}_{,i} - \Gamma^{\pm l}_{ij} J^\pm{}_l{}^k + \Gamma^{\pm k}_{il} J^\pm{}_j{}^l = 0, \tag{2.13}$$

where $\Gamma^{\pm k}_{ij}$ are connections with torsion given by

$$\Gamma^{\pm k}_{ij} = \left\{ {k \atop ij} \right\} \mp H^k{}_{ij} \tag{2.14}$$

where

$$H_{ijk} = \frac{1}{2}(b_{ij,k} + b_{ki,j} + b_{jk,i}) \tag{2.15}$$

and $\left\{ {k \atop ij} \right\}$ are the usual Christoffel symbols. In other words, $J^\pm{}_i{}^j$ are covariantly constant with respect to connections with torsion $\Gamma^{\pm k}_{ij}$, in contrast to the complex structure on a Kähler manifold which is covariantly constant with respect to the ordinary metric connection.

We can express the action for the twisted $N=2$ σ-model in a form analogous to Eq. (2.9) by making the additional assumption that the complex structures commute, in which case the tensor

$$\Pi_i{}^j = -J^+{}_i{}^k J^-{}_k{}^j \tag{2.16}$$

satisfies

$$\Pi_i{}^k \Pi_k{}^j = \delta_i{}^j, \tag{2.17}$$

which implies that $\Pi_i{}^j$ is what is called an almost product structure. We can pick co-ordinates in which Π, $J^\pm$ are constant and adopt the block-diagonal forms

$$\Pi_i{}^j = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$
$$J^+{}_i{}^j = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix},$$
$$J^-{}_i{}^j = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \tag{2.18}$$

where the rows correspond to co-ordinate types which we shall denote generically by p, $\bar{p}$, u, $\bar{u}$, repectively, from top to bottom (and similarly for the columns from left to right). We shall also denote p-type indices by $q \ldots t$, and u-type indices by $v \ldots z$, in what follows. The numbers of p-type and $\bar{p}$-type indices are of course equal, as are the numbers of u-type and $\bar{u}$-type, but there is no need for the numbers of p-type and u-type indices to coincide.

Owing to the hermiticity of the metric with respect to $J^\pm$ we have

$$\Pi_i{}^k g_{kj} = \Pi_j{}^k g_{ki} \tag{2.19}$$

so that the metric is also block-diagonal and can be written in the above co-ordinates as

$$g_{ij} = \begin{pmatrix} 0 & g_{p\bar{q}} & 0 & 0 \\ g_{\bar{p}q} & 0 & 0 & 0 \\ 0 & 0 & 0 & g_{u\bar{v}} \\ 0 & 0 & g_{\bar{u}v} & 0 \end{pmatrix}. \tag{2.20}$$

Defining

$$J^{\pm}{}_{ij} \equiv J^{\pm}{}_i{}^k g_{kj} = -J^{\pm}{}_{ji}$$
$$J^{\pm}{}_{ijk} \equiv \frac{1}{2} J^{\pm}{}_{[ij;k]} = \frac{1}{2} J^{\pm}_{[ij,k]} \tag{2.21}$$

we can show[10], using Eq.(2.13) and the complex structure properties of $J^{\pm}$, that

$$H_{ijk} = \mp\frac{1}{4} J^{\pm}{}_{[i}{}^l J^{\pm}{}_{jk]l} \tag{2.22}$$

and hence

$$J^{+}{}_{ijk} J^{+}{}_l{}^i J^{+}{}_m{}^j J^{+}{}_n{}^k = -J^{-}{}_{ijk} J^{-}{}_l{}^i J^{-}{}_m{}^j J^{-}{}_n{}^k, \tag{2.23}$$

from which we deduce that in the complex co-ordinate system corresponding to Eq. (2.18),

$$g_{p\bar{q},r} - g_{r\bar{q},p} = 0$$
$$g_{u\bar{v},w} - g_{w\bar{v},u} = 0. \tag{2.24}$$

It can then be shown[10], using Eqs. (2.15), (2.21), (2.22), and (2.24) that there is a single function K, which has the properties

$$g_{p\bar{q}} = K_{p\bar{q}} \qquad g_{u\bar{v}} = -K_{u\bar{v}}$$
$$H_{p\bar{q}u} = \frac{1}{2}K_{p\bar{q}u} \qquad H_{p\bar{q}\bar{u}} = -\frac{1}{2}K_{p\bar{q}\bar{u}}$$
$$H_{u\bar{v}p} = -\frac{1}{2}K_{u\bar{v}p} \qquad H_{u\bar{v}\bar{p}} = \frac{1}{2}K_{u\bar{v}\bar{p}}, \tag{2.25}$$

where

$$K_p = \frac{\partial}{\partial z^p} K$$

If we now take the action Eq. (2.10), and write it in the complex co-ordinate system with $g_{p\bar{q}}$ and $g_{u\bar{v}}$ given by Eq. (2.25) and b given by

$$b_{u\bar{p}} = K_{u\bar{p}}, \qquad b_{p\bar{u}} = K_{p\bar{u}}, \tag{2.26}$$

with all other components unrelated by symmetry vanishing, then the original action may be rewritten in a manifestly $N = 2$ supersymmetric form as

$$S = \int d^2x d^2\theta d^2\bar{\theta} K(\Phi^p, \bar{\Phi}^{\bar{q}}, \Phi^u, \bar{\Phi}^{\bar{v}}) \tag{2.27}$$

with Φ, $\bar{\Phi}$ complex superfields, and where Φ^u, $\bar{\Phi}^{\bar{v}}$ obey the usual chirality constraints Eq. (2.8), but Φ^p, $\bar{\Phi}^{\bar{q}}$ obey so-called *twisted* chiral constraints

$$D_+\bar{\Phi}^{\bar{p}} \equiv \frac{1}{2}(1+\gamma_5)D\bar{\Phi}^{\bar{p}} = 0 \qquad \bar{D}_+\Phi^q \equiv \frac{1}{2}(1+\gamma_5)\bar{D}\Phi^q = 0$$
$$\bar{D}_-\bar{\Phi}^{\bar{p}} \equiv \frac{1}{2}(1-\gamma_5)\bar{D}\bar{\Phi}^{\bar{p}} = 0 \qquad D_-\Phi^q \equiv \frac{1}{2}(1-\gamma_5)D\Phi^q = 0. \tag{2.28}$$

We shall describe K as the "twisted Kähler potential". Note that b as given by Eq. (2.26) is consistent with H as given by Eqs. (2.15) and (2.25). However, other consistent assignments of values for b are also possible. For instance, we may also take

$$b_{u\bar{p}} = \frac{1}{2}K_{u\bar{p}} \qquad b_{p\bar{u}} = \frac{1}{2}K_{p\bar{u}}$$
$$b_{up} = -\frac{1}{2}K_{up} \qquad b_{\bar{u}\bar{p}} = -\frac{1}{2}K_{\bar{u}\bar{p}}. \tag{2.29}$$

This corresponds to adding a total derivative term to the Lagrangian in Eq. (2.10); it is an instance of the general symmetry under

$$b_{ij} \to b_{ij} + \partial_{[i}\lambda_{j]} \tag{2.30}$$

In this paper we shall find it convenient to assume that b is given by Eq. (2.29).

3. Supergraph calculations for the twisted N = 2 σ-model

In this section we shall describe our calculation of the four-loop simple-pole corrections to the twisted Kähler potential for the twisted $N = 2$ σ-model, from which the 4-loop β-function can be obtained. First of all, however, it is necessary to say a few general words about perturbative calculations for σ-models. The usual perturbative calculation for a bosonic or $N = 1$ supersymmetric σ-model proceeds via a normal co-ordinate expansion [11] of the action (of the form Eq.(2.1), (2.2) or (2.10)) around some background configuration, and in terms of a quantum field proportional to the tangent to the geodesic joining the background field to some adjacent field configuration. This technique ensures that the counterterms will have a manifestly covariant expression in terms of tensor quantities. Let us consider Eq. (2.10) for definiteness. The divergences arising in the loop expansion are absorbed by replacing g_{ij} and b_{ij} by the corresponding bare quantities, defined by

$$g^B_{ij} = \mu^{-\epsilon}(g_{ij} + \sum_{m\leq l} \frac{T^{(l,m)}_{ij}}{\epsilon^m}) \tag{3.1}$$

(with a similar expression for b^B_{ij}) where $T^{(l,m)}_{ij}$ is chosen so as to cancel the divergences order by order in l, the number of loops. We are using dimensional regularization so that we work in $d = 2-\epsilon$ dimensions and the divergences appear as poles in ϵ.μ is the standard dimensional regularization mass parameter. The β-functions are then defined by

$$\beta^g_{ij} = \mu\frac{d}{d\mu}g_{ij}, \qquad \beta^b_{ij} = \mu\frac{d}{d\mu}b_{ij}. \tag{3.2}$$

Since the action Eq. (2.10) is invariant under

$$\delta g_{ij} = 2\nabla_{(i}v_{j)}, \qquad \delta b_{ij} = 2H_{ijk}v^k, \qquad \delta\phi^i = -v^i, \tag{3.3}$$

the β-functions have a corresponding arbitrariness. However, from the point of view of string theory the important quantity is the energy-momentum tensor. The β-functions appear there in the combinations [12]

$$B^g_{ij} = \beta^g_{ij} + 2\nabla_{(i}S_{j)}, \qquad B^b_{ij} = \beta^b_{ij} + 2H_{ijk}S^k \tag{3.4}$$

where S^i is a well-defined vector. These combinations are invariant since under a transformation of the form Eq. (3.3) we have

$$\delta\beta^g_{ij} = 2\nabla_{(i}V_{j)}, \qquad \delta\beta^b_{ij} = 2H_{ijk}V^k, \qquad \delta S_i = -V_i, \tag{3.5}$$

and thus the energy-momentum tensor is unambiguous. The condition for conformal invariance is then $B^g_{ij} = B^b_{ij} = 0$.

When performing perturbative calculations for an $N = 2$ σ-model it is more convenient to use the form Eq. (2.9) or (2.27) for the action in order to take advantage of the properties of $N = 2$ superspace. We then absorb divergences by replacing the Kähler potential by the bare potential K^B defined analogously to Eq. (3.1). The bare metric and antisymmetric tensor are then expressed in terms of the bare twisted Kähler potential exactly as for the renormalized quantities in Eq. (2.7), or in Eqs. (2.25), (2.29), whence the β-functions can be obtained by Eq. (3.2). However as we seen, the actions of Eqs. (2.9), (2.27) are simply specialized versions of the actions Eqs. (2.2), (2.10), obtained by choosing their target manifolds to have the particular properties described in Section 2. Hence we must be able to obtain equivalent results to those derived from Eqs. (2.9) or (2.27) by calculating with the actions of Eqs. (2.2) or (2.10) and then restricting to the appropriate manifold. Since the β-functions obtained from the latter actions, using the normal co-ordinate background field method described earlier, will have an explicitly covariant form with, of course, no appearance of the complex structure, the conformal invariance conditions obtained starting from the Kähler forms of the action must also be expressible in this form– even though the calculation itself is not explicitly covariant. This is the "universality" criterion[5, 13]. However, in view of the afore-mentioned ambiguity in the β-functions, the most that we can say of the β-functions obtained from the Kähler forms of the action is that they should be expressible ***up to a diffeomorphism*** in a covariant form with no use of the complex structure. These requirements provide a stringent consistency check on calculations using Eqs. (2.9) or (2.27).

The Feynman rules for the twisted $N = 2$ σ-model were first discussed by Buscher in Ref. [14], where the one-loop correction to the twisted Kähler potential was also calculated. The result is (in the notation of Eq. (3.1))

$$K^{(1,1)} = \frac{1}{2\pi}[\ell n det(g_{p\bar{q}}) - \ell n det(g_{u\bar{v}})] \tag{3.6}$$

However, the β-functions calculated using covariant methods starting from Eq. (2.10) are

$$\beta^g_{ij} = \frac{1}{2\pi}\mathcal{R}^+_{(ij)} \qquad \beta^b_{ij} = \frac{1}{2\pi}\mathcal{R}^+_{[ij]} \tag{3.7}$$

where $\mathcal{R}^{+i}{}_{jkl}$ is the Riemann tensor corresponding to the connection $\Gamma^{+}{}^k_{ij}$ in Eq. (2.14). If we calculate the β-functions from Eq. (3.6) using Eqs. (2.25), (2.29), they are not equal to those in Eq. (3.7). Buscher [14] showed that Eq. (3.7) can be recovered by making a diffeomorphism as in Eq. (3.5), with

$$V^i = \frac{1}{2\pi}\left\{ {i \atop jk} \right\} g^{jk} \tag{3.8}$$

Hence the one-loop results satisfy the universality requirement.

The formalism for performing higher loop twisted $N = 2$ calculations was elaborated by Grundberg *et al.* in Ref. [15]. They showed that there was no simple pole at two or three loops for the twisted $N = 2$ model and hence no contribution to the β-function at these orders. (They also showed that when the manifold is further specialized to give an $N = 4$ supersymmetry, the model becomes finite to all orders.) We shall explain their notation and formalism in some detail as we shall adopt it for the four-loop calculation.

We expand the action Eq. (2.29) around a background

$$\Phi \to \Phi_0 + \Phi \tag{3.9}$$

and write

$$\begin{aligned} S = \int d^2x d^2\theta d^2\bar{\theta}[&\Phi^u\bar{\Phi}^{\bar{v}}\delta_{uv} - \Phi^p\bar{\Phi}^{\bar{q}}\delta_{pq} \\ &+ (K_{u\bar{v}} - \delta_{uv})\Phi^u\bar{\Phi}^{\bar{v}} + (K_{p\bar{q}} + \delta_{pq})\Phi^p\bar{\Phi}^{\bar{q}} \\ &+ \frac{1}{2}K_{uv}\Phi^u\Phi^v + \ldots] \end{aligned} \tag{3.10}$$

The superspace propagators are extracted from the first two terms in Eq. (3.10):

$$\begin{aligned} \langle\Phi^u\bar{\Phi}^{\bar{v}}\rangle &= \frac{1}{p^2}\delta^4(\theta - \theta')\delta_{uv} \\ \langle\Phi^p\bar{\Phi}^{\bar{q}}\rangle &= -\frac{1}{p^2}\delta^4(\theta - \theta')\delta_{pq} \end{aligned} \tag{3.11}$$

All the other terms in the expansion Eq. (3.10), including the remaining quadratic ones, furnish vertices. There is a factor $D^2 \equiv -iD_+D_-$ ($\bar{D}^2 = -i\bar{D}_+\bar{D}_-$) for every Φ^u ($\bar{\Phi}^{\bar{v}}$) and $i\bar{D}_+D_-$ ($iD_+\bar{D}_-$) for every Φ^p ($\bar{\Phi}^{\bar{q}}$) field at each vertex. It is shown in Ref. [15] that the effect of these factors together with the quadratic vertices $(K_{u\bar{v}} - \delta_{uv})$ and $(K_{p\bar{q}} - \delta_{pq})$ is simply to produce effective propagators

$$\begin{aligned} \Delta_{u\bar{v}} &= \frac{1}{p^2}K^{u\bar{v}}\bar{D}^2\delta^4(\theta - \theta')D^2 \\ \Delta_{p\bar{q}} &= -\frac{1}{p^2}K^{p\bar{q}}\bar{D}_+D_-\delta^4(\theta - \theta')D_+\bar{D}_-. \end{aligned} \tag{3.12}$$

These propagators are denoted graphically as in Fig. 1.

$N=2$ supergraph power counting shows that the divergences are at most logarithmic. Hence graphs in which derivatives act on background fields are finite, and since we are only interested in ultra-violet divergent graphs we may integrate by parts ignoring the background fields. This immediately implies that we need only consider vertices involving at least one of the combinations $\Phi^p\bar{\Phi}^{\bar{q}}$ or $\Phi^u\bar{\Phi}^{\bar{v}}$. Using the anti-commutation relations of the D's,

$$\begin{aligned} \{D_\pm, \bar{D}_\pm\} &= \pm i\partial_\pm, \qquad \{D_\pm, \bar{D}_\mp\} = 0 \\ \{D_\pm, D_\pm\} &= 0, \qquad \{D_\pm, D_\mp\} = 0 \end{aligned} \tag{3.13}$$

Grundberg *et al.* [15] derive a set of identities which are useful in deciding which vertices need be considered and in simplifying the computation of Feynman graphs; writing

$$\begin{aligned} \mathbf{1} &\equiv \bar{D}^2 D^2, & \bar{\mathbf{1}} &\equiv D^2\bar{D}^2, \\ \mathbf{I} &\equiv \bar{D}_+ D_- D_+ \bar{D}_-, & \bar{\mathbf{I}} &\equiv D_+ \bar{D}_- \bar{D}_+ D_-, \\ \partial'_- &\equiv i\partial_- \bar{D}_+ D_+, & \bar{\partial}'_- &\equiv i\partial_- D_+ \bar{D}_+, \\ \partial'_+ &\equiv -i\partial_+ \bar{D}_- D_-, & \bar{\partial}'_+ &\equiv -i\partial_+ D_- \bar{D}_- \end{aligned} \tag{3.14}$$

it is easy to show

$$\begin{aligned} \mathbf{I} &= \mathbf{1} + \partial'_-, & \bar{\mathbf{I}} &= \bar{\mathbf{1}} + \bar{\partial}'_-, \\ \mathbf{I} &= \bar{\mathbf{1}} + \bar{\partial}'_+, & \bar{\mathbf{I}} &= \mathbf{1} + \partial'_+, \\ \mathbf{1} &= \bar{\mathbf{1}} + \bar{\partial}'_+ + \bar{\partial}'_- + \partial^2, & \bar{\mathbf{1}} &= \mathbf{1} + \partial'_+ + \partial'_- + \partial^2, \\ \mathbf{I} &= \bar{\mathbf{I}} - \bar{\partial}'_- - \partial'_+ - \partial^2, & \bar{\mathbf{I}} &= \mathbf{I} - \partial'_- - \bar{\partial}'_+ - \partial^2. \end{aligned} \tag{3.15}$$

We shall be using dimensional regularization combined with the minimal subtraction prescription. In this context, graphs with tadpoles, i.e. propagators which begin and end at the same point, do not have simple poles after taking into account necessary subtractions, and hence do not contribute to the β-functions; we shall therefore discard such graphs. If a ∂^2 is produced on a given line when Eq. (3.15) is applied, the propagator is cancelled and the points joined by the propagator are identified. If there are other propagators linking the same two points, they then form tadpoles and the graph can be ignored. Let us first consider four-point vertices. In our calculation, propagators emanating from four-point vertices will always be linked pairwise to other vertices, and hence by the above reasoning we may always ignore ∂^2 factors on such propagators. For a given four-point vertex, we use Eq. (3.15) to write all the factors $\mathbf{1}$, $\bar{\mathbf{1}}$, $\mathbf{I}$, $\bar{\mathbf{I}}$ associated with that vertex in terms of one selected element of that set. We then find by integrating by parts and using (from Eq. (3.13))

$$D_+^2 = D_-^2 = \bar{D}_+^2 = \bar{D}_-^2 = 0 \tag{3.16}$$

that the vertices depicted in Fig. 2 lead to no contribution to the β-function. The same applies to vertices obtained from those in Fig. 2 by reversing all arrows or by replacing solid by dotted lines and vice-versa. We call these additional vertices *concomitants* of the ones in Fig. 2. The vertices which potentially can appear in graphs contributing to the

β-function are then those shown in Fig. 3, together with their concomitants in the sense defined above. We call this full set of vertices the *admissible* vertices. Let us first consider graphs constructed from four-point vertices alone–such graphs have the topology shown in Fig. 4(a). We can write down all such graphs constructed solely from admissible vertices, and then we compute the Feynman graphs with propagators as given by Eq. (3.12), using Eq. (3.14) to simplify the D-algebra. After discarding tadpole graphs and performing all the θ-integrals, we find a final four-loop momentum integral which is the same as that considered in Ref. [3] (where it was denoted A_4). We shall denote by G_1 the residue of the simple pole in ϵ arising from this momentum integral together with all necessary subtractions, namely

$$G_1 = \frac{1}{8}\left(\frac{1}{2\pi}\right)^4\zeta(3). \tag{3.17}$$

It turns out that the graphs shown in Fig. 5 give no simple pole despite all their vertices being admissible. The remaining graphs all yield a contribution to $K^{(4,1)}$, i.e. the residue of the four-loop simple pole in K^B (defined analogously to Eq. (3.1)) of the form

$$\frac{n_s}{12}G_1(-1)^{n_p}K_{abcd}K_{efgh}K_{ijkl}K^{ae}K^{bf}K^{cj}K^{dj}K^{gk}K^{hl} \tag{3.18}$$

where $a\ldots l$ each represent one of the indices of type $p, \bar{p}, u, \bar{u}$, n_p is the number of $\Delta_{p\bar{q}}$ propagators, and n_s is a numerical factor.This numerical factor takes the value 6 for all graphs in which not all vertices are alike. In the case of graphs in which all vertices are alike, depicted in Fig. 6, n_s is 2 for Figs. 6(a)-(d) and 4 for Figs. 6(e), (f).

We now turn to graphs of the form Fig. 4(b). Using again Eq. (3.15) it is straightforward to prove the relations displayed in Fig. 7, together with their concomitants.(However, the identities formed by interchanging dashed and full lines acquire a relative minus sign on the right-hand side.) The loops on the right-hand sides in Fig. 7 are simple momentum-space loops–the θ, $\bar{\theta}$ integrals have been performed. What is more, all possible combinations not covered by Fig. 7 and its concomitants are inadmissible (in the sense defined above). Let us call a loop consisting of a pair of propagators joined at each end, as in Fig. 7, a "1-loop". After using Fig. 7 to perform the D-algebra in the top and bottom 1-loops of a graph of the form Fig. 4(b), we may apply Fig. 7 again to the combination of the final right-hand 1-loop of Fig. 4(b) and the single propagator on the left-hand side. So in fact the combinations of propagators which can appear in every 1-loop in a graph like Fig. 4(b), together with their orientation, are determined by the single propagator on the left-hand side. After performing all the D-algebra, the resultant momentum integral is easily cast into the form considered in Ref. [3] and hence evaluated. We find that the contribution to $K^{(4,1)}$ from every graph constructed from admissible 1-loops is of the form

$$\frac{n_s}{12}G_1(-1)^{n_p}(-1)^{n_l}F(K) \tag{3.19}$$

where $F(K)$ is a product of derivatives of K contracted together according to the topology of Fig. 4(b), (in a similar fashion to Eq. (3.18)), $n_s = 6$, n_p is the number of $\Delta_{p\bar{q}}$ propagators as before, and $n_l = 0$ or 1 according as the single propagator on the left-hand side of Fig. 4(b) is $\Delta_{u\bar{v}}$ or $\Delta_{p\bar{q}}$ respectively.

Finally we consider graphs of the form Figs. 4(c), (d). Here we find that all the single propagators (i.e. not in a 1-loop) in a given graph must be of the same kind and oriented in the same sense (i.e. all clockwise or all anti-clockwise). The pairs of propagators which can appear in 1-loops are then again those in Fig. 7 and their concomitants, with the same orientation relative to the single propagators. The contribution to $K^{(4,1)}$ is precisely of the form (3.19) for each of Figs. 4(c), (d), with n_l now being the number of single propagators of type $\Delta_{p\bar{q}}$. $F(K)$ is again determined by the topology of the graph and the types of propagators appearing, and $n_s = 6$ except for diagrams of type Fig. 4(d) with all 1-loops identical, in which case it takes the value 2. There is clearly a remarkable regularity in the types of terms contributing to the four-loop simple pole, and in their numerical factors. This will be explained later when we show that $K^{(4,1)}$ has a very concise expression in terms of the generalized curvature.

However, before we proceed further we should note that we have been unable to compute unambiguously graphs of the forms shown in Fig. 8. For instance consider a momentum integral represented by the graph of Fig. 9(a) (with a bar or a cross on a line representing a $\partial_\pm$ on that line respectively). It is straightforward to use integration by parts to reduce this graph either to the form Fig. 4(a) or alternatively to Fig. 10. However, the momentum integral for Fig. 10, computed in dimensional regularization, is completely different to that for Fig. 4(a)[16]. (In particular it contains no $\zeta(3)$ term.) The problem clearly resides in the fact that the decomposition of ∂_μ into light-cone components $\partial_\pm$ is only valid in two dimensions and hence there is a possible conflict with dimensional regularization. This difficulty is related to the ϵ-tensor problem which has been encountered previously in σ-models with torsion, since

$$k_-p_+ = k.p + \epsilon^{\mu\nu}k_\mu p_\nu, \tag{3.20}$$

where $\epsilon_{\mu\nu}$ is the two-dimensional alternating tensor. There are subtleties[17, 18] in extending away from 2 dimensions the relation

$$\epsilon_{\mu\nu}\epsilon_{\rho\sigma} = \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}, \tag{3.21}$$

especially when products of three or more ϵ-tensors are involved. The processes of integration by parts of $\partial_\pm$ and cancelling propagators using $\partial_+\partial_- = \partial^2$ implicitly involve manipulations of ϵ-tensors which are potentially ambiguous away from 2 dimensions. It is not difficult to convince oneself by trial and error that the graphs considered so far give unambiguous answers, in the sense that it is impossible to obtain different results by integrating by parts in different ways. The underlying reason for this is not clear, but it may be the high degree of symmetry of these graphs. In any case we have very compelling *a posteriori* reasons for believing that we have computed correctly all the graphs so far, since we shall show in Section 4 that the results for these graphs, several hundred in number, can all be summarised by a very compact expression in terms of the generalized curvature.

Finally there are graphs with the topology depicted in Fig. 11. It is easy to show that all such graphs can be reduced to graphs with tadpoles by integrating by parts, and hence they can be ignored.

4. General expression for the twisted Kähler potential

In this section we shall construct a general expression for $K^{(4,1)}$, the simple pole correction to the twisted Kähler potential at four loops.

As advertized earlier, we shall give an expression for $K^{(4,1)}$ in terms of the generalized Riemann tensor. There are two generalized Riemann tensors $\mathcal{R}^\pm$ at our disposal, constructed from either Γ^+ or Γ^- in Eq. (2.14), namely

$$\mathcal{R}^{\pm i}{}_{jkl} = \Gamma^{\pm i}{}_{kj,l} - \Gamma^{\pm i}{}_{lj,k} + \Gamma^{\pm m}_{kj}\Gamma^{\pm i}{}_{lm} - \Gamma^{\pm m}_{lj}\Gamma^{\pm i}{}_{km}. \tag{4.1}$$

However, since

$$\mathcal{R}^+{}_{ijkl} = \mathcal{R}^-{}_{klij}, \tag{4.2}$$

we can work with either. By analogy with Ref. [3], we shall be seeking a product of three generalized Riemann tensors, contracted together in some way, to reproduce our results The first thing to note in matching the results of Section 3 to an expression in terms of generalized Riemann tensors is that in view of Eq. (2.28), the factors of K with two derivatives, together with the factor $(-1)^{n_p}$, may be replaced by the corresponding metric tensors, with a positive overall sign. Suppose that this has been done. We can then write the contribution to $K^{(4,1)}$ for a general graph as

$$\frac{n_s}{12}G_1(-1)^{n_l}G(K) \tag{4.3}$$

where n_s is a numerical factor, n_l counts the number of single $\Delta_{p\bar{q}}$ propagators, and $G(K)$ is a product of Ks with three or four derivatives contracted with metric tensors. The most obvious problem which then confronts us is the differing behaviour of the calculated results, compared with a contracted product of three Riemann tensors, under interchange of p, $\bar{p}$ type with u, $\bar{u}$ type indices. For instance, the expressions corresponding to Fig. 4(a) retain the same sign under this interchange; however, these terms would need to be generated by picking the terms with four derivatives of K from a product of three generalized Riemann tensors, and it can be seen from Eq. (2.25) that these change their sign under the interchange. (See also Eq. (4.11).) It turns out that the solution to this dilemma is to include the complex structures $J^\pm$. The reason is the following: let $P^p{}_i{}^j$, $P^u{}_i{}^j$ etc. be projection operators onto Φ^p, Φ^u etc. Then we have

$$\begin{aligned} J^+{}_i{}^j &= i(P^p - P^{\bar{p}} + P^u - P^{\bar{u}})_i{}^j \\ J^-{}_i{}^j &= i(P^p - P^{\bar{p}} - P^u + P^{\bar{u}})_i{}^j \end{aligned} \tag{4.4}$$

We note that the action of J^- is thus of opposite sign on p, $\bar{p}$ co-ordinates relative to u, $\bar{u}$ co-ordinates respectively. However it is not sufficient to introduce J^- alone, since otherwise replacing all co-ordinates by their complex conjugates would give an unwanted sign change–we must have both J^- and J^+. Now, it turns out that J^- is naturally to be associated with the first pair of indices in $\mathcal{R}^-{}_{ijkl}$, and J^+ with the second. For we have

$$[\mathcal{D}_k^-, \mathcal{D}_l^-]V_j = \mathcal{R}^{-i}{}_{jkl}V_i - 2H^m{}_{kl}\mathcal{D}_m^- V_j, \tag{4.5}$$

where V is a general vector, and applying this to J^- and using Eq. (2.13) we find [13]

$$\mathcal{R}^-{}_{\bar{i}jkl} = -\mathcal{R}^-{}_{i\bar{j}kl} \tag{4.6}$$

introducing the notation

$$v^{\bar{i}} = J^{-i}{}_j v^j. \tag{4.7}$$

On the other hand, in view of Eq. (4.2), we also find, applying Eq. (4.5) to J^+,

$$\mathcal{R}^-{}_{ij\bar{k}l} = -\mathcal{R}^-{}_{ijk\bar{l}}, \tag{4.8}$$

where

$$v^{\bar{i}} = J^{+i}{}_j v^j. \tag{4.9}$$

In view of these facts, we postulate an expression for $K^{(4,1)}$ of the form

$$K^{(4,1)} = \frac{1}{12}\mathcal{R}^-{}_{i\bar{j}I}{}^{J}\mathcal{R}^{-ikIK}\mathcal{R}^-{}_k{}^j{}_K{}^J G_1. \tag{4.10}$$

where the summations over i, j, k, I, J, K range over all the index types $p, \bar{p}, u, \bar{u}$. We shall now prove that Eq. (4.10) does indeed correspond to the diagrammatic calculation. Our method will be to show that in expanding Eq. (4.10) in terms of derivatives of K, the rules for generating a non-zero term are isomorphic to the rules for obtaining a non-zero contribution to $K^{(4,1)}$ in Section 3.

Eqs. (4.4) and (4.6) imply that only pairs of indices of type $(p\bar{q})$, $(u\bar{v})$, (pu), or $(\bar{p}\bar{u})$ can appear as the first pair in $\mathcal{R}^-{}_{ijkl}$; similarly, Eqs. (4.4) and (4.8) imply that the second pair of indices must be of type $(p\bar{q})$, $(u\bar{v})$, $(p\bar{u})$, or $(\bar{p}u)$. In fact, by explicit calculation using Eqs. (2.25) and (4.1), we find that the non-zero components of $\mathcal{R}^-{}_{ijkl}$ are given by

$$\begin{aligned}
\mathcal{R}^-{}_{p\bar{q}\bar{l}k} &= K_{p\bar{q}\bar{l}k} + K^u{}_{p\bar{l}}K_{u\bar{q}k} - K^r{}_{pk}K_{r\bar{q}\bar{l}} \\
\mathcal{R}^-{}_{u\bar{v}w\bar{x}} &= K_{u\bar{v}w\bar{x}} + K^y{}_{uw}K_{y\bar{v}\bar{x}} - K^r{}_{u\bar{x}}K_{r\bar{v}w} \\
\mathcal{R}^-{}_{p\bar{q}\bar{v}u} &= K_{p\bar{q}\bar{v}u} + K^w{}_{p\bar{v}}K_{w\bar{q}u} - K^r{}_{pu}K_{r\bar{q}\bar{v}} \\
\mathcal{R}^-{}_{u\bar{v}p\bar{q}} &= K_{u\bar{v}p\bar{q}} + K^w{}_{up}K_{w\bar{v}\bar{q}} - K^r{}_{u\bar{q}}K_{r\bar{v}p} \\
\mathcal{R}^-{}_{pu\bar{v}q} &= K_{pu\bar{v}q} + K^w{}_{p\bar{v}}K_{wuq} - K^r{}_{pq}K_{ru\bar{v}} \\
\mathcal{R}^-{}_{upv\bar{q}} &= K_{upv\bar{q}} + K^w{}_{uv}K_{wp\bar{q}} - K^r{}_{u\bar{q}}K_{rpv} \\
\mathcal{R}^-{}_{upv\bar{w}} &= K_{upv\bar{w}} + K^x{}_{uv}K_{xp\bar{w}} - K^r{}_{u\bar{w}}K_{rpv} \\
\mathcal{R}^-{}_{pu\bar{r}q} &= K_{pu\bar{r}q} + K^w{}_{p\bar{r}}K_{wuq} - K^s{}_{pq}K_{su\bar{r}} \\
\mathcal{R}^-{}_{u\bar{w}v\bar{p}} &= K_{u\bar{w}v\bar{p}} + K^x{}_{uv}K_{x\bar{w}\bar{p}} - K^r{}_{u\bar{p}}K_{r\bar{w}v} \\
\mathcal{R}^-{}_{p\bar{r}\bar{u}q} &= K_{p\bar{r}\bar{u}q} + K^x{}_{p\bar{u}}K_{x\bar{r}q} - K^s{}_{pq}K_{s\bar{r}\bar{u}},
\end{aligned} \tag{4.11}$$

together with components related by complex conjugation of all indices or by the symmetries

$$\mathcal{R}^-{}_{ijkl} = \mathcal{R}^-{}_{[ij][kl]} \tag{4.12}$$

The generic form of the expressions for $\mathcal{R}^-$ in Eq. (4.11) is

$$\mathcal{R}^-{}_{i_1i_2i_3i_4} = K_{i_1i_2i_3i_4} + K^U{}_{i_1i_3}K_{Ui_2i_4} - K^P{}_{i_1i_4}K_{Pi_2i_3} \tag{4.13}$$

It is convenient for our discussion to call an ordered pair of index-types (I, J) a P-pair if there is a non-zero Riemann tensor such that when expressed in the form Eq. (4.13), i_2 is of type I and i_3 is of type J–in other words (I, J) is associated with P in the three-derivative terms on the right-hand side of Eq. (4.13). Similarly, we call an ordered pair (I, J) a $\bar{P}$-pair if i_1 is of type I and i_4 is of type J. (Note that for vectors V, W, we have $V^pW_p = V_{\bar{p}}W^{\bar{p}}$.) Moreover, we call (I, J) a U-pair if i_2 is of type I and i_4 is of type J, and a $\bar{U}$-pair if i_1 is of type I, i_3 of type J. It is easy to see from Eq. (4.11) that this is an unambiguous terminology. In fact, the P-pairs are the following: $(u, \bar{v})$, $(\bar{p}, \bar{q})$, $(\bar{p}, \bar{u})$, $(u, \bar{p})$, and the $\bar{P}$-pairs are their complex conjugates. The U-pairs are the following: $(p, \bar{q})$, $(\bar{u}, \bar{v})$, $(p, \bar{u})$, $(\bar{u}, \bar{p})$, and the $\bar{U}$-pairs are their complex conjugates.

It is clear from the definition and Eq. (4.13) that given a non-zero Riemann tensor $\mathcal{R}^-{}_{ijkl}$, (i, k) and (j, l) are either P and $\bar{P}$ or U and $\bar{U}$ in type. Moreover (i, k) and (j, l) are conjugacy related, i.e. if (i, k) is P-type then (j, l) is $\bar{P}$-type, etc. The same applies to (i, l) and (j, k). Conversely, given any P-type pair (i, k) and any $\bar{P}$-type pair (j, l), $\mathcal{R}^-{}_{ijkl}$ will appear in Eq. (4.11) or be related to a component in Eq. (4.11) by complex conjugation or symmetry, and the same for $P \to U$. We can then see that in Eq. (4.10), (i, I), (j, J) and (k, K) must either all be P and $\bar{P}$ or all U and $\bar{U}$ for a non-zero contribution. We can now begin to compare Eq. (4.10) with the explicit calculations of Section 3. We shall show that the rules for generating a non-zero term in Eq. (4.10) are equivalent to the rules found in Section 3 for constructing a non-zero diagram. It is immediately apparent that Figs. 4(a-d) correspond to terms in Eq. (4.10) obtained by taking the linear term in K from three, two, one or none of the Riemann tensors (as given by Eq. (4.11)) respectively, and taking the quadratic terms from the remaining Riemann tensors. Moreover to get the correct topology we must take the quadratic term in which the pair (i, I), (j, J) or (k, K) appear on the same K. The single propagators in Figs. 4(b-d) then correspond to the U or P indices in Eq. (4.13), and we identify the pairs of indices (i, I), (j, J) and (k, K) in Eq. (4.10) with the types of propagator in the 1-loops in the diagrams of Fig. 4. Of course in view of Eqs. (2.25), (3.12), if the indices at one end of a 1-loop form a P-pair, then those at the other end will form a $\bar{P}$-pair. With regard to the diagrams of Figs. 4(b-d), we see immediately that the admissible 1-loops in Fig. 7 (together with their concomitants) correspond precisely to the P-, $\bar{P}$-, U- and $\bar{U}$-pairs. In Fig. 7 and its concomitants, each of these pairs appears once and once only at the end of some 1-loop which is attached to the single propagator, and the index at that end of the propagator is then a p, $\bar{p}$, u, $\bar{u}$-type index accordingly. In other words, the 3-point vertices in Fig. 7 and their concomitants generate exactly the 3-derivative terms appearing in Eq. (4.11). The rule for Eq. (4.10) that all the pairs (i, I), (j, J), (k, K) must be P-, $\bar{P}$- or U-, $\bar{U}$-type pairs is equivalent to the rule found in Section 3 that the single propagators must all be the same type. It is easy to check from Eqs. (4.10), (4.11) that the fact that in $\mathcal{R}^-{}_{ijkl}$, (i, k) and (j, l) are related by conjugacy corresponds to the rule of Section 3 that the single propagators and all the 1-loops must all be oriented alike. Finally the fact that any P-pair can be combined with any $\bar{P}$ pair into a non-zero Riemann tensor component (and similarly for

$P \to U$) guarantees that we generate all diagrams with admissible vertices satisfying these requirements.

We now turn to Fig. 4(a). The non-zero diagrams of type Fig. 4(a), are all graphs of that type constructed using the vertices of Fig. 3 and their concomitants, but omitting the diagrams of Fig. 5. It is easy to see that, for a given diagram, either all the 1-loops have a P-pair at one end and a $\bar{P}$-pair at the other, or a U-pair and a $\bar{U}$-pair at opposite ends. Moreover at each 4-point vertex a P-pair alternates with a $\bar{P}$-pair (or a U-pair with a $\bar{U}$-pair). We have all possible diagrams satisfying those criteria. Again these are precisely the rules we noted from studying Eq. (4.10).

What we have shown so far is that the terms generated by Eq. (4.10) are in one-to-one correspondence with the terms produced in the explicit calculation. However we have still to demonstrate that the signs and numerical factors are given correctly. We have already accounted for the $(-1)^{n_p}$ factor which appeared in the results of Section 3 by writing Ks with two derivatives in terms of the metric. We now note that

$$(\tilde{j}, \hat{J}) = (j, J) \tag{4.14}$$

if (j, J) is a U-pair, while

$$(\tilde{j}, \hat{J}) = -(j, J) \tag{4.15}$$

if (j, J) is a P-pair. Now from Eq. (4.13), given $\mathcal{R}^-_{ijkl}$, if (i, k) and (j, l) are U, $\bar{U}$ pairs then both the linear term in K and the term $K^U{}_{ik} K_{Ujl}$ are positive. If on the other hand (i, k), (j, l) are P, $\bar{P}$ pairs then the linear term is negative, but the term $K^P{}_{ik} K_{Pjl}$ is positive. Consider the graph Fig. 4(a). We may call a given graph p-type or u-type according as the corresponding expression from Eq. (4.10) has all P, $\bar{P}$ pairs or all U, $\bar{U}$ pairs. In the p-type case, each $\mathcal{R}^-$ contributes a minus sign, but so do the $J^\pm$ according to Eqs. (4.14), (4.15), giving an overall positive sign. In the u-type case, each $\mathcal{R}^-$ gives a positive sign, as do the $J^\pm$, giving again a positive sign.

Now consider Fig. 4(b). Since the quadratic term in K gives a positive sign in both the p and u type cases, the Riemann tensors always give a positive sign. Hence the sign is determined by Eqs. (4.14) and (4.15) and is positive in the u-case and negative in the p-case, which agrees with Eq. (4.3). It is easy to see that in general there is a negative sign for a p-type graph relative to a u-type graph if we take an odd number of quadratic terms, i.e. if there is an odd number of single propagators, and hence we reproduce the sign of Eq. (4.3).

Finally we turn to the symmetry factor. We have shown that there is a one-to-one correspondence between types of term generated by Eq. (4.10) and types of diagram– however in general Eq. (4.10) will generate several terms for a given diagram. We have to identify a 4-point vertex, or a pair of adjacent 3-vertices connected by a single propagator, with a Riemann tensor in Eq. (4.10). If not all 1-loops in a diagram are the same, then in fact the corresponding Riemann tensor components will all be different. We then have 6 ways of identifying the Riemann tensors in Eq. (4.10) with either 4-point vertices or pairs of 3-point vertices. For these diagrams it is possible to decide whether they are p-type or u-type, and then there is only one way to identify (i, I), (j, J) and (k, K) in Eq. (4.10) with the propagators such that they become P-pairs or U-pairs as appropriate. Hence the overall symmetry factor is 6.

Now suppose that all the 1-loops are identical, and hence all the Riemann tensors must be the same component. We then have no choice in identifying Riemann tensors with 4-point vertices or pairs of 3-point vertices. However, in $\mathcal{R}^-{}_{ijIJ}$, we can choose whether (i, I) or (j, J) is to be associated with a given side of a vertex, giving a symmetry factor of 2. Moreover, in the case of Figs. 4(b) and 4(c) we have a further 3 ways of deciding which of the Riemann tensors are to furnish the pairs of 3-point vertices. Furthermore, in certain cases we cannot decide whether the diagram is p-type or u-type, and hence terms in Eq. (4.10) with either all P-, $\bar{P}$-pairs or all U-, $\bar{U}$-pairs can contribute to the same diagram. This is because $(\bar{p}, \bar{u})$ is a P-pair and $(\bar{u}, \bar{p})$ is a U-pair; similarly $(u, \bar{p})$ is a P-pair and $(\bar{p}, u)$ is a $\bar{U}$-pair. So Figs. 6 (e), (f) can be either p-type or u-type, which accounts for the factor of 2 in the symmetry factor relative to Figs. 6 (a)-(d). (Note that if we mix the 1-loops from Fig. 6 (e) with those from Fig. 6 (f), then we *can* distinguish u-type from p-type; this is because the first ambiguous pair above is the same conjugacy type however it is interpreted, while the second is of different conjugacy type depending on whether we interpret it as p-type or u-type. Since a vertex must have a P-pair on one side and a $\bar{P}$-pair on the other, or a U-pair on one side and a $\bar{U}$-pair on the other, there is bound to be a mismatch for one or other interpretation if we mix the 1-loops.) Of course in the case of Figs. 4 (b)-(d) there is no ambiguity since the single propagator defines the graph as p-type or u-type.

To summarise, the terms corresponding to Figs. 4(b) and 4(c) always have a symmetry factor of 6. The terms corresponding to Figs. 4(a) and 4(d) have a symmetry factor of 6 unless all the 1-loops are identical. In that case, terms corresponding to Fig. 4(d) have a symmetry factor of 2; in the case of Fig. 4(a), the terms corresponding to Figs. 6(a)-(d) have a factor of 2, while those corresponding to Figs. 6(e), (f) have a factor 4. Comparing with Section 3, we can now see that we have obtained precisely the correct symmetry factors. Hence Eq. (4.10) precisely reproduces the results for all the graphs computed in Section 3. We feel confident in asserting that Eq. (4.10) would also give the right results for the diagrams of Fig. 8 if an unambiguous prescription for calculating them could be devised.

5. Conclusions

Our main result is the succinct expression for the 4-loop correction to the twisted Kähler potential,

$$K^{(4,1)} = \frac{1}{96}\left(\frac{1}{2\pi}\right)^4 \zeta(3) \mathcal{R}^-{}_{ijIJ} \mathcal{R}^{-ikIK} \mathcal{R}^-{}_k{}^j{}_K{}^J. \tag{5.1}$$

It is remarkable that the results for several hundred graphs can be subsumed into such a compact form. This result is a natural generalization of that found in Ref. [3] for the ordinary $N = 2$ σ-model. Indeed an expression of the form Eq. (5.1), but with $\mathcal{R}^-$ replaced by the conventional Riemann tensor, and featuring the single complex structure appropriate to a Kähler manifold, arose naturally in Ref. [6] and was shown to be equivalent to the result of Ref. [3]. There are a number of issues which still warrant further investigation.

Firstly, there is the question of universality. The result Eq. (5.1) might be thought to violate this principle since the complex stuctures appear explicitly. However, the universality principle applies only to the β-functions which are obtained from $K^{(4,1)}$ by taking derivatives , as for g and b in Eqs. (2.25), (2.29). To be explicit,

$$\begin{aligned} \beta^{g(4)}_{p\bar{q}} &= 4K^{(4,1)}_{p\bar{q}} \qquad & \beta^{g(4)}_{u\bar{v}} &= 4K^{(4,1)}_{u\bar{v}} \\ \beta^{b(4)}_{u\bar{p}} &= 2K^{(4,1)}_{u\bar{p}} \qquad & \beta^{b(4)}_{p\bar{u}} &= 2K^{(4,1)}_{p\bar{u}} \\ \beta^{b(4)}_{up} &= -2K^{(4,1)}_{up} \qquad & \beta^{b(4)}_{\bar{u}\bar{p}} &= -2K^{(4,1)}_{\bar{u}\bar{p}}, \end{aligned} \tag{5.2}$$

with $K^{(4,1)}$ as in Eq. (5.1), the remaining non-zero components being those implied by the symmetry of β^g and the antisymmetry of β^b. It would be an interesting exercise to evaluate the β-functions themselves and show that they are expressible up to diffeomorphism in a covariant form free of the complex structure. This was achieved in the case of the untwisted $N = 2$ model in Refs. [6], [13]. However the calculation in the present instance would be more arduous because the cyclicity and Bianchi identities for the ordinary Riemann tensor, which played a crucial role in Refs. [6], [13], no longer hold for the generalized Riemann tensor. Moreover, in the untwisted case the partial derivatives acting on K as in Eq. (2.7) could be replaced by covariant derivatives–this is no longer valid in the twisted case. A final complication is that, as was found at one loop in Ref. [14], a universal expression might only be achieved by adding some non-covariant diffeomorphism to the β-function.

The second outstanding problem is the existence of an effective action which generates the conformal invariance conditions for the twisted model as its equations of motion. Such an action was constructed in Refs. [6], [7] in the untwisted case, by generalizing the 10-dimensional action, constructed in Ref. [9] from string amplitudes, to target manifolds of arbitrary dimension. (There is also an earlier discussion, specific to 10 dimensions, of the relation between the effective action of Ref. [9], and the β-functions of Ref. [3], in Ref. [8].) A superstring action incorporating an antisymmetric tensor background was constructed from string amplitudes in Refs. [19-21]. Again, this action is formulated for the 10-dimensional target manifold appropriate for strings propagating on a flat background. We have attempted, so far without complete success, to generalize it to target manifolds of arbitrary dimension, in order to compare with our results. However, an encouraging sign is that the superstring action of Refs. [19-21] is also constructed from $\mathcal{R}$, at least as far as terms in $\mathcal{R}$ linear in g and b.

We should also discuss the relation of our results to the β-function for the $N = 1$ σ-model with torsion, which has been calculated in Ref. [22]. The universal form of the β-function for the twisted $N = 2$ model might be expected to differ from the β-function for the $N = 1$ σ-model with torsion by a diffeomorphism and also by combinations of terms which vanish on the complex manifold on which the twisted model is defined. Bearing this in mind, it would in principle be possible to compare our results with those of Ref. [22], though it would be an arduous undertaking. This comparison, though, would be of interest in connection with a problem raised some time ago[23]. It is known that β^g and β^b vanish to all orders for a bosonic or supersymmetric σ-model with a parallelised manifold (i.e. one for which $\mathcal{R}$ vanishes) as target[24]. Moreover, it is known from non-perturbative calculations that the dilaton β-function for an $N = 1$ supersymmetric σ-model with such

a target manifold vanishes beyond one loop[25]. Now , the dilaton β-function for a general manifold is related to β^g and β^b by the Curci-Paffuti equation[18, 26], but in a non-trivial fashion, so that it is not at all obvious that the vanishing of β^g and β^b for a parallelized manifold would imply, or even be consistent with, the vanishing of the dilaton β-function. It would be useful to be able to elucidate this question. The $N = 2$ results we have have found here do not shed much light on this, inasmuch as they display the requisite properties in a rather trivial fashion: Any $K^{(4,1)}$ constructed from covariant quantities will be constant on a parallelized manifold, and hence the β-function, obtained by taking derivatives, will vanish; the fact that $K^{(4,1)}$ in Eq. (5.1) actually vanishes for a parallelized manifold is an added bonus but not essential. Moreover the dilaton β-function vanishes owing to $N = 2$ power counting[27]. However, there is no apparent connection between these properties and they do not give any information regarding the $N = 1$ case.

Perhaps the most striking feature of the present calculation is the way in which there is a precise correspondence between the rules for constructing graphs which contribute to the β-function, and the rules for obtaining non-vanishing components of the generalized Riemann tensor. One is led to conjecture that it might be possible to extend this observation into a general, all-orders proof that the β-function is constructed solely from the generalized Riemann tensor. However to succeed in this it might be necessary to confront more directly the problem of ambiguity in products of ϵ-tensors in dimensional regularization, alluded to at the end of Section 3.

We can also use the rules we have formulated, for constructing graphs which contribute to the β-function, to prove the vanishing of the β-function for the twisted $N = 4$ supersymmetric σ-model at four loops. The twisted $N = 4$ σ-model is obtained from the twisted $N = 2$ model[10] by imposing equality of the number of p-type and u-type fields and also the constraints

$$\begin{gathered} K_{p^i\bar{p}^j} + K_{u^i\bar{u}^j} = 0 \\ K_{p^i\bar{p}^j} = K_{p^j\bar{p}^i} \qquad K_{u^i\bar{u}^j} = K_{u^j\bar{u}^i} \end{gathered} \tag{5.3}$$

We can also[15] use a "gauge transformation" of the twisted Kähler potential[10] to ensure

$$K_{p[i}{}_{u j]} = K_{p[i}{}_{\bar{u} j]} = K_{\bar{p}[i}{}_{u j]} = K_{\bar{p}[i}{}_{\bar{u} j]} = 0. \tag{5.4}$$

We can use Eqs. (5.3), (5.4) to show that, given any Feynman diagram, if we take an arbitrary loop in the diagram and change the propagators in the loop according to the rules shown in Fig. 12 (together with their concomitants as usual), then the product of derivatives of K represented by the original diagram is the same as that for the altered diagram. We can then easily check that, given a 4-loop p-type diagram, there is (except for diagrams of type Fig. 4(a) with all 1-loops identical) one and only one loop in the diagram which, when transformed by the above rules, changes the diagram into a u-type diagram, and vice-versa. Hence the diagrams are divided into pairs, consisting of one p-type and one u-type diagram, such that the members of a given pair produce the same products of derivatives of K. However, it can be seen from Eq. (4.3) that the two members of the pair come with the same symmetry factor and opposite signs, and hence cancel. We have to consider separately diagrams of type Fig. 4(a) with all 1-loops identical, since

Figs. 6(e) and 6(f) can be either p-type or u-type, but it is easy to check the cancellation in this case also. Hence the twisted $N = 4$ β-function vanishes at four loops. So we see that the form of $K^{(4,1)}$ in Eq. (5.1) actually implies the finiteness of the twisted $N = 4$ σ-model at this order. It seems likely that one could extend this to higher orders and show that in general, if the $N = 2$ twisted Kähler potential can be expressed in terms of the generalized curvature, then the twisted $N = 4$ σ-model is finite. In fact, it has been shown already that the twisted $N = 4$ σ-model is finite to all orders, using $N = 4$ superspace power-counting[15]. Hence it would be very interesting if one could reverse the argument to show that $N = 4$ finiteness implies that the $N = 2$ twisted Kähler potential must be expressed in terms of the generalized curvature.

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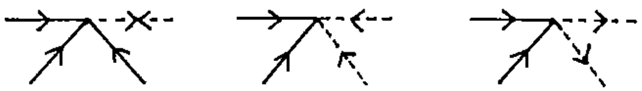
Figure Captions

Figures

Fig. 1

(a) (b)

Fig. 2

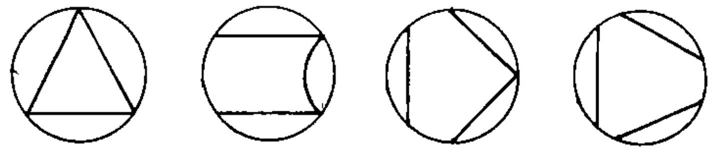


(a) (b) (c)

Fig. 3

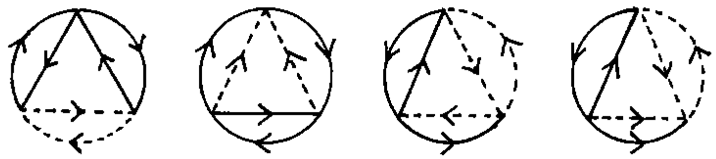
(a) (b) (c) (d)

Fig. 4



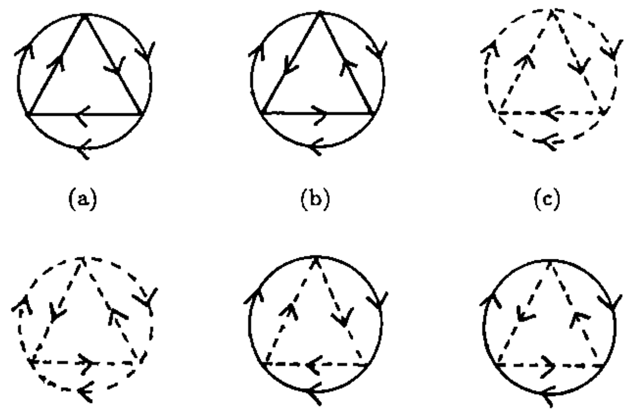
(a) (b) (c) (d)

Fig. 5



(a) (b) (c) (d)

Fig. 6



(a) (b) (c)

(d) (e) (f)

Fig. 7

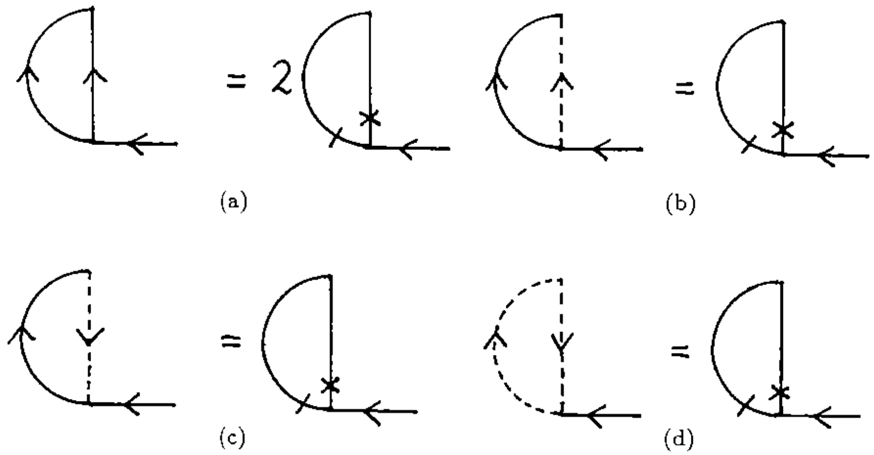


Fig. 8

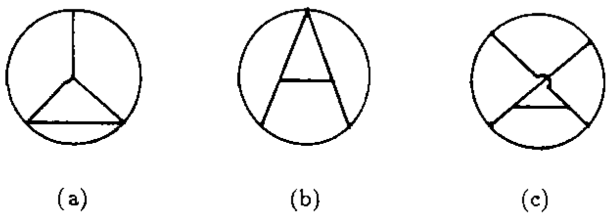


Fig. 9

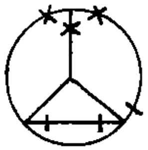


Fig. 10

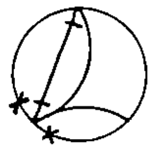
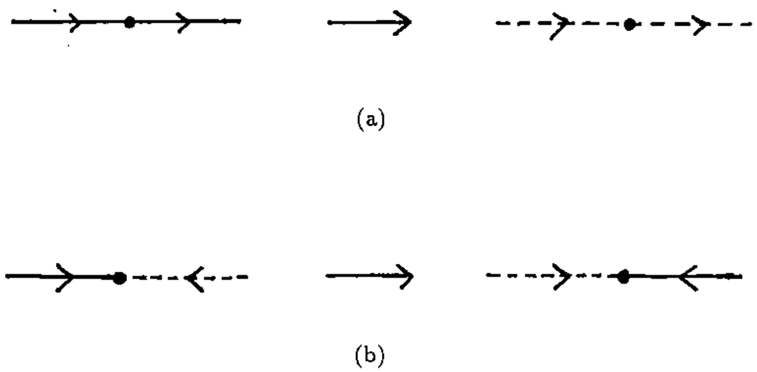


Fig. 11

Fig. 12



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