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Picard-Fuchs Equations and the Moduli Space of Superconformal Field Theories

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Abstract

We derive simple techniques which allow us to relate Picard-Fuchs differential equations for the periods of holomorphic p-forms on certain complex manifolds to their moduli space and its modular group (target space duality). For Calabi-Yau manifolds the special geometry of moduli space gives the Zamolodchikov metric and the Yukawa couplings in terms of the periods. For general $N = 2$ superconformal theories these equations exactly determine perturbed correlation functions of the chiral rings of primary fields.

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Introduction

Recently, stunning developments have occurred in the understanding of the moduli space of a class of superconformal field theories (SCFT), the $N = 2$ theories which are related to target-space supersymmetry [1]. The progress has been achieved in three different areas, namely Landau-Ginzburg [2] models, topological ($N = 2$ twisted) SCFT's [3, 4, 5, 6] and special geometry of Calabi-Yau (CY) moduli space [7]–[11]. In particular, with seemingly rather different techniques, the dynamics of SCFT's, which determines the evolution in coupling constant space of the correlation functions, has recently been encoded in certain differential equations for some functions of the "moduli" deformations which in turn determine all other correlation functions. This is rather transparent in SCFT's and in the special geometry of Calabi-Yau threefolds where all the dynamical information on the moduli evolution relies on a holomorphic function $\mathcal{F}(\psi)$ of the moduli coordinates (two functions in the case of CY threefolds due to the two independent spaces of moduli deformations). On the other hand, although there are strong similarities between topological theories and special geometry, their structure is rather different. For instance in topological field theories the space of deformations is locally flat, while this is not so for the special geometry. Moreover the formula [6]

$$< \phi_i \phi_j \phi_k >= \partial_i \partial_j \partial_k \mathcal{F} = C_{ijk} \tag{1}$$

is valid for any deformation (both marginal and non-marginal) in topological theories, irrespective of the value of the central charge c (related to the $U(1)$ charge). On the contrary in special geometry , equation **(1)** plays a particular role due to the special value of $c = 9$. In fact in the latter case C_{ijk} is related to the Yukawa couplings of the 27 and $\bar{27}$ families, as well as to the metric of the moduli space [7]. Such a metric is given by

$$G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$$

$$K = -log(2\mathcal{F} + 2\bar{\mathcal{F}} - (\psi^i - \bar{\psi}^i)(\mathcal{F}_i - \bar{\mathcal{F}}_i)) \tag{2}$$

In terms of the prepotential appearing in **(1)** the Riemann tensor satisfies the relation [11]

$$R_{i\bar{j}l\bar{k}} = G_{i\bar{j}} G_{l\bar{k}} + G_{i\bar{k}} G_{l\bar{j}} - (\exp 2K)\, \partial_i \partial_l \partial_k \mathcal{F}\ \partial_{\bar{j}} \partial_{\bar{k}} \partial_{\bar{p}} \bar{\mathcal{F}}\ G^{p\bar{p}} \tag{3}$$

Equation **(3)** has been derived both from $N = 2$ space-time supersymmetry and from the superconformal Ward identities. Interestingly it can also be derived from algebraic geometry, as an equation for the metric of the $(1, 1)$ and $(2, 1)$ moduli of CY threefolds given respectively by

$$G_{i\bar{j}} = \partial_i \partial_{\bar{j}} Y \qquad (4)$$
$$Y = -logV$$

where $V = \int J \wedge J \wedge J$ (for the Kähler class deformations) and $V = i \int \Omega \wedge \bar{\Omega}$ (for the complex structure deformations); J is the Kähler two-form and Ω is the deformed holomorphic three-form.

In this note we will show, based on a previous conjecture of one of the authors [12], that the equations defining the non-trivial moduli dependence of the correlators are nothing but the differential equations, known as Picard-Fuchs equations, for the periods associated to the defining polynomial $\mathcal{X}$ in CP^n which specifies the manifold of compactification. Although the method is completely general we will apply it to the case of two rather different manifolds which correspond to a $c = 3$ and a $c = 9$, $N = 2$ superconformal field theory, respectively, the (Z_3 orbifoldized) torus and the mirror manifold of $P_4(5)$. These manifolds are similarly manageable because they correspond to a defining cubic and quintic polynomial with one marginal deformation. However it should be noted that regarded as complex manifolds they are rather different, as shown by their moduli space and associated duality groups (discrete isometries). In principle the differential equations for the periods can be obtained explicitly by evaluating the periods as integrals of the deformed p-form along the b_p homology cycles (b_p is the p - Betti number). In the examples under consideration, $p = 1$, $b_1 = 2$ for the torus, and $p = 3$, $b_3 = 4$ for the CY manifold. The periods are given by

$$\omega_a(\psi) = \int_{A,B} \Omega_p(\psi); \quad a = 1, \dots b_p \qquad (5)$$

It is not an easy task to evaluate (5) for generic manifolds, although for the two examples at hand this was explicitly done [8, 9, 10]. A better strategy consists of writing a matrix first-order differential equation (for the case of one modulus), which only uses a special mapping from a certain polynomial basis associated to the defining polynomial and the cohomology spaces ($H^{2,1}$ and $H^{3,0}$ in the CY case). Using this method there is no real difference between the cubic and the quintic, with the exception that the matrix is 2×2 in one case and 4×4 in the other case, yielding precisely the second- and fourth-order Picard-Fuchs equations whose independent solutions are exactly the periods of the deformed one-and three-forms in each case [13].

Mathematical Preliminaries

In this section we intend to give a brief summary of the essential elements of the mathematical work that implements the program sketched above. Our objective is to give the reader a guide to the major results without presenting derivations or proofs, and then proceed to specialize to the two examples that we have been addressing. We base our results totally on the work completed in the 1960's by B. Dwork and N. Katz [14, 15, 16] and we do not pretend with this brief summary to do justice to such complete and rigorous papers.

An appropriate point to start is with the simple case of a non-singular projective curve $\mathcal{X}$ of genus g defined over a field K of characteristic zero. A meromorphic differential of the second kind, is determined by the fact that all its residues are zero. Define the quotient space of differentials of the second kind modulo exact differentials. The periods can then be obtained by integrating an element ω of the quotient space over a basis $(\gamma_i; i = 1 \ldots 2g)$ of $H_1(\mathcal{X}, K)$. The effect of this is to map the quotient isomorphically to K^{2g}. Then the quotient is dual to $H_1(\mathcal{X}, K)$ and identified with $H^1(\mathcal{X}, K)$, defining in this way a cohomology group that is defined over K.

In general $\mathcal{X}$ will depend on certain K parameters. It is natural to expect the periods to depend on such parameters and to be able to be differentiated with respect to them. Given a derivation D of K it is found that the quotient space has the structure of a module over the algebra of derivations of the base field K and that the periods satisfy the Picard-Fuchs equations

$$\sum_{i=0}^{2g} a_i D^i \int_{\gamma_j} \omega = 0$$

with $a_0, \ldots, a_{2g} \in K$.

One expects the same to occur for a non-singular $\mathcal{X}$ of higher dimension. However in this case the differentials of the second kind no longer give the cohomology over K as they do for curves. A closed meromorphic differential ω on $\mathcal{X}$ is of the second kind if there is an open set $\mathcal{X}^0$ where no coordinate vanishes and ω is holomorphic. The cohomology class on $\mathcal{X}^0$ determined by ω lies in the image of the restriction $H(\mathcal{X}) \to H(\mathcal{X}^0)$. Then in analogy to the curve, the differentiation of cohomology will give rise to Picard-Fuchs equations for the subspace of $H(\mathcal{X}^0)$ spanned by the differential of the second kind.

With these initial remarks completed we can proceed to give a general description of the program of Dwork and Katz. We will first introduce the various spaces and mappings in order to give an overview, and will proceed to define them in the next section.

Start with a projective hypersurface $\mathcal{X}$ in CP^n, defined over a field K of characteristic zero. Let $\mathcal{X}^0$ be the open subset where no coordinate vanishes. $H^{n-1}(\mathcal{X})$ and $H^{n-1}(\mathcal{X}^0)$ are the $(n-1)$ dimensional cohomology groups. As described above, $H^{n-1}(\mathcal{X})$ and $H^{n-1}(\mathcal{X}^0)$ are modules over the algebra of derivations of K. In general the hypersurface can be defined by a polynomial $f(X, \psi)$ of degree d with coefficients in $K(\psi)$ and where X stands for the coordinates of the embedding space. Associated to this polynomial Dwork and Katz develop a series of polynomial spaces and establish an isomorphism Θ of one of this spaces W^S, (described later), with the image of $H^{n-1}(\mathcal{X})$ in $H^{n-1}(\mathcal{X}^0)$. Furthermore, for any derivation D of the definition polynomial f, the equations of deformation of Dwork, given by the action of a certain operator $\mathcal{G}_D$ on W^S, are identified with the Picard-Fuchs equations on the image of $H^{n-1}(\mathcal{X})$ in $H^{n-1}(\mathcal{X}^0)$.

The mapping from polynomial to cohomology spaces

First let us define a set of symbols that will be useful for the purpose of indexing.

$$I = \{w = (w_0, w_1, \ldots, w_{n+1}) | w \in Z_+^{n+2}, dw_0 = w_1 + \ldots w_{n+1}\}$$

$$A = \{u \in I | 0 \le u_i < d, i = 1, \ldots, n+1\}$$

$$A' = \{u \in A | 0 < u_i < d, i = 1, \ldots, n+1\}$$

Let L_ψ be the space of polynomials of the type $\sum_{w \in I} B_w X^w$, with $B_w \in K(\psi)$ and the monomials $X^w = X_0^{w_0} \ldots X_{n+1}^{w_{n+1}}$. Some of the relevant subspaces are:

$$L^0, w_0 \ge 1; \;\; L^+, w_i \ge 0; \;\; L^{+0} = L^0 \bigcap L^+; \;\; L^S, w_i \ge 1$$

For a given polynomial form $f(X, \psi)$ and $f_i = X_i \frac{\partial f}{\partial X_i}$, the following operators can be defined on L_ψ.

$$D_0 = D_{X_0} = X_0 \frac{\partial}{\partial X_0} + X_0 f$$

$$D_i = D_{X_i} = X_i \frac{\partial}{\partial X_i} + X_0 f_i \quad (i = 1, \ldots, n+1) \tag{6}$$

For each derivation D of the field K we have a derivation, $\mathcal{G}_D = D + X_0 f^D$, of L_ψ where D acts only on the coefficients and f^D is the result of applying D to the coefficients of f.

The essence of the program is to construct a generic mapping from L^0 to $H^{n-1}(\mathcal{X}^0)$. It is found that corresponding to each element $X_0 X^{w'}$; $(w' = (w_1, \ldots, w_{n+1}))$ there is a differential form regular on $\mathcal{X}^0$ given in local coordinates $x_i = \frac{X_i}{X_{n+1}}$ as:

$$\frac{X^{w'} dx_1 \wedge \ldots \wedge dx_{n-1}}{X_n \frac{\partial f}{\partial X_n} x_1 \ldots x_{n-1}} \tag{7}$$

Focusing on the spaces that we intend to analyze, the form f will be given by

$$f(X, \psi) = \sum_{i=1}^{n+1} X_i^d + h(X, \psi) = \sum_{i=1}^{n+1} X_i^d + \psi h(X) \tag{8}$$

For the f given above , $\mathcal{G}_\psi = \frac{\partial}{\partial \psi} + X_0 h(X)$, so that L^0 considered as a module for the derivations of K is spanned by the elements of X_0 degree. Based on this fact a surjection is established such that

$$\Theta : L^0 / (D_0 L + \sum_i D_i L^0) \to H^{n-1}(\mathcal{X}^0) \tag{9}$$

With W^S given by

$$W^S = L^S / (L^S \bigcap (D_0 L^+ + \sum_{i=1}^{n+1} D_i L^{0+}) \tag{10}$$

Θ then establishes the isomorphism of W^S with the image of $H^{n-1}(\mathcal{X})$ in $H^{n-1}(\mathcal{X}^0)$.

Choice of basis and explicit calculations

Up to this point we have given very general statements about the mappings between the space of polynomials and the cohomology spaces. In order to actually construct the differential equations described earlier, we must enter deeper into the polynomial spaces.

Let $L_\psi^{(m)}$ be the space of all elements ξ of L_ψ whose X_0 degree is not greater than m. Let U be a subspace of $L^{(n)}$. For a given set of $\psi_0 \in K$ it is possible to write

$$L^{(n)} = U + \sum_{i=1}^{n+1} D_{i,\psi_0} L^{(n-1)}$$

When the defining polynomial $f(X,\psi)$ has the additional ψ dependence, the last expression becomes

$$L_\psi^{(n)} = U_\psi + \sum_{i=1}^{n+1} D_{i,\psi} L_\psi^{(n-1)} \tag{11}$$

where U_ψ is the vector space over $K(\psi)$.

It is possible to choose a basis of elements $\{\{\xi_\mu\}_{\mu\in A}\}$ for $\xi_\mu \in K[X,\psi]$ such that

$$L_\psi^{(n)} = \sum_{\mu\in A} \xi_\mu K(\psi) + \sum_{i=1}^{n+1} D_{i,\psi} L_\psi^{(n-1)}. \tag{12}$$

Let us define $\mathcal{A}_\psi$ to be the dual space to $L_\psi / \sum_{i=1}^{n+1} D_{i,\psi} L_\psi$. Given $u \in A$ there exists $\xi^*_{u,\psi} \in \mathcal{A}_\psi$ such that

$$< \xi^*_{u,\psi}, \xi_v >= \delta_{uv}. \tag{13}$$

For the given $f(X,\psi)$ let $R(\psi)$ be the resolution of the $(f_1,\ldots,f_{n+1})$. Let $\psi_0 \in K$ be such that $R(\psi_0) \neq 0$. For a given ψ_0 and if ψ is close enough to ψ_0, such that $|\psi-\psi_0| < 1$, there exists an isomorphism between $\mathcal{A}_{\psi 0}$ and $\mathcal{A}_\psi$ which can be described explicitly. Let $h(\psi_0,\psi,X) = -h(\psi,\psi_0.X) = f(X,\psi) - f(X,\psi_0)$. Construct

$$G(\psi_0,\psi,X) = \exp\left(X_0 h(\psi_0,\psi,X)\right) \tag{14}$$

With

$$\gamma_-(X^u) = \begin{array}{ll} X^u & \text{if } \ u_i \leq 0 \\ 0 & \text{otherwise} \end{array}$$

Dwork defines an operator $T_{\psi 0,\psi} = \gamma_- \circ G(\psi_0,\psi,X)^{-1}$ which provides an isomorphism from $\mathcal{A}_{\psi 0}$ to $\mathcal{A}_\psi$.

Defining $\mathcal{A}_\psi^S$ to be the annihilator of L^S in $\mathcal{A}_\psi$, the factor space $\mathcal{A}_\psi/\mathcal{A}_\psi^S$ is the dual space of $L^S/(L^S \bigcap \sum D_{i,\psi}L)$. It is possible then to also obtain an isomorphism between $\mathcal{A}_{\psi 0}/\mathcal{A}_{\psi 0}^S$ and $\mathcal{A}_\psi/\mathcal{A}_\psi^S$ given by the generator $T_{\psi 0,\psi}$, provided $R \neq 0$ for both ψ_0 and ψ.

Since it can be shown [14, 15] that a basis for $\mathcal{A}_\psi/\mathcal{A}_\psi^S$ is given by $\{\xi_u^*\}_{u\in A'}$ it is natural to calculate the matrix $C = C(\psi_0, \psi)$ for the operator T. Explicitly,

$$C_{uv}(\psi_0, \psi) = < T_{\psi 0,\psi}\ \ \xi^*_{u,\psi 0}, \xi_v > \tag{15}$$

for $u, v \in A'$.

Differentiating the last equation with respect to ψ one can obtain

$$\frac{\partial C}{\partial \psi} = CB(\psi) \tag{16}$$

with

$$B_{uv}(\psi) = < \xi^*_{u,\psi}, X_0 \frac{\partial f}{\partial \psi} \xi_v > \tag{17}$$

As will be shown, in the basis introduced above, one can find that the matrix equation **(16)** is precisely the Picard-Fuchs equation for the period matrix $P(\psi)$, where actually the matrix C is the transpose of the standard P matrix.

The Torus and the Quintic

The case of the torus was originally constructed by Dwork [15]. Here we report his result and then proceed to give our calculation of the mirror space to $P_4(5)$. Below $D_i = D_{i,\psi}$ and we take $\psi_0 = 0$ giving $|\psi| < 1$.

For the torus the defining polynomial is given as

$$f(X, \psi) = \sum_{i=1}^{3} X_i^3 - 3\psi X_1 X_2 X_3 \tag{18}$$

Here $h(X) = \frac{\partial f}{\partial \psi} = -3X_1X_2X_3$, and $A' = \{(1,1,1,1),(2,2,2,2)\}$, giving $\xi_1 = X_0X_1X_2X_3$; $\xi_2 = X_0^2X_1^2X_2^2X_3^2$.

$$X_0 h(X)\xi_1 = -3\xi_2$$

$$X_0 h(x)\xi_2 = -3X_0^3X_1^3X_2^3X_3^3$$

By applying the expansions of **(12)** to the last term above, one obtains the B matrix such that

$$\frac{\partial C}{\partial \psi} = C \begin{pmatrix} 0 & -\frac{\psi}{3(1-\psi^3)} \\ -3 & \frac{3\psi^2}{(1-\psi^3)} \end{pmatrix} \tag{19}$$

The matrix equation can be further analyzed, showing that the matrix C can be given as

$$C = \begin{pmatrix} \omega_a & -\frac{\omega_a^I}{3} \\ \omega_b & -\frac{\omega_b^I}{3} \end{pmatrix}$$

where the two periods w satisfy the second-order differential equation

$$(1-\psi^3)\omega^{II} - 3\psi^2\omega^I - \psi\omega = 0 \tag{20}$$

In this model, taking the ratio of the two periods, one gets the very same differential equation for the $\alpha(\psi)$ function related to the ratio of two three-point functions of the associated TSCFT. This equation was derived by other methods in [10, 17], using **(1)**, together with the associativity of fusion rule coefficients and the conservation of the $U(1)$ charge.

For the case of the quintic we have the defining polynomial given by

$$f(X,\psi) = \sum_{i=1}^{5} X_i^5 - 5\psi X_1X_2X_3X_4X_5 \tag{21}$$

Here $h(X) = -5X_1X_2X_3X_4X_5$ and the basis is given by $\xi_1 = X_0X_1X_2X_3X_4X_5$; $\xi_2 = (\xi_1)^2$; $\xi_3 = (\xi_1)^3$; $\xi_4 = (\xi_1)^4$.

$$X_0\xi_1 = -5\xi_2$$

$$X_0\xi_2 = -5\xi_3$$

$$X_0\xi_3 = -5\xi_4$$

$$X_0\xi_4 = -5X_0^5X_1^5X_2^5X_3^5X_4^5X_5^5$$

Again one must expand the last expression into the lower dimensional basis. The calculation is straightforward and just slightly longer than for the torus example. We get

$$\frac{\partial C}{\partial\psi} = C\begin{pmatrix} 0 & 0 & 0 & -\frac{\psi}{125(1-\psi^5)} \\ -5 & 0 & 0 & \frac{3\psi^2}{5(1-\psi^5)} \\ 0 & -5 & 0 & -5\frac{\psi^3}{(1-\psi^5)} \\ 0 & 0 & -5 & 10\frac{\psi^4}{(1-\psi^5)} \end{pmatrix} \tag{22}$$

As in the case of the last example one can find relations between the entries of the matrix C. We obtain that for each row, the coefficients that are identified with the periods obey the fourth-order differential equation

$$(1-\psi^5)\omega^{IV} - 10\psi^4\omega^{III} - 25\psi^3\omega^{II} - 15\psi^2\omega^I - \psi\omega = 0 \tag{23}$$

If one performs the change of variables $z = \psi^5$ the last equation becomes

$$(1-z)\omega^{IV} + \frac{(24-34z)}{5z}\omega^{III} + \frac{(96-241z)}{25z^2}\omega^{II} + \frac{(24-259z)}{125z^3}\omega^I - \frac{1}{625z^3}\omega = 0. \tag{24}$$

The solutions of the differential equation **(24)** can be given by the generalized hypergeometric function

$$z^{\frac{(k-1)}{5}}\ {}_4F_3\left(\frac{k}{5},\frac{k}{5},\frac{k}{5},\frac{k}{5};\frac{k+1}{5},\frac{k+2}{5},\frac{k+3}{5},\frac{k+4}{5};z\right) \tag{25}$$

with $k = 1, 2, 3, 4$. In the second set of parameters that parameter which is unity must be omitted. Equation (**25**) is precisely equation (3.13) of [8], obtained, essentially, from the direct calculation of the periods for the parameter region $|\psi| < 1$.

Relation with special geometry

It is clear that the very same method displayed here can in principle be generalized to other spaces and to the case of multidimensional varieties ($h_{2,1} > 1$). In that case one will generalize to partial differential equations for the periods giving rise to b_3 linearly independent solutions [1]. The interesting point is that these equations together with their monodromy will also determine the quantum duality group of the moduli space, which is a subgroup of $Sp(b_3; Z)$ and has a linear action on the periods [18]. For the torus this group trivializes to $SL(2, Z)$ and for the mirror $P_4(5)$ it was recently obtained in [8, 9]. In the case of the $c = 9$, the periods along the B cycles can be obtained from the periods along the A [7] cycles, in terms of derivatives of a "prepotential" [2]

$$F_a = i \int_B \Omega = \frac{\partial F(L^a)}{\partial L^a}, \quad L^a = \int_A \Omega \tag{26}$$

with the properties $F_a L^a = 2F, F = (L^0)^2 \mathcal{F}$.

Equations (**1**) and (**4**) can be rewritten in a general coordinate basis as[3]

$$G_{i,\bar{j}} = \partial_i L^A \partial_{\bar{j}} \bar{L}^B G_{A\bar{B}} \quad (i, \bar{j} = 1 \ldots h_{2,1}, A, B = 1 \ldots h_{2,1} + 1)$$

$$G_{A\bar{B}} = -\partial_A \partial_{\bar{B}} log(L^A \bar{F}_A + \bar{L}^A F_A)$$

$$C_{ijk} = \partial_i L^A \partial_j L^B \partial_k L^C F_{ABC}$$

where $iF_{AB} = i\partial_A \partial_B F$ is the period matrix Ω_{AB}. The determinant of the imaginary part of this matrix satisfies the property

$$\partial_i \partial_{\bar{j}} \log \det \mathrm{Im}\Omega = -C_{ipq} \bar{C}_{\bar{j}\bar{p}\bar{q}} G^{p\bar{p}} G^{q\bar{q}} \exp 2K$$

namely it can be expressed as a "norm" of the fusion rule coefficients C_{ijk} of the SCFT [19]. Remarkably this quantity determines a target space non-holomorphic modular anomaly [20]

[1] The two cohomology spaces $H^{2,1}$ and $H^{1,1}$ are related to each other by the mirror symmetry [22] of $N = 2$ SCFT. So it is generally sufficient to study the $H^{2,1}$ cohomology.

[2] It should be noted that the prepotential F is defined up to an $Sp(b_3, Z)$ rotation of the periods. This corresponds to a coordinate transformation which preserves the special geometry.

[3] The tensors C_{ijk} are holomorphic sections in $L \times L$ where L is a certain line bundle over the moduli space, ((1,1) and (2,1) deformations). The two tensors C with their conjugates correspond to fusion rule coefficients of the four chiral rings of TSCFT's. To get the Zamolodchikov metrics one has to fuse the two chiral rings of (chiral-chiral and chiral-antichiral) primary fields with their Hermitian conjugates.

in complete analogy with a similar quantity for genus g Riemann surfaces [21]. This anomaly has important physical implications in that it determines threshold effects in GUT's due to the infinitely many massive states of the string spectrum.

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