New Type of Bunch Lengthening with Cusp Catastrophe in Electron Storage Rings

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Using a simple wake function, it is shown that the equilibrium bunch length in electron storage rings with localized wake can have a cusp-catastrophe behavior. Contrary to the results of the conventional theory for a distributed wake force, the system becomes bistable (period 1 and period 2) and exhibits hysteresis in some region of the parameter space. These features are predicted by a Gaussian approximation and confirmed by multiparticle tracking.

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The electromagnetic interaction between particles and environment (i.e., vacuum chamber with its discontinuities) gives rise to a wake force that affects the particle distribution in a bunch.[1] Many kinds of wake sources are distributed in the ring, but each source should be regarded as a localized object. Usually, however, one averages the wake force over one turn and assumes that this averaged wake is distributed uniformly. The validity of this simplification is not clear: In the case of the external nonlinear field, the time averaging of the force makes the dynamics completely different. In the uniformly distributed case, the equilibrium bunch distribution at low current is a solution of the potential-well-distortion (PWD) equation.[2] A linear stability analysis around this static solution enables us to predict the threshold for turbulent bunch lengthening, provided we use an accurate PWD solution.[3] As for the localized case, however, this method tells us very little. The aim of this paper[4] is to investigate the case of a localized wake by considering a simple case.

Recently, one of the authors proposed a model to study the localized effect analytically.[5] He obtained an explicit expression for the equilibrium bunch distribution, but surveyed its stability only numerically and over a limited range of parameters. Here, we use the same model but, on the basis of a linear stability analysis, we extend the survey to a wider range of parameters. We will show that the model implies a new type of bunch lengthening, which was overlooked in Ref. 5. The equilibrium bunch length, or, more exactly, the synchrotron envelope matrix in the asymptotic state, behaves in the manner of the cusp catastrophe.[6] Depending on the parameters (strength of the wake force, synchrotron tune, and damping time), the equilibrium envelope is either in period-1 or in period-2 states: In some particular cases, it can jump from one state to the other, showing hysteresis. Such a behavior, implied by the model, will be confirmed with a more accurate simulation.

As in Ref. 5, we consider the case where there is only one localized wake source in the ring. Introducing the normalized synchrotron variables

$$x_1 = \frac{\text{longitudinal displacement}}{\text{nominal bunch length}}, \qquad x_2 = \frac{\text{energy deviation}}{\sigma_E}, \tag{1}$$

the motion of a particle in one turn is written as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to U \begin{pmatrix} x_1 \\ \Lambda x_2 + \hat{r}(1-\Lambda^2)^{1/2} - \phi(x_1) \end{pmatrix}. \tag{2}$$

Here σ_E is the nominal energy spread, U is the rotation matrix for the synchrotron oscillation,

$$U = \begin{pmatrix} \cos(2\pi\nu) & \sin(2\pi\nu) \\ -\sin(2\pi\nu) & \cos(2\pi\nu) \end{pmatrix},$$

ν being the synchrotron tune, $\Lambda = \exp(-2/T)$, T the synchrotron damping time measured in units of the revolution period, and $\hat{r}$ a Gaussian random variable with $\langle \hat{r} \rangle = 0$ and $\langle \hat{r}^2 \rangle = 1$. Without the wake force $\phi(x_1)$, Eq. (2) represents the one-turn effect of the synchrotron motion perturbed by radiation.[7] As long as the acceleration is considered in a linear approximation, a more accurate treatment would make little difference in what follows. The wake force $\phi(x_1)$ is given by

$$\phi(x_1) = F_0 \int_0^\infty \rho(x_1 - u)\, du\,, \tag{3}$$

where $\rho(x)$ is the charge density normalized to unity. Here we have employed a constant wake function as in Ref. 5 and F_0 is a dimensionless parameter defined by $F_0 = eQW_0/\sigma_E$, where e denotes the electron charge, Q the total charge in the bunch, and W_0 the wake strength. As an order-of-magnitude estimate, we can evaluate F_0 by identifying W_0 with twice the value of the loss parameter. For example, for the CERN LEP collider[8] at injection (20 GeV), a bunch current around 0.1 mA corresponds to $F_0 \sim 1$.

The above *stochastic* mapping is equivalent to an infinite hierarchy of *deterministic* mappings for the statistical quantities

$$\bar{x}_i = \langle x_i \rangle ,$$
$$\sigma_{ij} = \langle (x_i - \bar{x}_i)(x_j - \bar{x}_j) \rangle , \tag{4}$$

and so on, which are the moments of the phase-space distribution $\psi(x_1,x_2)$. In principle, this system should include an infinite number of equations for all higher-order moments. We now introduce the Gaussian approximation:[5] The distribution function in phase space is always approximated by a Gaussian even in the presence of the wake force; i.e., we consider only second-order moments. (We will not consider $\bar{x}_i$ hereafter, since it does not affect the higher-order moments.) The original mapping can be conveniently split into three parts: Radiation,

$$\sigma'_{11} = \sigma_{11} ,$$
$$\sigma'_{12} = \Lambda \sigma_{12} , \tag{5}$$
$$\sigma'_{22} = \Lambda^2 \sigma_{22} + (1 - \Lambda^2) ;$$

wake force,

$$\sigma'_{11} = \sigma_{11} ,$$
$$\sigma'_{12} = \sigma_{12} - \frac{F_0 \sigma_{11}^{1/2}}{2\sqrt{\pi}} , \tag{6}$$
$$\sigma'_{22} = \sigma_{22} - \frac{F_0 \sigma_{12}}{(\pi \sigma_{11})^{1/2}} + \frac{F_0^2}{12} ;$$

and synchrotron oscillation,

$$\sigma'_{ij} = \sum_{h,k=1}^{2} U_{ih} \sigma_{hk} U_{jk} . \tag{7}$$

We call the whole mapping for σ_{ij} the moment mapping and represent it as

$$\boldsymbol{\sigma}' = \mathbf{S}(\boldsymbol{\sigma}) ,$$

where $\boldsymbol{\sigma} = (\sigma_{11}, \sigma_{12}, \sigma_{22})^t$.

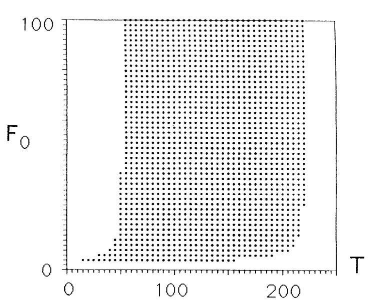

FIG. 1. The second region, where the period-1 fixed point is unstable, for $\nu = 0.2$. For smaller values of ν, the region is smaller. Note that the "first" region is not defined yet.

The period-1 fixed point of the moment mapping,

$$\boldsymbol{\sigma}_1^\infty = \mathbf{S}(\boldsymbol{\sigma}_1^\infty) ,$$

can be obtained explicitly.[5] The stability of this fixed point can be investigated by linearizing the mapping around $\boldsymbol{\sigma}_1^\infty$ and studying the eigenvalues of the linearized mapping

$$(\partial \mathbf{S} / \partial \boldsymbol{\sigma})_{\boldsymbol{\sigma}_1^\infty} .$$

By this eigenvalue analysis, we find that $\boldsymbol{\sigma}_1^\infty$ is unstable in some region of the parameter space, referred to as "the second region." This region is shown in Fig. 1 for $\nu = 0.2$.

What happens in the second region? A numerical tracking of the moment mapping shows that period-2 solutions $\boldsymbol{\sigma}_a^\infty$ and $\boldsymbol{\sigma}_b^\infty$ develop, which are solutions of

$$\boldsymbol{\sigma}_{a,b}^\infty = \mathbf{S}(\boldsymbol{\sigma}_{b,a}^\infty) .$$

Period-2 solutions $\boldsymbol{\sigma}_a^\infty$ and $\boldsymbol{\sigma}_b^\infty$ occur in alternating pairs. When F_0 is small enough, $\boldsymbol{\sigma}_1^\infty$ repeats itself every turn at equilibrium. When F_0 is increased and the parameters enter the second region, $\boldsymbol{\sigma}_1^\infty$ becomes unstable while $\boldsymbol{\sigma}_{a,b}^\infty$ become stable asymptotic states. Moreover, $\boldsymbol{\sigma}_{a,b}^\infty$ is stable not only in the second region but also in some region outside it. That is, in some region, both the period-1 and the period-2 fixed points are stable. Thus the parameter space is divided into three parts: *the first region*, where only the period-1 solution is stable; *the second region*, where only the period-2 solution is stable; and *the third region*, where both types of solutions are stable. The situation is illustrated in Fig. 2, where the three regions are shown with their numbers. In the third region, the state is not a single-valued function of the parameters. The actual state depends on the initial conditions.

Let us imagine a move along the line *a-b-c-d* in Fig. 2. At *a*, the asymptotic state of the system has period 1. At *b*, it shows period-doubling bifurcation and $\boldsymbol{\sigma}_{a,b}^\infty$ devel-

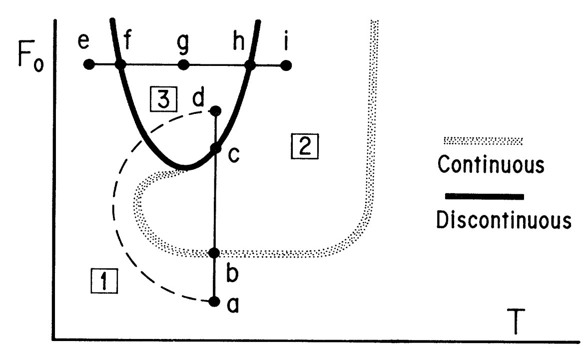

FIG. 2. An illustration of the three regions in parameter space. The second region is identical with the area shown in Fig. 1. The dashed line represents a path from *a* to *d* along which the system can always be in the period-1 state.

ops. This remains stable for larger values of F_0 (b-c-d) provided the change is adiabatic. From c to d, however, $\boldsymbol{\sigma}_1$ becomes stable again. Both kinds of states are possible. If the change of parameters is too rapid, or if we arrived at c from a by a path outside the second region (dashed line), the system chooses $\boldsymbol{\sigma}_1^\infty$ there. When we go down from d and the state is $\boldsymbol{\sigma}_1^\infty$ there, the state jumps to $\boldsymbol{\sigma}_{a,b}^\infty$ at c. The change of σ_{11}^∞ as a function of F_0 (i.e., along the line a-b-c-d in Fig. 2) is shown in Fig. 3. Points b and c in Fig. 2 correspond to $F_0 \sim 2$ and ~ 6 in Fig. 3, respectively.

When the parameters are changed slowly along a line e-f-g-h-i in Fig. 2, the system shows hysteresis. Let us start at e, where the state is uniquely period 1. If we move from e to i, $\boldsymbol{\sigma}_{ij}^\infty$ remains period 1 until h, where it jumps to period 2. If we start from i, the state is period 2 until f, where it jumps to period 1. The transitions are discontinuous and show hysteresis. The state structure is that of the cusp catastrophe.[6]

The above discussion was based on the Gaussian approximation. It is thus quite interesting to check whether this feature comes merely from a large simplification or it also occurs in a more accurate treatment. To this end, a multiparticle-tracking code was written, which tracks many particles (typically 1000) according to the map of Eq. (2). The calculation of the wake force ϕ experienced by a given particle is done by counting the number of particles preceding it. The results of the multiparticle tracking are shown and compared with the model prediction in Fig. 3. The agreement is quite satisfactory. The cusp catastrophe structure was also observed in the tracking.

The synchrotron phase-space distributions corresponding to period-1 and period-2 fixed points are shown in Figs. 4(a) and 4(b), 4(c), respectively, for the same set of parameters. In the tracking, it is a little difficult to confirm that the system has reached equilibrium. In particular, for the period-1 state shown in Fig. 4(a) the number of superparticles in the four islands seems to become eventually equal. However, one needs infinitely many turns to confirm this. It is also difficult to decide the exact threshold for the discontinuous change. Near threshold, the state easily becomes unstable as a consequence of a tiny fluctuation, which depends on the number of superparticles. Within these limitations, it can be

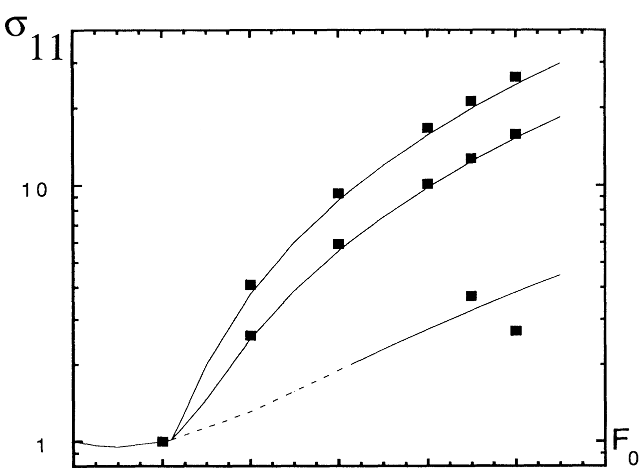

FIG. 3. The bunch length squared σ_{11} as a function of F_0, for $\nu = 0.2$ and $T = 25$. The dashed line denotes the unstable period-1 solution. The results of multiparticle tracking are also superimposed.

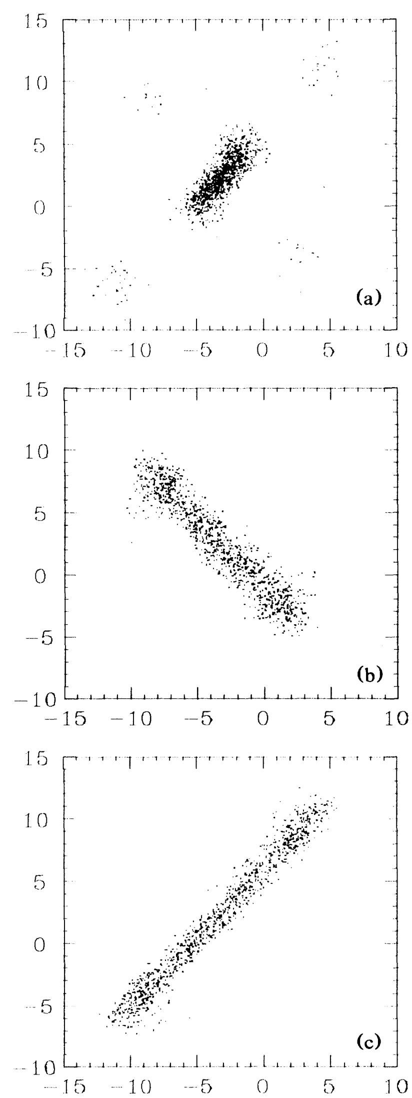

FIG. 4. Synchrotron phase-space distributions for (a) the period-1 and (b),(c) the period-2 cases at a Poincaré surface of section. In the period-2 case, the asymptotic state alternates the distributions (b) and (c) every turn. The parameters are $\nu = 0.2$, $T = 25$, and $F_0 = 9$.

said that the qualitative and quantitative features of the model were verified by multiparticle tracking.

Obviously, the present model is too simple and cannot be used to predict the performance of a particle ring. We have used the constant wake only because of its simplicity. The mapping method itself seems to have worked well. Despite the extreme simplification of the Gaussian approximation, the model showed quite good agreement with multiparticle-tracking results. It is, however, doubtful that a Gaussian approximation applies similarly well to more general wake functions. In order to extend the mapping method, we plan to use a Stratonovich expansion of the phase-space distribution,[9] which allows the introduction of higher-order moments of the particle distribution, although the manipulation of analytic expressions may become very cumbersome.

We have studied a localized wake. As discussed in Ref. 5, it is straightforward to extend this case to more general cases, even to uniformly distributed cases. The opposite is not true. We do so by introducing a periodicity N_s and letting it go to infinity. The eigenvalue analysis tells us that the period-1 fixed point is always stable in this limit. The limit

$$\lim_{N_s \to \infty} \sigma_{11}^{\infty}$$

corresponds to the solution of the PWD equation:[2] It is consistent with the known fact[5] that the solution of the PWD equation is always stable in the case of a constant wake function. The localized case is shown to be completely different: The dependence of the asymptotic state on the parameters is much more complicated. It is more dynamical and allows period-doubling bifurcation and bistability. For more general wake functions, we may expect more complicated dynamical features: successive period-doubling bifurcations and chaos, for example.[10] Localization effects become more important when ν is larger, T is smaller, and the wake force is stronger: These are the directions of the future high-luminosity storage rings.

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[6]For example, E. C. Zeeman, *Catastrophe Theory* (Addison-Wesley, Reading, MA, 1977).

[7]Despite its appearance, radiation is considered to apply continuously throughout the ring. The localized treatment used in the form of Eq. (2) is permissible when the tune ν is small, as in the case of synchrotron motion. When ν is large, as in the case of betatron motion, Eq. (2) would lead to erroneous results in general: See K. Hirata and F. Ruggiero, Part. Accel. **28**, 137 (1989).

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[9]K. Hirata, in *Proceedings of the 1989 IEEE Particle Accelerator Conference, Chicago, 1989,* edited by F. Bennett and J. Kopta (IEEE, New York, 1989), p. 1809.

[10]Imagine the state is period 1 in the third region after beam injection. As current decreases, this enters the second region where the bunch length (and the energy spread) is much larger: The lifetime of the beam is much shorter there. The current decreases rapidly until the state is in the first region. A similar phenomenon was observed: See A. Ogata, K. Nakajima, and N. Yamamoto, in *Proceedings of the European Particle Accelerator Conference, Rome, 1988,* edited by S. Tazzari (World Scientific, Singapore, 1989), p. 809. It was also observed that the bunch length changes every turn in a chaotic manner, when the lifetime is short.