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GEOMETRY OF N = $\frac{1}{2}$ SUPERSYMMETRY AND THE ATIYAH-SINGER INDEX THEOREM

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We present an explicit realization of a conjecture by Atiyah and Witten on the geometrical interpretation of $N = \frac{1}{2}$ supersymmetry. We apply our formalism to compute exactly (for all β) the path integral which yields the index of a Dirac operator for arbitrary gauge and gravitational background fields. Our approach is different from the original one by Alvarez-Gaume, and by Friedan and Windey in the sense that we use the co-adjoint orbit method to realize the gauge symmetry. In this way we avoid the need to introduce a projection operator to the desired representation of the gauge group.

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Introduction: The Atiyah-Singer index theorem [1] relates the zero modes of a Dirac operator on a compact manifold $\mathcal{M}$ to a topological invariant that characterizes the manifold. For arbitrary gauge and gravitational background fields a path integral evaluation of this topological invariant has been presented in [2,3]. The approach is based on $N = \frac{1}{2}$ supersymmetric quantum mechanics, with Dirac operator identified as the supersymmetry generator. By employing general arguments and using specific gauge and gravitational background fields the evaluation of the path integral can be reduced to its approximative evaluation at the high temperature ($\beta \to 0$) limit, and the final result coincides with the topological invariant of the Atiyah-Singer index theorem. A systematic WKB approach for an approximative evaluation of the path integral at the high temperature limit was subsequently developed in [4].

In [5] Atiyah discusses a conjecture originally due to Witten, on a geometrical interpretation of the path integral for the $N = \frac{1}{2}$ supersymmetric quantum mechanics with gravitational background fields. He argues, that if one assumes validity of the *degenerate* version of the finite dimensional Duistermaat-Heckman integration formula [6] in the infinite dimensional loop space, the path integral yields the Atiyah-Singer index theorem for a Dirac operator in a gravitational background. The approach is based on the conceptually interesting observation, that the fermionic part of the supersymmetry action can be interpreted as a (pre)symplectic two form in the bosonic loop space which turns the original bosonic path integral measure into a loop space Liouville measure. Furthermore, the approach suggests a technical advantage in the sense that it is unnecessary to consider the high temperature ($\beta \to 0$) limit.[1] A mathematical analysis of this conjecture was subsequently discussed in [8].

Recently, in a series of papers [9] a path integral generalization of the *nondegenerate* Duistermaat-Heckman integration formula has been developed. In this Letter we shall apply the ensuing geometrical formalism to present an exact path integral evaluation (for all β) of the Atiyah-Singer index theorem in an arbitrary gauge and gravitational background. Our construction verifies explicitly the conjectures in [5] and generalizes them to include an arbitrary background gauge field. In particular, in our approach it is not necessary to use a specific gauge condition, neither for the gauge nor for the gravitational background field. It is also unnecessary to consider the high-temperature ($\beta \to 0$) limit.[2] For a gauge field background our supersymmetric action is somewhat

[1] In the [2-4] approach it is technically very difficult to explicitly verify that $\mathcal{O}(\beta)$ corrections indeed vanish [7].

[2] In addition of conceptual interest in the case of Atiyah-Singer index theorems, the exact evaluation

different from that used in the original approach [2-4]. Instead of using anticommuting variables to realize canonically the action of the gauge group generators, we employ the recently developed co-adjoint orbit method [11]. This has the advantage, that the path integral yields directly the index theorem for the desired representation of the gauge group, and there is no need to introduce projection operators.

In the next section we shall realize the conjectures in [5] for a Dirac operator in an arbitrary gravitational background. This will be followed by a generalization to arbitrary background gauge fields. In the final section we outline the superfield formulation of our approach.

Gravitational Background: We shall first consider the Dirac operator on a compact even dimensional Riemannian manifold $\mathcal{M}$. We are interested in the zero modes $\mathcal{E} = 0$ of the eigenvalue equation

$$\gamma^\mu D_\mu \psi \;=\; \gamma^\mu(\partial_\mu + \frac{1}{8}\omega_{\mu jk}[\gamma^j, \gamma^k])\psi \;=\; \mathcal{E}\psi \tag{1}$$

Here $\gamma^i = e^i_\mu \gamma^\mu$ are local Dirac matrices, e^i_μ are components of the vielbein and $\omega_{\mu jk}$ are components of the spin connection. The Atiyah-Singer index theorem relates the zero modes $\mathcal{E} = 0$ of (1) to a topological invariant,

$$Index[\gamma^\mu D_\mu] \;=\; Tr\{\gamma^c e^{-\beta D^2}\} \tag{2}$$

where γ^c is the local chirality matrix on $\mathcal{M}$. General arguments suggest[3] that the *r.h.s.* of (2) is β-independent. In the low temperature ($\beta \to \infty$) limit only the zero modes contribute according to their chirality, and this yields the index of the Dirac operator. In the high temperature ($\beta \to 0$) limit the *r.h.s.* can be evaluated explicitly, either using a heat kernel expansion or using a supersymmetric path integral. The result is a topological invariant of the background fields, as stated by the Atiyah-Singer index theorem.

Here we are interested in an *exact* path integral evaluation of the *r.h.s.* of (2), for *all* values of β. For this, we need a canonical realization of the zero mode $\mathcal{E} = 0$ equation (1). We define a canonical structure on $\mathcal{M}$ by introducing the conjugate variable p_μ with Poisson brackets

$$\{p_\mu, x^\nu\} \;=\; \delta^\nu_\mu \tag{3}$$

of the path integral for *arbitrary* β is important in the case of odd dimensional index theorems where β-independence in general can not be assumed [10].

[3]Such arguments are incorrect in the case of odd-dimensional index theorems [10].

and realize the local Dirac algebra canonically using anticommuting variables ψ^i with graded Poisson brackets

$$\{\psi^i, \psi^j\} = \eta^{ij} \tag{4}$$

where η_{ij} is the local (flat) metric. The corresponding $\mathcal{M}$-dependent variables $\psi^\mu = E^\mu_i \psi^i$ then have the brackets

$$\{\psi^\mu, \psi^\nu\} = g^{\mu\nu} \tag{5}$$

where $g_{\mu\nu}$ is the Riemannian metric on $\mathcal{M}$. With these definitions we can realize the equation for zero modes of the Dirac operator (1) as a (graded) canonical constraint equation

$$\mathcal{S} = \psi^\mu(p_\mu + \frac{1}{4}\omega_{\mu jk}\psi^j\psi^k) = 0 \tag{6}$$

The pertinent constraint algebra is a first class constraint algebra, and coincides with the $N = \frac{1}{2}$ supersymmetry algebra

$$\{\mathcal{S}, \mathcal{S}\} = \mathcal{H} \quad \& \quad \{\mathcal{S}, \mathcal{H}\} = \{\mathcal{H}, \mathcal{H}\} = 0 \tag{7}$$

where, using identities of the Riemann tensor we have

$$\mathcal{H} = g^{\mu\nu}(p_\mu + \frac{1}{4}\omega_{\mu ij}\psi^i\psi^j)(p_\nu + \frac{1}{4}\omega_{\nu kl}\psi^k\psi^l) \tag{8}$$

The flat space path integral BRST quantization of the constraint algebra (7) has been discussed in [12]. Since the various ghost degrees of freedom only couple to the world line quantities and in particular do not couple to the metric structure on $\mathcal{M}$, we conclude that the results in [12] can be directly adopted to the present case. In particular, with the representation (6), (8) of the constraint algebra (7) the BRST operator constructed in [12] remains correct for an arbitrary Riemannian manifold $\mathcal{M}$. The pertinent BRST gauge fixed path integral

$$Z = \int [d(Liouville)] exp\{iS_\Psi\} \tag{9}$$

with S_Ψ the BRST gauge fixed action in [12] corresponding to the realization (6), (8) of the constraint algebra, describes the propagation of a Dirac particle on the manifold $\mathcal{M}$. The variables ψ^i in (4) yield the realization $\psi^i \sim \gamma^i$ of the Dirac matrices, hence with *periodic boundary conditions* the following version of the BRST gauge fixed path integral (9)

$$Z = \int [dx^\mu][d\psi^\mu] exp\{i \int_0^\beta \frac{1}{2} g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \frac{1}{2}\psi^\mu(g_{\mu\nu}\partial_t + \dot{x}^\rho g_{\mu\sigma}\Gamma^\sigma_{\rho\nu})\psi^\nu\} \tag{10}$$

is a path integral realization of the *r.h.s.* of (2). Notice that we have here deleted the contributions from the ghost variables and other variables that decouple from the background metric on $\mathcal{M}$, and contribute only to the normalization of (10).

We shall now proceed to an exact evaluation of the path integral (10) for all values of β. For this we first introduce the loop space exterior differential operator

$$d \;=\; \psi^{\mu}\frac{\delta}{\delta x^{\mu}} \tag{11}$$

We then find that if we interpret the fermionic part in (10) as a loop space two form,

$$\Omega \;=\; \frac{1}{2}\Omega_{\mu\nu}\psi^{\mu}\psi^{\nu} \;=\; \frac{1}{2}\psi^{\mu}(g_{\mu\nu}\partial_t + \dot{x}^{\rho}g_{\mu\sigma}\Gamma^{\sigma}_{\rho\nu})\psi^{\nu} \tag{12}$$

this two form is closed,

$$d\Omega \;=\; 0 \tag{13}$$

Consequently Ω determines a (pre)symplectic structure on the space of bosonic loops $x^{\mu}(t)$. In particular, if we combine the fermionic part of the action (10) with the bosonic measure $[dx^{\mu}]$ in (10) we get a (pre)symplectic *i.e.* Liouville measure on the loop space. The remaining bosonic part of the action in (10) can then be interpreted as a hamiltonian functional in the loop space, and its critical points are the classical loops,

$$\mathcal{X}_{\mu} \;=\; g_{\mu\nu}\ddot{x}^{\nu} + g_{\mu\sigma}\Gamma^{\sigma}_{\rho\nu}\dot{x}^{\rho}\dot{x}^{\nu} \;=\; 0 \tag{14}$$

Following [9] we identify (14) as the components of a one form in the loop space. The (pre)symplectic two form (12) then relates this one form to the corresponding hamiltonian vector field,

$$\mathcal{X}_{\mu} \;=\; \Omega_{\mu\nu}\mathcal{X}^{\nu} \tag{15}$$

Explicitly, we find

$$\mathcal{X}^{\mu} \;=\; -\dot{x}^{\mu}(t) \tag{16}$$

Hence the classical loops are constant (point) loops, and in particular the critical point set of the bosonic action coincides with the original manifold $\mathcal{M}$.

Following [9], we find it convenient to introduce variables $\bar{\psi}_{\mu}$ dual to the one forms ψ^{μ}, with the Poisson bracket

$$\{\bar{\psi}_{\mu}(t), \psi^{\nu}(t')\} \;=\; \delta^{\nu}_{\mu}(t-t') \tag{17}$$

Similarly, we introduce the conjugate loop variable

$$\{\pi_{\mu}(t), x^{\nu}(t')\} \;=\; \delta^{\nu}_{\mu}(t-t') \tag{18}$$

We then define an equivariant exterior differential operator by the canonical action of

$$Q \;=\; d + i_S \;=\; \pi_\mu \psi^\mu + \mathcal{X}^\mu \bar{\psi}_\mu \;=\; \pi_\mu \psi^\mu - \dot{x}^\mu \bar{\psi}_\mu \tag{19}$$

This operator generates the $N = \frac{1}{2}$ supersymmetry transformation

$$\delta x^\mu = \psi^\mu \;, \qquad \delta \psi^\mu = -\dot{x}^\mu \tag{20}$$

which leaves both the measure and the action in (10) invariant. The supersymmetry algebra is

$$\frac{1}{2}\{Q, Q\} \;=\; \mathcal{L} \;=\; -\dot{x}^\mu \pi_\mu - \dot{\psi}_\mu \bar{\psi}_\mu \tag{21}$$

where we identify $\mathcal{L}$ as the loop space Lie derivative along the hamiltonian vector field (16).

If ϑ is a one form in the subspace

$$\mathcal{L}\vartheta \;=\; 0 \tag{22}$$

we conclude from [9] that we can add to the action in (10) the Q-differential of ϑ,

$$S \;\rightarrow\; S + \{Q, \vartheta\} \tag{23}$$

and (22) ensures that the path integral (10) is independent of the functional ϑ. In order to construct the appropriate ϑ we use the metric tensor $g_{\mu\nu}$ on $\mathcal{M}$ to introduce a loop space metric tensor

$$G_{\mu\nu}(t, t') \;=\; g_{\mu\nu}(x)\delta(t - t') \tag{24}$$

Since the hamiltonian vector field (16) is a loop space Killing vector to (24),

$$\mathcal{L}G \;=\; 0 \tag{25}$$

we conclude that the loop space metric dual of the hamiltonian vector field (16)

$$\mathcal{X}^* \;=\; -\frac{1}{2} G_{\mu\nu} \dot{x}^\mu \psi^\nu \;=\; -\frac{1}{2} g_{\mu\nu} \dot{x}^\mu \psi^\nu \tag{26}$$

is a one form in the subspace (22). Identifying $\vartheta \;=\; \mathcal{X}^*_\lambda = \lambda \cdot \mathcal{X}^*$ with λ a parameter, we then conclude that the corresponding path integral (10), (23) is *independent* of λ.

We observe that we have here constructed the symplectic potential for the loop space symplectic two form (12),

$$\Omega \;=\; d(-\frac{1}{2} g_{\mu\nu} \dot{x}^\mu \psi^\nu) \;=\; d\mathcal{X}^* \tag{27}$$

Moreover, we find that the *entire* action in (10) can be represented as a Q-variation of the loop space symplectic potential $\mathcal{X}^*$,

$$\{Q, \mathcal{X}^*\} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \frac{1}{2} \psi^\mu (\partial_t + \dot{x}^\sigma g_{\mu\rho} \Gamma^\rho_{\sigma\nu}) \psi^\nu \tag{28}$$

From the results of [9] we then conclude that the original path integral (10) is equal to the path integral

$$Z_\lambda = \int [dx^\mu][d\psi^\mu] exp(i \int_0^\beta \{Q, \mathcal{X}^*_\lambda\})$$

$$= \int [dx^\mu][d\psi^\mu] exp\{i \int_0^\beta \frac{\lambda}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \frac{\lambda}{2} \psi^\mu (\partial_t + \dot{x}^\sigma g_{\mu\rho} \Gamma^\rho_{\sigma\nu}) \psi^\nu\} \tag{29}$$

independently of the parameter λ.[4]

In order to evaluate (29) we parametrize $x^\mu(t)$ and $\psi^\mu(t)$ by

$$x^\mu(t) = x^\mu_o + x^\mu_t \quad , \quad \psi^\mu(t) = \psi^\mu_o + \psi^\mu_t \tag{30}$$

where x^μ_o is the zero mode *i.e.* coordinates on the manifold $\mathcal{M}$, and the ψ^μ_o can be viewed as the corresponding basis of one forms. In particular, the path integral measure in (29) is

$$[dx^\mu][d\psi^\mu] = dx^\mu_o d\psi^\mu_o \prod_t dx^\mu_t d\psi^\mu_t \tag{31}$$

We introduce the following change of variables for the time dependent x^μ_t and ψ^μ_t in the decomposition (30), (31)

$$x^\mu_t \to \frac{1}{\sqrt{\lambda}} x^\mu_t \quad \& \quad \psi^\mu_t \to \frac{1}{\sqrt{\lambda}} \psi^\mu_t \tag{32}$$

The corresponding Jacobian in the path integral measure (31) is trivial, and for the action in (29) this yields

$$S \to \int_0^\beta \frac{1}{2} g_{\mu\nu}(x_o) \dot{x}^\mu_t \dot{x}^\nu_t + \frac{1}{2} \psi^i_t \eta_{ij} \partial_t \psi^j_t + \frac{1}{2} R_{ij\mu\nu}(x_o) \psi^i_o \psi^j_o \dot{x}^\mu_t \dot{x}^\nu_t + \mathcal{O}(\frac{1}{\sqrt{\lambda}}) \tag{33}$$

where we have used $\psi^i_t = \psi^\mu_t e^i_\mu(x_o) + \mathcal{O}(1/\sqrt{\lambda})$ with $R_{ij\mu\nu}(x_o)$ the Riemann tensor on the manifold $\mathcal{M}$.

By the λ-independence of (29) we can ignore the λ-dependent contributions to the action (33), *e.g.* by setting $\lambda \to \infty$. The evaluation of the remaining Gaussian path

[4]Notice that we can not *naively* set $\lambda = 0$ in (29).

integrals is then straightforward. We use ζ-function regularization, and accounting for the normalization factors involved the final result is

$$Z_\lambda = \int dx_o d\psi_o \sqrt{det \left| \frac{\frac{i}{4\pi}\mathcal{R}}{sinh(\frac{i}{4\pi}\mathcal{R})} \right|} = \int dx_o \hat{A}(\mathcal{R}) \tag{34}$$

where

$$\mathcal{R}_{\mu\nu} = \frac{1}{2} R_{ij\mu\nu}(x_o)\psi_o^i \psi_o^j \tag{35}$$

Here $\hat{A}(\mathcal{R})$ is the $\hat{A}$-genus [1] and in the last step we have used the identity

$$\int dx d\psi C_{\lambda\mu\nu...}\psi^\lambda \psi^\mu \psi^\nu ... = \int dx^\lambda \wedge dx^\mu \wedge dx^\nu ... C_{\lambda\mu\nu...} \tag{36}$$

The final result (34) coincides with the Atiyah-Singer index theorem for a Dirac operator on the Riemannian manifold $\mathcal{M}$.

Our evaluation of the path integral (29) is *exact* for all β, and explicitly verifies the conjectures presented in [5]. Since the arguments in [5] were based on the degenerate version of the finite dimensional Duistermaat-Heckman integration formula, our computation can be viewed as a derivation of the path integral generalization of the degenerate Duistermaat-Heckman integration formula in a special case.

Gauge Field Background: We shall now proceed to the generalization of (2) for a Dirac operator on the Riemannian manifold $\mathcal{M}$ with an arbitrary nonabelian background gauge field. For this, we need a canonical realization of the gauge group action. The realization used in [2-4] is based on anticommuting variables. This realization is highly reducible, and in order to get the index of the original Dirac operator it is necessary to introduce a projection onto the desired representation. Here we shall employ the recently developed co-adjoint orbit technique [11]. In this approach the path integral describes directly the index of the Dirac operator in the given representation of the gauge group, and there is no need to introduce projections.

The co-adjoint orbit of a simple Lie group is a homogeneous symplectic manifold, and there is a one-to-one correspondence between unitary irreducible representations of simple Lie groups and integral symplectic structures on the pertinent co-adjoint orbit. With ϕ^a local coordinates on the co-adjoint orbit and ω_{ab} the symplectic two form corresponding to the given representation of the gauge group, the Lie algebra can be represented canonically by functions $T^\alpha(\phi)$ on the co-adjoint orbit with

$$\{T^\alpha(\phi), T^\beta(\phi)\} = \partial_a T^\alpha \omega^{ab} \partial_b T^\beta = f^{\alpha\beta\gamma} T^\gamma(\phi) \tag{37}$$

The canonical realization of the zero mode equation for a Dirac operator in an arbitrary gauge and gravitational background is then a straightforward generalization of the constraint equation (6)

$$S \;=\; \psi^{\mu}(p_{\mu}+\frac{1}{4}\omega_{\mu jk}\psi^{j}\psi^{k}+A_{\mu}^{\alpha}T^{\alpha}) \;=\; 0 \tag{38}$$

and the pertinent first class constraint algebra realizes the $N=\frac{1}{2}$ supersymmetry algebra (7).

The BRST quantization proceeds in the same way as in the case of a pure gravitational background: Since the ghost degrees of freedom do not couple to the gauge and gravitational background fields, the BRST operator is of the functional form described in [12]. With *periodic boundary conditions* the following version of the BRST gauge fixed path integral (9) then yields a path integral representation of the *r.h.s.* in the Atiyah-Singer index theorem (2),

$$Z \;=\; \int [dx^{\mu}][d\psi^{\mu}][d\phi^{a}]\sqrt{det|\omega_{ab}|}exp\{iS\} \tag{39}$$

where the action is

$$S \;=\; \int_{0}^{\beta} \frac{1}{2}g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}+\theta_{a}\dot{\phi}^{a}+\dot{x}^{\mu}A_{\mu}^{\alpha}T^{\alpha}+\frac{1}{2}\psi^{\mu}(g_{\mu\nu}\partial_{t}+\dot{x}^{\rho}g_{\mu\sigma}\Gamma^{\sigma}_{\rho\nu}-F_{\mu\nu}^{\alpha}T^{\alpha})\psi^{\nu} \tag{40}$$

Here θ_a are components of the symplectic potential on the coadjoint orbit,

$$\omega_{ab} \;=\; \partial_{a}\theta_{b}-\partial_{b}\theta_{a} \tag{41}$$

We have again deleted the contribution from the ghost variables and other variables that decouple from the background and contribute only to the normalization of (39).

Since $F_{\mu\nu}\psi^{\mu}\psi^{\nu}$ is *not* a closed two form, the fermionic part in (40) also fails to be closed with respect to the exterior differential operator (11). Consequently the fermionic part in (40) can not be interpreted as a (pre)symplectic two form in the loop space $x^{\mu}(t)$. In order to generalize the construction of the previous section we first use (37) to decompose

$$F_{\mu\nu}^{\alpha}T^{\alpha}(\phi) \;=\; (\partial_{\mu}A_{\nu}^{\alpha}-\partial_{\nu}A_{\mu}^{\alpha})T^{\alpha}+A_{\mu}^{\alpha}A_{\nu}^{\beta}\partial_{a}T^{\alpha}\omega^{ab}\partial_{b}T^{\beta} \tag{42}$$

Following [9] we introduce an anticommuting variable c^a and exponentiate the square root of the symplectic determinant in (39). By redefining

$$c^{a} \;\rightarrow\; c^{a}+\omega^{ab}A_{\mu}^{\alpha}\partial_{b}T^{\alpha}\psi^{\mu} \tag{43}$$

we then get the action in (40) into the form

$$S = \int_0^\beta \Big\{ \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \theta_a \dot{\phi}^a + \dot{x}^\mu A^\alpha_\mu T^\alpha + \frac{1}{2} \psi^\mu (g_{\mu\nu}\partial_t + \dot{x}^\rho g_{\mu\sigma} \Gamma^\sigma_{\rho\nu} - (\partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu) T^\alpha) \psi^\nu$$

$$+ \frac{1}{2} \psi^\mu A^\alpha_\mu \partial_a T^\alpha c^a - \frac{1}{2} c^a A^\alpha_\mu \partial_a T^\alpha \psi^\mu - \frac{1}{2} c^a \omega_{ab} c^b \Big\} \quad (44)$$

If we now introduce the exterior differential operator

$$d = \psi^\mu \frac{\delta}{\delta x^\mu} + c^a \frac{\delta}{\delta \phi^a} \quad (45)$$

and interpret the fermionic part in (44) as a two form in the loop space of bosonic variables $x^\mu(t)$, $\phi^a(t)$,

$$\Omega = \Omega_{\mu\nu} \psi^\mu \psi^\nu + \Omega_{\mu a} \psi^\mu c^a + \Omega_{a\mu} c^a \psi^\mu + \Omega_{ab} c^a c^b$$

$$= \frac{1}{2} \psi^\mu (g_{\mu\nu}\partial_t + \dot{x}^\rho g_{\mu\sigma} \Gamma^\sigma_{\rho\nu} - (\partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu) T^\alpha) \psi^\nu + \frac{1}{2} \psi^\mu A^\alpha_\mu \partial_a T^\alpha c^a - \frac{1}{2} c^a A^\alpha_\mu \partial_a T^\alpha \psi^\mu - \frac{1}{2} c^a \omega_{ab} c^b \quad (46)$$

we find that this two form is closed,

$$d\Omega = 0 \quad (47)$$

The equation for the critical bosonic loops again defines a hamiltonian vector field in the bosonic loop space. This equation is a direct generalization of (17),

$$\Omega_{AB} \mathcal{X}^B = 0 \ ; \qquad A, B = (\mu, a) \quad (48)$$

with the hamiltonian vector field

$$\mathcal{X}^A = \begin{pmatrix} -\dot{x}^\mu \\ -\dot{\phi}^a \end{pmatrix} \quad (49)$$

Hence the critical loops *i.e.* solutions to the classical equations of motion are point loops, and the space of critical loops coincides with the direct sum of the original Riemannian manifold $\mathcal{M}$ and the co-adjoint orbit.

We again find it convenient to introduce the extended phase space (17), (18) with the additional variables

$$\{\rho_a(t), \phi^b(t')\} = \delta^b_a(t - t') \qquad \& \qquad \{\bar{c}_a(t), c^b(t')\} = \delta^b_a(t - t') \quad (50)$$

in the co-adjoint orbit sector. The equivariant exterior differential operator (19) then generalizes to

$$Q \;=\; d + i_S \;=\; \pi_\mu \psi^\mu - \dot{x}^\mu \bar{\psi}_\mu + \rho_a c^a - \dot{\phi}^a \bar{c}_a \tag{51}$$

and generates the $N = \frac{1}{2}$ supersymmetry transformations that leave the action (44) invariant. The even generator of this $N = \frac{1}{2}$ algebra coincides with the loop space Lie derivative along the hamiltonian vector field (49),

$$\mathcal{L} \;=\; \frac{1}{2}\{Q, Q\} \;=\; -\pi_\mu \dot{x}^\mu - \dot{\psi}^\mu \bar{\psi}_\mu - \rho_a \dot{\phi}^a - \dot{c}^a \bar{c}_a \tag{52}$$

and from the results of [9] we conclude that the corresponding path integral remains intact under a modification (23) of the action (44) provided ϑ is a one form in the subspace (22) with $\mathcal{L}$ defined by (52).

The hamiltonian vector field (49) remains a Killing vector for the loop space metric tensor (24), and we find that the one form

$$\mathcal{X}^*_\lambda \;=\; -\frac{1}{2\lambda} g_{\mu\nu} \dot{x}^\mu \psi^\nu - A^\alpha_\mu T^\alpha \psi^\mu - \theta_a c^a \tag{53}$$

is in the pertinent subspace (22). We observe that we have again constructed a symplectic potential for the loop space symplectic two form (46),

$$\Omega \;=\; d(-\frac{1}{2} g_{\mu\nu} \dot{x}^\mu \psi^\nu - A^\alpha_\mu T^\alpha \psi^\mu - \theta_a c^a) \;=\; d\mathcal{X}^*_{\lambda=1} \tag{54}$$

We also conclude that the *entire* action (44) is a Q-variation of this symplectic potential,

$$S \;=\; \{Q, \mathcal{X}^*_{\lambda=1}\} \tag{55}$$

From the results of [9] we then conclude that the one-parameter family of path integrals

$$Z_\lambda \;=\; \int [dx^\mu][d\psi^\mu][d\phi^a][dc^a] exp(i \int_0^\beta \{Q, \mathcal{X}^*_\lambda\}) \tag{56}$$

is *independent* of λ and coincides with the original path integral (40). Consequently for *any* value of λ (56) yields the topological invariant of the Atiyah-Singer index theorem.[5]

We shall now evaluate (56) exactly: For this we introduce the decomposition (30) and the corresponding decomposition (31) of the path integral measure. The Jacobian for the pertinent change of variables (32) is trivial, and using the λ independence of

[5] In general we can not set $\mathcal{X}^*_\lambda = \lambda \cdot \mathcal{X}^*_\lambda$ since $A^\alpha_\mu T^\alpha \psi^\mu$ and $\theta_a c^a$ may not be trivial in the cohomology; see [9].

(56) we are left with simple Gaussian integrals. These integrals can be evaluated *e.g.* using ζ-function regularization, and accounting for the normalization factors we finally get

$$Z = \int dx_o d\psi_o \sqrt{det \left| \frac{\frac{i}{4\pi}\mathcal{R}}{sinh(\frac{i}{4\pi}\mathcal{R})} \right|} \int [d\phi^a] exp\{i \int_0^\beta \theta_a \dot{\phi}^a - \psi_o^i \psi_o^j F_{ij}^\alpha(x_o) T^\alpha(\phi)\} \tag{57}$$

Here we recognize a co-adjoint orbit path integral of the form investigated in [11,9]. For simple Lie algebras such co-adjoint orbit path integrals can be evaluated exactly, either directly as explained in [11] or using the *nondegenerate* path integral version of the Duistermaat-Heckman integration formula as explained in [9]. Defining

$$\mathcal{F}^\alpha = \frac{1}{2} F_{ij}^\alpha(x_o) \psi_o^i \psi_o^j \tag{58}$$

and with τ^α the matrix realization of the Lie algebra elements T^α in the given representation of the gauge group, we finally get

$$Z = \int dx_o d\psi_o \sqrt{det \left| \frac{\frac{i}{4\pi}\mathcal{R}}{sinh(\frac{i}{4\pi}\mathcal{R})} \right|} Tr\{e^{\frac{i}{2\pi}\mathcal{F}^\alpha \tau^\alpha}\} = \int dx_o \hat{A}(\mathcal{R}) Ch(\mathcal{F}) \tag{59}$$

where in the last step we have applied (36). Here $Ch(\mathcal{F})$ is the Chern character [1], and the trace is over the matrix representation of the gauge group. The final result (59) is the standard formula for the Atiyah-Singer index.

Superfield Formulation: We shall now present the superfield formulation of the supersymmetric action (44). The superfield associated with x^μ is

$$Y^\mu = x^\mu + \eta \psi^\mu \tag{60}$$

where η is an anticommuting variable, and the superfield associated with ϕ^a is

$$\Phi^a = \phi^a + \eta c^a \tag{61}$$

The superfield metric tensor is

$$G_{\mu\nu}(Y) = g_{\mu\nu}(x + \eta\psi) = g_{\mu\nu}(x) + \eta \psi^\rho \partial_\rho g_{\mu\nu}(x) \tag{62}$$

and the superfield symplectic potential on the co-adjoint orbit is

$$\Theta_a(\Phi) = \theta_a(\phi + \eta c) = \theta_a(\phi) + \eta c^b \partial_b \theta_a \tag{63}$$

We also introduce the superderivative

$$D \; = \; \partial_\eta - \eta \partial_t \tag{64}$$

Its action on the superfields (60), (61) coincides with the canonical action of (51) on the component fields. With these definitions we find that the functional form of the superfield representation of the action (44) *coincides* with that of the bosonic part of (44),

$$S \; = \; \int_0^\beta dt \int d\eta \left[\frac{1}{2} G_{\mu\nu}(Y) D Y^\mu \partial_t Y^\nu + \Theta_a D \Phi^a + D Y^\mu A^\alpha_\mu(Y) T^\alpha(\Phi) \right] \tag{65}$$

One can now develop a supergeometric interpretation of our formalism. For example, the symplectic potential (54) is the $\eta = 0$ component of the superlagrangian (65),

$$\mathcal{X}^* \; = \; \int d\eta \cdot \eta \left[\frac{1}{2} G_{\mu\nu}(Y) D Y^\mu \partial_t Y^\nu + \Theta_a D \Phi^a + D Y^\mu A^\alpha_\mu(Y) T^\alpha(\Phi) \right] \tag{66}$$

Conclusions: We have presented an exact path integral evaluation of the topological invariant in the Atiyah-Singer index theorem for a Dirac operator in an arbitrary gauge and gravitational background. In particular, we have explicitly verified the conjectures presented in [5] and generalized them to include background gauge fields. Our approach differs from the original one [2-4] in that instead of anticommuting variables, we realize the canonical action of the gauge group generators using the recently developed co-adjoint method [11]. This has the advantage, that the path integral yields directly the index theorem for the desired representation of the gauge group, with no need for a projection. Instead of the co-adjoint orbit we could also consider the model space [11], which is the direct sum of each irreducible representation of the gauge group with multiplicity one. The path integral would then yield the Atiyah-Singer index theorem for *all* representations of the gauge group. Furthermore, we could also introduce a co-adjoint orbit or a model space for the Dirac matrices. In this way we obtain a path integral that describes the Atiyah-Singer index theorem for operators with arbitrary spin. Since the index theorems for other classical complexes can be obtained from the Atiyah-Singer index theorem for the Dirac operator, our approach can be directly generalized to these index theorems.

Finally, we observe that our path integrals (29), (56) define *topological quantum field theories* [14]. Indeed, we have found that these path integrals do describe topological invariants. It would be interesting to see, whether our formalism could be generalized *e.g.* to explicitly evaluate the path integral describing Donaldson's invariants [14]

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