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SPECIAL AND QUATERNIONIC ISOMETRIES: GENERAL COUPLINGS IN N=2 SUPERGRAVITY AND THE SCALAR POTENTIAL (*)

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Abstract
The general Lagrangian for N=2 matter coupled supergravity is obtained by gauging general isometries of quaternionic manifolds which can be coupled to supergravity. The resulting theories are purely geometrical and give an interplay between quaternionic and special Kähler geometry. The resulting scalar potential is expressed in terms of the two Killing prepotentials of the two geometries and it may be relevant to study transitions between different vacua in superstring theory. Furthermore, from the geometrical point of view the prepotentials are Hamiltonian functions yielding a Poissonian realization of the gauge algebra on both the special Kähler and the quaternionic manifolds. A possible cohomological obstruction to this realization is pointed out.

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1. Introduction

One of the most striking results of supersymmetric quantum field theories is the deep relation between geometry, topology and supersymmetry.

In particular global and local supersymmetric theories exhibit deep geometrical structures inherent to the non-linear interactions of matter multiplets. Almost all of these couplings can be rephrased in a geometrical language as couplings of some non-linear σ-model to gravity and gauge fields. [1,2,3,4,5,6,7]

The geometrical approach to supersymmetric field theories [8] has some distinct advantages with respect to alternative approaches such as tensor calculus and superspace. In particular the geometrical approach is especially powerful for $N > 1$ extended supergravities where alternative techniques offer only limited answers.

For instance using the geometrical approach one can construct general couplings for $N = 3, 4$ theories [6c,6d,8].

In this paper we complete the analysis of matter couplings in supergravity theory by focusing on the general coupling of $N = 2$ vector and hypermultiplets to $N = 2$ supergravity.

In geometrical language this study amounts to an investigation of the gauging of isometries of special Kähler and quaternionic manifolds. The first type of manifolds contains the scalars sitting in the vector multiplets [4c]; the second contains the scalars sitting in the hypermultiplets [3,4].

The non-linear σ-models associated with these two restricted geometries of $N = 2$ supersymmetric theories have been known since long ago both for hypermultiplets and vector multiplets [3,4a-c]. In the latter case, however, the coupling was obtained using a particular coordinate patch (special coordinates) and only recently it was generalized using a coordinate free formulation [4d, 4e].

Non-geometric formulations for the two separate couplings were also obtained from harmonic superspace techniques [9].

Furthermore in ref. [4b] a complete gauged Lagrangian was obtained for $N = 2$ matter-coupled supergravity in special coordinates; its validity, however, is restricted to the case where the hypermultiplets span the specific quaternionic

manifold $\mathbb{H}P(m) \equiv \frac{Sp(2m,2)}{Sp(2) \times Sp(2m)}$ or other classes of manifolds obtained by adding non propagating vector multiplets to the Lagrangian [13].

Moreover a general formula for the scalar potential, analogous to the general formula of standard $N = 1$ supergravity [2]:

$$\mathcal{V}(z,\bar{z}) = e^{G(z,\bar{z})}(\partial_i G \partial_{j^*} G g^{ij^*} - 3) \tag{1.1a}$$

$$G(z,\bar{z}) = J(z,\bar{z}) + ln\, |W(z)|^2 \tag{1.1b}$$

was so far missing in the case of $N = 2$ supergravity. We recall that in eqs. (1.1) $g_{ij^*} = \partial_i \partial_{j^*} J$ is the Kähler metric of the scalar manifold and the superpotential W is a section of a holomorphic line bundle $\mathcal{L}$ whose first Chern class $c_1(\mathcal{L})$ coincides with the Kähler class $[J]$. As we see $\mathcal{V}(z,\bar{z})$ is entirely determined in terms of geometrical objects. In this paper we provide the $N = 2$ analogue of formula (1.1). It is encoded in eqs. (4.4) and it is fully geometrical, being given in terms of natural objects defined over arbitrary special and quaternionic manifolds endowed with isometries.

There is a deep motivation for studying these geometries and the corresponding $N = 2$ supergravity theories. This is their relation with superstring theories and with the moduli space of superconformal fields theories in $D = 2$.

Special geometry is related with the moduli space of (2,2)-theories alternatively viewed as compactifications of superstrings on n-folds of $SU(n)$-holonomy. In particular for $n = 3$ one compactifies on Calabi-Yau manifolds of $SU(3)$-holonomy and obtains effective $N = 1$ supergravity Lagrangians, if one starts from heterotic superstrings, and $N = 2$ Lagrangians if one starts from type II superstrings. Quaternionic geometries are related with the moduli space of (4.4)-theories that yield $N = 2$ supergravity Lagrangians in the type II case.

As shown by the work of Seiberg [10], Cecotti, Ferrara and Girardello [11] and Dixon, Kaplunovski and Louis [12], one can obtain microscopic results on superconformal field theories by utilizing macroscopic methods, namely supergravity Lagrangians. From this point of view the absence of a general $N = 2$ Lagrangian was quite felt. We hope to have filled this gap. Indeed, in this paper we provide not only the scalar potential but the complete form of the general $N = 2$ supergravity Lagrangian up to 4-Fermi terms. Finally we note that in $N = 2$ theory the existence of the gauging is based on the vanishing of certain cohomological obstructions to the Poissonian realization of the gauge algebra on the special quaternionic manifolds [13] (momentum mapping) [14].

This raises further interesting questions of topological nature worth further studying.

2. Special and quaternionic geometrical Killing vectors: $SU(2) \otimes U(1)$ connection.

Our purpose is to determine the general form of four-dimensional $N = 2$ Supergravity coupled to:

i) n- vector multiplets gauging some Lie group G such that dim $G = n + 1$. (One gauge vector comes from the supergravity multiplet).

ii) $2m$-hypermultiplets supporting some linear or non-linear representation of G.

Such a theory contains $2n + 4m$ scalar fields interacting through a σ-model based on the following scalar manifold [3,4]:

$$\mathcal{M}_{\text{scalar}} = SK(n) \otimes \mathcal{Q}(m) \tag{2.1}$$

where $SK(n)$ is a special Kähler manifold with n-complex dimensions [4b-e] and $\mathcal{Q}(m)$ is a quaternionic manifold with m-quaternionic dimensions. ($dim\mathcal{Q}(m) = 4m$) .

The bosonic sector of the supergravity Lagrangian has the following general structure:

$$\begin{aligned}\mathcal{L}^{(N=2)}_{(\text{Bosonic})} = & \sqrt{-g}\{\mathcal{R} + g_{ij^*}(z,\bar{z})\nabla_\mu z^i \nabla_y \bar{z}^{j^*} g^{\mu\nu} + \\ & + h_{uv}(q)\nabla_\mu q^u \nabla_\nu q^v g^{\mu\nu} - 4Re\mathcal{N}_{\Lambda\Sigma}(z,\bar{z})F^\Lambda_{\mu\nu}F^{\Sigma\mu\nu} \\ & - \mathcal{V}(z,\bar{z},q)\} - 2iIm\mathcal{N}_{\Lambda\Sigma}(z,\bar{z})F^\Lambda_{\mu\nu}F^\Sigma_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma}\end{aligned} \tag{2.2}$$

where $z^i(i = 1,...,n)$ are the vector multiplet scalars and $q^u(u = 1,...,4m)$ are hypermultiplet scalars; $g_{ij^*}(z,\bar{z})$ is the Kähler metric on $SK(n)$ and $h_{uv}(q)$ is the

quaternionic metric on $\mathcal{Q}(m)$; the real function $\mathcal{V}(z,\bar{z},q)$ is the scalar potential and the $(n+1)\times(n+1)$ matrix $\mathcal{N}_{\Lambda\Sigma}(z,\bar{z})$ is the so-called vector kinetic metric.

2.a) *Special Kähler manifolds with Killing sectors*

The coordinate-free description of special Kähler manifolds was obtained in ref. [4d]. Let us recall their structure and discuss the properties of Killing vectors on such manifolds. In primis a special Kähler manifold is a Kähler manifold of a restricted type (Hodge manifold) [4e]. This means that it has a complex structure and a Hermitian metric

$$ds^2 = g_{ij^*}(z,\bar{z})dz^i \otimes d\bar{z}^{j^*} \tag{2.3}$$

such that the (1,1)-form

$$K = ig_{ij^*}(z,\bar{z})dz^i \wedge d\bar{z}^{j^*} \tag{2.4}$$

is closed ($dK = 0$). As is well known, K cannot be globally exact, yet it is certainly locally exact. Indeed in every coordinate patch we can find a real function $G(z,\bar{z})$ (named the Kähler potential) such that

$$g_{ij^*} = \partial_i\partial_{j^*}G(z,\bar{z}) \tag{2.5a}$$

$$K = dQ \tag{2.5b}$$

$$Q = -\frac{i}{2}(\partial_i G dz^i - \partial_{i^*} G d\bar{z}^{i^*}) \tag{2.5c}$$

Under a Kähler transformation

$$G \to G + f(z) + f(\bar{z}) \tag{2.6}$$

the 1-form Q transforms as

$$Q \to Q + d(Imf). \tag{2.7}$$

Therefore Q is a $U(1)$ connection.

The $U(1)$ covariant differential of a field $\Phi(z,\bar{z})$ of weight p is

$$\nabla\Phi = (d + ipQ)\Phi \tag{2.8}$$

or in components

$$\nabla_i\Phi = (\partial_i + \frac{1}{2}p\partial_i G)\Phi \tag{2.9a}$$

$$\nabla_{i^*}\Phi = (\partial_{i^*} - \frac{1}{2}p\partial_{i^*} G)\Phi \tag{2.9b}$$

A covariantly holomorphic field of weight p is defined by

$$\nabla_{i^*}\Phi = 0 \tag{2.10}$$

By a change of trivialization the real $U(1)$-bundle can be reduced to a holomorphic $U(1)$ line bundle $\mathcal{L}$. Indeed setting

$$\tilde{\Phi} = e^{-pG/2}\Phi \tag{2.11}$$

we have

$$\nabla_i\tilde{\Phi} = (\partial_i + p\partial_i G)\tilde{\Phi} \tag{2.12a}$$

$$\nabla_{i^*}\tilde{\Phi} = \partial_{i^*}\Phi \quad . \tag{2.12b}$$

In particular, if Φ is a covariantly holomorphic section with respect to the Q-connection, $\tilde{\Phi}$ is a holomorphic section with respect to the holomorphic connection $\partial_i G$.

If the $U(1)$ line bundle is such that the first Chern class $c_1(\mathcal{L})$ coincides with the Kähler class $[K]$ then the Kähler manifold is of restricted type or a Hodge manifold. In addition to the $U(1)$-holomorphic connection $\partial_i G$ one has the holomorphic Levi-Civita connection

$$\Gamma^i_j = \Gamma^i_{kj} dz^k \tag{2.13a}$$

$$\Gamma^i_{kj} = -g^{i\ell^*}(\partial_j g_{k\ell^*}) \tag{2.13b}$$

$$\Gamma^{i^*}_{j^*} = \Gamma^{i^*}_{k^* j^*} d\bar{z}^{k^*} \tag{2.13c}$$

$$\Gamma^{i^*}_{k^* j^*} = -g^{i^*\ell}(\partial_{j^*} g_{k^*\ell}) \tag{2.13d}$$

and its curvature

$$\mathcal{R}^i_j = \mathcal{R}^i_{jk^*\ell} d\bar{z}^{k^*} \wedge dz^\ell \tag{2.14a}$$

$$\mathcal{R}^i_{jk^*\ell} = \partial_{k^*}\Gamma^i_{j\ell} \tag{2.14b}$$

$$\mathcal{R}^{i^*}_{j^*} = \mathcal{R}^{i^*}_{j^* k\ell^*} dz^k \wedge d\bar{z}^{\ell^*} \tag{2.14c}$$

$$\mathcal{R}^{i^*}_{j^* k\ell^+} = \partial_k \Gamma^{i^*}_{j^*\ell^*} \tag{2.14d}$$

By definition a restricted Kähler manifold is special if and only if there exists a completely symmetric holomorphic 3-index section W_{ijk} of $(T^*)^3 \otimes \mathcal{L}^2$ (and its antiholomorphic conjugate $W_{i^*j^*k^*}$) such that

$$\partial_{m^*} W_{ijk} = 0 \quad \partial_m W_{i^*j^*k^*} = 0 \tag{2.15a}$$

$$\mathcal{R}_{i^*j\ell^*k} = -g_{\ell^*(j}g_{k)i^*} + \frac{1}{2}e^{2G} W_{i^*\ell^*s^*} W_{tkj} g^{s^*t} \tag{2.15b}$$

In equations (2.15) ∇ denotes the derivative covariant with respect to both the Levi-Civita and the $U(1)$ holomorphic connections.

In the case of the $W_{ijk}, p = 2$. In [4d-e] it was shown that on a special Kähler manifold $SK(n)$ one can always introduce a a $(n+1)$- dimensional holomorphic vector bundle whose holomorphic sections we denote by X^Λ $(\Lambda = 1, ..., n+1)$

$$\partial_{i^*} X^\Lambda = 0 \tag{2.16}$$

and a function $F(L)$ which is holomorphic and homogeneous of degree two in the transformed section

$$L^\Lambda(z,\bar{z}) = e^{1/2G(z,\bar{z})} X^\Lambda(z) \tag{2.17}$$

This means that $F(L) = e^{G(z,\bar{z})} F(X(z))$ so that $F(X)$ is a holomorphic section of $\mathcal{L}^2$.

For the $L^\Lambda(z,\bar{z})$ we have:

$$\nabla L^\Lambda = dL^\Lambda + iQL^\Lambda = \nabla_i L^\Lambda dz + \nabla_{i^*} L^\Lambda d\bar{z}^{i^*} \tag{2.18a}$$

$$\nabla_{i^*} L^\Lambda = \partial_{i^*} L^\Lambda - \frac{1}{2}\partial_{i^*} G L^\Lambda = 0 \tag{2.18b}$$

Eq. (2.18b) follows from eq. (2.16) and eq.(2.17).

The geometry of the special manifold is completely determined by the sections $\{X^\Lambda\}$ and by the analytic function $F(L)$. Define

$$F_{\Lambda_1\dots\Lambda_n} = \frac{\partial}{\partial\Lambda_1}\frac{\partial}{\partial\Lambda_2}\cdots\frac{\partial}{\partial\Lambda_n}F(L) \tag{2.19}$$

and set

$$N_{\Lambda\Sigma} = F_{\Lambda\Sigma} + F_{\Lambda\Sigma} \tag{2.20a}$$

$$f_i^\Lambda = \nabla_i L^\Lambda \equiv \partial_i L^\Lambda + \frac{1}{2}G_i L^\Lambda \equiv e^{G/2}(\delta^\Lambda_\Sigma - \frac{X^\Lambda (N\bar{X})_\Sigma}{XN\bar{X}})\partial_i X^\Sigma \tag{2.20b}$$

$$f_{i^\star}^\Lambda = \nabla_{i^\star}\bar{L}^\Lambda = \partial_{i^\star}\bar{L}^\Lambda + \frac{1}{2}G_{i^\star}\bar{L}^\Lambda \equiv e^{G/2}(\delta^\Lambda_\Sigma - \frac{\bar{X}^\Lambda (NX)_\Sigma}{XN\bar{X}})\partial_{i^\star}\bar{X}^\Sigma \tag{2.20c}$$

$$S = -\frac{1}{4}N_{\Lambda\Sigma}L^\Lambda L^\Sigma \tag{2.20d}$$

$$G = -ln N_{\Lambda\Sigma}X^\Lambda\bar{X}^\Sigma \tag{2.20e}$$

then one finds

$$g_{ij^\star} = -f_i^\Lambda f_{j^\star}^\Sigma N_{\Lambda\Sigma} = \partial_i X^\Lambda \partial_{j^\star}\bar{X}^\Sigma \partial_\Lambda \partial_{\bar{\Sigma}} G(X,\bar{X}) \tag{2.21a}$$

$$C_{ijk} \equiv e^G W_{ijk} = \nabla_i\nabla_j\nabla_k S = f_i^\Lambda f_j^\Sigma f_k^\Gamma F_{\Lambda\Sigma\Gamma} = $$

$$= e^G \partial_i X^\Lambda \partial_j X^\Sigma \partial_k X^\Gamma F_{\Lambda\Sigma\Gamma}(X) \tag{2.21b}$$

$$N_{\Lambda\Sigma}L^\Lambda\bar{L}^\Sigma = e^G N_{\Lambda\Sigma}X^\Lambda\bar{X}^\Sigma = 1 \tag{2.21c}$$

$$f_i^\Lambda \bar{L}^\Sigma N_{\Lambda\Sigma} = 0 \tag{2.21d}$$

$$f_{i^\star}^\Lambda L^\Sigma N_{\Lambda\Sigma} = 0 \tag{2.21e}$$

$$U^{\Lambda\Sigma} \equiv g^{ij^\star} f_i^\Lambda f_{j^\star}^\Sigma = -(N^{-1})^{\Lambda\Sigma} + L^\Lambda\bar{L}^\Sigma \tag{2.21f}$$

$$N_{\Lambda\Sigma}U^{\Sigma\Pi}N_{\Pi\Gamma} = e^G\partial_\Lambda\partial_{\bar{\Sigma}}G \tag{2.21g}$$

where $\partial_\Lambda \equiv \frac{\partial}{\partial X^\Lambda}, \partial_{\bar{\Lambda}} \equiv \frac{\partial}{\partial \bar{X}^\Lambda}$.

Actually it turns out that $\{X^\Lambda, \frac{\partial F}{\partial X^\Sigma}\}$ can be viewed as the cross-sections of a flat holomorphic, $Sp(2n+2)$- bundle [4e] but this fact will not be utilized in what follows. We just note that from eq.(2.15) we obtain a formula relating the Kähler potential G to the norm of the holomorphic section X^Λ in the $Sp(2n+2)$ flat bundle [4e]:

$$G = -ln \parallel X \parallel^2 = -\ln(N_{\Lambda\Sigma}X^\Lambda\bar{X}^\Sigma) \tag{2.22}$$

Let us now consider isometries of $SK(n)$. They are generated by holomorphic Killing vectors:

$$z^i \to z^i + \epsilon^\Lambda k^i_\Lambda(z) \tag{2.23a}$$

$$\bar{z}^{i^\star} \to z^{i^\star} + \epsilon^\Lambda k^{i^\star}_\Lambda(\bar{z}) \tag{2.23b}$$

that must be compatible with both the Kähler and the special structure of our manifold.

Let $\mathfrak{G}$ be the Lie algebra spanned by the Killing vectors under consideration:

$$\vec{k}_\Lambda = k^i_\Lambda\vec{\partial}_i + k^{i^\star}_\Lambda\vec{\partial}_{i^\star} \tag{2.24}$$

whose commutation relations

$$[\vec{k}_\Lambda, \vec{k}_\Sigma] = -f^\Gamma_{\Lambda\Sigma}\vec{k}_\Gamma \tag{2.25}$$

define the structure constants $f^\Gamma_{\Lambda\Sigma}$. In a general Kähler manifold $K(n)$ the dimension $d = dim\ \mathfrak{G}$ of the isometry algebra has no particular relation with the dimension n of the manifold. In a special Kähler manifold $SK(n)$ one has $d = n+1$ and the Killing vectors are in one-to-one correspondence with the components of the holomorphic cross-section $\{X^\Lambda(z)\}$. It is for this reason that they have been labelled with the same capital Greek index.

Invariance of the Kähler 2-form under the action of $\mathfrak{G}$ implies:

$$\ell_\Lambda K \equiv \mathrm{i}_\Lambda dK + d\mathrm{i}_\Lambda K = 0 \tag{2.26}$$

where $\ell_\Lambda = \ell_{\vec{k}_\Lambda}$ and $\mathrm{i}_\Lambda \equiv \mathrm{i}_{\vec{k}_\Lambda}$ denote, respectively, the Lie derivative and the contraction along $\vec{k}_\Lambda$. Since K is closed we conclude that $\mathrm{i}_\Lambda K$ is locally exact, so that in every coordinate patch we can find a real function $\mathcal{P}^\circ_\Lambda(z,\bar{z})$ such that:

$$\mathrm{i}_\Lambda K = -d\mathcal{P}^\circ_\Lambda \tag{2.27}$$

Using eqs.(2.3) and (2.24), from eq. (2.27) we get

$$k^i_\Lambda(z) = i g^{ij^*} \partial_{j^*} \mathcal{P}^\circ_\Lambda(z,\bar z) \tag{2.28a}$$
$$k^{i^*}_\Lambda(\bar z) = -i g^{i^* j} \partial_j \mathcal{P}^\circ_\Lambda(z,\bar z) \tag{2.28b}$$

The real function $\mathcal{P}^\circ_\Lambda(z,\bar z)$ is called the Killing vector prepotential [3,13,14] and, in the applications of Kähler geometry to $N=1$ supergravity, it is essentially the value taken by the auxiliary field D_Λ of the gauge vector multiplets.

From a geometrical point of view the prepotential $\mathcal{P}^\circ_\Lambda$ is the Hamiltonian function providing the Poissonian realization of the Lie algebra on the Kähler manifold.

Indeed the very existence of the closed 2-form K guarantees that every Kähler space is a symplectic manifold and that we can define a Poisson bracket.

Consider eqs. (2.28). To every generator of the abstract Lie algebra $\mathfrak{G}$ we have associated a function $\mathcal{P}^\circ_\Lambda$ on $SK(n)$; the Poisson bracket of $\mathcal{P}^\circ_\Lambda$ with $\mathcal{P}^\circ_\Sigma$ is defined as follows:

$$\{\mathcal{P}^\circ_\Lambda, \mathcal{P}^\circ_\Sigma\} \equiv 2K(\Lambda,\Sigma) \tag{2.29}$$

where $K(\Lambda,\Sigma) \equiv K(\vec k_\Lambda, \vec k_\Sigma)$ is the value of K along the pair of Killing vectors.

In ref. [13] the following lemma was proved. We recast both its enunciation and its proof in a slightly different way which is suitable for our later purposes.

Lemma 1

The following identity is true:

$$\{\mathcal{P}^\circ_\Lambda, \mathcal{P}^\circ_\Sigma\} = f^\Gamma_{\Lambda\Sigma} \mathcal{P}^\circ_\Gamma + C_{\Lambda\Sigma} \tag{2.30}$$

where $C_{\Lambda\Sigma}$ is a constant fulfilling the cocycle condition

$$f^\Gamma_{\Lambda\Pi} C_{\Gamma\Sigma} + f^\Gamma_{\Pi\Sigma} C_{\Gamma\Lambda} + f^\Gamma_{\Sigma\Lambda} C_{\Gamma\Pi} = 0 \tag{2.31}$$

Proof: Let us set $f_{\Lambda\Sigma} \equiv K(\Lambda,\Sigma)$. Using eq. (2.26) we get:

$$\begin{aligned} 2f_{\Lambda\Sigma} &= 2f_{\Sigma\Lambda} = \mathrm{i}_\Sigma \mathrm{i}_\Lambda K = -\mathrm{i}_\Sigma d\mathcal{P}^\circ_\Lambda = \mathrm{i}_\Lambda d\mathcal{P}^\circ_\Sigma = \\ &= \frac{1}{2}(\ell_\Lambda \mathcal{P}^\circ_\Sigma - \ell_\Sigma \mathcal{P}^\circ_\Lambda) \end{aligned} \tag{2.32}$$

Let us now calculate $df_{\Lambda\Sigma}$. Since the exterior derivative commutes with the Lie derivative we find

$$\begin{aligned} df_{\Lambda\Sigma} &= \frac{1}{4}(\ell_\Lambda d\mathcal{P}^\circ_\Sigma - \ell_\Sigma d\mathcal{P}^\circ_\Lambda) = \\ &= \frac{1}{4}(-\ell_\Lambda \mathrm{i}_\Sigma K + \ell_\Sigma \mathrm{i}_\Lambda K) \end{aligned} \tag{2.33}$$

Using now the identity

$$[\mathrm{i}_\Lambda, \ell_\Sigma] = \mathrm{i}_{[\Lambda,\Sigma]} \tag{2.34}$$

and eq.(2.25), from eq.(2.32) we obtain

$$\begin{aligned} df_{\Lambda\Sigma} &= \frac{1}{2}\mathrm{i}_{[\Lambda,\Sigma]} K = \frac{1}{2} f^\Gamma_{\Lambda\Sigma} \mathrm{i}_\Gamma K = \\ &= \frac{1}{2} f^\Gamma_{\Lambda\Sigma} d\mathcal{P}^\circ_\Gamma \end{aligned} \tag{2.35}$$

It follows that the difference

$$C_{\Lambda\Sigma} = \{\mathcal{P}^\circ_\Lambda, \mathcal{P}^\circ_\Sigma\} - f^\Gamma_{\Lambda\Sigma} \mathcal{P}^\circ_\Gamma \tag{2.36}$$

is a constant since we have shown that its exterior derivative vanishes: $dC_{\Lambda\Sigma} = 0$. The cocycle condition (2.29) follows from the Jacobi identities fulfilled by the Poisson bracket (2.29). This concludes the proof of the lemma.

If the Lie Algebra $\mathfrak{G}$ has a trivial second cohomology group $H^2(\mathfrak{G}) = 0$ the cocycle $C_{\Lambda\Sigma}$ is a coboundary, namely we have :

$$C_{\Lambda\Sigma} = f^\Gamma_{\Lambda\Sigma} C_\Gamma \tag{2.37}$$

where C_Γ are suitable constants. Hence, assuming $H^2(\mathfrak{G}) = 0$ we can reabsorb C_Γ in the definition of $\mathcal{P}^\circ_\Lambda$:

$$\mathcal{P}^\circ_\Lambda \to \mathcal{P}^\circ_\Lambda + C_\Lambda \tag{2.38}$$

(note that eq.(2.27) fixes $\mathcal{P}^\circ_\Lambda$ up to constant) and we obtain the stronger equation

$$\{\mathcal{P}^\circ_\Lambda, \mathcal{P}^\circ_\Sigma\} = f^\Gamma_{\Lambda\Sigma} \mathcal{P}^\circ_\Gamma \tag{2.39}$$

Note that $H^2(G) = 0$ is true for all semisimple Lie Algebras.

Using (2.29) eq. (2.39) can be rewritten in components as follows:

$$\frac{i}{2} g_{ij^*}(k^i_\Lambda k^{j^*}_\Sigma - k^i_\Sigma k^{j^*}_\Lambda) = -\frac{1}{2} f^\Gamma_{\Lambda\Sigma} \mathcal{P}^\circ_\Gamma \tag{2.40}$$

As we are going to see, eq.(2.39) has an analogue on quaternionic manifolds. Such an identity will be crucial in proving the supersymmetry invariance of the $N = 2$ supergravity Lagrangian.

All the Killing vector properties proved so far are true on any Kähler manifold. In the case of a special Kähler manifold there is an additional condition to be imposed on $\vec{k}_\Lambda$. We must require that the cross-sections $\{X^\Lambda\}$ of the holomorphic $(n+1)$-bundle should transform in the adjoint representation of $\mathfrak{G}$:

$$\ell_\Lambda X^\Gamma = k^i_\Lambda \partial_i X^\Gamma = -f^\Gamma_{\Lambda\Sigma} X^\Sigma \tag{2.41}$$

In the sequel we shall require that the Kähler potential is exactly invariant under the Lie Algebra $\mathfrak{G}$

$$\ell_\Lambda G \equiv k^i_\Lambda \partial_i G + k^{i^*}_\Lambda \partial_{i^*} G = 0 \tag{2.42}$$

Under this hypothesis eq. (2.40) is equivalent to:

$$\ell_\Lambda L^\Gamma \equiv k^i_\Lambda \partial_i L^\Gamma + k^{i^*}_\Lambda \partial_{i^*} L^\Gamma = -f^\Gamma_{\Lambda\Sigma} L^\Sigma \tag{2.43}$$

This condition is obvious when one introduces the L^Λ's in the supersymmetry algebra (see eq.(3.7 c)).

Using the definitions (2.20 b,c) and eq. (2.18 b) from (2.43) we find:

$$k^i_\Lambda f^\Gamma_i = \frac{1}{2}(k^i_\Lambda \partial_i G - k^{i^*}_\Lambda \partial_{i^*} G) L^\Gamma - f^\Gamma_{\Lambda\Sigma} L^\Sigma \tag{2.44}$$

or, equivalently:

$$k^i_\Lambda f^\Gamma_i = i\mathcal{P}^0_\Lambda L^\Gamma - f^\Gamma_{\Lambda\Sigma} L^\Sigma \tag{2.45}$$

Indeed from eqs. (2.28) and (2.42) it follows that the prepotential $\mathcal{P}^0_\Lambda$ has the following general form:

$$\mathcal{P}^0_\Lambda = -ik^i_\Lambda \partial_i G + C_\Lambda = ik^{i^*}_\Lambda \partial_{i^*} G + C_\Lambda \tag{2.46}$$

where C_Λ is a real constant; eq.(2.45) then follows.

We can now prove the following relations:

$$k^i_\Lambda L^\Lambda = k^{i^*}_\Lambda \bar{L}^\Lambda = 0 \tag{2.47}$$

$$\mathcal{P}^0_\Lambda L^\Lambda = \mathcal{P}^0_\Lambda \bar{L}^\Lambda = 0 \tag{2.48}$$

$$v^\Lambda \equiv f^\Lambda_i k^i_\Sigma \bar{L}^\Sigma = -(v^\Lambda)^* \tag{2.49}$$

which will be essential for deriving a consistent solution of the $N = 2$ supergravity Bianchi identities in the next section.

To prove eqs. (2.47) let us contract eq.(2.41) with X^Λ; we find:

$$X^\Lambda \ell_\Lambda X^\Gamma = X^\Lambda k^i_\Lambda \partial_i X^\Gamma = 0 \tag{2.50}$$

where we have used the antisymmetry of the structure constants $f^\Gamma_{\Lambda\Sigma}$ in the lower indices.

Eq. (2.50) implies

$$X^\Lambda k^i_\Lambda \partial_i X^\Gamma \partial_{j^*} \bar{X}^\Pi \partial_\Gamma \partial_{\bar{\Pi}} G = 0 \tag{2.51}$$

and, using eq. (2.21a):

$$g_{ij^*} X^\Lambda k^i_\Lambda = 0 \tag{2.52}$$

Since g_{ij^*} is non-degenerate, we find:

$$X^\Lambda k^i_\Lambda = 0 = \bar{X}^\Lambda k^{i^*}_\Lambda \tag{2.53}$$

where the second equality follows by complex conjugation. Eqs. (2.47) are then a consequence of the definition (2.17).

To prove eq.(2.48) we contract both sides of eq. (2.45) by the matrix $N_{\Gamma\Sigma}\bar{L}^\Sigma L^\Lambda$; then:

$$L^\Lambda k^i_\Lambda N_{\Gamma\Sigma} \bar{L}^\Sigma f^\Gamma_i = iL^\Lambda \mathcal{P}^0_\Lambda N_{\Gamma\Sigma} \bar{L}^\Sigma L^\Gamma \tag{2.54}$$

Using the identities (2.21 c) and (2.21 d) we find the first of eqs. (2.48) ($\mathcal{P}^0_\Lambda \bar{L}^\Lambda = 0$ follows by complex conjugation).

Let us note that by contracting eqs. (2.47) with $\partial_i G(\partial_{i^\star} G)$ and using (2.46) we also find:

$$(\mathcal{P}^0_\Lambda - C_\Lambda)L^\Lambda = \quad (\mathcal{P}^0_\Lambda - C_\Lambda)\bar{L}^\Lambda = 0 \tag{2.55}$$

Therefore the constant vector C_Λ must in any case be orthogonal to $L^\Lambda, \bar{L}^\Lambda$

$$C_\Lambda L^\Lambda \equiv C_\Lambda \bar{L}^\Lambda = 0 \tag{2.56}$$

in order that eqs. (2.55) be consistent with eqs. (2.47).

Finally, to prove eq. (2.49) we observe that from eq. (2.45) it follows:

$$k^i_\Lambda f^\Gamma_i \bar{L}^\Lambda = i\mathcal{P}^0_\Lambda L^\Gamma \bar{L}^\Lambda - f^\Gamma_{\Lambda\Sigma} L^\Sigma \bar{L}^\Lambda. \tag{2.57}$$

Hence, using eq.(2.48), one obtains:

$$(v^\Lambda)^* \equiv k^{i^\star}_\Sigma f^\Lambda_{i^\star} L^\Sigma = -f^\Lambda_{\Sigma\Pi} L^\Sigma \bar{L}^\Pi = -k^i_\Sigma f^\Lambda_i L^\Sigma = -v^\Lambda \tag{2.58}$$

This concludes our discussion of special Kähler manifolds with Killing vectors.

2b) *Quaternionic Kähler Manifolds with Killing vectors*

We summarize the definition and the properties of a quaternionic Kähler manifold $\mathcal{Q}(m)$ following Galicki [13a].

$\mathcal{Q}(m)$ is a $4m$-dimensional real manifold endowed with a metric h:

$$ds^2 = h_{uv}(q) dq^u \otimes dq^v \tag{2.59}$$

and three complex structures $(J^x)^u_v$ $(x = 1, 2, 3)$ that satisfy the quaternionic algebra

$$J^x J^y = -\delta^{xy} + \epsilon^{xyz} J^z \tag{2.60}$$

Defining the three 2-forms

$$\Omega^x = \Omega^x_{uv} dq^u \wedge dq^v \tag{2.61a}$$

$$\Omega^x_{uv} = \lambda h_{uw} (J^x)^w_v \tag{2.61b}$$

where λ is some real parameter, we complete the definition of a quaternionic Kähler manifold by assuming that Ω^x are covariantly closed

$$\nabla \Omega^x \equiv d\Omega^x + \epsilon^{xyz} \omega^y \wedge \Omega^z = 0 \tag{2.62}$$

with respect to an $SU(2)$ connection ω^x such that Ω^x is its field strength:

$$d\omega^x + \frac{1}{2}\epsilon^{xyz}\omega^y \wedge \omega^z = \Omega^x \tag{2.63}$$

As a consequence of this structure the manifold $\mathcal{Q}(m)$ has a holonomy group $\mathrm{Hol}(\mathcal{Q}(m)) = SU(2) \otimes \mathcal{H}$ where $\mathcal{H} \subset Sp(2m)$ is some subgroup of the sympletic group in $2m$-dimensions. This means that introducing flat indices $\{A, B, C = 1, 2\}, \{\alpha, \beta, \gamma = 1, .., 2m\}$ that run, respectively, in the fundamental representations of $SU(2)$ and $Sp(2m)$, we can find a vielbein 1-form

$$\mathcal{U}^{A\alpha} = \mathcal{U}^{A\alpha}_u(q) dq^u \tag{2.64}$$

such that

$$h_{uv} = \mathcal{U}^{A\alpha}_u \mathcal{U}^{B\beta}_v \mathbb{C}_{\alpha\beta}\epsilon_{AB} \tag{2.65}$$

where $\mathbb{C}_{\alpha\beta} = -\mathbb{C}_{\beta\alpha}$ and $\epsilon_{AB} = -\epsilon_{BA}$ are, respectively, the flat $Sp(2m)$ and $Sp(2) \sim SU(2)$ invariant metrics.

The vielbein $\mathcal{U}^{A\alpha}$ is covariantly closed with respect to the $SU(2)$-connection ω^z and to some $Sp(2m)$-Lie Algebra valued connection $\Delta^{\alpha\beta} = \Delta^{\beta\alpha}$:

$$\begin{aligned} \nabla \mathcal{U}^{A\alpha} \equiv d\mathcal{U}^{A\alpha} &+ \frac{i}{2}\omega^x (\epsilon\sigma_x\epsilon^{-1})^A{}_B \wedge \mathcal{U}^{B\alpha} \\ &+ \Delta^{\alpha\beta} \wedge \mathcal{U}^{A\gamma}\mathbb{C}_{\beta\gamma} = 0 \end{aligned} \tag{2.66}$$

where $(\sigma^x)_A{}^B$ are the standard Pauli matrices. Furthermore $\mathcal{U}^{A\alpha}$ satisfies the reality condition:

$$\mathcal{U}_{A\alpha} \equiv (\mathcal{U}^{A\alpha})^* = \epsilon_{AB}\mathbb{C}_{\alpha\beta}\mathcal{U}^{B\beta} \tag{2.67}$$

Eq.(2.67) defines the rule to lower the sympletic indices by means of the flat sympletic metrics ϵ_{AB} and $\mathbb{C}_{\alpha\beta}$.

More specifically we can write the stronger version of eq. (2.65)

$$(\mathcal{U}^{A\alpha}_u\mathcal{U}^{B\beta}_v + \mathcal{U}^{A\alpha}_v\mathcal{U}^{B\beta}_u)\mathbb{C}_{\alpha\beta} = h_{uv}\epsilon^{AB} \tag{2.68a}$$

$$(\mathcal{U}^{A\alpha}_u\mathcal{U}^{B\beta}_v + \mathcal{U}^{A\alpha}_v\mathcal{U}^{B\beta}_u)\epsilon_{AB} = h_{uv}\frac{1}{m}\mathbb{C}^{\alpha\beta} \tag{2.68b}$$

originally introduced in the physical literature by Bagger and Witten in [3].

We have also the inverse vielbein $\mathcal{U}^u_{A\alpha}$ defined by the equation

$$\mathcal{U}^u_{A\alpha}\mathcal{U}^{A\alpha}_v = \delta^u_v \tag{2.69}$$

Flattening a pair of indices of the Riemann tensor $\mathcal{R}^{uv}{}_{ts}$ we obtain

$$R^{uv}{}_{ts}\mathcal{U}^{\alpha A}_u\mathcal{U}^{\beta B}_v = \Omega^x_{ts}\frac{i}{2}(\epsilon^{-1}\sigma_x)^{AB}\mathbb{C}^{\alpha\beta} + \mathbb{R}^{\alpha\beta} \tag{2.70}$$

where $\mathbb{R}^{\alpha\beta}_{ts}$ is the field strength of the $Sp(2m)$ connection:

$$d\Delta^{\alpha\beta} + \Delta^{\alpha\gamma}\wedge\Delta^{\delta\beta}\mathbb{C}_{\gamma\delta} \equiv \mathbb{R}^{\alpha\beta} = \mathbb{R}^{\alpha\beta}_{ts}dq^t\wedge dq^s \tag{2.71}$$

Eq. (2.70) is the explicit statement that the Levi-Civita connection associated with the metric h has a holonomy group contained in $SU(2)\otimes Sp(2m)$. Consider now eqs. (2.60) and (2.61). We easily deduce the following relation:

$$h^{st}\Omega^x_{us}\Omega^y_{tw} = -\lambda^2\delta^{xy}h_{uw} + \lambda\epsilon^{xyz}\Omega^z_{uw} \tag{2.72}$$

Eq. (2.72) implies that the intrinsic components of the 2-form Ω^x yield a representation of the quaternion algebra. Hence we can set

$$\Omega^x_{A\alpha,B\beta} \equiv \Omega^x_{uv}\mathcal{U}^u_{A\alpha}\mathcal{U}^v_{B\beta} = -i\lambda\mathbb{C}_{\alpha\beta}(\sigma^x\epsilon)_{AB} \tag{2.73}$$

Alternatively eq.(2.73) can be rewritten in an intrinsic form as

$$\Omega^x = i\lambda\mathbb{C}_{\alpha\beta}(\sigma^x\epsilon^{-1})_{AB}\mathcal{U}^{\alpha A}\wedge\mathcal{U}^{\beta B} \tag{2.74}$$

wherefrom we also get:

$$\frac{i}{2}\Omega^x(\sigma_x)_A{}^B = \lambda\mathcal{U}_{A\alpha}\wedge\mathcal{U}^{B\alpha} \tag{2.75}$$

Note that the $SU(2)\otimes U(1)$ connections provided by ω^x and $\mathcal{Q}$ are related to the $SU(2)\otimes U(1)$ auxiliary fields of $N=2$ superconformal tensor calculus. Moreover in the case where $N=2$ supergravity can be interpreted as the effective theory of a $N=2$ heterotic superstring, these connections are related to the $SU(2)\otimes U(1)$ Kac-Moody algebra of the $n=4$ superconformal algebra [15,16,17,18].

All the above formulas are crucial in solving the Bianchi identities in superspace. Apart from a few changes in notations they are already discussed in the literature [3]. We turn now to a discussion of isometries of the manifold $\mathcal{Q}(m)$.

For applications to $N=2$ supergravity we must assume that on $\mathcal{Q}(m)$ we have an action by quaternionic isometries of the same $(n+1)$-dimensional Lie group G that acts on the special Kähler manifold $SK(n)$. This means that on $\mathcal{Q}(m)$ we have Killing vectors

$$\vec{k}_\Lambda = k^u_\Lambda\frac{\vec{\partial}}{\partial q^u} \tag{2.76}$$

satisfying the same Lie algebra (2.25) as the corresponding Killing vectors on $SK(n)$. In different words

$$\hat{\vec{k}}_\Lambda = \hat{\vec{k}}_\Lambda = k^i_\Lambda\vec{\partial}_i + k^{i^\star}_\Lambda\vec{\partial}_{i^\star} + k^u_\Lambda\vec{\partial}_u \tag{2.77}$$

is a Killing vector of the block diagonal metric:

$$\hat{g} = \begin{pmatrix} g_{ij^\star} & 0 \\ 0 & h_{uv}\end{pmatrix} \tag{2.78}$$

defined on the product manifold (2.1).

In order to be compatible with the quaternionic structure of the manifold, the Killing vectors must leave invariant the 2-form Ω^x up to $SU(2)$ gauge transformations. Namely:

$$\ell_\Lambda \Omega^x = \epsilon^{xyz}\Omega^y W^z_\Lambda \tag{2.79a}$$

$$\ell_\Lambda \omega^x = \nabla W^x_\Lambda \tag{2.79b}$$

where W^x_Λ is an $SU(2)$ compensator associated with the Killing vector k^u_Λ. The compensator W^x_Λ necessarily fulfils the cocycle condition:

$$\ell_\Lambda W^{x\Sigma} - \ell_\Sigma W^x_\Lambda + \epsilon^{xyz} W^y_\Lambda W^z_\Sigma = f^\Gamma_{\Lambda\Sigma} W^x_\Gamma \tag{2.80}$$

(For a complete discussion of compensators see for instance [19]).

In a fully analogous way to the case of Kähler manifolds, to each quaternionic Killing vector we can associate a triplet $\mathcal{P}^x_\Lambda(q)$ of 0-form prepotentials.

Indeed on $\mathcal{Q}(m)$ eq.(2.27) is replaced by

$$\mathrm{i}_\Lambda \Omega^x = -\nabla \mathcal{P}^x_\Lambda \equiv -(d\mathcal{P}^x_\Lambda + \epsilon^{xyz}\omega^y \mathcal{P}^z_\Lambda) \tag{2.81}$$

where ∇ denote the $SU(2)$ covariant exterior derivative.

In the quaternionic case $\mathcal{P}^x_\Lambda$ is explicitly constructed in terms of the compensator W^x_Λ. Indeed we can write

$$\mathcal{P}^x_\Lambda = H^x_\Lambda - W^x_\Lambda \tag{2.82}$$

where we have set

$$H^x_\Lambda = \mathrm{i}_\Lambda \omega^x \tag{2.83}$$

Eq. (2.82) was shown by Galicki [13a]. We recall the very elementary proof. From eq. (2.79a) we get

$$\begin{aligned}\epsilon^{xyz}\Omega^y W^z_\Lambda &= \mathrm{i}_\Lambda d\Omega^x + d(\mathrm{i}_\Lambda \Omega^x) = \\ &= \mathrm{i}_\Lambda(-\epsilon^{xyz}\omega^y \wedge \Omega^z) - d(\nabla \mathcal{P}^x_\Lambda) = \\ &= -\epsilon^{xyz} H^y_\Lambda \Omega^z - \nabla^2 \mathcal{P}^x_\Lambda = \\ &= -\epsilon^{xyz}(H^y_\Lambda - \mathcal{P}^y_\Lambda)\Omega^z\end{aligned} \tag{2.84}$$

that proves eq. (2.82).

Going now one step further with respect to the results obtained by Galicki, [13a] we can define a Poisson bracket also on the quaternionic prepotentials $\mathcal{P}^x_\Lambda$.

In analogy with eq.(2.29) we set

$$\{\mathcal{P}_\Lambda, \mathcal{P}_\Sigma\}^x \equiv 2\Omega^x(\Lambda,\Sigma) - \frac{1}{2}\epsilon^{xyz}\mathcal{P}^y_\Lambda \mathcal{P}^z_\Sigma, \tag{2.85}$$

where $\Omega^x(\Lambda,\Sigma) \equiv \Omega^x(\vec{k}_\Lambda, \vec{k}_\Sigma)$.

To see that eq.(2.85) does indeed define an operation on the prepotentials $\mathcal{P}^x_\Lambda$ it suffices to show that eq. (2.81) can be solved for the Killing vectors in terms of the prepotentials:

$$k^u_\Lambda = \frac{1}{\lambda^2}\sum_{x=1}^{3} h^{vw}(\nabla_v \mathcal{P}^x \Omega^x_{wt}) h^{tu} \tag{2.86}$$

We have also the analogue of lemma 1.

Lemma 2

<u>The following identity is true</u>

$$\{\mathcal{P}_\Lambda, \mathcal{P}_\Sigma\}^x = f^\Gamma_{\Lambda\Sigma}\mathcal{P}^x_\Gamma + C^x_{\Lambda\Sigma} \tag{2.87}$$

<u>where $C^x_{\Lambda\Sigma}$ is covariantly constant</u>

$$\nabla C^x_{\Lambda\Sigma} = 0 \tag{2.88}$$

<u>and fulfils the cocycle condition (2.31).</u>

<u>*Proof:*</u> It is fully analogous to the proof of lemma 1: one has just to utilize covariant exterior derivatives and covariant Lie derivatives instead of the ordinary ones (for a general discussion of covariant Lie derivatives we refer to [19]). In our case, for any $SU(2)$ vector T^x, we have:

$$\mathcal{L}_\Lambda T^x = \ell_\Lambda T^x + \epsilon^{xyz} W^y_\Lambda T^z \tag{2.89}$$

where W^y_Λ is the compensator introduced in eq.(2.79). With this definition eq.(2.79) can be simply rewritten as

$$\mathcal{L}_\Lambda \Omega^x = 0 \tag{2.90}$$

Furthermore, as shown in [19] the covariant Lie derivative commutes with the covariant exterior derivative:

$$[\nabla, \mathcal{L}_\Lambda] = 0 \tag{2.91}$$

Having set these preliminaries, in analogy with the proof of lemma 1 we define

$$f^x_{\Lambda\Sigma} = \Omega^x(\Lambda,\Sigma) - \frac{1}{4}\epsilon^{xyz}\mathcal{P}^y_\Lambda \mathcal{P}^z_\Sigma \tag{2.92}$$

and then we note that

$$\begin{aligned}\Omega^x(\Lambda,\Sigma) &= \frac{1}{4}(\mathrm{i}_\Lambda \nabla \mathcal{P}^x_\Sigma - \mathrm{i}_\Sigma \nabla \mathcal{P}^x_\Lambda) = \\ &= \frac{1}{4}(\ell_\Lambda \mathcal{P}^x_\Sigma - \ell_\Sigma \mathcal{P}^x_\Lambda + \epsilon^{xyz} H^y_\Lambda \mathcal{P}^z_\Sigma - \epsilon^{xyz} H^y_\Sigma \mathcal{P}^z_\Lambda)\end{aligned} \tag{2.93}$$

Inserting (2.93) into (2.92) we obtain

$$f^x_{\Lambda\Sigma} = \frac{1}{4}(\mathcal{L}_\Lambda \mathcal{P}^x_\Sigma - \mathcal{L}_\Sigma \mathcal{P}^x_\Lambda) \tag{2.94}$$

which is fully analogous to eq. (2.32). With identical steps from eq. (2.94) we obtain

$$\nabla f^x_{\Lambda\Sigma} = -\frac{1}{2} f^\Gamma_{\Lambda\Sigma} \nabla \mathcal{P}^x_\Gamma \tag{2.95}$$

which shows that $C^x_{\Lambda\Sigma}$ as defined by eq.(2.87) is covariantly constant. The lemma is proved.

This shows that in some sense $C^x_{\Lambda\Sigma}$ is a cocycle in a quaternionic-valued cohomology of the Lie Algebra $\mathfrak{G}$. If we assume that the second of these cohomology groups is trivial then we have

$$C^x_{\Lambda\Sigma} = f^\Gamma_{\Lambda\Sigma} C^x_\Gamma \tag{2.96}$$

and C^x_Γ can be reabsorbed in the definition of $\mathcal{P}^x_\Lambda$. In this case and only in this case we obtain the analogue of eq. (2.40):

$$\Omega^x_{uv} k^u_\Lambda k^v_\Sigma - \frac{1}{2}\epsilon^{xyz}\mathcal{P}^y_\Lambda \mathcal{P}^z_\Sigma + \frac{1}{2} f^\Gamma_{\Lambda\Sigma} \mathcal{P}^x_\Gamma = 0 \tag{2.97}$$

As we are going to see, eq.(2.97) and the analogous one for $SK(m)$, eqs. (2.29) and (2.30), are the essential Ward Identities of $N = 2$ supersymmetry, ensuring that in the supergravity Lagrangian the variation of the bilinear Fermi terms is cancelled by the variation of a suitable scalar potential. If eq. (2.97) or (2.37) do not hold, no scalar potential can do the work and $N = 2$ Supergravity cannot be gauged.

We see that SUSY Ward identities have a root in Lie Algebra cohomology with complex and quaternionic coefficients.

2 c) *Gauging of the composite connection*

On the scalar manifold (2.1) we have introduced several connection 1-forms related with different bundles. In particular we have the Levi-Civita connection (2.13) and the $SU(2) \otimes U(1)$ connection defined in eqs. (2.5c) and (2.62), (2.63).

Gauging the corresponding supergravity theory is done by gauging these composite connections of the underlying σ-model. There is a general procedure introduced first by de Wit and Nicolai [5] in the case of $N = 8$ supergravity for the coset manifold $E_{7+}/SU(8)$. A systematic procedure for the extension to all other cases, including manifolds which are not cosets, was given in refs. [3,6,8].

For the Levi-Civita connection the gauging is standard on any scalar manifold $\mathcal{M}$. Let $\phi^i(x)$ be the scalar fields and $k^i_\Lambda(\phi)$ be the Killing vectors. The ordinary differential $d\phi^i$ is replaced by the covariant one defined below:

$$d\phi^i \to \nabla\phi^i = d\phi^i + gA^\Lambda k^i_\Lambda(\phi) \tag{2.98}$$

where g is the gauge coupling constant and A^Λ is the gauge 1-form.

At the same time the Levi-Civita connection $\Gamma^i_j = \Gamma^i_{jk} d\phi^k$ is replaced by its gauged counterpart defined as follows:

$$\Gamma^i_j \to \hat{\Gamma}^i_j \equiv \Gamma^i_{jk} \nabla\phi^k + gA^\Lambda \partial_j k^i_\Lambda \tag{2.99}$$

A lengthy but straighforward calculation yields the following result for the gauged curvature 2-form

$$\hat{R}^i_j \equiv d\hat{\Gamma}^i{}_j + \hat{\Gamma}^i{}_k \wedge \hat{\Gamma}^k_j = \\ = R^i{}_{jk\ell} \nabla\phi^k \wedge \nabla\phi^\ell + g\mathcal{F}^\Lambda \partial_j k^i_\Lambda \tag{2.100}$$

where $R^i{}_{jk\ell}$ is the Riemann tensor of the ungauged connection and where

$$\mathcal{F}^\Lambda = dA^\Lambda + \frac{1}{2} g f^\Lambda{}_{\Sigma\Gamma} A^\Sigma \wedge A^\Gamma \tag{2.101}$$

is the gauge field-strength.

For a Kähler manifold eqs. (2.98), (2.99) and (2.100) become

$$dz^i \to \nabla z^i = dz^i + gA^\Lambda k^i_\Lambda(z) \tag{2.102a}$$
$$d\bar{z}^{i*} \to \nabla \bar{z}^{i*} = d\bar{z}^{i*} + gA^\Lambda k^{i^*}_\Lambda(z) \tag{2.102b}$$
$$\Gamma^i{}_j \to \hat{\Gamma}^i{}_j = \Gamma^i_{jk} \nabla z^k + gA^\Lambda \partial_j k^i_\Lambda \tag{2.102c}$$
$$\Gamma^{i^*}{}_{j^*} \to \hat{\Gamma}^{i^*}{}_{j^*} = \Gamma^{i^*}{}_{j^*k} \nabla \bar{z}^{k^*} + gA^\Lambda \partial_{j^*} k^{i^*}_\Lambda \tag{2.102d}$$

$$\hat{\mathcal{R}}^i{}_j = d\hat{\Gamma}^i_j + \hat{\Gamma}^i{}_k \wedge \Gamma^k{}_j = \\ = \mathcal{R}^i{}_{jk^*\ell} \nabla \bar{z}^{k^*} \wedge \nabla z^\ell + g\mathcal{F}^\Lambda \partial_j k^i_\Lambda \tag{2.103}$$

In an analogous way the gauging of the $Sp(2m)$ connection, $\Delta^{\alpha\beta} \to \hat{\Delta}^{\alpha\beta}$, gives

$$\hat{\Delta}^{\alpha\beta} = \Delta^{\alpha\beta} + gA^\Lambda \partial_u k^v_\Lambda \mathcal{U}^{u|\alpha A} \mathcal{U}^\beta_{v|A} \tag{2.104}$$

and the associated gauged curvature $\hat{\mathbb{R}}^{\alpha\beta}$ becomes

$$\hat{\mathbb{R}}^{\alpha\beta} = \mathbb{R}^{\alpha\beta}{}_{uv} \nabla q^u \wedge \nabla q^v + g\mathcal{F}^\Lambda \partial_u k^v_\Lambda \mathcal{U}^{u|\alpha A} \mathcal{U}^\beta_{v|A} \tag{2.105}$$

For the $SU(2) \times U(1)$ connection, the existence of the Killing vector prepotentials allows the following definitions:

$$\mathcal{Q} \to \hat{\mathcal{Q}} = \mathcal{Q} + gA^\Lambda \mathcal{P}^\circ_\Lambda \tag{2.106a}$$
$$\omega^x \to \hat{\omega}^x = \omega^x + gA^\Lambda \mathcal{P}^x_\Lambda \tag{2.106b}$$

Eq.(2.106a) is easily retrieved by replacing $dz^i \to \nabla z^i$ in (2.5c) and using (2.46). Eq. (2.106b) is an obvious extension of the previous one.

Indeed by computing the associated gauged curvatures:

$$\hat{K} = d\hat{\mathcal{Q}} \tag{2.107a}$$
$$\hat{\Omega}^x = d\hat{\omega}^x + \frac{1}{2}\epsilon^{xyz} \hat{\omega}^y \wedge \hat{\omega}^z \tag{2.107b}$$

one finds the gauge covariant expressions

$$\hat{K} = d\hat{\mathcal{Q}} = ig_{ij^*} \nabla z^i \wedge \nabla z^{j^*} + g\mathcal{F}^\Lambda \mathcal{P}^\circ_\Lambda \tag{2.108a}$$
$$\hat{\Omega}^x = d\hat{\omega}^x + \frac{1}{2}\epsilon^{xyz} \hat{\omega}^y \wedge \hat{\omega}^z = \Omega^x_{uv} \nabla q^u \wedge \nabla q^v + g\mathcal{F}^\Lambda \mathcal{P}^x_\Lambda \tag{2.108b}$$

where we have used eqs. (2.28), (2.40), (2.81) and (2.97).

These formulae will be crucial in solving the superspace Bianchi identities.

3. The solution of superspace Bianchi identities in the presence of gauging

In reference [4d] the superspace Bianchi identities were solved for the case of N=2 supergravity coupled to vector multiplets without gauging. The emphasis there was on deriving the constraints of the special Kähler geometry in a covariant way, independently of the choice of special coordinates.

In the present section we extend the results of ref. [4d] by studying the Bianchi identities in the most general case, that is by gauging an arbitrary group G and furthermore by adding to the supergravity-vector multiplet system an arbitrary number m of hypermultiplets. Our primary goal is to determine the extra terms in the "fermionic curvatures" due to the gauging: indeed these "fermionic shifts", which are proportional to the gauge coupling constant g, are all we need in order to reconstruct the full scalar potential $\mathcal{V}$.

The derivation of the scalar potential together with the determination of the N=2 Lagrangian (up to four-fermion terms) is the subject of the next section.

We begin by writing down the superspace parametrization of the hypermultiplet "curvatures" in the ungauged case.

Given the quaternionic coordinates $q^u, u = 1, ..., 4m$, and the positive and negative chirality superpartners $\zeta_\alpha, \zeta^\alpha, \alpha = 1, ..., 2m$, we define the "generalized curvature" (or field-strengths), of the corresponding superfields as the (covariant)-differential in superspace, namely dq^u and $\nabla\zeta_\alpha(\nabla\zeta^\alpha)$, respectively. The derivative ∇ is covariant with respect to Lorentz, symplectic and U(1)-Kähler transformations:

$$\nabla\zeta_\alpha \equiv d\zeta_\alpha - \frac{1}{4}\gamma_{ab}\omega^{ab}\zeta_\alpha - \Delta_\alpha{}^\gamma\zeta_\gamma + \frac{i}{2}\mathcal{Q}\zeta_\alpha \tag{3.1a}$$

$$\nabla\zeta^\alpha \equiv d\zeta^\alpha - \frac{1}{4}\gamma_{ab}\omega^{ab}\zeta_\alpha - \Delta^\alpha{}_\gamma\zeta^\gamma + \frac{i}{2}\mathcal{Q}\zeta^\alpha \tag{3.1b}$$

where $\Delta_\alpha{}^\gamma \equiv \Delta^{\beta\gamma}C_{\beta\alpha}, \Delta^\alpha{}_\gamma \equiv C_{\gamma\beta}\Delta^{\alpha\beta}$, $\Delta^{\alpha\beta}$ being the $Sp(2m)$ connection.

It is convenient to convert the world index of the "curvature" dq^u into a flat index (A, α) by means of the quaternionic vielbein

$$\mathcal{U}^{A\alpha} \equiv \mathcal{U}^{A\alpha}{}_u dq^u \tag{3.2}$$

defined in eqs. (2.64-65).

Using $\mathcal{U}^{A\alpha}$, the "Bianchi identity" $d^2q^u \equiv 0$ translates into the statement of zero torsion for the quaternionic manifold $\mathcal{Q}_m$, i.e. eq. (2.66).

The Bianchi identity for $\nabla\zeta_\alpha(\nabla\zeta^\alpha)$ is

$$\nabla\nabla\zeta_\alpha = -\frac{1}{4}\gamma_{ab}R^{ab}\zeta_\alpha - \mathbb{R}_\alpha{}^\beta(\Delta)\zeta_\beta + \frac{i}{2}K\zeta_\alpha \tag{3.3a}$$

$$\nabla\nabla\zeta^\alpha = -\frac{1}{4}\gamma_{ab}R^{ab}\zeta^\alpha - \mathbb{R}^\alpha{}_\beta(\Delta)\zeta^\beta + \frac{i}{2}K\zeta^\alpha \tag{3.3b}$$

where the $Sp(2m)$-curvature $\mathbb{R}_\alpha{}^\beta = C_{\alpha\gamma}\mathbb{R}^{\gamma\beta}$ was defined in eq. (2.71).

The superspace parametrizations of $\mathcal{U}^{A\alpha}$ and $\nabla\zeta_\alpha(\nabla\zeta^\alpha)$ are now easily fixed. We set:

$$\mathcal{U}^{A\alpha} = \mathcal{U}^A_a V^a + \epsilon^{AB}C^{\alpha\beta}\bar{\psi}_B\zeta_\beta + \bar{\psi}^A\zeta^\alpha \tag{3.4}$$

Eq. (3.4) just fixes the supersymmetry transformation law of the quaternionic coordinate fields q^u.

For $\nabla\zeta_\alpha$ and $\nabla\zeta^\alpha$ we write:

$$\nabla\zeta_\alpha = \nabla_a\zeta_\alpha V^a + i\gamma_1\mathcal{U}^{B\beta}_a\gamma^a\psi^A\epsilon_{AB}C_{\alpha\beta} \tag{3.5a}$$

$$\nabla\zeta^\alpha = \nabla_a\zeta^\alpha V^a + i\gamma_2\mathcal{U}^A_\alpha\gamma^a\psi_A \tag{3.5b}$$

and using the torsion equation (2.66) one easily finds $\gamma_1 = \gamma_2 = 1$.

The Bianchi identities (3.3) will not be analyzed since they simply give known results for the coupling of the hypermultiplets to N=2 supergravity [3]. The parametrizations of the curvatures associated to the gravitational and vector multiplet superfields in the ungauged case and in the absence of hypermultiplets were computed in reference [4d].

With hypermultiplets there are some modifications both in the definition of the curvatures and in their explicit superspace parametrization. We briefly consider them. The supergravity curvatures are

$$R^a = \mathcal{D}V^a - i\bar{\psi}_A \wedge \gamma^a\psi^A \tag{3.6a}$$

$$\rho_A = d\psi_A - \frac{1}{4}\gamma_{ab}\omega^{ab} \wedge \psi_A + \frac{i}{2}\mathcal{Q} \wedge \psi_A + \omega_A{}^B \wedge \psi_B \equiv \nabla\psi_A \tag{3.6b}$$

$$\rho^A = d\psi^A - \frac{1}{4}\gamma_{ab}\omega^{ab} \wedge \psi^A + \frac{i}{2}\mathcal{Q} \wedge \psi^A + \omega^A{}_B \wedge \psi^B \equiv \nabla\psi^A \tag{3.6c}$$

$$R^{ab} = d\omega^{ab} - \omega^a{}_c \wedge \omega^{cb} \tag{3.6d}$$

where V^a are the superspace bosonic vielbein and ψ_A, ψ^A the superspace fermionic vielbein of positive and negative chirality, respectively; ω^{ab} is the spin-connection 1-form, $\mathcal{Q}$ and $\omega_A{}^B \equiv \frac{i}{2}(\sigma_x)_A{}^B\omega^x$ $(\omega^A{}_B \equiv \epsilon^{AL}\omega_L{}^M\epsilon_{MB})$ are the $U(1) \times SU(2)$ composite connection, defined in the previous section ($\mathcal{D}$ is the Lorentz covariant differential).

For the vector multiplet we define, together with the differentials dz^i, $d\bar{z}^{i^*}$, ("curvatures" of z^i, $\bar{z}^{i^*}$) the following superspace field-strengths:

$$\nabla\lambda^{iA} \equiv d\lambda^{iA} - \frac{1}{4}\gamma_{ab}\omega^{ab}\lambda^{iA} - \frac{i}{2}\mathcal{Q}\lambda^{iA} - \Gamma^i{}_j\lambda^{jA} + \omega^A{}_B\lambda^{iB} \tag{3.7a}$$

$$\nabla\lambda^{i^*}_A \equiv d\lambda^{i^*}_A - \frac{1}{4}\gamma_{ab}\omega^{ab}\lambda^{i^*}_A + \frac{i}{2}\mathcal{Q}\lambda^{i^*}_A - \Gamma^{i^*}{}_{j^*}\lambda^{j^*}_A - \omega_A{}^B\lambda^{i^*}_B \tag{3.7b}$$

$$F^\Lambda \equiv dA^\Lambda + \bar{L}^\Lambda\bar{\psi}_A \wedge \psi_B\epsilon^{AB} + L^\Lambda\bar{\psi}^A \wedge \psi^B\epsilon_{AB} \tag{3.7c}$$

where λ^{iA}, $\lambda^{i^*}_A$ are the spin 1/2 components of the vector multiplet (gaugino's) of positive and negative chirality respectively, $\Gamma^i{}_j$, $\Gamma^{i*}{}_{j^*}$ are the (1,0) and (0,1)-parts of the Levi-Civita connection on the Kähler manifold defined in eqs. (2.13), and A^Λ is the gauge connection 1-form. The index Λ runs from 0 to n, the value 0 being associated to the graviphoton (the gauge vector of the supergravity multiplet) and the values $1...n$ being associated with the vectors of the n vector multiplets. Finally the quantities L^Λ, $\bar{L}^\Lambda$ are a priori arbitrary functions of z^i and $\bar{z}^{i*}$ on the Kähler manifold, but, as derived in ref. [4d], the Bianchi identities constrain them in such a way that they coincide with the objects defined in eqs. (2.17) in terms of which the whole special Kähler geometry can be derived.

We note that the $U(1)$-weights of the fermion fields are $[\psi_A] = [\lambda^{i^*}_A] = -[\psi^A] = -[\lambda^{i^*}] = 1/2$. It follows that $[L^\Lambda] = -[\bar{L}^\Lambda] = 1$.

Comparing eqs. (3.6) and (3.7) with the analogous ones given in ref. [4d] one sees that the only modifications due to the presence of the hypermultiplets appear in the definition of the ∇-covariant differential of the fermionic superfields $\{\psi_A, \bar{\psi}^A\}$ and $\{\lambda^{iA}, \lambda^{i^*}_A\}$ which now contain the extra SU(2)-connection $\omega_A{}^B, \omega^A{}_B$.

From (3.6) and (3.7) one derives the following Bianchi identities:

$$\mathcal{D}R^{ab} = 0 \tag{3.8a}$$

$$\mathcal{D}R^a + R^{ab} \wedge V_b - i\bar{\psi}^A \wedge \gamma_a\rho_A + i\bar{\rho}^A \wedge \gamma_a\psi_A = 0 \tag{3.8b}$$

$$\nabla\rho_A + \frac{1}{4}\gamma_{ab}R^{ab} \wedge \psi_A - \frac{i}{2}K \wedge \psi_A - \frac{i}{2}R_A{}^B \wedge \psi_B = 0 \tag{3.8c}$$

$$\begin{aligned} dF^\Lambda &- \epsilon_{AB}\nabla L^\Lambda \wedge \bar{\psi}^A \wedge \psi^B + 2\epsilon_{AB}L^\Lambda\bar{\psi}^A \wedge \rho^B - \epsilon^{AB}\nabla\bar{L}^\Lambda \wedge \bar{\psi}_A \wedge \psi_B \\ &+ 2\epsilon^{AB}\bar{L}^\Lambda\bar{\psi}_A \wedge \rho_B = 0 \end{aligned} \tag{3.8d}$$

$$\nabla^2\lambda^{iA} + \frac{1}{4}R^{ab}\gamma_{ab}\lambda^{iA} + \frac{i}{2}K\lambda^{iA} + R^i{}_j(\Gamma)\lambda^{jA} - \frac{i}{2}R^A{}_B(\omega)\lambda^{iB} = 0 \tag{3.8e}$$

$$d^2z^i = d^2\bar{z}^{i^*} = 0 \tag{3.8f}$$

and similar equations for the chirality reversed fermions ρ^A, $\nabla\lambda^{i^*}_A$. Here $R_A{}^B \equiv \frac{i}{2}(\sigma_x)_A{}^B\Omega^x$, $R^A{}_B = \epsilon^{AC}R_C{}^D\epsilon_{DC}$ and Ω^x is the SU(2) curvature defined in (2.61). The complete set of Bianchi identities is given by eqs. (2.66), (3.3) and (3.8).

Since the solution of eqs. (3.8) was already found in the absence of hypermultiplets ($\omega_A{}^B = R_A{}^B = 0$) in ref. [4d], it is easy to find the extra terms in the parametrization of the new curvatures (3.6) and (3.7) due to the presence of hypermultiplets. Of course the hypermultiplets do not affect the constraints on the Kähler geometry implied by the Bianchi identities (3.8).

For completeness, in the following we write down both the complete parametrization of the curvatures for the ungauged theory and the constraints on the geometry, already derived from the Bianchi identities in ref. [4d], which restrict the general Kähler-Hodge manifold to a special Kähler manifold. The general solution of the Bianchi identities (3.3) and (3.8) in the ungauged case is:

$$R^a = 0 \tag{3.9}$$

$$\begin{aligned} \rho_A =& \rho_{A|ab}V^a \wedge V^b + \{A^B{}_{A\,|b}\eta^{ab} + A'^B{}_{A\,|b}\gamma^{ab})\psi_B + \\ &+ [iS_{AB}\eta^{ab} + \epsilon_{AB}(T^+_{ab} + iT^{+'}_{ab} - iF^+_{ab})]\gamma^a\psi^B\} \wedge V^b \end{aligned} \tag{3.10a}$$

$$\begin{aligned} \rho^A =& \rho^A_{|ab}V^a \wedge V^b + \{(\bar{A}^A{}_{B|b}\eta^{ab} + \bar{A}'^A{}_{B|b}\gamma^{ab})\psi_B \\ &+ [i\bar{S}^{AB}\eta^{ab} + \epsilon^{AB}(T^-_{ab} - iT^{-'}_{ab} + iF^-_{ab})]\gamma^a\psi^B\} \wedge V^b \end{aligned} \tag{3.10b}$$

$$R^{ab} = R^{ab}{}_{cd} V^c \wedge V^d - i(\bar\psi_A \theta_c^{A|ab} + \bar\psi^A \theta^{ab}_{A|c}) \wedge V^c + \\ + \epsilon^{abcf} \bar\psi^A \wedge \gamma_f \psi_B (A'^B{}_{A|c} - \bar A'^B{}_{A|c}) - \bar\psi_A \wedge \gamma^{ab} \psi_B \bar S^{AB} - \bar\psi^A \wedge \gamma^{ab} \psi^B S_{AB} + \\ + i\epsilon^{AB} \bar\psi_A \wedge \psi_B (T^{ab} - iT'^{ab} + iF^{-ab}) - i\epsilon_{AB} \bar\psi^A \wedge \psi^B (T^{ab} + iT'^{ab} - iF^{+}_{ab}) \tag{3.11}$$

$$F^\Lambda = F^\Lambda_{ab} V^a \wedge V^b + (i f^\Lambda_i \bar\lambda^{iA} \gamma^a \psi^B \epsilon_{AB} + i f^\Lambda_{i^\star} \bar\lambda^{i^\star}_A \gamma^a \psi_B \epsilon^{AB}) \wedge V_a \tag{3.12}$$

$$\nabla \lambda^{iA} = \nabla_a \lambda^{iA} V^a + i Z^i_a \gamma^a \psi^A + G^{+i}_{ab} \gamma^{ab} \psi_B \epsilon^{AB} + Y^{iAB} \psi_B \tag{3.13a}$$

$$\nabla \lambda^{i^\star}_A = \nabla_a \lambda^{i^\star}_A V^a + i \bar Z^{i^\star}_a \gamma^a \psi_A + G^{-i}_{ab} \gamma^{ab} \psi^B \epsilon_{AB} + Y^{i^\star}{}_{AB} \psi^B \tag{3.13b}$$

$$dz^i = Z^i_a V^a + \bar\psi_A \lambda^{iA} \tag{3.14a}$$

$$d\bar z^{i^\star} = Z^{i^\star}_a V^a + \bar\psi^A \lambda^{i^\star}_A \tag{3.14b}$$

$$K \equiv i g_{ij^\star} dz^i \wedge d\bar z^{j^\star} = i g_{ij^\star} Z^i_a \bar Z^{j^\star}_b V^a \wedge V^b + \\ + i g_{ij^\star} (Z^i_a \bar\psi^A \lambda^{j^\star}_A + \bar Z^{j^\star}_a \bar\psi_A \lambda^{iA}) \wedge V^a \tag{3.15}$$

where

$$A^B{}_{A\,|b} = -\frac{i}{4} g_{k^\star \ell} (\bar\lambda^{k^\star}_A \gamma_a \lambda^{\ell B} - \delta^B_A \bar\lambda^{k^\star}_C \gamma_a \lambda^{\ell C}) \tag{3.16a}$$

$$A'^B{}_{A\,|a} = \frac{i}{4} (\bar\lambda^{k^\star}_A \gamma_a \lambda^{\ell B} - \frac{1}{2} \delta^B_A \bar\lambda^{k^\star}_C \gamma_a \lambda^{C\ell}) + \frac{1}{4} \lambda \delta^B_A \bar\zeta^\alpha \gamma_a \zeta_\alpha \tag{3.16b}$$

$$S_{AB} = \bar S^{AB} = 0 \tag{3.17}$$

$$\theta^{ab}_{c|A} = 2\gamma^{[a} \rho^{b]c}_A + \gamma^c \rho^{ab}_A; \quad \theta^{ab|A}_c = 2\gamma^{[a} \rho^{b]cA} + \gamma^c \rho^{abA} \tag{3.18}$$

$$T^+_{ab} - iT'^+_{ab} = -\frac{i}{4} \lambda \mathbb{C}^{\alpha\gamma} \bar\zeta \gamma_{ab} \zeta_\alpha \tag{3.19a}$$

$$T^-_{ab} + iT'^-_{ab} = \frac{i}{4} \lambda \mathbb{C}^{\alpha\gamma} \bar\zeta^\gamma \gamma_{ab} \zeta^\alpha \tag{3.19b}$$

$$T^+_{ab} + iT'^+_{ab} - iF^+_{ab} = \frac{i}{4} \bar L^\Sigma N_{\Lambda\Sigma} \bar S^{-1} \left\{ F^{+\Lambda}_{ab} + \frac{1}{8} \nabla_i f^\Lambda_j \bar\lambda^{iA} \gamma_{ab} \lambda^{jB} \epsilon_{AB} \right\} \tag{3.19c}$$

$$T^-_{ab} + iT'^-_{ab} + iF^-_{ab} = -\frac{i}{4} L^\Sigma N_{\Lambda\Sigma} S^{-1} \left\{ F^{-\Lambda}_{ab} + \frac{1}{8} \nabla_{i^\star} f^\Lambda_{j^\star} \bar\lambda^{i^\star}_A \gamma_{ab} \lambda^{j^\star}_B \epsilon^{AB} \right\} \tag{3.19d}$$

$$G^{i+}_{ab} = \frac{1}{2} g^{ij^\star} f^\Gamma_{j^\star} N_{\Lambda\Sigma} \left(-\delta^\Sigma_\Lambda - \frac{1}{4} \bar S^{-1} N_{\Gamma\Delta} \bar L^\Lambda \bar L^\Sigma \right) \times \\ \times \left(F^{+\Lambda}_{ab} + \frac{1}{8} \nabla_i f^\Lambda_i \bar\lambda^{iA} \gamma_{ab} \lambda^{jB} \epsilon_{AB} \right) \tag{3.20a}$$

$$G^{i-}_{ab} = \frac{1}{2} g^{i^\star j} f^\Gamma_j N_{\Lambda\Sigma} \left(-\delta^\Sigma_\Lambda - \frac{1}{4} S^{-1} N_{\Gamma\Delta} L^\Delta L^\Sigma \right) \times \\ \times \left(F^{-\Lambda}_{ab} + \frac{1}{8} \nabla_{i^\star} f^\Lambda_{j^\star} \bar\lambda^{i^\star}_A \gamma_{ab} \lambda^{j^\star}_B \epsilon^{AB} \right) \tag{3.20b}$$

$$Y^{ABi} = g^{ij^\star} C_{j^\star k^\star \ell^\star} \bar\lambda^{k^\star}_C \lambda^{\ell^\star}_D \epsilon^{AC} \epsilon^{BD} \tag{3.21a}$$

$$Y^{i^\star}_{AB} = g^{i^\star j} C_{ik\ell} \bar\lambda^{kC} \lambda^{\ell D} \epsilon_{AC} \epsilon_{BD} \tag{3.21b}$$

where S has been defined in eq. (2.20d) and the upper indices + and − on the antisymmetric tensor T_{ab}, T'_{ab}, F^Λ_{ab} and G^i_{ab} mean their selfdual and antiselfdual projections respectively.

The special geometry objects $L^\Lambda, \bar L^\Lambda, f^\Lambda; f^\Lambda_i, f^\Lambda_{i^\star}$ and the weight $p = \pm, 2$ tensors C_{ijk}, and $C_{i^\star j^\star k^\star}$ turn out to be constrained as follows:

$$\nabla_{i^\star} L^\Lambda = \nabla_i \bar L^\Lambda = 0 \tag{3.22}$$

$$f_i^\Lambda = \nabla_i L^\Lambda; \quad f_{i^\star}^\Lambda = \nabla_{i^\star} L^\Lambda \tag{3.23}$$

$$\nabla_{\ell^\star} C_{ijk} = \nabla_\ell C_{i^\star j^\star k^\star} = 0 \tag{3.24a}$$

$$\nabla_{[\ell} C_{i]jk} = \nabla_{[\ell^\star} C_{i^\star]j^\star k^\star} = 0 \tag{3.24b}$$

$$g^{i\ell^\star} f_{\ell^\star}^\Lambda C_{ijk} = \frac{1}{2} \nabla_{(j} f_{k)}^\Lambda \tag{3.25}$$

The solution for C_{ijk} is best expressed in terms of the degree 2 homogeneous and holomorphic function $F(L^\Lambda)$ introduced in sect. 2. Using (2.21a), from (3.25) one finds:

$$C_{ijk} = -\frac{1}{2} N_{\Lambda\Sigma} f_i^\Lambda \nabla_{(j} f_{k)}^\Sigma \tag{3.26}$$

where $N_{\Lambda\Sigma}$ is given by eq. (2.20a).

One also finds the following alternative expression:

$$C_{ijk} \equiv \nabla_i \nabla_j \nabla_k S \equiv -\frac{1}{2} f_i^\Lambda f_j^\Sigma f_k^\Pi F_{\Lambda\Sigma\Pi} \tag{3.27}$$

where S is given by (2.20d). The expressions (3.27) for C_{ijk} were proved in ref. [4d] and they show that C_{ijk} is actually a completely symmetric tensor. (This can also be shown directly from (3.26) using eqs. (2.21).)

Inserting (3.26) into (3.24a) one obtains:

$$R_{i^\star jk^\star \ell} = -g_{i^\star(j} g_{\ell)k^\star} + \frac{1}{2} C_{i^\star k^\star s^\star} C_{tj\ell} g^{s^\star t} \tag{3.28}$$

which again implies that $C_{ijk}(C_{i^\star j^\star k^\star})$ are completely symmetric tensors. The constraints (3.24b) are then consequences of the Bianchi identities of the Riemann tensor $R_{i^\star jk^\star \ell}$. They imply [11] that C_{ijk} can be written as the third covariant derivative of a weight-2 function which turns out to be S (see eq. (3.27)).

Since C_{ijk} and $C_{i^\star j^\star k^\star}$ satisfy (3.24a) with U(1)-Kähler weight $p = \pm 2$ respectively, we may define the holomorphic and antiholomorphic sections

$$W_{ijk} = e^G \, C_{ijk} \tag{3.29a}$$

$$W_{i^\star j^\star k^\star} = e^G \, C_{i^\star j^\star k^\star} \tag{3.29b}$$

in terms of which eqs. (3.24a) and (3.28) become identical to eqs. (2.15) defining the special Kähler geometry.
Eqs. (3.9-3.21) and (3.22-3.25) exhaust the content of the Bianchi identities in the ungauged case.

(In ref. [4d] the solution of (3.25) was written as follows

$$C_{ijk} = \frac{1}{2} g_{i\ell^\star} f_I^{\ell^\star} \nabla_{(j} f_{k)}^I$$

where $I = 1, \ldots, n$ runs on the vector multiplets only so that f_i^I is a squared matrix and $f_I^{\ell^\star} \equiv (f_{\ell^\star}^I)^{-1}$. One can then show that the equation (3.25) with $\Lambda = 0$ is satisfied by virtue of the special Kähler geometry constraints. It is of course preferable to have all the $n+1$-values of the $\Lambda, \Sigma, \ldots$, indices on the same footing: indeed this shows the covariance of the formalism under $Sp(2n+2)$ transformations acting on the basic $\left(L^\Lambda, F_\Lambda \equiv \frac{\partial}{\partial L^\Lambda} F\right)$ periods of the unique (3,0)-form of the Calabi-Yau 3-fold on which the underlying theory is compactified (see refs. [20,21]). The same kind of considerations also hold for eqs. (3.19), (3.20) which were written in ref. [4e] in a $Sp(2n+2)$ non-covariant way.)

Let us next consider the modifications to the previous results when the theory is gauged, that is when the group G is non-Abelian. In that case we must redefine the generalized curvatures (3.1), (3.2), (3.6), (3.7) according to the rules discussed in sect. 2c. In particular for the scalar fields we replace $dz^i, d\bar{z}^{i^\star}$ and dq^u by

$$\nabla z^i = dz^i + g A^\Lambda k_\Lambda^i(z) \tag{3.30a}$$

$$\nabla \bar{z}^{i^\star} = d\bar{z}^{i^\star} + g A^\Lambda k_\Lambda^{i^\star}(\bar{z}) \tag{3.30b}$$

$$\nabla q^u = dq^u + g A^\Lambda k_\Lambda^u(q) \tag{3.30c}$$

Eq. (3.30c) implies that the gauged quaternionic vielbein $\hat{\mathcal{U}}^{A\alpha}$ is given by

$$\hat{\mathcal{U}}^{A\alpha} = \mathcal{U}^{A\alpha}{}_{uv} \nabla q^u \wedge \nabla q^v \tag{3.31}$$

In eqs. (2.66), (3.1), (3.6) and (3.7) we replace $\Gamma^i{}_j$, $\omega_A{}^B \equiv \frac{i}{2}(\sigma_x)_A{}^B \omega^x$, $\Delta^{\alpha\beta}$ and Q, by the corresponding gauged connections $\hat{\Gamma}^i{}_j$, $\hat{\omega}_A{}^B \equiv \frac{i}{2}(\sigma_x)_A{}^B$, $\hat{\omega}^x$, $\hat{\Delta}^{\alpha\beta}$ and $\hat{Q}$ defined in eqs. (2.102), (2.104) and (2.106).

Furthermore, we have to replace, in eq. (3.7c), dA^Λ by the complete gauge curvature (2.101) so that (3.7c) is replaced by

$$F^\Lambda = \mathcal{F}^\Lambda + \bar{L}^\Lambda \bar{\psi}_A \wedge \psi_B \epsilon^{AB} + L^\Lambda \bar{\psi}^A \wedge \psi^B \epsilon_{AB} \tag{3.32}$$

In an analogous way the gauged Bianchi identities are obtained from (3.3) and (3.8) by replacing the curvature of the composite connections with their gauged expressions. Specifically, in eqs. (3.3) and (3.8 c,e) we substitute

$$R^i{}_j \rightarrow \hat{R}^i{}_j \tag{3.33a}$$

$$\mathbb{R}^{\alpha\beta} \rightarrow \hat{\mathbb{R}}^{\alpha\beta} \tag{3.33b}$$

$$K \rightarrow \hat{K} \tag{3.33c}$$

$$R_a{}^B \rightarrow \hat{R}_A{}^B \equiv \frac{i}{2}(\sigma_x)_A{}^B \hat{\Omega}^x \tag{3.33d}$$

where $\hat{R}^i{}_j, \hat{K}, \hat{\mathbb{R}}^{\alpha\beta}$ and $\hat{\Omega}^x$ are given in eqs. (2.105), (2.107) and (2.108).

Finally the equations (3.8f) and (2.66), taking into account (3.30-31), become:

$$\nabla^2 z^i = g \mathcal{F}^\Lambda k^i{}_\Lambda \tag{3.34a}$$

$$\nabla^2 \bar{z}^{i^*} = g \mathcal{F}^\Lambda k^{i^*}_\Lambda \tag{3.34b}$$

$$\nabla \hat{\mathcal{U}}^{A\alpha} = g \mathcal{F}^\Lambda k^u_\Lambda \hat{\mathcal{U}}^{A\alpha}_u \tag{3.34c}$$

Let us rewrite here the complete set of Bianchi identities in the gauged case:

$$\mathcal{D} R^{ab} = 0 \tag{3.35a}$$

$$\mathcal{D} R^a + R^{ab} \wedge V_b - i \bar{\psi}^A \wedge \gamma_a \hat{\rho}_A + i \hat{\bar{\rho}}^A \wedge \gamma_a \psi_A = 0 \tag{3.35b}$$

$$\hat{\nabla} \hat{\rho}_A + \frac{1}{4} R^{ab} \wedge \psi_A - \frac{i}{2} \hat{R}_A{}^B \wedge \psi_B = 0 \tag{3.35c}$$

$$\begin{aligned} &\nabla F^\Lambda - \epsilon_{AB} \nabla L^\Lambda \wedge \bar{\psi}^A \wedge \psi^B + 2\epsilon_{AB} L^\Lambda \bar{\psi}^A \wedge \hat{\rho}^B - \epsilon^{AB} \nabla \bar{L}^\Lambda \wedge \psi_B \\ &+ 2\epsilon^{AB} \bar{L}^\Lambda \bar{\psi}_A \wedge \hat{\rho}_B = 0 \end{aligned} \tag{3.35d}$$

$$\nabla^2 \lambda^{iA} + \frac{1}{4} R^{ab} \gamma_{ab} \lambda^{iA} + \frac{i}{2} \hat{K} \lambda^{iA} + \hat{R}^i{}_j(\hat{\Gamma}) \lambda^{jA} - \frac{i}{2} \hat{R}^A{}_B(\hat{\omega}) \lambda^{iB} = 0 \tag{3.35e}$$

$$\nabla^2 z^i = g(F^\Lambda - \bar{L}^\Lambda \bar{\psi}_A \wedge \psi_B \epsilon^{AB} - L^\Lambda \bar{\psi}^A \wedge \psi^B \epsilon_{AB}) k^i_\Lambda(z) \tag{3.35f}$$

$$\hat{\nabla} \mathcal{U}^{A\alpha} = g(F^\Lambda - \bar{L}^\Lambda \bar{\psi}_A \wedge \psi_B \epsilon^{AB} - L^\Lambda \bar{\psi}^A \wedge \psi^B \epsilon_{AB}) k^u_\Lambda(q) \mathcal{U}^{A\alpha}_u \tag{3.35g}$$

$$\hat{\nabla}^2 \zeta_\alpha + \frac{1}{4} \gamma_{ab} R^{ab} \zeta_\alpha + \hat{\mathbb{R}}_\alpha{}^\beta(\hat{\Delta}) \zeta_\beta - \frac{i}{2} \hat{K} \zeta_\alpha = 0 \tag{3.35h}$$

where we have used eq. (3.32) in eqs. (3.35 f,g). Similar expressions hold for the chirality reversed spinorial curvature $\hat{\rho}^A \equiv \hat{\nabla} \psi^A$; $\hat{\nabla} \lambda^{i^*}$; $\hat{\nabla} \zeta^\alpha$ and for $\nabla \bar{z}^{i^*}$. The hat on the covariant derivatives and on $\hat{\rho}_A = \hat{\nabla} \psi_A$ means that with respect to the previous definitions (2.66), (3.1) and (3.7), the hatted covariant derivatives contain the gauged connections.

Furthermore $\hat{\nabla}$ is also covariant with respect to gauge transformations when acting on gauge-indexed quantities. For example

$$\hat{\nabla} F^\Lambda \equiv dF^\Lambda + f^\Lambda{}_{\Sigma\Gamma} A^\Sigma \wedge F^\Gamma \tag{3.36a}$$

$$\hat{\nabla} L^\Lambda \equiv (dL^\Lambda + i\hat{Q}L^\Lambda) + f^\Lambda_{\Sigma\Gamma} A^\Sigma L^\Gamma \tag{3.36b}$$

The solutions of the gauged Bianchi identities (3.35) will modify the ungauged parametrizations in two ways: first of all the old parametrizations on the r.h.s. of eqs. (3.9-21) have to be replaced by analogous ones where we substitute $\nabla \to \hat{\nabla}$: that is, e.g.:

$$\rho_{A|ab} \equiv \nabla_{[a}\psi_{b]A} \to \hat{\nabla}_{[a}\psi_{b]A} \equiv \hat{\rho}_{ab|A} \tag{3.37a}$$

$$Z^i_a \equiv \partial_a z^i \to \hat{\nabla}_a z^i \equiv \hat{Z}^i_a \tag{3.37b}$$

$$\nabla_a \lambda^{iA} \to \hat{\nabla}_a \lambda^i{}_A \tag{3.37c}$$

$$\nabla_j f^\Lambda_i \to \hat{\nabla}_j f^\Lambda_i \tag{3.37d}$$

Secondly, the new parametrizations will contain extra terms with respect to the old ones which are proportional to the gauge coupling constant g and that we name "shifts". As is well known, in the study of extended supergravity theories these shifts occur in the parametrizations of the fermionic curvatures $\hat{\rho}_A, \hat{\rho}^A$; $\hat{\nabla}\lambda^{iA}$, $\hat{\nabla}\lambda^{i^*}_A$; $\hat{\nabla}\zeta_\alpha$, $\hat{\nabla}\zeta^\alpha$. Taking into account the previous observations we write the parametrization of the gauged curvatures as follows:

$$R^a = 0 \tag{3.38a}$$

$$\hat{\rho}_A = \hat{\rho}_A^{(old)} + igS_{AB}\gamma_a\psi^B \wedge V_a \tag{3.38b}$$

$$\hat{\rho}^A = \hat{\rho}^{A(old)} + ig\bar{S}^{AB}\gamma_a\psi_B \wedge V^a \tag{3.38c}$$

$$\hat{R}^{ab} = \hat{R}^{ab(old)} - \bar{\psi}_A \wedge \gamma^{ab}\psi_B g\bar{S}^{AB} - \bar{\psi}^A \wedge \gamma^{ab}\psi^B gS_{AB} \tag{3.38d}$$

$$F^\Lambda = F^{\Lambda(old)} \tag{3.38e}$$

$$\hat{\nabla}\lambda^{iA} = \hat{\nabla}\lambda^{iA(old)} + gW^{iAB}\psi_B \tag{3.38f}$$

$$\hat{\nabla}\lambda^{i^*}_A = \hat{\nabla}\lambda^{i^*(old)}_A + gW^{i^*}_{AB}\psi^B \tag{3.38g}$$

$$\hat{\nabla}z^i = \hat{\nabla}z^{i(old)} \tag{3.38h}$$

$$\hat{\nabla}\bar{z}^{i^*} = \hat{\nabla}\bar{z}^{i^*(old)} \tag{3.38i}$$

$$\hat{\mathcal{U}}^{\alpha A} = \hat{\mathcal{U}}^{\alpha A(old)} \tag{3.38j}$$

$$\hat{\nabla}\zeta_\alpha = \hat{\nabla}\zeta_\alpha^{(old)} + g\mathcal{N}^A_\alpha\psi_A \tag{3.38k}$$

$$\hat{\nabla}\zeta^\alpha = \hat{\nabla}\zeta^{\alpha(old)} + g\mathcal{N}^\alpha{}_A\psi^A \tag{3.38l}$$

where the suffix "old" refers to the parametrizations (3.9)-(3.15) in which the previously indicated replacements ($\nabla \to \hat{\nabla}$) etc. have been performed. The new superfields S_{AB}, $\bar{S}^{AB}$; W^{iAB}, $W^{i^*}_{AB}$; $\mathcal{N}^A_\alpha$, $\mathcal{N}^\alpha_A$ define the fermionic shifts. (Notice that the shifts S_{AB}, $\bar{S}^{AB}$ were already introduced in eqs. (3.10), (3.11), but their value in the absence of gauging was zero (see eq. (3.17)).

The computation of the shifts presents no difficulty; $S_{AB}(\bar{S}^{AB})$ can be computed from the (0,3)-sector of the gauged Bianchi identities (3.35c); one finds

$$S_{AB} = \frac{i}{2}(\sigma_x)_A{}^C \epsilon_{BC}\mathcal{P}^x_\Lambda L^\Lambda \tag{3.39a}$$

$$\bar{S}^{AB} = \frac{i}{2}(\sigma_x)_C{}^B \epsilon^{CA}\mathcal{P}^x_\Lambda \bar{L}^\Lambda \tag{3.39b}$$

The non-zero value of S_{AB},$(\bar{S}^{AB})$ is due essentially to the presence of a gauged curvature $\hat{R}_A{}^B$ $(\hat{R}^A{}_B)$ which contains the extra term $g\mathcal{F}^\Lambda\mathcal{P}^x_\Lambda$ (see eq. (2.110b)).

Indeed, taking into account eq. (3.32), the r.h.s. of the Bianchi identity (3.35c) contains the following extra contribution proportional to g:

$$\frac{i}{2} g \mathcal{F}^\Lambda \mathcal{P}^x_\Lambda (\sigma_x)_A \psi_c \equiv g(F^\Lambda - L^\Lambda \bar{\psi}_A \wedge \psi_B \epsilon^{AB} - \bar{L}^\Lambda \bar{\psi}_B \epsilon_{AB}) \mathcal{P}^x_\Lambda (\sigma_x)_A{}^C \psi_C \quad (3.40)$$

Since $F^\Lambda_{(0,2)} = 0$, (*) (see eq. (3.12)), these extra terms give rise in the (0,3)-sector of eq. (3.35c) to contributions proportional to $\mathcal{P}^x_\Lambda L^\Lambda$ which are compensated by the shifts (3.39a,b).

We note that an analogous term, namely $g\mathcal{F}^\Lambda P^0_\Lambda$, is also generated in the same sector by the gauged curvature $\hat{K}$, eq. (2.110a). In this case, however, according to eq. (3.39b) we have:

$$P^0_\Lambda L^\Lambda = P^0_\Lambda \bar{L}^\Lambda = 0 \quad (3.41)$$

so that no modification to the gravitino curvature is needed. This is welcome since it is easy to see that a non-zero value for $P^0_\Lambda L^\Lambda$ (and its complex conjugate) would be inconsistent with the Bianchi identities.

The computation of the shifts $\mathcal{N}^A_\alpha$, $\mathcal{N}^\alpha_A$ is also very easy: it is sufficient to look at the (0,2)-sector of the Bianchi's (3.35g); one finds immediately

$$\mathcal{N}^A_\alpha = 2\mathcal{U}^A_{\alpha|u} k^u_\Lambda \bar{L}^\Lambda \quad (3.42a)$$

$$\mathcal{N}^\alpha_A = -2\mathcal{U}^\alpha_{A|u} k^u_\Lambda L^\Lambda \quad (3.42b)$$

Finally, the determination of the gaugino shifts W^{iAB} and $W^{i^*}_{AB}$ requires a little more labour. First we decompose the shifts into the symmetric and antisymmetric parts in the A, B indices:

$$W^{iAB} = W^{i[AB]} + W^{i(AB)} \quad (3.43a)$$

(*) Given a $(p+q)$-form by Ω, we denote $\Omega_{(p,q)}$, ((p,q)-sector of Ω) that part of Ω which, when expanded in terms of the supervielbein basis $\{V^a, \psi_A(\psi^A)\}$, contains p V^a's and q ψ_A's (or (ψ^A's)).

$$W^{i^*}_{AB} = W^{i^*}_{[AB]} + W^{i^*}_{(AB)} \quad (3.43b)$$

The antisymmetric parts are immediately determined from the (0,2)- sector of the Bianchi (3.35e). One obtains:

$$W^{i[AB]} = \epsilon^{AB} k^i{}_\Lambda \bar{L}^\Lambda \quad (3.44a)$$

$$W^{i^*}_{[AB]} = \epsilon_{AB} k^{i^*}_\Lambda L^\Lambda \quad (3.44b)$$

Notice that this contribution due to the gauging depends only on the vector multiplets. The symmetric parts are determined by solving the Bianchi's (3.35d) in the (1,2)-sector. Taking into account only the terms proportional to g one obtains, as a coefficient of $\bar{\psi}^A_\Lambda \gamma_a \psi_B \wedge V^a$, the following equation:

$$\begin{aligned} &2\epsilon^{AB} S_{BC} \bar{L}^\Lambda + 2\epsilon_{CB} \bar{S}^{BA} L^\Lambda = \\ &\epsilon^{AB} f^\Lambda_{i^*} W^{i^*}_{(BC)} + \epsilon_{CB} f^\Lambda_i W^{i(BA)} \end{aligned} \quad (3.45)$$

Setting

$$W^{iAB} = -i(\sigma_x)_C{}^B \epsilon^{CA} \mathcal{P}^x_\Sigma t^{i\Sigma} \quad (3.46a)$$

$$W^{i^*}_{AB} = -i(\sigma_x)_A{}^C \epsilon_{BC} \mathcal{P}^x_\Sigma t^{i^*\Sigma} \quad (3.46b)$$

and using eqs. (3.39), eq. (3.45) becomes:

$$\bar{L}^\Lambda L^\Sigma - L^\Lambda \bar{L}^\Sigma = f^\Lambda{}_i t^{i\Sigma} - f^\Lambda_{i^*} t^{i^*\Sigma} \quad (3.47)$$

Using eq. (2.21f) one finds the solution:

$$t^{i\Sigma} = g^{ij^*} f^\Pi_{j^*} K_\Pi{}^\Sigma \quad (3.48a)$$

$$t^{I^*\Sigma} = g^{i^*j} f^\Pi_j \bar{K}_\Pi{}^\Sigma \quad (3.48)$$

where

$$K_\Pi{}^\Sigma = \gamma\delta_\Pi{}^\Sigma + (1-\gamma)\frac{1}{1-16S\bar{S}}N_{\Pi\Delta}\bar{L}^\Delta L^\Sigma + (1-\gamma)\frac{4S}{1-16S\bar{S}}N_{\Pi\Delta}\bar{L}^\Pi\bar{L}^\Sigma \tag{3.49}$$

and γ is an arbitrary real parameter.

However, from the analysis of the Bianchi identities (3.35e) in the (0,2)-sector one concludes that $\gamma = 1$. Therefore the symmetric shifts $W^{i(AB)}$ and $W^{i^*}_{(AB)}$ are given by

$$W^{i(AB)} = -i(\sigma_x)_C{}^B\epsilon^{CA}\mathcal{P}^x_\Sigma g^{ij^*} f^\Sigma_{j^*} \tag{3.50a}$$

$$W^{i^*}_{(AB)} = -i(\sigma_x)_A{}^C\epsilon_{BC}\mathcal{P}^x_\Sigma g^{i^*j} f^\Sigma_j \tag{3.50b}$$

Equations (3.39), (3.42), (3.43), (3.44) and (3.50) inserted in eqs. (3.38) fix the complete parametrization of the superspace curvatures in the gauged case, and hence, according to the well -known rules, the transformation laws of the physical fields.

As we mentioned at the beginning of this section, the fermionic shifts are all we need in order to reconstruct the full scalar potential for the N=2 complete Lagrangian. This is done in the next section.

4. The Lagrangian and the scalar potential

In this section, relying on the solution of Bianchi identities encoded in eqs. (3.4-5), (3.9-15), (3.38-39), (3.43-44) and (3.50), we derive the N=2 bosonic Lagrangian and we obtain the N=2 analogue of eq. (1.1), namely the general formula for the scalar potential. As a by-product of our derivation we get also the complete N=2 Lagrangian up to 4-Fermi terms.

Let us begin by stating our result. The bosonic sector of N=2 supergravity is described by the following Lagrangian:

$$\begin{aligned}\mathcal{L}^{N=2}_{bosonic} = \sqrt{-g}\Big\{&\mathcal{R} + g_{ij^*}(z,\bar{z})\nabla_\mu z^i\nabla_\nu \bar{z}^{j^*}g^{\mu\nu} \\ &+ h_{uv}(q)\nabla_\mu q^u\nabla_\nu q^v g^{\mu\nu} - 4Re\mathcal{N}_{\Lambda\Sigma}(z,\bar{z})F^\Lambda_{\mu\nu}F^{\Sigma\mu\nu} \\ &- \mathcal{V}(z,\bar{z},q)\Big\} - 2i\, Im\mathcal{N}_{\Lambda\Sigma}(z,\bar{z})F^\Lambda_{\mu\nu}F^\Sigma_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma}\end{aligned} \tag{4.1}$$

where

$$\nabla_\mu z^i = \partial_\mu z^i + gA^\Lambda_\mu k^i_\Lambda(z) \tag{4.2a}$$

$$\nabla_\mu \bar{z}^{j^*} = \partial_\mu \bar{z}^{j^*} + gA^\Lambda_\mu k^{j^*}_\Lambda(\bar{z}) \tag{4.2b}$$

$$\nabla_\mu q^u = \partial_\mu q^u + gA^\Lambda_\mu k^u_\Lambda(q) \tag{4.2c}$$

are the covariant differentials of the special and quaternionic scalars, respectively;

$$\mathcal{N}_{\Lambda\Sigma} = -\bar{F}_{\Lambda\Sigma} + \frac{1}{N_{\Gamma\Delta}L^\Gamma L^\Delta}N_{\Lambda\Pi}L^\Pi N_{\Sigma\Delta}L^\Delta \tag{4.3}$$

is the kinetic metric of the vectors constructed in terms of the L-sections of the $Sp(2n+2)$-bundle (see eqs. (2.20) (2.21)) and where the scalar potential has the following expression

$$\begin{aligned}\mathcal{V}(z,\bar{z},q) = g^2\Big[&\Big(g_{ij^*}k^i_\Lambda k^{j^*}_\Sigma + h_{uv}k^u_\Lambda k^v_\Sigma\Big)\bar{L}^\Lambda L^\Sigma \\ &- \Big((N^{-1})^{\Lambda\Sigma} - \bar{L}^\Lambda L^\Sigma\Big)\mathcal{P}^x_\Lambda\mathcal{P}^x_\Sigma - 3\bar{L}^\Lambda L^\Sigma\mathcal{P}^x_\Lambda\mathcal{P}^x_\Sigma\Big]\end{aligned} \tag{4.4}$$

in terms of the Killing vectors and of their prepotentials (see eqs. (2.27) and (2.81)). The last term in eq. (4.4) represents the contribution of the gravitino shift, while the other terms give the contributions of the vector-multiplet and hypermultiplet shifts. Before deriving these results we note that from (4.4) we can show the $N=2$ analogue of the Fayet-Iliopoulos [22] mechanism. To do this let us delete the hypermultiplets ($\mathcal{Q}(m) = \mathcal{Q}(0) = 0$) and let us restrict our attention to a theory containing only vector multiplets. This implies $k^u_\Lambda = 0$ and $\Omega^x_{uv} = 0$, yet the quaternionic prepotentials $\mathcal{P}^x_\Lambda$ can be chosen constant rather than zero provided eq. (2.97) is satisfied. We achieve this by setting $\mathcal{P}^1_\Lambda = \mathcal{P}^2_\Lambda = 0$ $\mathcal{P}^3_\Lambda = g_\Lambda$ where g_Λ is a constant fulfilling the condition $f^\Gamma_{\Lambda\Sigma}g_\Gamma = 0$.

In this way one retrieves the result of ref. [23]. Note however that $f^\Gamma_{\Lambda\Sigma}g_\Gamma = 0$ means that g_Γ are the coordinates of a Lie algebra element commuting with all the generators, in other words a non-trivial element of the centre $\mathbf{Z}(\mathbf{G})$. We conclude that a cosmological-like term can be introduced only if the Lie algebra $\mathbf{G}$ is not semisimple and has a non-vanishing centre.

Let us now derive (4.1) and (4.4). The potential $\mathcal{V}(z,\bar{z},q)$ is determined by the following supergravity Ward identity:

$$K^A_B = \frac{2}{3}\delta^A_B \mathcal{V} \tag{4.5}$$

where the 2×2 matrix K^A_B is a quadratic form in the shifts of the fermion fields (3.39), (3.42), (3.44) and (3.50):

$$K^A_B \equiv 2\delta_1 \bar{S}^{MA} S_{MB} + \delta_2 g_{ij^\star} W^{iMA} W^{j^\star}_{MB} + \delta_3 N^A_\alpha N^\alpha_B \tag{4.6}$$

In eq. (4.6) δ_i denote the coefficients of the bilinear fermionic terms in the Lagrangian with the following structure:

$$\delta_i \; Gravitino \cdot (Fermi)_i \cdot (Shift)_i$$

Eqs. (4.5)-(4.6) are the particular N=2 case of a general Ward identity appearing in all N-extended supergravities. For a general discussion the reader is referred to the book [8] or to the original papers [6a,6b,6c].

If we substitute eqs. (3.39), (3.42), (3.44) and (3.50) into (4.6), after some algebra we find

$$K^A_B = \frac{1}{2}\delta^A_B K + i(\epsilon\sigma_x\epsilon)^A_B \Delta^x \tag{4.7}$$

$$\begin{aligned}\frac{1}{g^2}K = &-\delta_1 \mathcal{P}^x_\Lambda \mathcal{P}^x_\Sigma \bar{L}^\Lambda L^\Sigma + 2\,\delta_2 g^{i^\star j} f^\Lambda_i f^\Sigma_{j^\star} \mathcal{P}^x_\Lambda \mathcal{P}^x_\Sigma \\ &+ 2\,\delta_2 g_{ij^\star} k^i_\Lambda k^{j^\star}_\Sigma \bar{L}^\Lambda L^\Sigma + 4\,\delta_3 h_{uv} k^u_\Lambda k^v_\Sigma \bar{L}^\Lambda L^\Sigma\end{aligned} \tag{4.8a}$$

$$\begin{aligned}\frac{1}{g^2}\Delta^x = \bar{L}^\Lambda L^\Sigma \Big\{ &\frac{2\delta_3}{\lambda}\Omega^x_{uv} k^u_\Lambda k^v_\Sigma + \Big(\frac{1}{2}\delta_1 - \delta_2\Big)\epsilon^{xyz}\mathcal{P}^z_\Lambda \mathcal{P}^y_\Sigma \\ &- 2\delta_2 f^\Gamma_{\Lambda\Sigma} \mathcal{P}^x_\Gamma \Big\}\end{aligned} \tag{4.8b}$$

Comparing eq. (4.8) with eq. (2.98) we see that if the coefficients $\delta_1, \delta_2, \delta_3$ are in the following ratios

$$\frac{2}{\lambda}\delta_3 = a\,; \quad \Big(\frac{1}{2}\delta_1 - \delta_2\Big) = -1/2a\,; \quad 2\delta_2 = \frac{1}{2}a \tag{4.9}$$

a being some constant, then $\Delta^x = 0$ and the SUSY Ward identity (4.5) is fulfilled. Obviously, since $\delta_1, \delta_2, \delta_3$ are determined by the cancellation of other terms in the

supersymmetry variation of the Lagrangian they are bound to come out in the correct ratio (4.9), and the vanishing of Δ^x is just a check on the calculations. The important point, however, is that this vanishing is based on eq. (2.97) whose cohomological nature has already been pointed out.

Let us now briefly sketch the derivation of the Lagrangian. We just follow the standard construction of the rheonomy approach (see the book [8] or ref.[6]) and we organize the most general ansatz for the action as follows:

$$\mathcal{A}^{(N=2)} = \int_{M_4} \mathcal{L} \tag{4.10}$$

where

$$\mathcal{L} = \mathcal{L}^0 + \Delta\mathcal{L}_{gauging} \tag{4.11}$$

is the sum of the ungauged Lagrangian ($g=0$) plus the shift due to the gauging. Explicitly one has:

$$\mathcal{L}^0 = \mathcal{L}_{(Kin)} + \mathcal{L}_{(Pauli)} + \mathcal{L}_{(Torsion)} + \mathcal{L}_{(4-Fermi)} + \mathcal{L}^{(4,0)}_{(4-Fermi)} \tag{4.12a}$$

$$\Delta\mathcal{L}_{gauging} = \Delta\mathcal{L}^{(4,0)}_{(Potential)} + \Delta\mathcal{L}_{(2-Fermi)} + \Delta\mathcal{L}^{(4,0)}_{(2-Fermi)} \tag{4.12b}$$

where(*)

(*) The curvatures and connections appearing in eqs. (4.13) refer to the gauged theory; we have suppressed for simplicity the hats introduced in the previous section.

$$\mathcal{L}_{(Kin)} = \epsilon_{abcd} R^{ab} \wedge V^c \wedge V^d - 4(\bar{\psi}^A \wedge \gamma_a \rho_A - \bar{\psi}_A \wedge \gamma_a \rho^A) \wedge V^a$$
$$+ \alpha_1 \epsilon_{AB} \mathbb{C}_{\alpha\beta} \mathcal{U}_a^{A\alpha} (\mathcal{U}^{B\beta} - \bar{\psi}\zeta^\beta - \epsilon^{BC} \mathbb{C}^{\beta\gamma} \bar{\psi}_C \zeta_\gamma) \wedge V_b \wedge V_c \wedge V_d \epsilon^{abcd}$$
$$\beta_1 g_{ij^*} \left[Z_a^i (\nabla \bar{z}^{j^*} - \bar{\psi}\lambda_A^{j^*}) + Z_a^{j^*} (\nabla z^i - \bar{\psi}_A \lambda^{iA}) \right] \wedge V^b \wedge V^c \wedge V^d \epsilon_{abcd}$$
$$+ \left[i\alpha_2 (\bar{\zeta}^\alpha \gamma_a \nabla \zeta_\alpha + \bar{\zeta}_\alpha \gamma_a \nabla \zeta^\alpha) + i\beta_2 g_{ij^*} (\bar{\lambda}^{iA} \gamma_a \nabla \lambda_A^{j^*} + \bar{\lambda}_A^{j^*} \gamma_a \nabla \lambda^{iA}) \right] \wedge V_b \wedge V_c \wedge V_d \epsilon^{abcd}$$
$$+ i\beta_3 (\mathcal{N}_{\Lambda\Sigma} F_{ab}^{+\Lambda} - \bar{\mathcal{N}}_{\Lambda\Sigma} F_{ab}^{-}) \wedge \left[F^\Sigma - (i f_i^\Sigma \epsilon_{AB} \bar{\lambda}^{iA} \gamma_m \psi^B + \right.$$
$$\left. i f_{i^*}^\Sigma \epsilon^{AB} \bar{\lambda}_A^{i^*} \gamma_m \psi_B) \wedge V^m \right] \wedge V^a \wedge V^b$$
$$- \left[\frac{1}{8} \alpha_1 \mathcal{U}_a^{A\alpha} \mathcal{U}_a^{B\beta} \epsilon_{AB} \mathbb{C}_{\alpha\beta} + \frac{1}{4} \beta_1 g_{ij^*} Z_a^i \bar{Z}_a^{j^*} \right.$$
$$\left. + \frac{1}{24} \beta_3 (\mathcal{N}_{\Lambda\Sigma} F_{ab}^{+\Lambda} F_{ab}^{+\Sigma} - \bar{\mathcal{N}}_{\Lambda\Sigma} F_{ab}^{-\Lambda} F_{ab}^{-\Sigma}) \right] \epsilon_{c_1 \dots c_4} V^{c_1} \wedge \dots \wedge V^{c_4}. \qquad (4.13a)$$

$$\mathcal{L}_{(Pauli)} = \beta_5 F^\Lambda \wedge (\mathcal{N}_{\Lambda\Sigma} L^\Sigma \epsilon_{AB} \bar{\psi}^A \wedge \psi^B + \bar{\mathcal{N}}_{\Lambda\Sigma} \bar{L}^\Sigma \epsilon^{AB} \bar{\psi}_A \wedge \psi_B)$$
$$+ i\beta_6 F^\Lambda \wedge (\mathcal{N}_{\Lambda\Sigma} f_i^\Sigma \bar{\lambda}^{iA} \gamma_a \psi^B \epsilon_{AB} + \bar{\mathcal{N}}_{\Lambda\Sigma} f_{i^*}^\Sigma \bar{\lambda}_A^{i^*} \gamma_a \psi_B \epsilon^{AB}) \wedge V^a$$
$$+ \alpha_5 F^\Lambda \wedge (\mathcal{N}_{\Lambda\Sigma} L^\Sigma \bar{\zeta}^\alpha \gamma_{ab} \zeta^\beta \mathbb{C}_{\alpha\beta} + \bar{\mathcal{N}}_{\Lambda\Sigma} \bar{L}^\Sigma \bar{\zeta}_\alpha \gamma_{ab} \zeta_\beta \mathbb{C}^{\alpha\beta}) \wedge V^a \wedge V^b$$
$$+ \beta_7 F^\Lambda \wedge (\mathcal{N}_{\Lambda\Sigma} \nabla_i f_j \bar{\lambda}^{iA} \gamma_{ab} \lambda^{jB} \epsilon_{AB} + \bar{\mathcal{N}}_{\Lambda\Sigma} \nabla_{i^*} f_{j^*}^\Sigma \bar{\lambda}_A^{i^*} \gamma_{ab} \lambda_B^{j^*} \epsilon^{AB}) \wedge V^a \wedge V^b$$
$$+ \alpha_3 (\bar{\zeta}^\alpha \gamma_{ab} \psi^A \wedge \mathcal{U}_{\alpha A} + \bar{\zeta}_\alpha \gamma_{ab} \psi_A \wedge \mathcal{U}^{\alpha A}) \wedge V_c \wedge V_d \epsilon^{abcd}$$
$$+ \beta_8 g_{ij^*} (\bar{\lambda}^{iA} \gamma_{ab} \psi_A \wedge \nabla \bar{z}^{j^*} + \bar{\lambda}_A^{j^*} \gamma_{ab} \psi^A \wedge \nabla z^i) \wedge V_c \wedge V_d \epsilon^{abcd} \qquad (4.13b)$$

$$\mathcal{L}_{(Torsion)} = R_a \wedge V^a \wedge (\alpha_4 \bar{\zeta}^\alpha \gamma_b \zeta_\alpha + \beta_9 \bar{\lambda}^{iA} \gamma_b \lambda_A^{j^*} g_{ij^*}) V^b \qquad (4.13c)$$

$$\Delta\mathcal{L}_{(2-Fermi)} = -i\delta_1 (S_{AB} \bar{\psi}^A \wedge \gamma_{ab} \psi^B - \bar{S}^{AB} \bar{\psi}_A \wedge \gamma_{ab} \psi_B) \wedge V^a \wedge V^b$$
$$+ i\delta_2 g_{ij^*} (W^{iAB} \bar{\lambda}_A^{j^*} \gamma_a \psi_B + W_{AB}^{j^*} \bar{\lambda}^{iA} \gamma_a \psi^B) V_b \wedge V_c \wedge V_d \epsilon^{abcd} \qquad (4.13d)$$
$$+ i\delta_3 (N_\alpha^A \bar{\zeta}^\alpha \gamma_a \psi_A + N_A^\alpha \bar{\zeta}_\alpha \gamma_a \psi^A) \wedge V_b \wedge V_c \wedge V_d \epsilon^{abcd}$$

$$\Delta\mathcal{L}_{(Potential)} = -g^2 \frac{\mathcal{V}}{6} \epsilon_{abcd} V^a \wedge V^b \wedge V^c \wedge V^d \qquad (4.13e)$$

$\mathcal{L}^{(4,0)}_{(4-Fermi)}$, $\mathcal{L}^{(2,2)}_{(4-Fermi)}$ and $\Delta\mathcal{L}^{(4,0)}_{(2-Fermi)}$ are not given since they are not needed. (The superscripts $(4,0)$ and $(2,2)$ refer to the superspace (p,q) sectors as explained

in the footnote of the previous section). The coefficients α_i, β_i, δ_i are determined by considering the variational equations on superspace and substituting into them the rheonomic parametrizations (3.4-5) (3.38-39) (3.42-44) and (3.50).

Consistency of those parametrizations with the variational equations implies

$$\alpha_1 = -\frac{4}{3}\lambda\,; \quad \alpha_2 = \frac{2}{3}\lambda\,; \quad \alpha_3 = 2\lambda\,; \quad \alpha_4 = -2\lambda \qquad (4.14a)$$

$$\beta_1 = \frac{2}{3}\,; \quad \beta_2 = -\frac{1}{3}\,; \quad \beta_3 = -4\,; \quad \beta_5 = \dots$$
$$\beta_6 = 4i\,; \quad \beta_7 = \dots\,; \quad \beta_8 = -1\,; \quad \beta_9 = 1 \qquad (4.14b)$$

$$\delta_1 = 4\,; \quad \delta_2 = \frac{2}{3}\,; \quad \delta_3 = -\frac{4}{3}\lambda \qquad (4.14c)$$

plus the identification (4.3). The empty dots in eq. (4.13) correspond to coefficients that were not calculated since they were not needed to determine $\delta_1, \delta_2, \delta_3$. Substituting the values (4.13) into the consistency condition (4.9) we see that it is satisfied with $a = 8/3$.

The Ward identity (4.5)-(4.6) is derived by considering the SUSY variation of the Lagrangian and in this variation isolating the contributions of the following form

$$\delta\Delta\mathcal{L}_{(2-Fermi)} = -2\delta_1\, g\bar{S}^{AB} \bar{\psi}_A \wedge \gamma_{ab} \delta\psi_B \wedge V^a \wedge V^b$$
$$- i\delta_2\, g_{ij^*} g W^{iAB} \bar{\psi}_B \wedge \gamma_a \delta\lambda_A^{j^*} V_b \wedge V_c \wedge V_d \epsilon^{abcd} \qquad (4.15a)$$
$$- i\,\delta_3\, g N_\alpha^A \bar{\psi}_A \wedge \gamma_a \delta\zeta^\alpha V_b \wedge V_c \wedge V_d \epsilon^{abcd}$$

$$\delta\Delta\mathcal{L}_{(Potential)} = -\frac{2}{3} g^2 \mathcal{V} \delta V^a \wedge V^b \wedge V^c \wedge V^d \epsilon_{abcd} \qquad (4.15b)$$

where

$$\delta\psi_A = \dots + igS_{AB}\gamma_a \epsilon^B V^a \qquad (4.16a)$$

$$\delta\lambda_A^{j^*} = \dots + gW_{AB}^{j^*} \epsilon^B \qquad (4.16b)$$

$$\delta\zeta^\alpha = \dots + gN_A^\alpha \epsilon^A \qquad (4.16c)$$

$$\delta V^a = ... - i\bar{\psi}_A \gamma^a \epsilon^A \tag{4.16d}$$

In eq.s (4.15) we have just written the terms which give contributions proportional to g^2 in the supersymmetry variation of the Lagrangian.
Cancellation of $\delta\Delta\mathcal{L}_{(2-Fermi)}$ against $\delta\Delta\mathcal{L}_{(Potential)}$ is achieved if eq. (4.5) holds.

As already stressed, this is a fully general mechanism applying to all N-extended supergravities.

At this point we isolate the bosonic sector of the action (4.10), we go over to second order formalism by replacing the auxiliary fields $z^i_a, \mathcal{U}^{A\alpha}_a$ and $F^{\pm\Lambda}_{ab}$ with their on-shell values and we substitute

$$V^a \wedge V^b \wedge V^c \wedge V^d \to -\frac{1}{4}(det\ V) d^4x \epsilon^{abcd} \tag{4.17}$$

The resulting bosonic Lagrangian is

$$\begin{aligned}\mathcal{L}^{N=2} &= \sqrt{-g}\Big\{R + \frac{3}{2}\beta_1 g_{ij^\star} \nabla_\mu z^i \nabla_\nu \bar{z}^{j^\star} g^{\mu\nu} - \frac{3}{4}K \\ &+ \frac{3}{4}\alpha_1 h_{uv}(q) \nabla_\mu q^u \nabla_\nu q^v g^{\mu\nu} + \\ &+ \beta_3 Re\mathcal{N}_{\Lambda\Sigma} F^\Lambda_{\mu\nu} F^{\Sigma\mu\nu}\Big\} + \frac{1}{2}\beta_3 Im\ \mathcal{N}_{\Lambda\Sigma} F^\Lambda_{\mu\nu} F^\Sigma_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma}\end{aligned} \tag{4.18}$$

where K was defined in eq. (4.8a) and where we have used eq. (2.65). In order to have a canonical and physical normalization of the quaternionic scalars we must fix $\lambda = -1$ which implies $\alpha_1 = \frac{4}{3}$. This is just the result of Bagger and Witten [3], fixing the curvature of the quaternionic manifold to a prescribed negative value ($\lambda = -1$ in our normalizations). Substituting this value in eqs. (4.14) and (4.8a),and using eq. (2.21f), the potential $\mathcal{V} = 3/4K$ reduces to the expression (4.4) and the bosonic Lagrangian (4.18) to the standard form (4.1).

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