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(2, 2) Vacuum configurations for type IIA superstrings: $N = 2$ supergravity Lagrangians and algebraic geometry.

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Abstract

The compactification of $D = 10$ type IIA supergravity, (the point-field theory limit of the type IIA superstring), on a Calabi-Yau manifold ((2, 2) vacuum) is performed. The resulting $D = 4, N = 2$ effective supergravity Lagrangian gives rise to an explicit construction of special Kähler manifolds for the (1, 1) moduli and dual quaternionic manifolds for the (2, 1) moduli. In both cases the scalar manifolds are characterized by a homogeneous holomorphic function of degree two. In the case of the (1, 1) moduli, the holomorphic function is given by a cubic polynomial, a result which is valid only in the classical large volume limit, (compared to the string scale), of the internal manifold. In contrast for the (2, 1) moduli the holomorphic function, which is related to the "periods" of a holomorphic three form, is unrestricted and due to non-renormalization theorems, should coincide with the exact (tree-level) string result.

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1 Introduction

The effort to understand compactified string theories has lead to the study of Calabi-Yau spaces [1], since these appear to be the appropriate backgrounds for which the compactifications can give reasonable results for four dimensional physics. A lot of the interest has been focused on understanding the moduli space of Calabi-Yau threefolds [1-6]. In particular it has been found that the moduli space of the Kähler class and complex structure deformations can be shown to be locally a product space of two special Kähler manifolds: $M_1(h_{1,1}) \times M_2(h_{2,1})$ of complex dimension $h_{(1,1)}$ and $h_{(2,1)}$ respectively. $h_{(1,1)}$ and $h_{(2,1)}$ are the two independent Hodge numbers which describe the cohomology of a given Calabi-Yau manifold [1].

From the point of view of an internal $(2,2)$ superconformal field theory it can be seen that the two types of moduli correspond to chiral-chiral or chiral-antichiral primary fields. In the type II theories the original moduli pair up with extra bosonic fields coming from the Ramond-Ramond sector. For the type IIA the $(2,1)$ moduli pair up with Ramond-Ramond scalars, while the $(1,1)$ moduli pair up with the R-R vectors. It is found that the $(1,1)$ sector parametrizes a special Kähler manifold while the enhanced $(2,1)$ sector parametrizes a quaternionic manifold. A similar result is obtained for the type IIB theory, with the roles of the $(1,1)$ and $(2,1)$ moduli reversed. Given such a relation, a map can be established between the IIA and IIB theories. The c-map [3] takes the special Kähler manifold parametrized by one type of moduli in one of the theories, onto the quaternionic manifold parametrized by the same type of moduli in the other theory.

Since 10D supergravity can be regarded as the large volume limit of the effective theory of critical superstrings compactified on an internal $(2,2)$ superconformal theory, we have pursued the Kaluza Klein program for the type IIA supergravity and given an explicit construction of the special Kähler and quaternionic manifolds by compactifying on an arbitrary Calabi-Yau internal space. In a previous work [9] we had investigated the vector sector for both the type IIA and IIB theories. In this case the moduli parametrize special Kähler manifolds with a curvature satisfying

$$R_{a\bar{b}c\bar{d}} = G_{a\bar{b}}G_{c\bar{d}} + G_{a\bar{d}}G_{c\bar{b}} - e^{2k}\, C_{acp}\bar{C}_{\bar{b}\bar{d}\bar{q}}G^{p\bar{q}}$$

where $G_{a\bar{b}}$ is the Kähler metric, k the Kähler potential and $e^k C_{abc}$ is a covariantly holomorphic tensor with respect to a Kähler connection $k_i = -\bar{\partial}_{\bar{i}} k$. This means that C_{abc} can be considered as a holomorphic section of the line bundle specified by the gauge field k_i and can be identified as the Yukawa coupling of the 27 and $\bar{27}$ families in the corresponding heterotic theory compactified on the same CY manifold. [1,3,6,7,8].

Since we are focusing on the type IIA theory we will construct the special Kähler manifold in terms of the $(1,1)$ moduli, corresponding to the vector sector, and the quaternionic manifold for the hypermultiplets from the special Kähler geometry of the $(2,1)$ moduli. We give in this way a realization of the c-map (s-map) in terms of the $(2,1)$ moduli, by relating the special Kähler manifold of the type IIB theory, described in reference [9], with

the quaternionic manifold encountered here for the type IIA theory. We should note that our result is obtained purely as a product of the compactification from ten to four dimensions and does not hinge on a dimensional reduction from D=4 to D=3 [10]. Since the cohomology theory of $(2,1)$ forms and its moduli spaces do not receive string corrections, our results, unlike the case for the $(1,1)$ forms, have general validity and explicitly realize the dual quaternionic structure of the string theory.

The organization of the paper is as follows. In the preliminary section we will present the 11D, N=1 supergravity Lagrangian given by Cremmer, Julia and Scherk [11] and then perform a dimensional reduction to 10D to obtain the effective type IIA, non-chiral N=2 supergravity Lagrangian. We will proceed with an analysis of the massless scalars on which the ten dimensional fields will be expanded. As a last entry in the preliminary section, we will give a series of useful definitions for the harmonic forms on a Calabi-Yau space together with some of their integrals and relations. We intend then to divide the compactification into sectors determined by the type of moduli being considered, or alternatively the cohomology group to which the harmonic forms in the CY space belong to. (The H^n sector will deal with fields expanded in harmonic forms (p,q) ; $p+q=n$). Section 3 will analyze the gravitational and the H^0 sectors of the theory. We will proceed to section 4 and study the compactification of the vector sector associated to the $(1,1)$ moduli. We will show the expected structure of the couplings in terms of the topological integrals of $(1,1)$ harmonic forms. We expect that our answer in the limit $h_{(1,1)} = 1$ should reduce to that obtained via the dimensional reduction (SU(3) truncation) of the type IIA supergravity [12]. Note that the result of the truncation coincides with a true compactification over a Calabi-Yau space, with $h_{(1,1)} = 1$, only for large values of the deformations, since for small values non-perturbative string corrections are important. Recently these corrections have been exactly computed for a particular CY space [13] using the mirror symmetry which relates the perturbative to the tree level world-sheet phenomena [1]. In section 5 we will finally focus on the scalar sector associated to the $(2,1)$ moduli. We will expand the fields in the Dolbeault cohomology basis of a general Calabi-Yau manifold, imposing an invariance under the symplectic transformations $Sp(2h_{2,1}+2, Z)$ that will fix the coefficients of the expansion [9,16]. After this we will proceed with the compactification expressing the couplings of the fields in terms of topological integrals over Calabi-Yau space. The final expression will be recast in such a way as to compare it with the result of reference [10], that based on three dimensional duality gives a general parametrization for a quaternionic manifold in terms of the same holomorphic function that characterizes the corresponding special Kähler manifold under the s-map. Such construction was originally suggested in reference [3].

[1] Non-perturbative corrections in the untwisted moduli using the quantum symmetries of $(2,2)$ orbifolds and their implications for supersymmetric effective Lagrangians have been discussed in reference [14]

2 Preliminaries

2.1 The Lagrangian

The type IIA 10 dimensional Lagrangian can be obtained via dimensional reduction from the 11 dimensional $N = 1$ supergravity Lagrangian[2] given by Cremmer et. al. [11]. The bosonic sector is given by the vierbein $\bar{e}_{\bar{\mu}}^{\bar{a}}$ and the three form $\bar{A}_{\bar{\mu}\bar{\nu}\bar{\rho}}$.

The vierbein can always be put in the upper triangular form as follows:

$$\bar{e}_{\bar{\mu}}^{\bar{a}} = \begin{pmatrix} e_{\hat{\mu}}^{\hat{a}} & \phi Z_{\hat{\mu}} \\ 0 & \phi \end{pmatrix} \tag{1}$$

where

$$\begin{aligned} \bar{a}, \bar{\mu} &= 1 \ldots 11 \\ \hat{a}, \hat{\mu} &= 1 \ldots 10 \end{aligned}$$

The Lagrangian is given by

$$\mathcal{L}_{11} = -\frac{1}{2}\bar{e}\bar{R} - \frac{1}{48}\bar{e}(\bar{F}_{\bar{\mu}_1\bar{\mu}_2\bar{\mu}_3\bar{\mu}_4})^2 + \frac{\sqrt{2}}{(12)^4}\varepsilon^{\bar{\mu}_1,\ldots\bar{\mu}_{11}}\bar{F}_{\bar{\mu}_1\ldots\bar{\mu}_4}\bar{F}_{\bar{\mu}_5\ldots\bar{\mu}_8}\bar{A}_{\bar{\mu}_9\bar{\mu}_{10}\bar{\mu}_{11}} \tag{2}$$

In the dimensional reduction we follow the convention of reference [12] and establish the following relations:

$$\begin{aligned} F_{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3\hat{\mu}_4} &= 4\partial_{[\hat{\mu}_1}\bar{A}_{\hat{\mu}_2\hat{\mu}_3\hat{\mu}_4]}. \\ B_{\hat{\mu}\hat{\nu}} &= \bar{A}_{\hat{\mu}\hat{\nu}11} \\ H_{\hat{\mu}\hat{\nu}\hat{\rho}} = 3\partial_{[\hat{\mu}}B_{\hat{\nu}\hat{\rho}]}&, \;\; Z_{\hat{\mu}\hat{\nu}} = 2\partial_{[\hat{\mu}}Z_{\hat{\nu}]} \\ \bar{A}_{\hat{\mu}\hat{\nu}\hat{\rho}} &= A_{\hat{\mu}\hat{\nu}\hat{\rho}} + 3Z_{[\hat{\mu}}B_{\hat{\nu}\hat{\rho}]}. \end{aligned} \tag{3}$$

Substituting the expressions above into the 11D lagrangian and performing a Weyl rescaling we obtain the 10D lagrangian.

$$\begin{aligned} \mathcal{L}_{10} = &-\frac{1}{2}e_{10}R_{10} - \frac{1}{8}e_{10}\phi^{\frac{9}{4}}(Z_{\hat{\mu}\hat{\nu}})^2 - \frac{9}{16}e_{10}(\partial_{\hat{\mu}}\log\phi)^2 - \frac{1}{48}e_{10}\phi^{\frac{3}{4}}(F_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} + 6Z_{[\hat{\mu}\hat{\nu}}B_{\hat{\rho}\hat{\sigma}]})^2 \\ &+ \frac{\sqrt{2}}{(48)^2}\varepsilon^{\hat{\mu}_1\ldots\hat{\mu}_{10}}(F_{\hat{\mu}_1\ldots\hat{\mu}_4} + 6Z_{\hat{\mu}_1\hat{\mu}_2}B_{\hat{\mu}_3\hat{\mu}_4})\,F_{\hat{\mu}_5\ldots\hat{\mu}_8}\,B_{\hat{\mu}_9\hat{\mu}_{10}} \end{aligned} \tag{4}$$

We should note, as it will be applied later, that in the above result we have used for the Weyl rescaling the formula (valid in any dimension D)

$$\sqrt{g}\,\mathcal{V}^{D-2}R \rightarrow \sqrt{g}\,(R + (D-1)(D-2)(\partial_\mu \log \mathcal{V})^2) \tag{5}$$

with

$$g_{\mu\nu} \rightarrow \mathcal{V}^{-2}g_{\mu\nu}; \;\; e_\mu^a \rightarrow \mathcal{V}^{-1}e_\mu^a.$$

In the case of the reduction from 11D to 10D we have $\mathcal{V} = \phi^{\frac{1}{8}}$.

[2]This technique was used to obtain the point theory limit of the heterotic superstring in references [15].

2.2 The zero modes

To perform the compactification from ten dimensions to $M_4 \times K$, (M_4 being Minkowski space), we must follow the Kaluza Klein program and expand the 10D fields in harmonics on the internal manifold K. A Calabi-Yau space (K), is a compact Kähler manifold of vanishing first Chern class and complex dimension three. It is characterized by the Hodge numbers $h_{p,q}$ which count the number of independent (p,q) harmonic forms that can exist on the manifold. In the case of the Calabi-Yau manifolds the non-vanishing cohomology groups have dimensions $h_{0,0} = h_{3,3} = 1, h_{3,0} = 1, h_{1,1} = h_{2,2}$ and $h_{2,1}$. By convention the $(0,0)$ form can be taken as a constant set to one, the $(3,0)$ form is a covariantly constant form and the $(1,1)$ and $(2,1)$ forms are related to the zero modes of the metric and therefore to the deformation parameters of the Kähler form and complex structure respectively.

Before starting with the analysis of the modes let us make a comment about the coordinates of the ten dimensional space splitting into four Minkowski components and six components in the Calabi-Yau manifold.

$w^{\hat{\mu}}$ is the ten dimensional coordinate $\hat{\mu} = 1, \ldots, 10$.

x^μ is the four dimensional coordinate $\mu = 1, \ldots, 4$.

y^a is the six dimensional real coordinate $a = 1, \ldots, 6$.

In the calculations that follow ahead, we will find more useful a complex set of coordinates for the CY space. They can be defined as follows: $\xi_1 = \frac{y_1+iy_2}{\sqrt{2}}$ $\xi_2 = \frac{y_3+iy_4}{\sqrt{2}}$ $\xi_3 = \frac{y_5+iy_6}{\sqrt{2}}$, together with their corresponding complex conjugates. $(\xi^i, \bar{\xi}^{\bar{i}};\; i, \bar{i} = 1,2,3)$. The relation between the integration measures can be given as, $d^6\xi = d^3\xi d^3\bar{\xi} = -id^6y$.

After this short detour, we can proceed to analyze the zero modes in the theory. Let us first start with those modes that arise from the fluctuations of the gravitational field [17,18]. The metric in ten dimensions $g_{\hat{\mu}\hat{\nu}}(w)$, can be decomposed into components in M_4 and K as follows: $g_{\mu\nu}, g_{\mu a}, g_{ab}$. The first term $g_{\mu\nu}(w) = g_{\mu\nu}(x)$ has all indices in the space M_4 and corresponds to the four dimensional graviton. The component $g_{\mu a}$ would be a vector gauge boson from the four dimensional standpoint and can be shown to be zero since a Killing vector field must be covariantly constant and this is incompatible with $SU(3)$ holonomy. Finally let us analyze the $g_{ab} \rightarrow g_{ij}, g_{i\bar{j}}$ zero modes. Any deformation of the metric of the Calabi-Yau manifold that respects $SU(3)$ holonomy must be a zero mode of the wave operator and therefore a massless field from the four dimensional point of view. The metric variations precisely correspond to deformations of the Kähler class and the complex structure and can be expanded in a basis of harmonic $(1,1)$ and $(2,1)$ forms with the four dimensional scalar moduli as their coefficients. We have

$$\begin{aligned} i\delta g_{i\bar{j}} &= \sum_{A=1}^{h_{1,1}} \delta M^A\, V_{i\bar{j}}^A \\ \delta g_{ij} &= \sum_{\alpha=1}^{h_{2,1}} \delta \bar{Z}^\alpha \bar{b}_{\alpha\, ij} \end{aligned} \tag{6}$$

where $V^A_{i\bar{j}}$ are the $(1,1)$ harmonic forms and $\bar{b}_{\alpha\, ij}$ can be related to the $(2,1)$ $\Phi_{\alpha\, ij\bar{k}}$, and $(3,0)$ Ω_{ijk} forms by

$$\bar{b}_{\alpha\, ij} = \frac{i}{||\Omega||^2}\, \Omega_i^{\bar{l}\bar{k}}\, \bar{\Phi}_{\alpha\, \bar{l}\bar{k}j} \tag{7}$$

with $||\Omega||^2 = \frac{1}{3!}\Omega_{ijk}\bar{\Omega}^{ijk}$.

In the expansion for the metric variations the $M^A(x)$ are real and the $Z^I(x)$ are complex massless scalar fields with vanishing potential in Minkowski space.

Now let us proceed with the other bosonic zero modes coming from the dilaton ϕ, the second rank tensor $B_{\hat{\mu}\hat{\nu}}$, the vector $Z_{\hat{\mu}}$ and the third rank tensor $A_{\hat{\mu}\hat{\nu}\hat{\rho}}$.

We will start by giving a complete list of all the possible bosonic field components and then from the conditions given by the non-vanishing harmonic forms select those modes that survive.

$$\phi(w) = \phi(x)$$
$$B_{\hat{\mu}\hat{\nu}}(w)dw^{\hat{\mu}} \wedge dw^{\hat{\nu}} = B_{\mu\nu}(x)dx^\mu \wedge dx^\nu + B_{\mu a}(w)dx^\mu \wedge dy^a + B_{ab}(w)dy^a \wedge dy^b$$
$$\to B_{\mu\nu}dx^\mu \wedge dx^\nu + a^A(x)V^A$$
$$Z_{\hat{\mu}}(w)dw^{\hat{\mu}} = Z_\mu(x)dx^\mu + Z_a(w)dy^a$$
$$\to Z_\mu dx^\mu$$
$$A_{\hat{\mu}\hat{\nu}\hat{\rho}}(w)dw^{\hat{\mu}} \wedge dw^{\hat{\nu}} \wedge dw^{\hat{\rho}} =$$
$$A_{\mu\nu\rho}(x)dx^\mu \wedge dx^\nu \wedge dx^\rho + A_{\mu ab}(w)dx^\mu \wedge dy^a \wedge dy^b + A_{abc}(w)dy^a \wedge dy^b \wedge dy^c$$
$$\to A_{\mu\nu\rho}dx^\mu \wedge dx^\nu \wedge dx^\rho + A^A_\mu(x)V^A + A_{abc}(w)dy^a \wedge dy^b \wedge dy^c \tag{8}$$

where the discussion of the expansion of $A_{abc}(w)$ into harmonics will be postponed until section 5.

2.3 Harmonic forms and integrals over the CY space

To close the present section of preliminary information, we think it will be useful to give some relevant formulae that will be encountered in many places in the future development. Let us first recall the non-vanishing harmonic forms:

$$V^A = V^A_{i\bar{j}}d\xi^i \wedge d\bar{\xi^j} \quad A = 1, \ldots, h_{1,1}$$
$$\Phi_I = \frac{1}{2!}\Phi_{I\, ij\bar{k}}d\xi^i \wedge d\xi^j \wedge d\bar{\xi}^k \quad I = 0, \ldots, h_{2,1}$$
$$\Omega = \frac{1}{3!}\Omega_{ijk}d\xi^i \wedge d\xi^j \wedge d\xi^k \tag{9}$$

The Kähler form J will be given as:

$$J = ig_{i\bar{j}}d\xi^i \wedge d\bar{\xi^j} = M^A(x)V^A \tag{10}$$

and with it the volume of the space can be calculated

$$\mathcal{V} = \frac{1}{3!}\int J \wedge J \wedge J = i\int \sqrt{g}\, d^6\xi = \int \sqrt{g}d^6y. \tag{11}$$

Some of the expressions that will be found in the compactification process are given below, following closely the definitions of reference [16].

For the H^2 sector involving the $(1,1)$ we have,

$$G_{AB} = \frac{i}{2}\frac{\int V^A \wedge^* V^B}{\mathcal{V}} \tag{12}$$

with $\int V^A \wedge^* V^B = \int V^A_{i\bar{j}}V^{B\, i\bar{j}}\sqrt{g}d^6\xi$. Also

$$\mathcal{K} = \int J \wedge J \wedge J$$
$$\mathcal{K}_A = \int V^A \wedge J \wedge J$$
$$\mathcal{K}_{AB} = \int V^A \wedge V^B \wedge J$$
$$\mathcal{K}_{ABC} = \int V^A \wedge V^B \wedge V^C \tag{13}$$

With

$$^*V^A = i\left(J \wedge V^A - \frac{3}{2}\frac{\mathcal{K}_A}{\mathcal{K}}J \wedge J\right) \tag{14}$$

we have

$$G_{AB} = -3\left(\frac{\mathcal{K}_{AB}}{\mathcal{K}} - \frac{3}{2}\frac{\mathcal{K}_A\mathcal{K}_B}{\mathcal{K}^2}\right) \tag{15}$$

Now for the H^3 sector in terms of the $(2,1)$ and $(3,0)$ harmonic forms we can define

$$(\Phi_I, \bar{\Phi}_J) = \int \Phi_I \wedge \bar{\Phi}_J = -\frac{1}{2!}\int \Phi_{I\, ij\bar{k}}\, \bar{\Phi}_J^{ij\bar{k}}\sqrt{g}d^6\xi$$
$$= -\frac{1}{(2!)^2}\int \Phi_{I\, ij\bar{n}}\, \bar{\Phi}_{J\, \bar{l}\bar{m}k}\epsilon^{ijk}\, \epsilon^{\bar{l}\bar{m}\bar{n}}\, d^6\xi \tag{16}$$
$$(\Omega, \bar{\Omega}) = \int \Omega \wedge \bar{\Omega} = \frac{1}{3!}\int \Omega_{ijk}\, \bar{\Omega}^{ijk}\, \sqrt{g}d^6\xi$$
$$= -i||\Omega||^2\mathcal{V}$$
$$= \frac{1}{(3!)^2}\int \Omega_{ijk}\, \bar{\Omega}_{\bar{l}\bar{m}\bar{n}}\epsilon^{ijk}\epsilon^{\bar{l}\bar{m}\bar{n}}d^6\xi \tag{17}$$
$$\int b_\alpha \wedge \bar{b}_\beta = \int b_{\alpha\, \bar{i}\bar{j}}\bar{b}_\beta^{\bar{i}\bar{j}}\, \sqrt{g}d^6\xi = -\frac{1}{||\Omega||^2}\int \Phi_\alpha \wedge \bar{\Phi}_\beta \tag{18}$$
$$G_{IJ} = -\frac{\int \Phi_I \wedge \bar{\Phi}_J}{\int \Omega \wedge \bar{\Omega}} \tag{19}$$
$$G_{\alpha\bar{\beta}} = -\frac{i}{4}\frac{\int b_\alpha \wedge \bar{b}_\beta}{\mathcal{V}} \tag{20}$$

3 The gravitational and H^0 sectors

As stated in the introduction we are going to divide the compactification of the ten dimensional Lagrangian into sectors according to the expansion of the various fields into different harmonic forms. The motivation for this choice of presentation is hopefully to facilitate the reading of the paper, and to highlight the important characteristics of each type of moduli.

In the ten dimensional Lagrangian we have to evaluate the Ricci curvature $R_{10}(w)$ in terms of the moduli scalar fields discussed in the last section. Using that information and since in a CY manifold $g_{ij} = g_{\bar{i}\bar{j}} = 0$, the only non-vanishing components of the affine connection apart from the 4D $\Gamma^{\rho}_{\mu\nu}$ are:

$$\begin{aligned}
\Gamma^{j}_{\mu i} &= \frac{-i}{2} g^{j\bar{k}} \partial_\mu M^A V^A_{\bar{k}i} \\
\Gamma^{\bar{j}}_{\mu i} &= \frac{1}{2} g^{\bar{j}k} \partial_\mu \bar{Z}^\alpha \bar{b}_{\alpha\, ki} \\
\Gamma^{i}_{\mu \bar{j}} &= \frac{1}{2} g^{i\bar{k}} \partial_\mu Z^\alpha b_{\alpha\, \bar{k}\bar{j}} \\
\Gamma^{\bar{i}}_{\mu \bar{j}} &= \frac{-i}{2} g^{\bar{i}k} \partial_\mu M^A V^A_{k\bar{j}} \\
\Gamma^{\mu}_{ij} &= -\frac{1}{2} \partial^\mu \bar{Z}_\alpha \bar{b}_{\alpha\, ij} \\
\Gamma^{\mu}_{i\bar{j}} &= \frac{i}{2} \partial^\mu M^A V^A_{i\bar{j}} \\
\Gamma^{\mu}_{\bar{i}\bar{j}} &= -\frac{1}{2} \partial^\mu Z_\alpha b_{\alpha\, \bar{i}\bar{j}}
\end{aligned} \quad (21)$$

Since the CY manifold is Ricci flat we have $R_{ij} = R_{i\bar{j}} = R_{\bar{i}\bar{j}} = 0$, and the curvature will be given by

$$R_{10}(w) = R_4(x) + g^{\mu\nu}(R^{i}_{\mu i\nu} + R^{\bar{i}}_{\mu\bar{i}\nu}) \quad (22)$$

After a straightforward calculation we get for the last equation:

$$\begin{aligned}
-\frac{R_{10}}{2} &= -\frac{R_4}{2} + \frac{3}{4} \partial_\mu M^A \partial^\mu M^B V^A_{i\bar{j}} V^{Bi\bar{j}} \\
&- \frac{1}{2} \partial_\mu M^A \partial^\mu M^B V^A_{i\bar{j}} V^B_{\bar{i}k} g^{i\bar{j}} g^{\bar{i}k} - \frac{1}{4} \partial_\mu Z_\alpha \partial^\mu \bar{Z}_\beta b_{\alpha\, \bar{i}\bar{l}} \bar{b}_{\beta\, jk} g^{\bar{i}j} g^{\bar{l}k}
\end{aligned} \quad (23)$$

Substituting into the integral we have for the gravity action

$$\begin{aligned}
S_{grav} = \int d^4x e \int i d^6\xi \sqrt{g} [&- \frac{R_4}{2} - \frac{9}{16} (\partial_\mu \log \phi)^2 + \partial_\mu M^A \partial^\mu M^B (\frac{1}{4} V^A_{i\bar{j}} V^{Bi\bar{j}} + \frac{1}{2} \frac{\mathcal{K}_{AB}}{\mathcal{V}}) \\
&- \frac{1}{4} \partial \mu Z_\alpha \partial^\mu \bar{Z}_\beta b_{\alpha\, ij} \bar{b}^{\bar{i}\bar{j}}_{\beta}]
\end{aligned} \quad (24)$$

Performing the integrals over the Calabi-Yau space following the definitions given in the preliminary section we have

$$S_{grav} = \int d^4x e [-\mathcal{V}\frac{R}{2} - \mathcal{V}\frac{9}{16}(\frac{\partial_\mu \phi}{\phi})^2 + \mathcal{V}\partial_\mu Z_\alpha \partial^\mu \bar{Z}_\beta G_{\alpha\bar{\beta}} + \partial_\mu M^A \partial^\mu M^B [\mathcal{V}\frac{G_{AB}}{2} + \frac{1}{2}\mathcal{K}_{AB}] \quad (25)$$

Under Weyl rescaling for the case $\Omega = \mathcal{V}^{\frac{1}{2}}$, following equation (5), we obtain

$$\begin{aligned}
S_{grav} = \int d^4x e [&- \frac{R}{2} - \frac{9}{16}(\frac{\partial_\mu \phi}{\phi})^2 - \frac{3}{4}(\frac{\partial_\mu \mathcal{V}}{\mathcal{V}})^2 \\
&+ \partial_\mu Z_\alpha \partial^\mu \bar{Z}_\beta G_{\alpha\bar{\beta}} + \partial_\mu M^A \partial^\mu M^B \frac{1}{2}(G_{AB} + \frac{\mathcal{K}_{AB}}{\mathcal{V}})]
\end{aligned} \quad (26)$$

Substituting the expression for the volume and using the equation (15) we have

$$S_{grav} = \int d^4x e [-\frac{R}{2} - \frac{9}{16}(\frac{\partial_\mu \phi}{\phi})^2 + G_{\alpha\bar{\beta}} \partial_\mu Z_\alpha \partial^\mu \bar{Z}^\beta - \partial_\mu M^A \partial^\mu M^B (\frac{1}{2} G_{AB} + \frac{9}{4} \frac{\mathcal{K}_A \mathcal{K}_B}{\mathcal{K}^2})] \quad (27)$$

Let $M_A \rightarrow \beta v^A \phi^{-\frac{3}{4}}$ with $\beta = constant$. With this change the various couplings transform as:

$$\begin{aligned}
G_{AB}(M) &\rightarrow \beta^{-2} G_{AB}(v) \phi^{\frac{3}{2}} \\
\mathcal{K}_{ABC}(M) &\rightarrow \mathcal{K}_{ABC}(v) \\
\mathcal{K}_{AB}(M) &\rightarrow \beta \mathcal{K}_{AB}(v) \phi^{-\frac{3}{4}} \\
\mathcal{K}_{A}(M) &\rightarrow \beta^2 \mathcal{K}_{AB}(v) \phi^{-\frac{3}{2}} \\
\mathcal{K}(M) &\rightarrow \beta^3 \mathcal{K}(v) \phi^{-\frac{9}{4}}
\end{aligned} \quad (28)$$

Finally, substituting the expression for M^A in terms of v^A

$$\partial_\mu M^A \partial^\mu M^B = \beta^2 \phi^{-\frac{3}{2}} [\partial_\mu v^A \partial^\mu v^B - \frac{3}{4} \partial_\mu v^A v^B (\frac{\partial_\mu \phi}{\phi}) - \frac{3}{4} \partial_\mu v^B v^A (\frac{\partial_\mu \phi}{\phi}) + \frac{9}{16} (\frac{\partial_\mu \phi}{\phi})^2 v^A v^B]$$

and using

$$\begin{aligned}
G_{AB} \partial_\mu v^A v^B &= \frac{3}{2} \partial_\mu v^A \frac{\mathcal{K}_A}{\mathcal{K}} \\
G_{AB} v^A v^B &= \frac{3}{2}
\end{aligned}$$

we have for the gravity action

$$S_{grav} = \int d^4x e [-\frac{R}{2} - \frac{G_{AB}}{2} \partial_\mu v^A \partial^\mu v^B - \frac{1}{4} \frac{\partial(\mathcal{V}\phi^{-3})^2}{(\mathcal{V}\phi^{-3})^2} + G_{\alpha\bar{\beta}} \partial_\mu Z^\alpha \partial^\mu \bar{Z}^\beta] \quad (29)$$

Now we are in a position to analyze the modes that correspond to $(0,0)$ harmonic forms in the expansion of the fields. Our objective is to extract the terms that involve only

$\phi, B_{\mu\nu}, A_{\mu\nu\rho}$ and $\mathcal{Z}_\mu$, when they are not mixed with the scalars corresponding to the $(1,1)$ or $(2,1)$ sector. As it is expected these terms will only describe the field strengths of the particular fields.

$$S_{H^0(\mathcal{Z})} = -\frac{1}{8}\int d^4xe_4 \int id^6\xi\sqrt{g}\phi^{\frac{9}{4}}\mathcal{Z}^2_{\hat{\mu}\hat{\nu}} \,|_{H^0}$$

Where the $|_{H^0}$ specifies that we are limiting the contribution to the scalars corresponding to harmonic forms in H^0. After Weyl rescaling and following with a substitution of the M^A in terms of v^A and ϕ we obtain

$$S_{H^0(\mathcal{Z})} = -\frac{\beta^3}{8}\int d^4xe\mathcal{V}(v)\mathcal{Z}^2_{\mu\nu} \tag{30}$$

We continue with the other term

$$S_{H^0(A_{\mu\nu\rho})} = -\frac{1}{48}\int d^4e_4 \int id^6\xi\sqrt{g}$$

$$\phi^{\frac{3}{4}}[F_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} + 6\mathcal{Z}_{[\hat{\mu}\hat{\nu}}B_{\hat{\rho}\hat{\sigma}]}]^2 \,|_{H^0} \tag{31}$$

after Weyl rescaling

$$S_{H^0(A_{\mu\nu\rho})} = -\frac{1}{48}\int d^4xe\mathcal{V}^3(v)\phi^{\frac{3}{4}}[F_{\mu\nu\rho\sigma} + 6\mathcal{Z}_{[\mu\nu}B_{\rho\sigma]}]^2$$

Finally

$$S_{H^0(H_{\mu\nu\rho})} = -\frac{1}{12}\int d^4xe \int id^6\xi\sqrt{g}\phi^{-\frac{3}{2}}(H_{\hat{\mu}\hat{\nu}\hat{\rho}})^2 \,|_{H^0} \tag{32}$$
$$= -\frac{\beta^6}{12}\int d^4xe(\mathcal{V}(v)\phi^{-3})^2H^2_{\mu\nu\rho}$$

Before leaving this section, let us combine a contribution from the gravity sector with the latter expression and define an action that will be of use in analyzing the $(2,1)$ scalars.

$$S_{H^0(H_{\mu\nu\rho},\phi)} = -\int d^4xe[\frac{1}{4}(\frac{\partial_\mu\tilde{\phi}}{\tilde{\phi}})^2 + \frac{1}{6}\tilde{\phi}^2(H_{\mu\nu\rho})^2] \tag{33}$$

with $\tilde{\phi} = 2\mathcal{V}(v)\phi^{-3}$, when $\beta = \sqrt{2}$.

4 The H^2 sector

Following the prescription introduced in the last section we can now proceed to perform the compactification for the case in which the fields are expanded in terms of $(1,1)$ forms.

$$S_{H^2(A^A_\mu,a^A)} = -\frac{1}{4}\int d^4xe \int id^6\xi\sqrt{g}V^A_{i\bar{j}}V^{Bi\bar{j}}\phi^{\frac{3}{4}}[f^A_{\mu\nu} + a^A\mathcal{Z}_{\mu\nu}][f^{B\mu\nu} + a^B\mathcal{Z}^{\mu\nu}] \tag{34}$$

with $f^A_{\mu\nu} = 2\partial_{[\mu}A^A_{\nu]}$. After Weyl rescaling and substituting $M^A \to v^A$

$$S_{H^2(A^A_\mu,a^A)} = -\frac{\beta}{2}\frac{1}{3!}\int d^4xe\mathcal{K}G_{AB}[f^A_{\mu\nu} + a^A\mathcal{Z}_{\mu\nu}][f^{B\mu\nu} + a^B\mathcal{Z}^{\mu\nu}] \tag{35}$$

Similarly

$$S_{H^2(a^A)} = -\frac{1}{12}\int d^4xe \int id^6\xi\sqrt{g}\phi^{-\frac{3}{2}}(H_{\hat{\mu}\hat{\nu}\hat{\rho}})^2 \,|_{H^2}$$
$$= -\frac{1}{2}\int d^4xe\phi^{\frac{-3}{2}}\partial_\mu a^A\partial^\mu a^B \int id^6\xi\sqrt{g}V^A_{i\bar{j}}V^B_{i\bar{j}}$$
$$= -\beta^{-2}\int d^4xeG_{AB}(v)\partial_\mu a^A\partial^\mu a^B \tag{36}$$

Finally the topological integral yields when specified for the $(1,1)$ moduli the following expression

$$S_{H^2(top)} = i\frac{\sqrt{2}}{48}\int d^4x\varepsilon^{\mu\nu\rho\lambda}\int V^A_{i\bar{j}}V^B_{l\bar{k}}V^C_{m\bar{n}}\epsilon^{ilm}\epsilon^{\bar{j}\bar{k}\bar{n}}d^3\xi d^3\bar{\xi}$$

$$2[a^Aa^Ba^C\mathcal{Z}_{\mu\nu}\mathcal{Z}_{\rho\lambda} + \frac{3}{2}a^Aa^Bf^C_{\mu\nu}\mathcal{Z}_{\rho\lambda} + \frac{3}{2}a^Aa^B\mathcal{Z}_{\rho\lambda}f^C_{\mu\nu} + 3a^Af^B_{\mu\nu}f^C_{\rho\lambda}] \tag{37}$$

Adding the three actions derived in this section, with $\beta = \frac{1}{\sqrt{2}}$, together with the contribution from the $(1,1)$ moduli coming from the gravity sector, we have after some reorganization,

$$S_{H^2} = \int d^4xe[-\frac{1}{2}G_{AB}\partial_\mu W^A\partial^\mu W^B + \frac{\sqrt{2}}{2}\mathcal{Z}^2_{\mu\nu}(-\frac{1}{12}\mathcal{K} + \frac{1}{2}(\mathcal{K}_{AB} - \frac{3}{2}\frac{\mathcal{K}_A\mathcal{K}_B}{\mathcal{K}})a^Aa^B)$$
$$+\frac{\sqrt{2}}{2}(\mathcal{K}_{AB} - \frac{3}{2}\frac{\mathcal{K}_A\mathcal{K}_B}{\mathcal{K}})f^A_{\mu\nu}a^B\mathcal{Z}^{\mu\nu} + \frac{\sqrt{2}}{4}(\mathcal{K}_{AB} - \frac{3}{2}\frac{\mathcal{K}_A\mathcal{K}_B}{\mathcal{K}})f^A_{\mu\nu}f^{\mu\nu B}$$
$$+i\int d^4xe[\frac{\sqrt{2}}{12}\mathcal{K}_{ABC}a^Aa^Ba^C\tilde{\mathcal{Z}}_{\mu\nu}\mathcal{Z}^{\mu\nu} + \frac{\sqrt{2}}{8}\mathcal{K}_{ABC}a^Aa^B\tilde{\mathcal{Z}}_{\mu\nu}f^{C\mu\nu}$$
$$+\frac{\sqrt{2}}{8}\mathcal{K}_{ABC}a^Aa^B\mathcal{Z}^{\mu\nu}\tilde{f}^C_{\mu\nu} + \frac{\sqrt{2}}{4}\mathcal{K}_{ABC}a^Af^B_{\mu\nu}\tilde{f}^{C\mu\nu}] \tag{38}$$

where $W^A = a^A + iv^A$ and $e\tilde{\mathcal{Z}}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\lambda}\mathcal{Z}_{\rho\lambda}$

Following references [4,16] we can define a function

$$f(W) = \frac{i}{3!}\mathcal{K}_{ABC}\frac{W^AW^BW^C}{W^0} \tag{39}$$

with W^0 introduced for convenience since $f(W)$ is expected to be homogeneous of degree two. W^0 will be set to 1 after taking derivatives. It is now possible to recast the result of the various integrals in terms of the function $f(W)$. Since our objective is to compare the result of the compactification with that of a theory with $N = 2$ supergravity coupled to $h_{(1,1)}$ vector multiplets we can immediately identify $f(W)$ with the standard homogeneous holomorphic function of second degree. Trying to reproduce the results of De Witt and Van Proeyen [19] we proceed now to calculate the various quantities in which the Lagrangian is written.

The real symmetric matrix $N_{IJ} = \frac{1}{4}(f_{IJ} + \bar{f}_{IJ})$ with $(I = 0, ..., h_{1,1})$ will have entries

$$N_{AB} = -\frac{1}{2}\mathcal{K}_{AB}$$
$$N_{A0} = \frac{1}{2}\mathcal{K}_{AB}a^B$$
$$N_{00} = \frac{1}{3!}\mathcal{K} - \frac{1}{2}\mathcal{K}_{AB}a^Aa^B \qquad (40)$$

With this matrix we can calculate also

$$(NW)_0 = \frac{1}{3!}\mathcal{K} + \frac{i}{2}\mathcal{K}_Aa^A$$
$$(NW)_A = -\frac{i}{2}\mathcal{K}_A$$

and

$$(\bar{W}NW) = -\frac{1}{3}\mathcal{K} \quad (WNW) = \frac{2}{3}\mathcal{K}. \qquad (41)$$

Also

$$\mathcal{N}_{IJ} = \frac{1}{4}\bar{f}_{IJ} - \frac{(NW)_I(NW)_J}{(WNW)}$$

will have components

$$Re\mathcal{N}_{00} = \frac{1}{24}\mathcal{K} + (-\frac{1}{4}\mathcal{K}_{AB} + \frac{3}{8}\frac{\mathcal{K}_A\mathcal{K}_B}{\mathcal{K}})a^Aa^B$$
$$Re\mathcal{N}_{0A} = \frac{1}{4}(\mathcal{K}_{AB} - \frac{3}{2}\frac{\mathcal{K}_A\mathcal{K}_B}{\mathcal{K}})a^B$$
$$Re\mathcal{N}_{AB} = -\frac{1}{4}(\mathcal{K}_{AB} - \frac{3}{2}\frac{\mathcal{K}_A\mathcal{K}_B}{\mathcal{K}})$$
$$Im\mathcal{N}_{00} = -\frac{1}{12}\mathcal{K}_{ABC}a^Aa^Ba^C$$
$$Im\mathcal{N}_{0A} = \frac{1}{8}\mathcal{K}_{ABC}a^Aa^B$$
$$Im\mathcal{N}_{AB} = -\frac{1}{4}\mathcal{K}_{ABC}a^A \qquad (42)$$

Return now to the action and choose the following normalizations

$$W^A \to -\frac{1}{\sqrt{2}}W^A;\ a^A \to -\frac{1}{\sqrt{2}}a^A;\ v^A \to -\frac{1}{\sqrt{2}}v^A$$

This gives in consequence

$$\mathcal{K}_{AB} \to -\frac{1}{\sqrt{2}}\mathcal{K}_{AB}$$
$$\mathcal{K}_A \to \frac{1}{2}\mathcal{K}_A$$
$$\mathcal{K} \to -\frac{1}{2\sqrt{2}}\mathcal{K}$$
$$G_{AB} \to 2G_{AB} \qquad (43)$$

We can then rewrite the action as

$$S_{H^2} = \int d^4xe[-\frac{1}{2}G_{AB}\partial_\mu W^A\partial^\mu\bar{W}^B$$
$$+\frac{1}{4}(2Z^2_{\mu\nu}Re\mathcal{N}_{00} + 4\sqrt{2}Z_{\mu\nu}f^{A\mu\nu}Re\mathcal{N}_{0A} + 4f^A_{\mu\nu}f^{B\mu\nu}Re\mathcal{N}_{AB})]$$
$$+ \int d^4xe\frac{1}{4}[2Z^{\mu\nu}\tilde{Z}_{\mu\nu}Im\mathcal{N}_{00} + 2\sqrt{2}\tilde{Z}^{\mu\nu}f^A_{\mu\nu}Im\mathcal{N}_{0A} + 2\sqrt{2}Z^{\mu\nu}\tilde{f}^A_{\mu\nu} + 4f^{A\mu\nu}\tilde{f}^B_{\mu\nu}Im\mathcal{N}_{AB}] \quad (44)$$

Finally we can recast the action as

$$S_{H^2} = \int d^4xe[\partial^\mu W^A\partial_\mu\bar{W}^B\frac{\mathcal{M}_{AB}}{Y} + \frac{1}{4}(F^I_{\mu\nu}F^{\mu\nu J}Re\mathcal{N}_{IJ} + i\tilde{F}^I_{\mu\nu}\tilde{F}^{\mu\nu J}Im\mathcal{N}_{IJ})] \quad (45)$$

with

$$F^0_{\mu\nu} = \sqrt{2}Z_{\mu\nu}$$
$$F^A_{\mu\nu} = 2f^A_{\mu\nu}$$

and

$$Y = (\bar{W}NW)$$
$$\mathcal{M}_{AB} = N_{AB} - \frac{(N\bar{W})_A(NW)_B}{(\bar{W}NW)} = -\frac{1}{2}G_{AB}.$$

So we have successfully reproduced the bosonic sector of a theory for $h_{1,1}$ vector multiplets coupled to $N = 2$ supergravity, based on a homogeneous function $f(W)$ of $h_{1,1}+1$ variables. As expected the scalars parametrize a special Kähler manifold. It is important to note that the proof has been dependent on the cubic form of the function $f(W)$ and does not necessarily generalize to an arbitrary choice of holomorphic function. This situation will differ for the case of the $(2,1)$ moduli that will be treated in the next section.

5 The H^3 Cohomology Sector

5.1 Expansion in a real cohomology basis

Before starting the analysis of the compactification of the elements in the Lagrangian that have forms lying in the H^3 cohomology let us review some useful definitions that will be fundamental in the forthcoming work. We will follow closely references [9,16] in this description. We can use the real cohomology basis for H^3 given by the mutually dual three forms α_I,β_I,$I = 0, ..., h_{2,1}$, to expand the fundamental $(3,0)$ and $(2,1)$ forms. For the $(3,0)$ we have

$$\Omega(Z) = Z^I\alpha_I + iF_I(Z)\beta^I$$
$$\Omega(\lambda Z) = \lambda\Omega(Z)\,;\; F(\lambda Z) = \lambda^2 F(Z) \qquad (46)$$

where $F(Z)$ is a holomorphic function, λ is a constant, and $F_I = \partial_I F$. The $(2,1)$ forms can be given as

$$\Phi_I = \Omega_I - \frac{(\Omega_I,\bar\Omega)}{(\Omega,\bar\Omega)}\Omega$$
$$Z^I\Phi_I = 0 \qquad (47)$$

and

$$\Omega_I = \partial_I\Omega = \alpha_I + iF_{IJ}\beta^J \qquad (48)$$

Other relevant expressions are:

$$K_I = \frac{(N\bar Z)_I}{(\bar Z N Z)}$$
$$N_{IJ} = \frac{1}{4}(F_{IJ} + \bar F_{IJ})$$
$$\mathcal{N}_{IJ} = \frac{1}{4}\bar F_{IJ} - \frac{(NZ)_I(NZ)_J}{(ZNZ)}$$
$$R_{IJ} = Re[\mathcal{N}_{IJ}]$$
$$(R^{-1})^{IJ} = 2(N^{-1}(I - \bar K\bar Z - KZ))^{IJ} \qquad (49)$$

and finally the integrals of equations (16) and (17) can be evaluated as

$$(\Omega,\bar\Omega) = -4i(\bar Z N Z)$$
$$(\Phi_I,\bar\Phi_J) = -4i\mathcal{M}_{IJ} = -4i(N_{IJ} - \frac{(N\bar Z)_I(NZ)_J}{(\bar Z N Z)})$$
$$(\Omega_I,\bar\Omega) = -4i(N\bar Z)_I$$
$$K_I = \frac{(\Omega_I,\bar\Omega)}{(\Omega,\bar\Omega)} \qquad (50)$$

As explained in the preliminary section, the general expansion for the three form A, in terms of the $(2,1)$ and $(3,0)$ forms has been postponed until now in order to make use of the real cohomology bases (α_I, β^I). The motivation behind this choice has been to impose the invariance of A under symplectic transformations $Sp(2h_{(2,1)}+2)$. This technique proved to be successful in fixing the arbitrary coefficients of the expansion in a previous work [9] that studied the H^3 cohomology sector for the type IIB theory.

A symplectic transformation for the (α,β) bases is given by

$$\delta\alpha_I = -B_I^{TJ}\alpha_J - C_{IJ}\beta^J$$
$$\delta\beta^I = D^{IJ}\alpha_J + B^I_J\beta^J \qquad (51)$$

with B a constant real matrix and C, D two constant real symmetric matrices. In order to have an invariant expansion for Ω, $(-iF_I, Z^I)$ must transform as the bases elements and from there we can derive the transformations of the other expressions as follows:

$$\delta Z = (B - iD\mathcal{F})Z$$
$$\delta\mathcal{F} = -iC - B^T\mathcal{F} - \mathcal{F}B + i\mathcal{F}D\mathcal{F}$$
$$\delta K = -(B^T - i\mathcal{F}D)K$$
$$\delta N = -(B^T + \frac{i}{2}(\bar{\mathcal{F}} - \mathcal{F})D)N - N(B + \frac{i}{2}D(\bar{\mathcal{F}} - \mathcal{F}))$$
$$\delta N^{-1} = N^{-1}(B^T - \frac{i}{2}(\mathcal{F} - \bar{\mathcal{F}})D) + (B - \frac{i}{2}D(\mathcal{F} - \bar{\mathcal{F}}))N^{-1}$$
$$\delta R^{-1} = R^{-1}(B^T - \frac{i}{2}(\mathcal{F} - \bar{\mathcal{F}})D) + (B - \frac{i}{2}D(\mathcal{F} - \bar{\mathcal{F}}))R^{-1} + 4iD(\bar K\bar Z - KZ) + 4i(\bar Z\bar K - ZK)D \qquad (52)$$

where indices have been omitted for simplicity and $\mathcal{F}$ stands for F_{IJ}.

Let us now focus on the expansion for the real three form A. In general we have:

$$A = \Psi_I(x)(a^{IJ}(Z)\alpha_J + b^I_J(Z)\beta^J) + \bar\Psi_I(x)(\bar a^{IJ}(Z)\alpha_J + \bar b^I_J(Z)\beta^J) \qquad (53)$$

From the results of reference [10], that describes an $N = 2$ Lagrangian in $D = 3$, with the scalars parametrizing a quaternionic manifold, we expect the real combination of the fields $Re\Psi = \frac{1}{2}(\Psi + \bar\Psi)$ to appear with coefficients that depend on the moduli, and in multiplication with itself. In contrast the imaginary combination $Im\Psi = -i\frac{1}{2}(\Psi - \bar\Psi)$, is expected to have coefficients that do not depend on the moduli and does not appear in multiplication with itself. Since $\int\alpha_I\wedge\beta^J = \delta_I^J$, and after rewriting the expansion in terms of the real and imaginary combinations of the scalar fields we have for A:

$$A = (Re\Psi)(2a)\alpha + ((Re\Psi)(b + \bar b) + i(Im\Psi)(b - \bar b))\beta \qquad (54)$$

where we have chosen $a = \bar a$.

Rewriting the expression above in terms of the $(2,1)$ and $(3,0)$ forms we obtain

$$A = \frac{1}{4}\Psi((a\bar{\mathcal{F}} - ib)N^{-1}(\Phi + K\Omega) + (a\mathcal{F} + ib)N^{-1}(\bar{\Phi} + \bar{K}\bar{\Omega})) + h.c. \quad (55)$$

Returning once more to the case of the H^3 sector in the type IIB theory, we recall that the invariant combination involved an expansion on $(\Phi, \bar{\Omega})$ or $(\bar{\Phi}, \Omega)$, excluding any other possibility. We use this result as an ansatz and impose the following conditions

$$(a\mathcal{F} + ib)N^{-1}\bar{\Phi} = 0 \; ; \; (a\mathcal{F} + ib)N^{-1}\bar{K}\bar{\Omega} \propto \bar{K}\bar{\Omega} \quad (56)$$

By equations (47) and (49) the above conditions can by satisfied by setting

$$(a\mathcal{F} + ib)N^{-1} = d\bar{K}\bar{Z},$$

where d is an arbitrary real matrix that depends on the moduli. Substituting for b in terms of a and d we obtain

$$A = \frac{1}{4}\Psi((4a - d\bar{K}\bar{Z})(\Phi + K\Omega) + d\bar{K}\bar{\Omega}) + h.c. \quad (57)$$

We can proceed now and require

$$(4a - d\bar{K}\bar{Z})K = 0$$

with a possible solution given by

$$(4a - d\bar{K}\bar{Z}) = -d(I - KZ)$$

Finally we get for arbitrary d

$$A = \frac{1}{4}\Psi d(-\Phi + \bar{K}\bar{\Omega}) + \frac{1}{4}\bar{\Psi}d(-\bar{\Phi} + K\Omega) \quad (58)$$

Substituting into the original expansion a and b in terms of d and the moduli, we have for A

$$A = Re\Psi(-\frac{d}{2}[I - KZ - \bar{K}\bar{Z}])\alpha - (iRe\Psi\frac{d}{4}[(I - 2KZ)\mathcal{F} - (I - 2\bar{K}\bar{Z})\bar{\mathcal{F}}] - Im\Psi dN)\beta. \quad (59)$$

Recalling our requirement to have the coefficient in front of $Im\Psi$ not dependent on the moduli, and since d is a real matrix, it can be absorbed in a redefinition of Ψ. As a matter of later convenience we choose $d = -2(2^{\frac{1}{4}})N^{-1}$. A final rewriting of A can be done in the form

$$A = f^T\alpha + h^T\beta \quad (60)$$

where

$$f^T = \frac{2^{\frac{1}{4}}}{2}Re\Psi R^{-1}$$

$$h^T = \frac{2^{\frac{1}{4}}}{2}(iRe\Psi[(I - 2KZ)\mathcal{F} - (I - 2\bar{K}\bar{Z})\mathcal{F}] - 4Im\Psi) \quad (61)$$

Now the invariance of A under the symplectic transformation of the basis given by (51) can be assured provided the coefficients f and h transform as

$$\delta h = -B^T h + Cf$$
$$\delta f = -Dh + Bf \quad (62)$$

From the expression above we can deduce the transformation of the scalar field Ψ:

$$\delta\Psi_I = -\Psi_J(B_I^J + 4iD^{JL}N_{LI}) \quad (63)$$

The expression for $\delta\Psi$ together with those displayed in equation (52) can be used to show the invariance under symplectic transformations of the Lagrangian later obtained in the compactification.

Before we close this section we think it will be of interest to quote a result from our earlier work on the type IIB theory. We found that the general invariant expansion for the vector field strength was given in that case by

$$F^-((I - \frac{(N\bar{Z})\bar{Z}}{(\bar{Z}N\bar{Z})})\Phi - \frac{(N\bar{Z})}{(\bar{Z}N\bar{Z})}\bar{\Omega}) + h.c.$$

with

$$\delta F^{-I} = F^{-J}(B_J^{TI} - 4i\bar{N}_{JL}D^{LI})$$

Although there is a remarkable similarity between the variation for Ψ and the variation for F^-, they are not the same, as it should be expected given that the coefficients of the $(2,1)$ and $(0,3)$ forms in the expansion are different in the two cases. Furthermore, the transformations of the vector field strength can be viewed as generalizations of the duality transformations in electromagnetism, while in the present case we do not have a comparable physical interpretation for the transformation of the scalar Ψ.

5.2 Compactification of the Lagrangian

Let us recall from the preliminary section the terms in the original $10D$ Lagrangian that will include the H^3 sector of the three form A

$$\mathcal{L}_{H^3} = -\frac{1}{48}e_{10}\phi^{\frac{3}{4}}(F_{\hat{\mu}_1\ldots\hat{\mu}_4})^2 + \frac{\sqrt{2}}{48}\varepsilon^{\hat{\mu}_1\ldots\hat{\mu}_{10}}F_{\hat{\mu}_1\ldots\hat{\mu}_4}F_{\hat{\mu}_5\ldots\hat{\mu}_8}B_{\hat{\mu}_9\hat{\mu}_{10}} \quad (64)$$

Any other combinations contribute only to the integrals involving $A_{\mu\nu\rho}$, as discussed earlier or simply vanish because there are no surviving integrals that mix $(2,1)$ and $(3,0)$ forms with the $(1,1)$ forms. The field strength defined as $F = dA = \partial_\mu A\, dx^\mu$ yields, after some simplifications:

$$F_\mu = \partial_\mu A$$

$$= \frac{2^{\frac{1}{4}}}{2}[\partial_\mu \Psi N^{-1} - (\Psi+\bar{\Psi})N^{-1}(K\partial_\mu Z + \frac{1}{4}\bar{\mathcal{G}}\partial_\mu \bar{Z} N^{-1})]_I \Phi^J$$

$$+ \frac{2^{\frac{1}{4}}}{2}[\partial_\mu \Psi^I (N^{-1}\bar{K})_I - (\Psi+\bar{\Psi})^I (N^{-1})\frac{\partial \bar{K}_J}{\partial Z^L}\partial^\mu Z^L]\bar{\Omega} + h.c. \quad (65)$$

where $\bar{\mathcal{G}}$ stands for F_{IJK} and we have used

$$\partial_\mu N^{-1} = -N^{-1}\partial_\mu N N^{-1}$$

Let us divide the Lagrangian in equation (64) into the two natural elements and treat them separately. For the non-topological integral we have

$$S_{H^3(ntop)} = -\frac{1}{48}\int d^4x e\phi^{\frac{3}{4}} \int i d^6\xi \sqrt{g}(24F_{\mu i\bar{j}\bar{n}}F_{\nu \bar{l} \bar{m} k} + 8F_{\mu ijk}F_{\nu \bar{l}\bar{m}\bar{n}})g^{\mu\nu}g^{i\bar{l}}g^{j\bar{m}}g^{k\bar{n}} \quad (66)$$

By expressing the forms Φ and Ω into their components in the equation for F_μ and performing the integrals following equations (16) and (17) we get

$$S_{H^3(ntop)} = \sqrt{2}\int d^4x e\phi^{\frac{3}{4}}(\partial_\mu \Psi N^{-1} - (\Psi+\bar{\Psi})N^{-1}(K\partial_\mu Z + \frac{1}{4}\bar{\mathcal{G}}\partial^\mu \bar{Z} N^{-1}))_I$$

$$(\partial^\mu \bar{\Psi} N^{-1} - (\Psi+\bar{\Psi})N^{-1}(\bar{K}\partial^\mu \bar{Z} + \frac{1}{4}\mathcal{G}\partial^\mu Z N^{-1}))_J \mathcal{M}_{IJ}$$

$$-\sqrt{2}\int d^4x e\phi^{\frac{3}{4}}|\partial_\mu \Psi(N^{-1}\bar{K}) - (\Psi+\bar{\Psi})N^{-1}\frac{\partial \bar{K}}{\partial Z}\partial^\mu Z|^2(\bar{Z}NZ) \quad (67)$$

Combining the two parts of the action together with the same Weyl rescaling performed in the earlier sections

$$S_{H^3(ntop)} = \int d^4x \frac{e}{2\tilde{\phi}}(\partial_\mu \Psi - \frac{1}{2}(\Psi+\bar{\Psi})R^{-1}\partial_\mu \mathcal{N})R^{-1}(\partial^\mu \bar{\Psi} - \frac{1}{2}\partial_\mu \bar{\mathcal{N}}R^{-1}(\Psi+\bar{\Psi})) \quad (68)$$

where $\tilde{\phi} = 2\mathcal{V}(v)\phi^{-3}$.

We have made use of the following expressions in order to recombine the various terms

$$\frac{\partial \bar{K}_I}{\partial Z^J} = \frac{N_{IJ}}{(\bar{Z}NZ)} - \bar{K}_I K_J$$

$$R^{-1}\partial_\mu \mathcal{N} R^{-1} = \frac{1}{\bar{Z}NZ}[-(\bar{Z}\partial_\mu Z + \partial_\mu Z\bar{Z}) + K\partial_\mu Z(Z\bar{Z} + \bar{Z}Z)] - \frac{1}{4}N^{-1}\bar{\mathcal{G}}N^{-1}\partial^\mu \bar{Z} \quad (69)$$

Let us now return to the second term in Lagrangian (64), involving the topological integral.

$$S_{H^3(top)} = \frac{\sqrt{2}}{(48)^2}\int d^4x \varepsilon^{\mu\nu\rho\lambda}B_{\mu\nu}\int_K i d^6\xi \varepsilon^{ikm}\varepsilon^{\bar{j}\bar{l}\bar{n}}[16F_{\rho ikm}F_{\lambda\bar{j}\bar{l}\bar{n}} + (12)^2 F_{\rho i\bar{j}\bar{n}}F_{\lambda \bar{l} m k}] \quad (70)$$

Just as in the case of the non-topological integral we can give the component expansion for the field strength form F and by using the integrals displayed in the preliminary section

calculate the expression above. It can be seen immediately that we obtain under the integral over CY manifold the same expression encountered for $S_{H^3(ntop)}$.

$$S_{H^3(top)} = \frac{1}{2}\int d^4x \varepsilon^{\mu\nu\rho\lambda}B_{\mu\nu}(\partial_\lambda \Psi - \frac{1}{2}(\Psi+\bar{\Psi})R^{-1}\partial_\lambda \mathcal{N})R^{-1}(\partial_\rho \bar{\Psi} - \frac{1}{2}\partial_\rho \bar{\mathcal{N}}R^{-1}(\Psi+\bar{\Psi})) \quad (71)$$

From this point on follows a series of manipulations of the previous equation for $S_{H^3(top)}$ as to be able to recast it in the standard form. Let us first expand the integrand as to treat the various pieces independently.

$$S_{H^3(top)} = \frac{1}{2}\int d^4x \varepsilon^{\mu\nu\rho\lambda}B_{\mu\nu}[\partial_\lambda \Psi R^{-1}\partial_\rho \bar{\Psi} - \frac{1}{2}\partial_\lambda \Psi R^{-1}\partial_\rho \bar{\mathcal{N}}R^{-1}(\Psi+\bar{\Psi})$$

$$-\frac{1}{2}(\Psi+\bar{\Psi})R^{-1}\partial_\lambda \mathcal{N}R^{-1}\partial_\rho \bar{\Psi} + \frac{1}{4}(\Psi+\bar{\Psi})R^{-1}\partial_\lambda \mathcal{N}R^{-1}\partial_\rho \bar{\mathcal{N}}R^{-1}(\Psi+\bar{\Psi})] \quad (72)$$

We will focus initially on the last term.

$$S_{H^3(last)} = \frac{1}{4}\int d^4x \varepsilon^{\mu\nu\rho\lambda}B_{\mu\nu}(\Psi+\bar{\Psi})R^{-1}\partial_\lambda \mathcal{N}R^{-1}\partial_\rho \bar{\mathcal{N}}R^{-1}(\Psi+\bar{\Psi}) \quad (73)$$

Recalling that $R = \frac{\mathcal{N}+\bar{\mathcal{N}}}{2}$ and using

$$R^{-1}\partial_\rho R R^{-1} = -\partial_\rho R^{-1}$$

we get

$$S_{H^3(last)} = \frac{1}{4}\int d^4x \varepsilon^{\mu\nu\rho\lambda}B_{\mu\nu}((\Psi+\bar{\Psi})R^{-1}\partial_\lambda \bar{\mathcal{N}}\partial_\rho R^{-1}(\Psi+\bar{\Psi}) + (\Psi+\bar{\Psi})\partial_\lambda R^{-1}\partial_\rho \mathcal{N}R^{-1}(\Psi+\bar{\Psi})) \quad (74)$$

where any other contributing terms have cancelled due to the antisymmetry of $\varepsilon^{\mu\nu\rho\lambda}$. Integrating by parts we get:

$$S_{H^3(last)} = -\frac{1}{4}\int d^4x \varepsilon^{\mu\nu\rho\lambda}[B_{\mu\nu}((\Psi+\bar{\Psi})R^{-1}\partial_\lambda \bar{\mathcal{N}}R^{-1}\partial_\rho(\Psi+\bar{\Psi}) + \partial_\lambda(\Psi+\bar{\Psi})R^{-1}\partial_\rho \mathcal{N}R^{-1}(\Psi+\bar{\Psi}))$$

$$+\frac{1}{2}(\Psi+\bar{\Psi})R^{-1}\partial_\lambda \bar{\mathcal{N}}R^{-1}(\Psi+\bar{\Psi})\partial_\rho B_{\mu\nu} + \frac{1}{2}(\Psi+\bar{\Psi})R^{-1}\partial_\rho \mathcal{N}R^{-1}(\Psi+\bar{\Psi})\partial_\lambda B_{\mu\nu}] \quad (75)$$

Adding this result to the other elements in the action and integrating by parts any remaining terms with $B_{\mu\nu}$,

$$S_{H^3(top)} = \frac{1}{(4)(3)}\int d^4x \varepsilon^{\mu\nu\rho\lambda}H_{\mu\nu\rho}[(\Psi+\bar{\Psi})R^{-1}\partial_\lambda \bar{\Psi} - (\Psi+\bar{\Psi})R^{-1}\partial_\lambda \bar{\mathcal{N}}R^{-1}(\Psi+\bar{\Psi})$$

$$-(\Psi+\bar{\Psi})R^{-1}\partial_\lambda \Psi - \frac{1}{4}(\Psi+\bar{\Psi})R^{-1}\partial_\lambda \mathcal{N}R^{-1}(\Psi+\bar{\Psi})] \quad (76)$$

with $H_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]}$.

At this stage we are ready to add to the last expression the contributions from the gravitational and H^0 sectors that involve $H_{\mu\nu\rho}$, and perform a duality transformation on the $B_{\mu\nu}$ field. Following reference [12] this is equivalent to adding a lagrange multiplier $i\partial_\lambda D H^\lambda$ with $H^\lambda = \frac{1}{6}\epsilon^{\mu\nu\rho\lambda}H_{\mu\nu\rho}$.

We have then for the sum of the two actions,

$$S_{H^3(top)} + S_{H^0(H_{\mu\nu\rho},\phi)} = \int d^4x e[-\tilde{\phi}^2(H_\lambda)^2 - \frac{1}{4}(\frac{\partial_\mu \tilde{\phi}}{\tilde{\phi}})^2 + H^\lambda(-i\partial_\lambda D + \frac{1}{2}(\bar{a}_\lambda - a_\lambda))] \quad (77)$$

where

$$a_\lambda = (\Psi + \bar{\Psi})R^{-1}\partial_\lambda \Psi - \frac{1}{4}(\Psi + \bar{\Psi})\partial_\lambda \mathcal{N} R^{-1}(\Psi + \bar{\Psi})$$

Integrating over H_λ and by utilizing the following relations

$$a_\lambda + \bar{a}_\lambda = \frac{1}{2}\partial_\lambda((\Psi + \bar{\Psi})R^{-1}(\Psi + \bar{\Psi}))$$

$$|\partial_\lambda \tilde{\phi} + i\partial_\lambda D - \frac{1}{2}(a_\lambda + \bar{a}_\lambda) + a_\lambda|^2 = (\partial_\lambda \tilde{\phi})^2 - (-i\partial_\lambda D + \frac{1}{2}(\bar{a}_\lambda - a_\lambda))^2 \quad (78)$$

we can finally write

$$S_{H^3(top)} + S_{H^0(H_{\mu\nu\rho},\phi)} = -\int d^4x e \frac{1}{4\tilde{\phi}^2}$$

$$|\partial_\mu \tilde{\phi} + i\partial_\mu D - \frac{1}{4}\partial_\mu[(\Psi + \bar{\Psi})R^{-1}(\Psi + \bar{\Psi})] + (\Psi + \bar{\Psi})R^{-1}\partial_\mu \Psi - \frac{1}{4}(\Psi + \bar{\Psi})R^{-1}\partial_\mu \mathcal{N} R^{-1}(\Psi + \bar{\Psi})|^2 \quad (79)$$

choosing

$$S = \tilde{\phi} + iD - \frac{1}{4}((\Psi + \bar{\Psi})R^{-1}(\Psi + \bar{\Psi}))$$

we can add the above result to the one obtained in (68) for S_{H^3} non-topological as:

$$S_{H^3} + S_{H^0(H_{\mu\nu\rho},\phi)} = -\int d^4x e[\frac{|\partial_\mu S + (\Psi + \bar{\Psi})R^{-1}\partial_\mu \Psi - \frac{1}{4}(\Psi + \bar{\Psi})R^{-1}\partial_\mu \mathcal{N} R^{-1}(\Psi + \bar{\Psi})|^2}{[S + \bar{S} + \frac{1}{2}(\Psi + \bar{\Psi})R^{-1}(\Psi + \bar{\Psi})]^2}$$

$$- \frac{[\partial_\mu \Psi - \frac{1}{2}(\Psi + \bar{\Psi})R^{-1}\partial_\mu \mathcal{N}]R^{-1}[\partial_\mu \bar{\Psi} - \frac{1}{2}\partial_\mu \bar{\mathcal{N}} R^{-1}(\Psi + \bar{\Psi})]}{[(S + \bar{S}) + \frac{1}{2}(\Psi + \bar{\Psi})R^{-1}(\Psi + \bar{\Psi})]}] \quad (80)$$

This expression is precisely the one encountered in the reduction from $D = 4$ to $D = 3$ of the $N = 2$ supergravity coupled to n-vector multiplets, and where the fields ranged over $(n + 1)$ variables. In our case the number of fields range over $h_{(2,1)} + 1$ variables, moduli, and we can conclude following the results of reference [10] that the scalars parametrize a quaternionic manifold of real dimension $4(h_{(2,1)} + 1)$. Since this proof uses the $(2, 1)$ forms cohomology, equation (80) should give the exact string (tree-level) effective Lagrangian for the hypermultiplet sector. The vector coupling for $(2, 1)$ forms in the type IIB theory, was already derived in reference [9] using the same methods of cohomology theory, as has been explained earlier.

In contrast the cubic form for the $f(W)$ function of the H^2 sector, is actually an artifact of the point-fixed theory limit of 10D superstrings, which gives the correct compactification only in the large volume limit. f has to be regarded as an asymptotic formula that is related to a Peccei-Quinn symmetry that is however affected by world-sheet non-perturbative effects [8]. This is not the case for the $(2, 1)$ forms for which the point-field limit should coincide with the exact (tree-level) string calculation. [8,20,21]. This result has recently been used to compute non-perturbative effects for the $(1, 1)$ moduli, by using classical results for the mirror manifolds where the even and odd cohomologies are interchanged [13,22,23].

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