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SYMMETRIES OF DUAL-QUATERNIONIC MANIFOLDS

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ABSTRACT

Classical superstring vacua corresponding to $c = 9$, $N = (2,2)$ superconformal theories lead to Kähler or quaternionic nonlinear sigma models that are characterized in terms of a holomorphic homogenous function of second degree. The Kähler manifolds may be invariant under the generalized duality invariances known from supergravity. Here we study the symmetry structure of the corresponding quaternionic manifolds, which have a symmetry group that is considerably larger. It turns out that the symmetry structure is encoded in a single function invariant under duality transformations. We discuss a number of examples and exhibit the existence of new symmetries.

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The classical superstring vacua corresponding to $c = 9$, $N = (2,2)$ superconformal theories lead to low-energy effective field theories with space-time supersymmetry in four dimensions. Under certain conditions the moduli space of the superconformal field theories is just the target space of the nonlinear sigma model that describes the scalar fields of the effective four-dimensional theory [1]. Recently it has been shown that this target space factorizes into a number of separate subspaces [2,3]. These subspaces are of a restricted type, in the sense that they are each determined in terms of a holomorphic function which is homogeneous and of second-degree. For type II strings, where one has $N = 2$ space-time supersymmetry, the effective field theory contains two nonlinear sigma models, one a restricted Kähler manifold (as required by $N = 2$ local supersymmetry [4]) associated with vector supermultiplets, the other a quaternionic manifold associated with scalar (hyper) supermultiplets, which is also restricted in the sense that it is determined in terms of a homogeneous holomorphic function. The latter restriction is not implied by local supersymmetry and is a consequence of string theory. These restricted manifolds are called dual-quaternionic [2] because they follow from the reduction to three dimensions of a four-dimensional $N = 2$ supergravity theory with vector multiplets upon a duality transformation that converts the three-dimensional vector fields into scalar fields. Depending on the type of string theory, the two homogeneous holomorphic functions describe two Kähler spaces or one Kähler and one dual-quaternionic space. Furthermore these two holomorphic functions completely determine the low-energy theory.

The reduction of a $d = 4$, $N = 2$ supergravity theory to three dimensions combined with a duality transformation thus defines a map between a restricted Kähler manifold of complex dimension n and a dual-quaternionic one of real dimension $4(n+1)$. This map was denoted by s_n in [2].The four-dimensional supergravity field equations (but not necessarily the action) often exhibit a symmetry that acts on the vector fields by means of generalized duality transformations [5,6]. Under the s-map these symmetries are preserved and are now realized as symmetries of the Lagrangian. Therefore the dual-quaternionic spaces must exhibit the same invariance. In addition, the conversion of (abelian) vector fields into scalars induces a number of extra symmetries (two for each of the $n+1$ vector fields, and furthermore two symmetries related to the fields that follow from the dimensional reduction of the graviton field). Therefore the dual-quaternionic spaces are at least invariant under the generalized duality transformations of the corresponding Kähler space and $2n + 4$ additional transformations [7]. Together these transformations constitute a non-semisimple group.

The purpose of this letter is to study the full isometry group of the dual-quaternionic spaces. It turns out that all information on the symmetry structure is encoded in a single function which is invariant under

duality transformations. Extra symmetries are possible if and only if the dependence of this function on the coordinates of the original Kähler manifold is restricted in a certain way. In particular, if the function does not depend on the Kähler coordinates at all, then the maximal number of extra symmetries is realized. This number is equal to $2n + 3$. However, this is only possible if the Kähler manifold is symmetric, and it represents the only case where the group of motions of the quaternionic manifold can be semisimple. The resulting quaternionic manifold is then symmetric as well.

We follow the standard notation for $N = 2$, $d = 4$ vector multiplets coupled to supergravity [8]. In order to couple n vector multiplets to $N = 2$ Poincaré supergravity, one starts from $n+1$ vector multiplets labelled by indices $I = 0,\ldots,n$. The corresponding $n+1$ scalar fields X^I appear only in the Lagrangian through the ratios $z^A \equiv X^A/X^0$ $(A = 1,\ldots,n)$. (For alternative parametrizations, see [9].) The action is determined by a holomorphic function $F(X)$ which is homogeneous of second degree in X. Therefore it satisfies identities such as $F = \frac{1}{2}F_I X^I$, $F_I = F_{IJ}X^J$, $X^I F_{IJK} = 0$, where the subscripts $I, J, \ldots$ denote differentiation with respect to X^I, X^J, etc. The following tensors, which depend only on the fields z^A, play an important role,

$$\begin{aligned} N_{IJ} &= \frac{1}{4}\left(F_{IJ} + \bar F_{IJ}\right), \\ \mathcal{M}_{I\bar J} &= N_{IJ} - \frac{(N\bar X)_I (NX)_J}{XN\bar X}, \\ \mathcal{N}_{IJ} &= \frac{1}{4}\bar F_{IJ} - \frac{(NX)_I (NX)_J}{XNX}, \end{aligned} \qquad (1)$$

where some contracted indices have been suppressed. For instance, the tensor $\mathcal{M}_{IJ}(zN\bar z)^{-1}$, which has a null vector proportional to X^I, appears as the metric in the kinetic term for the scalar fields, while the third tensor enters in the kinetic term for the vector fields.

After reduction to three dimensions and a duality transformation to replace the (abelian) vector fields by scalars, the Lagrangian acquires the form

$$\begin{aligned} e^{-1}\mathcal{L} &= \mathcal{M}_{A\bar B}\,\partial_\mu z^A\,\partial^\mu \bar z^B (zN\bar z)^{-1} + \frac{1}{4}\phi^{-1}(\mathcal{N} + \overline{\mathcal{N}})_{IJ}\, W^I_\mu\, \overline{W}^{J\mu} \\ &\quad -\frac{1}{4}\phi^{-2}\left\{\left(\partial_\mu\sigma - \frac{1}{2}A^I \overleftrightarrow{\partial}_\mu B_I\right)^2 + (\partial_\mu\phi)^2\right\}, \end{aligned} \qquad (2)$$

where

$$W^I_\mu = \left(\mathcal{N} + \overline{\mathcal{N}}\right)^{-1\,IJ}\left[2\overline{\mathcal{N}}_{JK}\partial_\mu A^K - i\partial_\mu B_J\right] \qquad (3)$$

is directly related to the field strengths of the original vector fields. This result was already obtained in [7]. The fields σ and ϕ originate from the four-dimensional graviton field, A^I from the component of the vector fields in the direction of the contracted coordinate, and B_I from the remaining three components after the duality transformation.

As discussed above the Lagrangian has a number of invariances. When the field equations of the original four-dimensional theory are invariant under generalized duality transformation, then the Lagrangian (2) will exhibit the same invariance. The generalized duality transformations constitute a subgroup of $Sp(2n+2,\mathbb{R})$. On generic $2(n+1)$ dimensional vectors (u^I, v_J) they act by means of real matrices $\mathcal{O}$ with a unit determinant satisfying

$$\mathcal{O}^{-1} = \Omega\,\mathcal{O}^{\rm T}\,\Omega, \;\; \text{where } \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \qquad (4)$$

Infinitesimal transformations can thus be decomposed as

$$\begin{aligned} \delta u^I &= B^I{}_J\, u^J - D^{IJ}\, v_J, \\ \delta v_I &= C_{IJ}\, u^J - B^J{}_I\, v_J, \end{aligned} \qquad (5)$$

where $B^I{}_J$, C_{IJ} and D^{IJ} are real constant $(n+1)$-by-$(n+1)$ matrices (note that C and D are symmetric). Both $(X^I, -\frac{1}{2}iF_J)$ and (A^I, B_J) transform under duality transformations according to (5). The fact that the transformation of F_I is already determined by the transformation of X^I poses restrictions on the duality transformations. This is expressed by the consistency equation [6]

$$iC_{IJ}\,X^I X^J - B^I{}_J\,F_I X^J - \frac{1}{4} iD^{IJ}\,F_I F_J = 0. \qquad (6)$$

We note that the transformation rules for A^I and B_I are in accordance with the fact that the "field strengths" $(W^I_\mu, 2i\mathcal{N}_{IJ}W^J_\mu)$ transform also according to (5). The fields ϕ and σ are invariant under the duality transformations.

In addition there are $2n+4$ additional invariances which originate from the gauge invariances of the original four-dimensional Lagrangian. We distinguish $2n+3$ shifts,

$$\begin{aligned} \delta A^I &= \alpha^I, & \delta\phi &= 0, \\ \delta B_I &= \beta_I, & \delta\sigma &= \tfrac{1}{2}\left(\alpha^I B_I - \beta_I A^I\right) + \epsilon^+. \end{aligned} \qquad (7)$$

and a scale transformation,

$$\begin{aligned} \delta A^I &= -\tfrac{1}{2}\epsilon^0 A^I, & \delta\phi &= -\epsilon^0\phi, \\ \delta B_I &= -\tfrac{1}{2}\epsilon^0 B_I, & \delta\sigma &= -\epsilon^0\sigma. \end{aligned} \qquad (8)$$

The root lattice corresponding to all these symmetries consists of the root lattice of the duality invariance extended with one dimension associated with the eigenvalue of the roots under the scale symmetry (8). This

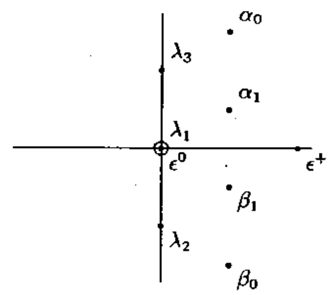

Figure 1: Root lattice corresponding to the $SU(1,1)$ duality invariance of the $n=1$ Kähler manifold based on $F(X) = i(X^1)^3/X^0$, extended with the roots belonging to the shifts (7) and the scale transformation (8). The $SU(1,1)$ transformations associated with the roots λ_1, λ_2 and λ_3 have been worked out in detail in [10]

leads to a root lattice such as shown in Fig. 1, where we have exhibited the case where the duality invariances constitute a group of rank 1 (namely $SU(1,1)$) and $n=1$. In the general case we obtain a similar diagram after projecting all the roots on a suitably chosen plane. The algebra $\mathcal{V}$ corresponding to these roots, which is obviously non-semisimple, can generally be decomposed into eigenspaces of the generator associated with (8) in the adjoint representation. We have

$$\mathcal{V} = \mathcal{V}_0 + \mathcal{V}_{\frac{1}{2}} + \mathcal{V}_1, \qquad (9)$$

where the generators contained in $\mathcal{V}_a$ thus satisfy $[e_0, \mathcal{V}_a] = a\,\mathcal{V}_a$. Here e_0 denotes the generator of the scale transformation (with corresponding parameter ϵ_0). As emphasized in [2] the inverse s-map plays a role in the classification of *normal* quaternionic spaces as given by Alekseevskiĭ [11]. Normal quaternionic spaces are quaternionic spaces that admit a transitive completely solvable group of motions. As conjectured in [11], the homogeneous quaternionic spaces consist of compact symmetric quaternionic and normal quaternionic spaces. The algebra corresponding to the group of solvable motions in the latter case exhibits the same decomposition as in (9) (in this analysis e_0 is defined as a one-dimensional invariant subalgebra). According to Alekseevskiĭ there are two different types of normal quaternionic spaces characterized by their so-called canonical quaternionic subalgebra. The first type with subalgebra C_1^1 turns out to correspond to the quaternionic projective spaces $Sp(1,n+1)/(Sp(1)\otimes Sp(n+1))$. The solvable algebra decomposes as in (9), where $\mathcal{V}_0$ contains only the generator e_0, while $\mathcal{V}_{\frac{1}{2}}$ and $\mathcal{V}_1$ have dimension $4n$ and 3, respectively. The second type has a canonical subalgebra denoted by A_1^1 and the structure of the solvable algebra is as follows: $\mathcal{V}_0$ contains e_0 and a solvable subalgebra of dimension $2n$, $\mathcal{V}_{\frac{1}{2}}$ has dimension $2n+2$ and $\mathcal{V}_1$ has dimension 1. Alekseevskiĭ then determines the possible structure of the so-called principal Kählerian subalgebra, which consists of $\mathcal{V}_0 + \mathcal{V}_1$. This subalgebra corresponds to a normal Kähler space, i.e. a Kähler space which admits a solvable transitive group of motions. Application of the s-map yields again the full normal quaternionic space. Hence for normal Kählerian spaces the s-map is closely related to the work in [11]. These observations were used in [12] to classify the normal Kählerian spaces that can be coupled to $N=2$ supergravity and to determine their corresponding holomorphic function $F(X)$.

In this paper we want to investigate whether extra symmetries are possible, beyond the ones found above. It turns out that this is indeed the case, at least if certain conditions are satisfied. The extra generators have all negative eigenvalues with respect to e_0. For instance, in the example introduced in Fig. 1, the roots corresponding to the extra symmetries are all located on the left half-plane, as we shall exhibit later. The general structure of the algebra associated with the full group of isometries is

$$\mathcal{V} = \mathcal{V}_{-1} + \mathcal{V}_{-\frac{1}{2}} + \mathcal{V}_0 + \mathcal{V}_{\frac{1}{2}} + \mathcal{V}_1, \tag{10}$$

where, for symmetric spaces, the dimension of $\mathcal{V}_{-1}$ and $\mathcal{V}_{-\frac{1}{2}}$ is equal to 1 and $2n+2$, respectively. Otherwise the dimension of $\mathcal{V}_{-1}$ is 0, and the dimension of $\mathcal{V}_{-\frac{1}{2}}$ is less than or equal to $n+1$. The fact that there are no new generators with positive eigenvalues shows that $\mathcal{V}_1$ is always of dimension 1. Therefore the quaternionic projective spaces (which comprise the class of normal quaternionic spaces with canonical subalgebra C_1^1) are *not* in the image of the s-map. This confirms a result derived in [2] on the basis of different arguments. Furthermore, one can prove that each additional root corresponds not only to an extra isometry, but also leads to a new isotropy transformation of the manifold. Therefore it follows from a simple counting argument that homogeneous Kähler manifolds will always lead to homogeneous quaternionic manifolds, and vice versa.

In the attempt to determine all isometries of the Lagrangian (2) it is convenient to first introduce the function

$$\Delta \equiv \delta\sigma + \tfrac{1}{2}B_I\,\delta A^I - \tfrac{1}{2}A^I\,\delta B_I \tag{11}$$

linear in the symmetry variations of the fields σ, A^I and B_I. Requiring the vanishing of those variations of the Lagrangian quadratic in $\partial_\mu\phi$ and $\partial_\mu\sigma$ implies that

$$\Delta = (\sigma^2 - \phi^2)\Delta_2 + \sigma\Delta_1 + \Delta_0, \tag{12}$$

where the Δ_2, Δ_1 and Δ_0 are functions of A^I, B_I and z^A only, while the variation of ϕ is given by

$$\frac{\delta\phi}{\phi} = \frac{\partial\Delta}{\partial\sigma} = 2\sigma\Delta_2 + \Delta_1. \tag{13}$$

Subsequently one considers the variations linear in $\partial_\mu A^I$ or $\partial_\mu B_I$ and in $\partial_\mu\sigma$ or $\partial_\mu\phi$. This leads to a full determination of δA^I and δB_I in terms of the above functions and shows that Δ_2 does not depend on A^I and B_I. From the variations quadratic in $\partial_\mu A^I$ and $\partial_\mu B_I$ one infers that Δ_1 depends at most linearly on A^I and B_I. One also finds an expression for the variation of $\mathcal{N}$, which turns out to be independent of ϕ and σ. As variations of z^A must necessarily produce a nonzero variation of $\mathcal{N}$ (this follows from the analysis presented in [6]) we conclude that δz^A is independent of ϕ and σ. Therefore all the variations of the Lagrangian proportional to $\partial_\mu\phi\,\partial^\mu z^A$ and $\partial_\mu\sigma\,\partial^\mu z^A$ are known, from which we derive that Δ_2, Δ_1 and Δ_0 are independent of z^A. Hence we can parametrize Δ_2 and Δ_1 by

$$\Delta_2 = \epsilon^- \,, \qquad \Delta_1 = \hat\alpha^I B_I - \hat\beta_I A^I - \epsilon^0, \tag{14}$$

where we have introduced $2n+3$ parameters corresponding to possible new transformations, ϵ^-, $\hat\alpha^I$ and $\hat\beta_I$. Observe that the parameter ϵ^0 has already been introduced before and corresponds to the scale transformation (8).

The transformation rules for A^I and B_I now take the form

$$\begin{aligned} \delta A^I &= (\Delta_2\sigma + \tfrac{1}{2}\Delta_1)\,A^I + \partial^I f, \\ \delta B_I &= (\Delta_2\sigma + \tfrac{1}{2}\Delta_1)\,B_I - \partial_I f, \end{aligned} \tag{15}$$

where ∂^I and ∂_I denote the derivatives with respect to B_I and A^I, respectively, and

$$\begin{aligned} f &= \sigma\Delta_1 + \Delta_0 - \tfrac{1}{2}\phi\,\mathcal{D}\left(W^I(\mathcal{N} + \bar{\mathcal{N}})_{IJ}\bar W^J\right), && (16)\\ \mathcal{D} &= \epsilon^- + \hat\alpha^I\partial_I + \hat\beta_I\partial^I, && (17)\\ W^I &= \left(\mathcal{N} + \bar{\mathcal{N}}\right)^{-1\,IJ}\left[2\bar{\mathcal{N}}_{JK}A^K - iB_J\right]. && (18) \end{aligned}$$

The transformation law found for $\mathcal{N}_{IJ}$ takes the form implied by a duality transformation (cf. (3.5) of [6]),

$$\delta\mathcal{N}_{IJ} = -\frac{1}{2}iC_{IJ} - 2\mathcal{N}_{K(I}B^K{}_{J)} + 2i\mathcal{N}_{IK}D^{KL}\mathcal{N}_{LJ}, \tag{19}$$

with z^A-independent matrices

$$\begin{aligned} C_{IJ} &= -\tfrac{1}{2}\mathcal{D}(B_IB_J) - (\Delta_0)_{IJ}, \\ D^{IJ} &= -\tfrac{1}{2}\mathcal{D}(A^IA^J) - (\Delta_0)^{IJ}, \\ B^I{}_J &= -\tfrac{1}{2}\mathcal{D}(A^IB_J) + (\Delta_0)^I{}_J. \end{aligned} \tag{20}$$

We can then use the analysis in [6] to conclude that X^I must transform according to the corresponding duality transformation (up to a term proportional to X^I itself)

$$\begin{aligned}\delta X^I &= \left(B^I{}_J + \tfrac{1}{2}iD^{IK}F_{KJ}\right)X^J + X^I\Lambda,\\ &= -2i\frac{\partial}{\partial \mathcal{B}_I}\frac{\partial\Delta_0}{\partial\bar{\mathcal{B}}_J}N_{JK}X^K - \frac{1}{2}\mathcal{D}\left(A^I\mathcal{B}_J X^J\right) + X^I\Lambda. \end{aligned}\tag{21}$$

where Λ is undetermined. We introduced here the quantities

$$\mathcal{B}_I = B_I + \tfrac{1}{2}iF_{IJ}A^J = N_{IJ}\mathcal{B}^J. \tag{22}$$

Furthermore the matrices (20) must satisfy the consistency equation (6) while the variation of the Lagrangian proportional to $\partial_\mu z^A\,\partial^\mu z^B$ and $\partial_\mu z^A\,\partial^\mu \bar{z}^B$ cancels.

On the other hand we can also determine the variation δX^I (up to terms proportional to X^I) from the remaining variations of the Lagrangian proportional to $\partial_\mu z^A\,\partial^\mu A^I$ and $\partial_\mu z^A\,\partial^\mu B_I$. This leads to the following expression

$$\delta X^I = \mathcal{D}\left(-\frac{i}{2}\bar{\mathcal{B}}^I(X^J\mathcal{B}_J) + \frac{i}{16}(N^{-1})^{IJ}\mathcal{B}^K\bar{F}_{JKL}\mathcal{B}^L(XN\bar{X})\right) + X^I\Lambda' + \delta^0X^I, \tag{23}$$

where Λ' is undetermined and δ^0X^I is the variation independent of A^J and B_J.

We have now three equations to determine Δ_0. For convenience we first introduce the notation

$$\begin{aligned} D &\equiv X^I N_{IJ}\frac{\partial}{\partial\bar{\mathcal{B}}_J} = \frac{i}{2}X^I\partial_I + \frac{1}{4}F_I\partial^I,\\ D_I &\equiv \left.\frac{\partial}{\partial X^I}\right|_{A,B} = \left.\frac{\partial}{\partial X^I}\right|_{\mathcal{B},\bar{\mathcal{B}}} + \frac{1}{4}F_{IJK}\left(\mathcal{B}^K - \bar{\mathcal{B}}^K\right)\frac{\partial}{\partial\bar{\mathcal{B}}_J}. \end{aligned}\tag{24}$$

The only nonvanishing commutator of D_I, $\bar{D}_I$, D, $\bar{D}$ and $\mathcal{D}$ is

$$[D_I, D] = N_{IJ}\frac{\partial}{\partial\bar{\mathcal{B}}_J}, \tag{25}$$

and its complex conjugate. The first equation follows from combining (21) and (23),

$$\begin{aligned}\left(\frac{\partial}{\partial\mathcal{B}_I} + \frac{\partial}{\partial\bar{\mathcal{B}}_I}\right)D\Delta_0 &= \frac{i}{2}\delta^0X^I + X^I\Lambda''\\ &+\mathcal{D}\left(\frac{1}{8}(\mathcal{B}^I + \bar{\mathcal{B}}^I)(X^J\mathcal{B}_J) - \frac{1}{32}(N^{-1})^{IJ}\bar{F}_{JKL}\mathcal{B}^K\mathcal{B}^L(XN\bar{X})\right),\end{aligned}\tag{26}$$

(where Λ'' is again undetermined). The second one is the consistency equation following from (6). This gives

$$D^2\Delta_0 = \tfrac{1}{8}\mathcal{D}\left(\mathcal{B}_J X^J\right)^2. \tag{27}$$

The third one follows from the fact that Δ_0 is independent of X^I,

$$D_I\Delta_0 = 0. \tag{28}$$

Acting with D_ID_J on (27) and using (26) and (28) gives

$$\begin{aligned}\left(N\frac{\partial}{\partial\bar{\mathcal{B}}}\right)_I\left(N\frac{\partial}{\partial\bar{\mathcal{B}}}\right)_J\Delta_0 &= -\frac{i}{8}F_{IJK}\,\delta^0X^K\\ &+\frac{1}{8}\mathcal{D}\left(\mathcal{B}_I\mathcal{B}_J - \frac{1}{2}F_{IJK}\bar{\mathcal{B}}^KX^L\mathcal{B}_L + \frac{1}{16}F_{IJK}N^{-1\,KL}\bar{F}_{LMN}\mathcal{B}^M\mathcal{B}^N(XN\bar{X})\right).\end{aligned}\tag{29}$$

This can be integrated to

$$\begin{aligned}\Delta_0 &= -\mathcal{D}h - \left(\frac{i}{16}F_{IJK}\bar{\mathcal{B}}^I\bar{\mathcal{B}}^J\delta^0X^K + h.c.\right)\\ &+\theta^{IJ}\mathcal{B}_I\bar{\mathcal{B}}_J + \left(\gamma^I\mathcal{B}_I + h.c.\right) + \epsilon^+,\end{aligned}\tag{30}$$

where θ^{IJ}, γ^I, and ϵ^+ are integration "constants" corresponding to parameters of possible symmetries, and h is equal to

$$\begin{aligned}h &= -\frac{1}{16}\Big\{(\mathcal{B}_I\bar{\mathcal{B}}^I)^2 - \frac{1}{6}\left[(F_{IJK}\bar{\mathcal{B}}^I\bar{\mathcal{B}}^J\bar{\mathcal{B}}^K)(X^L\mathcal{B}_L) + h.c.\right]\\ &+\frac{1}{16}(XN\bar{X})\bar{\mathcal{B}}^I\bar{\mathcal{B}}^JF_{IJK}N^{-1\,KL}\bar{F}_{LMN}\mathcal{B}^M\mathcal{B}^N\Big\}.\end{aligned}\tag{31}$$

We note that h is a real function, which is homogeneous of zeroth degree in X and $\bar{X}$ separately, and invariant under duality transformations.

Imposing (26, 27, 28) separately, one can verify that all terms in (30) with exception of the first term containing h, pertain to the known symmetries: θ^{IJ} is related to the duality transformations (with δ^0X^I the corresponding variation) and γ^I, $\bar{\gamma}^I$ and ϵ^+ to the shift transformations. The only remaining condition is that $\mathcal{D}h$ must be independent of X^I. If this is the case for a certain choice of ϵ^-, $\hat{\alpha}^I$ and $\hat{\beta}_I$, there are additional symmetries. To analyze this question we first evaluate the derivative of h with respect to X^I,

$$\begin{aligned}D_Ih &= \frac{1}{16}\Big\{-\frac{2}{3}\mathcal{B}^J\bar{\mathcal{B}}^K\bar{\mathcal{B}}^L\bar{\mathcal{B}}^M\bar{D}_JC_{IKLM}\\ &+\frac{1}{4}\mathcal{B}^J\mathcal{B}^K\bar{\mathcal{B}}^L\bar{\mathcal{B}}^M\bar{F}_{JKN}N^{-1NP}C_{ILMP}\Big\}.\end{aligned}\tag{32}$$

where C_{IJKL} is the symmetric tensor

$$C_{IJKL} \equiv \frac{1}{4}F_{IJKL}(XN\bar{X}) + (N\bar{X})_{(I}F_{JKL)} - \frac{3}{16}F_{M(IJ}N^{-1\,MN}F_{KL)N}(XN\bar{X}), \tag{33}$$

It satisfies the relations

$$\begin{aligned}X^IC_{IJKL} &= 0,\\ \bar{X}^M\frac{\partial}{\partial\bar{X}^M}C_{IJKL} &= -X^M\frac{\partial}{\partial X^M}C_{IJKL} = C_{IJKL}.\end{aligned}\tag{34}$$

The tensor (33) is proportional to the tensor $\mathcal{C}_{ABCD}$ introduced in [13], whose vanishing is a necessary and sufficient condition for the Kähler space to be symmetric. The precise relation is

$$\mathcal{C}_{ABCD}(z,\bar z) \equiv \frac{(X^0)^2}{XN\bar X}\, C_{ABCD}. \qquad (35)$$

In order that $\mathcal{D}h$ be independent of X^I one proves that the following equations must be satisfied,

$$\epsilon^- C_{IJKL} = \bar\xi^M C_{MIJK} = \xi^M \frac{\partial}{\partial \bar X^M} C_{IJKL} = 0, \qquad (36)$$

where ξ^I is defined by

$$\xi_I = \hat\beta_I + \tfrac12 i F_{IJ}\hat\alpha^J = N_{IJ}\xi^I. \qquad (37)$$

In addition to the duality transformations, the shifts (7) and the scale transformation (8), there are thus at most $2n+3$ additional symmetries characterized by the parameters ϵ^-, $\hat\alpha^I$ and $\hat\beta_I$. These parameters are subject to the conditions (36) and are only independent for symmetric Kähler spaces where $C_{IJKL}=0$. The corresponding transformation rules follow directly from combining the previous results, and take the form

$$\begin{aligned}
\delta\phi &= \phi\left(2\sigma\epsilon^- + \hat\alpha^I B_I - \hat\beta_I A^I\right),\\
\delta\sigma &= (\sigma^2-\phi^2)\epsilon^- + \tfrac12\sigma\left(\hat\alpha^I B_I - \hat\beta_I A^I\right) + \mathcal{D}h, \qquad (38)\\
\delta A^I &= \sigma\hat\alpha^I + \left(\epsilon^-\sigma + \tfrac12\hat\alpha^J B_J - \tfrac12\hat\beta_J A^J\right)A^I - \partial^I\mathcal{D}\left(h+\tfrac12\phi\mathcal{Z}_2\right),\\
\delta B_I &= \sigma\hat\beta_I + \left(\epsilon^-\sigma + \tfrac12\hat\alpha^J B_J - \tfrac12\hat\beta_J A^J\right)B_I + \partial_I\mathcal{D}\left(h+\tfrac12\phi\mathcal{Z}_2\right),\\
\delta X^I &= \mathcal{D}\left(-\tfrac12 i\bar{\mathcal{B}}^I X^J B_J + \tfrac{1}{16} i\, N^{-1\,IJ}\bar F_{JKL}B^K B^L (XN\bar X)\right) + X^I\Lambda,
\end{aligned}$$

where $\mathcal{D}$ and h were defined before and

$$\mathcal{Z}_2 \equiv W^I(\mathcal{N}+\overline{\mathcal{N}})_{IJ}\overline{W}^J. \qquad (39)$$

To illustrate these results, let us discuss a few examples. First the example with $n=1$ introduced in Fig. 1, which is based on the function $F(X) = i(X^1)^3/X^0$ and corresponds to the Kähler manifold $SU(1,1)/U(1)$ with an $SU(1,1)$ group of duality transformations. As this is a symmetric space, the function h should be independent of X. Already in this simple example the explicit determination of h involves a few hundred terms. Using REDUCE we found that these terms combine and give the simple result

$$h = \frac14 (A^0)^2(B_0)^2 - \frac{1}{12}(A^1)^3(B_1)^2 + \frac12 A^0A^1B_0B_1 - \frac12 (A^1)^3 B_0 + \frac{2}{27}A^0(B_1)^3, \qquad (40)$$

which is indeed independent of X. We also verified that this expression is invariant under the duality transformations found in [10].

Fig. 2 shows the root lattice corresponding to the full group of symmetries of the quaternionic space. Compared to the root lattice shown in Fig.1 there are $2n+3=5$ extra symmetries which extend the symmetry group to $G_{2,2}$. Furthermore they extend the isotropy group $U(1)$, which is already present in the original Kähler manifold, to $SU(2)\otimes SU(2)$. We are thus dealing with the quaternionic manifold $G_{2,2}/(SU(2)\otimes SU(2))$ (the coupling of the corresponding nonlinear sigma model to $N=2$ supergravity was derived from dimensional reduction of type-IIB supergravity in [14]; in the context of harmonic superspace it was constructed in [15]).

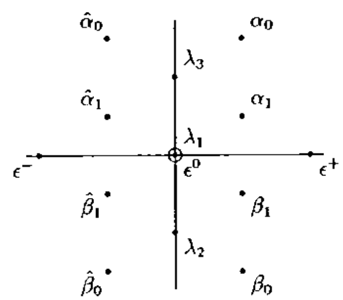

Figure 2: Extension of the root lattice introduced in Fig. 1, corresponding to the full group of motions of the quaternionic manifold. The isometry group is now extended to $G_{2,2}$.

Another example is the case of minimal coupling corresponding to the function $F(X) = X^I\eta_{IJ}X^J$, where η is a constant diagonal matrix with eigenvalues $(1,-1,\ldots,-1)$. The Kähler manifold is the complex projective space $U(1,n)/(U(1)\otimes U(n))$. The function h takes a simple form,

$$h = -\frac{1}{16}(A\eta A + B\eta B)^2, \qquad (41)$$

and is invariant under the $U(n,1)$ group of duality transformations. The resulting quaternionic manifold corresponds to $SU(2,n+1)/S(U(2)\otimes U(n+1))$ and is invariant under $(n+1)^2$ duality transformations, $2n+3$ shifts (7), the scale transformation (8) and $2n+3$ extra isometries, while the isotropies consist of n^2+1 duality transformations and $2n+3$ extra isotropies. Note that for $n=1$ the Kähler space is $SU(1,1)/U(1)$, just as in the previous example. However, the group of duality transformations contains an extra

$U(1)$ group (which does not act on the coordinates of the Kähler space). Therefore the group of isometries is of higher rank in this case and equal to $SU(2,2)$.

One may also consider the commutation relations of the isometries. It turns out that most of these relations are universal and do not depend on the function h (of course, whether or not the symmetries exist may still depend on h). Only those commutators that involve the duality transformations depend on the details of these transformations and on the function h. The function $\mathcal{D}h$, which is independent of X^I and $\bar{X}^I$, is just a third- or fourth-order polynomial in A^I and B_I. Its coefficients are related to the structure constants of the algebra associated with $\left[\mathcal{V}_{-\frac{1}{2}}, \mathcal{V}_{\frac{1}{2}}\right] \subset \mathcal{V}_0$. The commutation relations contain a lot of useful information. For instance, the commutator of two transformations in $\mathcal{V}_{-\frac{1}{2}}$ is given by

$$[\delta(\hat{\alpha}^I), \delta(\hat{\beta}_J)] = \delta(\epsilon^- = -\tfrac{1}{2}\hat{\alpha}^I\hat{\beta}_I). \tag{42}$$

This equation implies that if there are two extra symmetries characterized by parameters $\hat{\beta}_I$ and $\hat{\alpha}^I$, respectively, with $\hat{\alpha}^I\hat{\beta}_I \neq 0$ then the ϵ^- symmetry should be realized and the space is symmetric. This argument leads to the result quoted previously that the maximal dimension of $\mathcal{V}_{-\frac{1}{2}}$ is $n+1$ for non-symmetric spaces. Furthermore the commutators of the $\mathcal{V}_{-\frac{1}{2}}$ symmetries with duality transformations (characterized by matrices B, C, D) give rise to new $\mathcal{V}_{-\frac{1}{2}}$ symmetries with parameters

$$\hat{\alpha}'^I = B^I{}_J\,\hat{\alpha}^J - D^{IJ}\,\hat{\beta}_J\,; \qquad \hat{\beta}'_I = C_{IJ}\,\hat{\alpha}^J - B^J{}_I\,\hat{\beta}_J. \tag{43}$$

This may allow us to establish the presence of more $\mathcal{V}_{-\frac{1}{2}}$ symmetries once one such symmetry and the duality transformations are known.

Finally let us discuss the class of functions

$$F(X) = i d_{ABC}\frac{X^A X^B X^C}{X^0}. \tag{44}$$

Supergravity theories based on these functions can lead to flat potentials, as was shown in [10]. Furthermore they appear in the low-energy sector of superstring compactifications on (2,2) superconformal theories and may exhibit Peccei-Quinn-like symmetries [2]. This class of functions also contains all the normal Kähler spaces that can be coupled to $N = 2$ supergravity [12].

It turns out that the tensor C_{ABCD} defined in (35) depends only on $x^A \equiv i(z^A - \bar{z}^A)$. In fact it is a homogeneous function of the x^A of degree -1. Furthermore it satisfies $x^A C_{ABCD} = 0$. Imposing the conditions (36) it turns out that $\hat{\beta}_0$ remains unrestricted, whereas either $\hat{\alpha}^0$ or C_{ABCD} must vanish. Therefore, the quaternionic spaces corresponding to (44) have *always* an extra isometry associated with the parameter $\hat{\beta}_0$. Of course, to determine

all other possible symmetries requires more work. However, as we explained above, in certain cases where one knows the duality transformations, one can deduce from (43) that other isometries of the same type should exist as well. The general analysis for the duality transformations for (44) was given in [10] (see (6.21-26)). In particular we found that, if there exist a symmetric tensor C^{ABC} and constants a_A such that

$$d_{E(AB}C^{EFG}d_{CD)F}a_G = a_{(A}d_{BCD)}, \tag{45}$$

then there are duality transformations with $B^0{}_A = a_A$ and $D^{AB} = -\frac{4}{9}C^{ABC}a_C$. So from (43) we can then deduce, using the existence of β_0, that there must be extra symmetries with $\hat{\beta}_A = a_A$. Applying these arguments to the class of normal Kähler manifolds denoted by $K(p,q)$ in [12], we found two such solutions for a_A leading to duality transformations and thus to the existence of two more isometries associated with parameters $\hat{\beta}_A$. A more explicit discussion of this derivation is outside the scope of this letter, but we hope to return to this subject in the near future.

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