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ON THE IMPACT OF LINEAR COUPLING ON NONLINEAR DYNAMICS

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Abstract The impact of linear coupling interfering with sextupole fields on beam dynamics in HERA is studied analytically and by simulation. The analysis shows that linear coupling causes as another quality the excitation of skew resonances ($nQ_x + mQ_z = integer$,m odd) by nonlinear fields with midplane symmetry. The normal type resonances (m even) are reduced in strength. These effects occur in the vicinity of the coupling resonance $Q_x - Q_z = integer$. Broadening of the width of nonlinear resonance clusters is a consequence of the split in the eigenfrequencies. The dynamic aperture is reduced by a small but significant amount along the main tune diagonal. This depends on the size of the tune split and, in the case of randomly excited coupling, on the random distortions of the lattice functions. All these effects can easily be controlled by compensating the coupling resonance using skew quadrupoles. Our conclusion is that full linear coupling when operating on the coupling resonance does not significantly enhance nonlinear effects of sextupole fields.

INTRODUCTION

Usually, linear coupling between horizontal and vertical betatron oscillations is to be avoided on account of the operational problems due to the more complicated motion of the beam around the accelerator. Very little is known about the interference of linear coupling with the nonlinearities in an accelerator. In the HERA electron-proton collider this question is of special interest because of two reasons. The main dipole field generated by the superconducting magnets of the proton ring exhibits a considerable random skew quadrupole component ($a_2^{rms}/b_1 = 4.7 \cdot 10^{-4}$ @ $r = 25mm$ at injection fields[1] which excites the coupling resonance with an rms width of $\kappa_- = 0.031$. Only a single pair of skew quadrupoles was planned for HERA and there is some concern that residual effects of the skew quadrupole will further reduce the dynamic aperture which is limited by strong, persistent current dominated, nonlinear field errors.

The second reason why it is interesting to investigate this question for HERA is that a naturally flat electron beam will collide with a round proton beam. In $\bar{p} - p$ collisions it has been observed that when $\bar{p}$ and p beam sizes at the collision point are different, the large amplitude particles in the larger beam are lost quickly[2]. Therefore in HERA we expect that the protons with vertical oscillation amplitudes exceeding the electron beam size will be lost. Therefore, very much in contrast to $e^+ - e^-$-colliders, a round beam is desirable in the HERA electron ring for colliding beam operation. A convenient way of generating a large vertical emittance is linear coupling. The interference of linear coupling with the nonlinearities of the machine has therefore to be investigated.

We start our investigation by the formulation of nonlinear accelerator theory for coupled betatron oscillations. In particular, explicit expressions for the widths of nonlinear resonances will be derived for linearly coupled motion. In the case that coupling is generated by random errors, we are able to give expectation values for the width of nonlinear resonances. Using these expressions, the motion of the particles in a beam in the vicinity of resonances can be explained. In parallel, numerical particle tracking has been performed for HERA in order to check the results obtained by analytical treatment.

NONLINEAR RESONANCES IN THE PRESENCE OF LINEAR COUPLING

The motion of a particle in a circular accelerator with linear fields including skew quadrupoles and solenoids is described by the eigenvectors of the revolution matrix $\vec{v}_{\pm k}$ (where $k = 1,2$ denotes the two eigenmodes of the coupled system)

$$\vec{z}(s) = \sum_{k=1,2} \{A_k \cdot \vec{v}_k(s) + A_{-k}\vec{v}_{-k}\}.$$

The components of the eigenvectors are expressed by the linear lattice functions for the motion in the horizontal (x) and vertical (y) plane $z_{\pm k}(s) = \sqrt{\beta_{zk}(s)}e^{i\pm\phi_{zk}(s)}$ $(z = x, y)$. The complex expansion coefficients A_k are written in the form $A_{\pm k}(s) = \sqrt{2I_k}e^{\pm i\psi_k(s)}$, where I_k and ψ_k are constants of motion. If a small nonlinear distortion described by a potential $\mathcal{H} = \sum_{nm} a_{nm}x^n y^m$ is added to the system, these constants of motion will be allowed to vary $(I_k(s),\ \psi_k(s))$. Inserting this ansatz in the equation of motion including the distortion and making use of the properties of a symplectic matrix $(\vec{v}_l^{\,+}\underline{S}\vec{v}_k = \pm i\delta_{kl})$ (see ref[4]), we find that the perturbed constants obey the differential equations

$$\frac{\partial I_k}{\partial s} = -\frac{\partial \mathcal{H}}{\partial \psi_k}$$
$$\frac{\partial \psi_k}{\partial s} = \frac{\partial \mathcal{H}}{\partial I_k}.$$

Thus we obtain a result which is similar to the well known uncoupled case. In order to calculate nonlinear effects like the width of nonlinear resonances or nonlinear distortions we proceed in the usual way by expressing $\mathcal{H}$ by the perturbed constants and expanding it in a Fourier series. In order to study the impact of nonlinear resonances, one restricts the series to the slowly varying terms having chosen the tunes close to a single resonance $n_1Q_1 + n_2Q_2 = integer$. This yields an integrable system described by a Hamiltonian

$$\begin{aligned}\mathcal{K} = \quad & \Delta_1 I_1 + \Delta_2 I_2 \\ & + \sum_{nm}\sum_{k=0}^{n}\sum_{l=0}^{m}\sum_{\kappa=0}^{k}\sum_{\nu=-n+k}^{n-k}\sum_{\lambda=-l}^{l}\sum_{\mu=-m+l}^{m-l} B_{\bar{n}}\sqrt{I_1^{k+l}I_2^{n+m-k-l}} \\ & \times \cos(n_1\varphi_1 + n_2\varphi_2 + \varphi_{\bar{n}})\end{aligned}$$

with $\varphi_k = \psi_k + n_k\Delta_k 2\pi s/L$, s the path length along the beam orbit, L the circumference, $n_1 = \kappa+\lambda$, $n_2 = \nu+\mu$, $\Delta_k = n_k/(n_1^2+n_2^2)\cdot\Delta$, $\Delta = n_1Q_1+n_2Q_2+q$, q Fourier index, Q_1, Q_2 the tunes, and $\bar{n} = (n, m, k, l, \kappa, \nu, \lambda, \mu, q)$. The indices $\kappa, \lambda, \nu, \mu$ extend over either even or odd integers. The coefficients $B_{\bar{n}}$ are either driving or detuning terms $(n_1 = n_2 = q = 0)$

$$\begin{aligned} B_{\bar{n}} \quad & = \frac{1}{\pi}\binom{n}{k}\binom{m}{l}\binom{k}{\frac{k-\kappa}{2}}\binom{n-k}{\frac{n-k-\nu}{2}}\binom{l}{\frac{l-\lambda}{2}}\binom{m-l}{\frac{m-l-\mu}{2}} \\ & \Big| \oint ds a_{nm}\sqrt{\left(\frac{\beta_{x1}}{2}\right)^k\left(\frac{\beta_{x2}}{2}\right)^{n-k}\left(\frac{\beta_{y1}}{2}\right)^l\left(\frac{\beta_{y2}}{2}\right)^{m-l}} \\ & e^{i(\kappa\phi_{x1}(s) + \lambda\phi_{y1}(s) + \nu\phi_{x2}(s) + \mu\phi_{y2}(s) + \Delta 2\pi s/L)}\Big| \end{aligned}$$

The amplitude dependent width of the resonance is then calulated from the fixed point condition

$$\partial\mathcal{K}/\partial I_1 = \partial\mathcal{K}/\partial I_2 = \partial\mathcal{K}/\partial\varphi_1 = \partial\mathcal{K}/\partial\varphi_2 = 0$$

under the condition that $c = n_2 I_1 - n_1 I_2$ is a constant of motion, yielding Δ, the width of the nonlinear resonance for given amplitudes $\sqrt{I_1}, \sqrt{I_2}$. These formulae show that, for fields with midplane symmetry (m = *even*), normal (n_2 = *even*) and skew (n_2 = *odd*) type resonances are excited. In addition the number of terms which drive these resonances increases drastically. For normal sextupole fields, we find 28 different driving terms which drive 8 resonances. Without coupling, only 4 resonances are driven by 5 driving terms. In order to calculate the impact of these driving terms we have to insert the coupled linear lattice functions β_{zk}.

COUPLED LINEAR LATTICE FUNCTIONS

Linear lattice functions in the coupled case need to be provided to calculate the width of nonlinear resonances. They are easily calculated from the eigenvectors of the 4×4 revolution matrix which is a standard method implemented in beam optics codes such as PETROS[3]. In order to provide some qualitative understanding about the impact of coupling we prefer to derive the following useful formulae, which describe the coupled lattice functions in terms of the uncoupled linear lattice functions, in first order in the coupling generating fields. Here we want to restrict ourselves to skew quadrupole fields. The treatment is different for choosing the working points close to or far from the coupling resonance.

Far from the Coupling Resonance: We expand the eigenvectors in the coupled case in terms of the eigenvectors of the uncoupled case and insert this ansatz in the eigenvalue equation retaining only first order corrections. Using the orthogonality of the eigenvectors of a symplectic matrix we find the corrections for the eigenvectors. The lattice functions β_{x2}, β_{y1} which are zero in the uncoupled case are simply given by $\beta_{zk} = z_k^2 + z_{-k}^2, zk = y1, x2$. For randomly generated coupling, the corrections can be expressed in terms of the width of the coupling- and the sum resonance κ_-, κ^+

$$\kappa_\pm = \left| \frac{1}{2\pi} \oint ds N \sqrt{\beta_x \beta_y} \exp i(\phi_x \pm \phi_y) \right|$$

($N = \frac{e}{p}\partial B_x/\partial x$, skew quadrupole strength, B=magnetic field,e=elementary charge and p=particle momentum) and the uncoupled lattice functions $\beta_{x,y}, \phi_{x,y}$

$$\beta_{zk}^{rms} = \beta_z \cdot \pi^2 \left[\left(\frac{\kappa_-^{rms}}{\sin \pi(Q_x - Q_y)} \right)^2 + \left(\frac{\kappa_+^{rms}}{\sin \pi(Q_x + Q_y)} \right)^2 \right]$$

Usually, in circular accelerators, uncompensated coupling- or sum resonances are in the order of 10^{-2} or less. This shows, that perturbative coupling cannot have a large impact on nonlinear dynamics far from the sum or the coupling resonance.

On the Coupling Resonance: On the sum resonance, the motion becomes unstable which is qualitatively expressed by the above approximate formulae. Close to the coupling resonance, the result is definitely not valid. For this special case, we perform perturbation theory for two degenerated modes. We write the perturbed eigenvector as a linear combination of the eigenvectors of the unperturbed degenerated modes. Inserting this ansatz in the eigenvalue equation yields a homogeneous linear equation system. Solving for the characteristic determinant yields the split of the perturbed eigenvalues and the distorted eigenvectors. As a final result, we obtain the coupled lattice functions as a function of the distance from the resonance $\Delta = Q_x - Q_y$ and the width of the coupling resonance κ_-

$$\beta_{x2} = \beta_x \frac{4\kappa_-^2}{(\sqrt{\Delta^2 + 4\kappa_-^2} + \Delta)^2 + 4\kappa_-^2}$$

$$\beta_{x1} = \beta_x \frac{4\kappa_-^2}{(\sqrt{\Delta^2 + 4\kappa_-^2} - \Delta)^2 + 4\kappa_-^2}$$

On the coupling resonance, $\Delta = 0$, the mode 1 and mode 2 β–functions have the same value corresponding to exactly one half of the original unperturbed β–function $\beta_{z1} = \beta_{z2} = \frac{1}{2}\beta_z$. Moving away from the resonance, these formulae give the same result as those in the previous section if the contribution from the sum resonance term is neglected. For the nonlinear effects is it important to notice that on the coupling resonance, the coupled beta functions become independent of the strengths of the coupling. More details of the procedures described here are found in ref[4].

APPLICATION TO HERA AND COMPARISON WITH SIMULATION

The results have been applied to the HERA proton ring where the nonlinear field errors are dominated by random sextupolar field errors from the superconducting magnets. According to the result of superconducting magnet measurements[1] Gaussian distributded sextupole field errors with a standard deviation of $b_3^{rms} = 3 \cdot 10^{-4}$ at $r = 25mm$ referring to a bend angle of $\theta = 15mrad$ have been assumed. Higher order multipoles which are also present have been omitted in this study. Random and systematic coupling is introduced in the system by tilts of the quadrupole magnets in the arc. An rms tilt of $1mrad$ causes a tune split of $\kappa_- = |Q_1 - Q_2| = .01$ We first calculate the width of the resonances in the third order resonance cluster near the main diagonal for the case with and without coupling and compare it with numerical tracking calculations. For the coupled case we use, according to the formulae in the previous section, $\beta_{x1} = \beta_{x2} = \frac{1}{2}\beta_x$ and $\beta_{y1} = \beta_{y2} = \frac{1}{2}\beta_y$. For the width of the $\frac{1}{3}$-integer resonance for randomly distributed sextupole errors one obtains without coupling

$$\Delta = \frac{1}{8\pi\sqrt{2}} \frac{b_3^{rms}\theta}{r^2} \sqrt{M} \sqrt{< \beta_x^3 >} \sqrt{\varepsilon_1} \tag{1}$$

(ε_1, mode 1, horizontal beam emittance) and with coupling

$$\Delta = \frac{1}{32\pi} \frac{b_3^{rms}\theta}{r^2} \sqrt{M} \sqrt{< \beta_x^3 > +9 < \beta_x \beta_y^2 >} \sqrt{\varepsilon_k} \tag{2}$$

(brackets denote averaging over the dipole magnets M is the number of sextupole). Skew and normal resonances have the same width. TABLE I summarizes the width of the $3rd$-order resonances on the diagonal.

TABLE I: Comparison of 3rd order Resonance Widths for $b_3^{rms}\theta/r^2 = 0.02m^{-2}$ and $\varepsilon_1 = \varepsilon_2 = 1\pi mradmm$

resonance	Δ (without coupling)	Δ (with coupling)
$Q_1 - Q_2$	0.0000	0.0300
$3Q_1$	0.0162	0.0098
$Q_1 + 2Q_2$	0.0323	0.0316
$3Q_2$	0.0000	0.0098
$Q_2 + 2Q_1$	0.0000	0.0316

Tracking calculations have been performed for 5 different seeds of random numbers. The simulations and extraction of the acceptance or dynamic aperture are done the same way as described in references[5] The width of the coupling resonance has been adjusted to $\kappa_- = 0.03$. Fig 1a shows the tracking results and Fig 1b the analytical results. Since one compares an analytical expectation value with the result from a particular random seed, the widths of the resonance don't agree absolutely. Higher order detuning effects have not yet been taken into account. However a certain broadening of the resonance cluster due to coupling shows well on both analytic and numerical results. This broadening is due

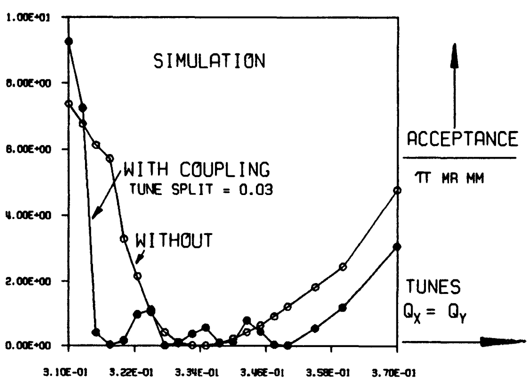

Figure 1a: Acceptance of HERA-P for $b_3^{rms}\theta/r^2 = 0.02$ near $Q_1 = Q_2 \simeq 0.033$ particle tracking. open circles: without coupling, full circles: with coupling, resonance width $\kappa_- = 0.03$.

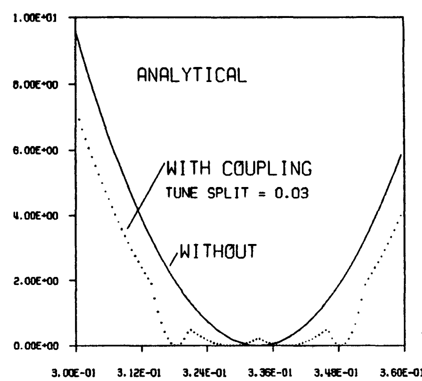

Figure 1b: Analytical Calculation of the Acceptance for the same conditions. The lines are without coupling, the dots are with a coupling resonance of strength $\kappa_- = 0.03$.

to the additional skew type resonances which do not completely overlap with the normal type resonances because the tunes Q_1 and Q_2 are split due to coupling. Tracking calculations around $\frac{1}{4}$-integer and $\frac{2}{7}$-integer resonances show that the broadening is occurs also at higher order resonances. It is important to notice, that the octupolar detuning as given by the formulae of the previous section, increases by about 50% due to coupling (for HERA-p). The simulations show that the second order sextupolar detuning is not changed significantly. Besides the broadening of major resonances, the tracking calculations for HERA show a small but significant reduction of the dynamic aperture of about $20 - 30\%$, if random coupling is switched on. A large part of this reduction is produced by the random distortion of the lattice functions due to the random skew quadrupoles. A similar reduction is a consequence of optical distortions by random normal gradient errors.

Consequently the perturbational part vanishes and the dynamic aperture improves if the random quadrupole tilts are replaced by systematic ones which produce the same tune split. There is still a residual 10 – 20% reduction of dynamic aperture due to coupling which is the effect of the higher density of resonance lines outside the main tune diagonal. If the strength of the coupling is reduced, the tunes can be adjusted closer to the diagonal. In this situation, although the motion is still fully coupled, the dynamic aperture is improving monotonically with decreasing width of the coupling resonance.

If the randomly generated coupling is compensated with distributed skew quadrupoles the effect of the optical distortion still leads to a reduction of the dynamic aperture. If only one pair of skew quadrupoles is used, this effect can be even enhanced by the lumped compensation. For HERA we find for some seeds a drastic reduction of the dynamic aperture if compensation is turned on. Off the tune diagonal, the coupling which would produce a tune spread of $\kappa_- = 0.03$, does not have an important effect on the beam dynamics in HERA-p. However, at some tunes, not too far from the diagonal, the influence of additional skew resonances is noticeable.

CONCLUSION

Nonlinear coupling introduces as a new quality the excitation of skew resonances by fields with midplane symmetry. Far away from the the main tune diagonal this is only of academic interest since the coupling fields in real accelerators are too weak to produce a significant effect. On the main diagonal, coupling causes a split of the tunes. Therefore, the normal resonances and the additional skew resonances do not completely overlap which leads to effective broadening of a resonance cluster near the main diagonal. As far as the strength of the resonances is concerned, the reduction of β-functions by a factor of two is balanced by the fact that there are more driving terms present, such that the single resonance widths are only insignificantly reduced. Two orthogonal families of skew quadrupoles are sufficient to control this effect. Due to the strong skew quadrupole component of the HERA dipole magnet, some difficulty in compensating the coupling effects with only two single skew quadrupoles in the HERA proton is possible.

If one wants to use the coupling to increase the vertical emittance of an electron beam in order to produce round beam cross sections for beam beam interaction, it is essential that the width of the coupling resonance is carefully minimized. Therefore, if one wants to make use of coupling, installation of skew quadrupoles is necessary. Under optimized conditions, operating on the coupling resonance with minimum width of the coupling resonance (which depends on how well one can control the tunes) linear coupling does not lead to a disadvantageous situation for beam dynamics with sextupoles.

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