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A GENERAL THEORY OF BEAM LOADING

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Abstract. A matrix formalism is developed for describing the interaction of the beam-bunch fundamental harmonic component with a radio-frequency accelerating cavity. The amplitude and phase modulations form the components of a vector. The adoption of a matrix notation systematises and, to some degree, automates the derivation of the characteristic polynomials which determine system stability. The method is applied to derive analogues of the Robinson criterion for complex systems including combinations of phase and radial loops, quadrupole-mode damping, and fast feed-back around the cavity.

BEAM-LOADING TRANSFER MATRICES

The first step is to find the beam current phase and amplitude response to modulations of the cavity gap voltage and phase. Figure 1 shows the phasor diagram.

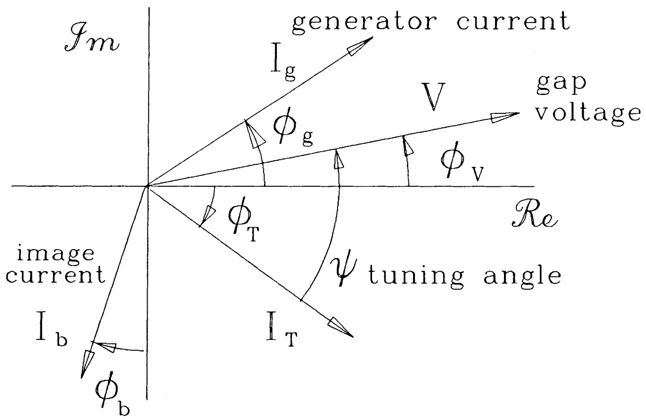

FIGURE 1 Co-relation of current vectors and accelerating voltage.

Let ϕ_b be the equilibrium bunch phase. The Laplace transform of the dipole mode equation is :

$$[(s/\Omega)^2 + \cos\phi_b]\Delta\phi_b \; + \; (\sin\phi_b, \;\cos\phi_b)\cdot(\Delta V, \; V\Delta\phi_V)/V = 0\;. \qquad (1)$$

Here s is the complex modulation frequency and Ω is the unperturbed synchrotron (angular) frequency. Let θ_0 be the equilibrium bunch half-length. The quadrupole mode equation is :

$$[(s/\Omega)^2 + 4\cos\phi_b]\Delta\theta \; + \; \theta_0(\cos\phi_b, \;0\;)\cdot(\Delta V, \; V\Delta\phi_V)/V = 0\;. \qquad (2)$$

Now $I_b f(\theta_0)\Delta\theta = \Delta I_b$ where I_b is the beam current fundamental and the form factor $f(\theta)$ is derived in the Appendix. Thus we may write (1) and (2) in matrix form :

$$\begin{pmatrix} \Delta I_b \\ I_b \Delta\phi_b \end{pmatrix} = (-)\Omega^2 \frac{I_b}{V} \begin{pmatrix} \frac{\theta_0 f(\theta_0)\cos\phi_b}{s^2+4\Omega^2\cos\phi_b} & 0 \\ \frac{\sin\phi_b}{s^2+\Omega^2\cos\phi_b} & \frac{\cos\phi_b}{s^2+\Omega^2\cos\phi_b} \end{pmatrix} \begin{pmatrix} \Delta V \\ V\Delta\phi_V \end{pmatrix} . \tag{3}$$

In an obvious symbolic notation we summarise (3) by $\mathbf{i_b} = \mathbf{Yv}$. Note (3) is diagonal for a non-accelerating beam ($\phi_b = 0$). The matrix equation relating cavity output voltage modulations to input current modulations is :

$$\begin{pmatrix} \Delta V \\ V\Delta\phi_V \end{pmatrix} = \frac{V}{I_T} \begin{pmatrix} G_C & G_S \\ -G_S & G_C \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \Delta I_T \\ I_T \Delta\phi_T \end{pmatrix} . \tag{4}$$

In symbolic notation : $\mathbf{v} = \mathbf{Zi_T}$. This is the dynamic cross-coupling. Explicit expressions for G_C and G_S are given below. The rotation matrix which pre-multiplies the total current vector occurs because the senses of rotation of ϕ_T and ϕ_V, as defined in Fig. 1, are opposite.

Close to resonance the cavity behaves like a lumped parallel resonance circuit with shunt resistance R, quality factor $Q = R\sqrt{C/L}$, and resonant (angular) frequency $\omega_0 = 1/\sqrt{LC}$. The cavity time constant is $\tau_c = (2Q/\omega_0)$. Steady state beam-loading compensation is achieved by detuning. The drive frequency is $\omega_c = \omega_0 + \Delta\omega$, where $\Delta\omega$ is negative below transition energy. Suppose $\tau_c\Omega << 1$. Let us define the denominator $D = (1+s\tau_c)^2 + \tan^2\psi$. Thence the modulation transfer functions[1] are :

$$G_C(s) = [(1+s\tau_c) + \tan^2\psi]/D \qquad \text{and} \qquad G_S(s) = +s\tau_c \tan\psi/D . \tag{5}$$

Under the condition of minimum power the detuning angle ψ is given by :

$$\tan\psi = (-)\Delta\omega \times \frac{2Q}{\omega_0} = \frac{I_b}{I_0}\cos\phi_b \qquad \text{where } I_0 \equiv \frac{V}{R} . \tag{6}$$

To the accelerating cavity, the generator current and the beam image current are indistinguishable – and so the 'drive' signal is their phasor sum, as shown in figure 1. This is the geometric cross-coupling. For small displacements, the superposition principle applies and so the total current modulation induced by simultaneous modulations of beam and generator currents is :

$$\begin{pmatrix} \Delta I_T \\ I_T\Delta\phi_T \end{pmatrix} = \begin{pmatrix} S_b & -C_b \\ C_b & S_b \end{pmatrix} \begin{pmatrix} \Delta I_b \\ I_b\Delta\phi_b \end{pmatrix} + \begin{pmatrix} C_g & -S_g \\ -S_g & -C_g \end{pmatrix} \begin{pmatrix} \Delta I_g \\ I_g\Delta\phi_g \end{pmatrix} \tag{7}$$

$$\text{where} \quad C_b = \cos(\phi_T - \phi_b) \quad \text{and} \quad S_b = \sin(\phi_T - \phi_b) . \tag{8a}$$

$$\text{where} \quad C_g = \cos(\phi_T + \phi_g) \quad \text{and} \quad S_g = \sin(\phi_T + \phi_g) . \tag{8b}$$

In an obvious symbolic notation : $\mathbf{i_T} = \mathbf{R_b i_b} + \mathbf{R_g i_g}$. Note, under optimal detuning $\phi_T = \psi$ and $I_0 = I_T \cos\psi$.

ROBINSON STABILITY

Consider the system block-diagram which shows the beam positive feedback loop.

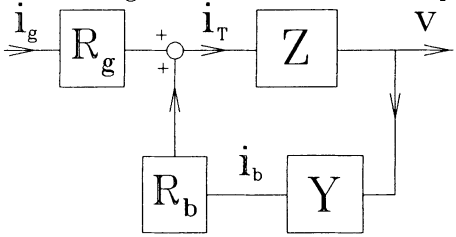

FIGURE 2 Beam and generator currents interact with cavity

The gap voltage modulation is given by :

$$\mathbf{v} = \mathbf{Z}\mathbf{i_T} = \mathbf{Z}(\mathbf{R_g}\mathbf{i_g} + \mathbf{R_b}\mathbf{i_b}) = \mathbf{Z}(\mathbf{R_g}\mathbf{i_g} + \mathbf{R_b}\mathbf{Y}\mathbf{v}) \; . \tag{9}$$

Rearranging and pre-multiplying by an inverse, we find : $\mathbf{v} = [\mathbf{I} - \mathbf{Z}\mathbf{R_b}\mathbf{Y}]^{-1}\mathbf{Z}\mathbf{R_g}\mathbf{i_g}$. The system is unstable if the determinant $|\mathbf{I} - \mathbf{Z}\mathbf{R_b}\mathbf{Y}| = 0$, which leads to a characteristic equation. Since there are s^2 resonance terms from each of the cavity, the dipole-mode, and the quadrupole mode, the polynomial contains terms up to s^6. To find the instability threshold we substitute $s \equiv 0$. This is a powerful method, and derives from the (auxiliary) Routh-Hurwitz condition that all coefficients appearing in the characteristic polynomial must be greater than zero for stability. In the limit $s \to 0$, $G_C \to 1$ and $G_S \to 0$. Hence a necessary condition for stability of the system of coupled motions is :

$$\cos\phi_b - \frac{\mathrm{I_b}}{\mathrm{I_T}}\left[\sin\psi + \frac{\theta_0}{4}|f(\theta_0)|\sin(\psi - \phi_b)\cos\phi_b\right] + \left(\frac{\mathrm{I_b}}{\mathrm{I_T}}\right)^2\frac{\theta_0}{4}|f(\theta_0)|\cos\phi_b > 0 \; . \tag{10a}$$

The form factor $f(\theta)$ depends on the bunch shape, but in the limit of short elliptic bunches goes roughly as $(-)\theta_0/4$. Taking the limit $f(\theta) \to 0$ gives the usual Robinson[2] stability criterion for oscillations of the bunch centre, namely $\mathrm{I_b}/\mathrm{I_0} < \sin(2\psi)/2\cos\phi_b$.

A complete stability analysis requires consideration of the Routh determinants. The quartic equation for the dipole mode has been studied by Cooper[3]; and the sextic for the dipole-quad' hybrid by Wang[4], who shows that a further restriction exists on the stable beam-current :

$$\frac{\mathrm{I_b}}{\mathrm{I_0}} < \frac{3\tan\psi}{\theta_0|f(\theta)|} \; . \tag{10b}$$

This is confirmed by simulations,[5,6] which show that for moderate (but not small) tuning angles the coupled system is more stable than the dipole mode in isolation. This is because energy can be shared between the coupled modes so as to reduce the initial perturbation of a single mode. Example form factors $f(\theta)$ are given in the Appendix.

BEAM DAMPING LOOPS

The naming of phase and amplitude loops is perilous, since we need to distinguish if beam or if cavity information is conveyed in the feedback circuit. If it is beam amplitude and phase then we have quadrupole and dipole mode damping, respectively. If it is the cavity amplitude and phase then we have the automatic gain compensation (AGC) and automatic phase compensation (AΦC), respectively. With a suitable choice of the transfer matrix **P** either system can be represented (naively) by the digram below.

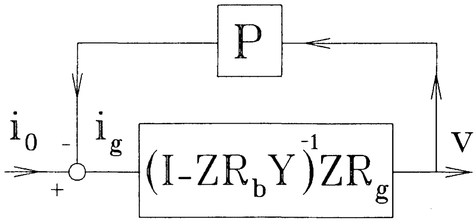

FIGURE 3 Schematic of general feedback.

The system is unstable if for any frequency s the determinant $|\mathbf{I}-\mathbf{Z}(\mathbf{R_b}\mathbf{Y}+\mathbf{R_g}\mathbf{P})|$ equals zero. Stability is guaranteed if $-\mathbf{P} = (\mathbf{R_g}^{-1}\mathbf{P}\mathbf{Y})$, but this is only of academic interest.

For the beam damping loops our intention is to deliberately modulate the generator current according to $\Delta\phi_g = (+)sK_p\Delta\phi_b$ and $\Delta \mathrm{I_g} = (+\mathrm{I_g})sK_a\Delta\theta$, which requires sensing of the bunch centroid and length. Inevitably there are band-width and delay limitations so that the transfer matrix is :

$$\mathbf{P} = \begin{pmatrix} sK_a/(1+sT_a) & 0 \\ 0 & sK_p/(1+sT_p) \end{pmatrix} \mathbf{Y}\,, \tag{11}$$

where $K_{a,p}$ are feedback gains and $T_{a,p}$ the response time constants. We may substitute $s = 0$ into the determinant, and immediately find that the threshold current for instability is identical with (10). However, below threshold the damping rates are increased compared with the case of no feedback provided[1] that $\left[K_p\cos(\psi+\phi_b)+\theta_0 K_a\cos(\psi)\right] > 0$ which poses a constraint on combinations of bunch phase and tuning angle.

FAST FEEDBACK

The most widely advocated method for increasing the instability threshold beyond that given in (10) is 'fast feedback'[7]. The feedback does *not* change the equilibrium detuning. However, the 'drive' voltage V_0 is increased by $\sim g \times b$.

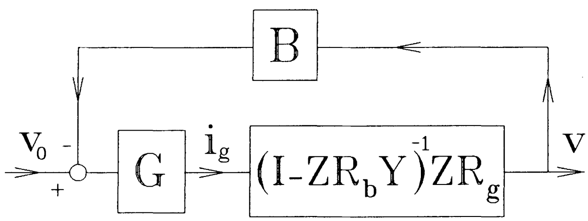

FIGURE 4 Schematic of cavity with fast feedback.

Consider the block diagram (Fig. 4) for a system with high-power feedback around the accelerating cavity. We have represented the combination of power-tube and pre-amplifier by the matrix $\mathbf{G}$, and the feedback path (with delay T) by an attenuation matrix $\mathbf{B}$. Assuming sufficiently broad-band components, the transfer functions are real and the transfer matrices diagonal.

$$\text{Then} \quad \mathbf{G} = g\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = be^{-sT}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{12}$$

Note that in the diagram we have replaced the beam-cavity loop (of Fig. 2) by a single transfer function : $\mathbf{H} = [\mathbf{I} - \mathbf{Z}\mathbf{R_b}\mathbf{Y}]^{-1}\mathbf{Z}\mathbf{R_g}$. The matrix relation between control signal ($\mathbf{v_0}$) and gap-voltage is :

$$\mathbf{v} = [\mathbf{I} + \mathbf{HGB}]^{-1}\mathbf{HGv_0} = [\mathbf{I} + \mathbf{HGB}]^{-1}[\mathbf{I} - \mathbf{Z}\mathbf{R_b}\mathbf{Y}]^{-1}\mathbf{Z}\mathbf{R_g}\mathbf{G}\mathbf{v_0}$$

We use the matrix relation between inverses $A^{-1}B^{-1} = (BA)^{-1}$ and substitute a second time for $\mathbf{H}$ to give :

$$\mathbf{v} = [\mathbf{I} + \mathbf{Z}(\mathbf{R_g}\mathbf{GB} - \mathbf{R_b}\mathbf{Y})]^{-1}\mathbf{Z}\mathbf{R_g}\mathbf{G}\mathbf{v_0} . \tag{13}$$

This clearly shows the 'competition' between the beam feedback ($-\mathbf{R_b}\mathbf{Y}$) and the dedicated high-power feedback ($\mathbf{R_g}\mathbf{GB}$). Let us define $\mathbf{A} = \mathbf{GB}$. The system will be unstable if the determinant $|\mathbf{I} + \mathbf{Z}(\mathbf{R_g}\mathbf{A} - \mathbf{R_b}\mathbf{Y})| = 0$. The instability threshold is found by setting $s = 0$ in the characteristic equation. We find the condition analogous to the Robinson criterion (14) :

$$\frac{\mathrm{I_b}}{\mathrm{I_T}}\left[\frac{\sin\psi}{\cos\phi_b} + \frac{\theta_0}{4}|f(\theta_0)|\sin(\psi - \phi_b)\right] < \left[1 + 2A\cos\psi + A^2\right] + \left(\frac{\mathrm{I_b}}{\mathrm{I_T}}\right)^2\frac{\theta_0}{4}|f(\theta_0)|\left[A\sin\phi_b + \frac{\mathrm{I_b}}{\mathrm{I_T}}\right] .$$

Provided that $A > 1$ and $\psi < \pi/2$ the system is much more stable than the dipole mode in isolation.

APPENDIX

We here show the relation of changes in bunch length to variations in the beam-current harmonics. Let the bunch shape be written $\lambda(\theta, x)$ where x is rf-phase and θ is the bunch half-length in radians of rf-phase. We define the normalised bunch shape as :

$$\lambda^*(\theta, x) \equiv \lambda(\theta, x)\Big/\int_{-\pi}^{+\pi}\lambda(\theta, x)dx . \tag{15}$$

Under the conditions $\lambda(\theta, x) = \lambda(\theta, -x)$ and $\lambda(\theta, \theta) = 0$, we find to first order that $\Delta\mathrm{I_b} = \mathrm{I_b}f(\theta)\Delta\theta$ where

$$f(\theta) = \int_0^\theta \frac{\partial\lambda^*(\theta, x)}{\partial\theta}\cos(nx)dx\Big/\int_0^\theta \lambda^*(\theta, x)\cos(nx)dx . \tag{16}$$

Note that lengthening the bunch also flattens it out, so $f(\theta)$ must be negative.

Parabolic bunch

For the bunch shape $\lambda(\theta, x) = (\theta^2 - x^2)$ for $|x| \leq \theta$, we find the form-factor :

$$f(\theta) \approx (-)\theta/5 \tag{17a}$$

The approximation is good for $\theta < 2$ radian.

Elliptic bunch

For the bunch shape $\lambda(\theta, x) = \sqrt{(\theta^2 - x^2)}$ for $|x| \leq \theta$, we find the form-factor :

$$f(\theta) \approx (-)\theta/4 \tag{17b}$$

Rectangular bunch

For the bunch shape $\lambda(x) = 1$ for $|x| \leq \theta$, we find the form-factor :

$$f(\theta) \approx (-)\theta/3 \tag{17c}$$

The approximation is good for $\theta < \frac{1}{2}$ radian.

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