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Geometry of σ-models with heterotic supersymmetry

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Abstract

We apply the method of harmonic superspace for solving locally the torsion constraints for the manifolds of (4,0) supersymmetric σ-models. The solution is given in terms of an unconstrained harmonic-analytic prepotential with a curved vector index. This prepotential also determines the form of the most general off-shell (4,0) σ-model action. The torsion-free (hyper-Kähler) case is recovered if one takes the gradient of a scalar analytic superfield as the vector potential.

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1 Introduction

In [1], B. Zumino showed that extended supersymmetry for a non-linear σ-model in two dimensions puts some restrictions on the manifold from which the σ-model is built. Since that time, these constraints have been studied for many supersymmetry algebras [2-12].

Considering for instance the (2,2) case (2 left-handed and 2 right-handed supersymmetries) without torsion, we know that the manifold has to be Kähler. Moreover, the superspace action is determined directly by the Kähler potential, which may be considered as the prepotential of Kähler geometry. Indeed, the Kähler potential is the unconstrained object which determines the constrained geometrical quantities (here, the metric). It turns out that in all known cases, the superspace action of supersymmetric σ-models is determined by the prepotentials of the geometry.

In the hyper-Kähler case, corresponding to (4,4) supersymmetry without torsion, the superspace action was given in [13], and subsequently shown to be determined by the prepotential of hyper-Kähler geometry [14]. This prepotential is a scalar object living in an enlarged space including harmonic variables.

In this article, we shall focus our attention on the (2,0) and (4,0) heterotic supersymmetries. The action includes a Wess-Zumino term, which leads to a connection including torsion on the target manifold [3]. For (2,0) supersymmetry, the prepotential carries a vector index [6]. Kähler geometry appears as a subcase, when the vector potential is the gradient of a scalar (Kähler) potential.

In [15], an off-shell action for (4,0) supersymmetric σ-models was proposed, and it was shown that the corresponding supergraphs are ultraviolet finite. As in the (4,4) case, the object which determines this action is an analytic superfield defined in harmonic space, but this time this object carries a vector index. What we are going to show is that this vector-like object is the prepotential of the (4,0) geometry. This will be done by solving locally the constraints corresponding to this geometry. Moreover, again the hyper-Kähler case appears as a subcase, when the vector potential is the gradient of a scalar potential.

The essential meaning of these results is that all the (4,0) σ-models are parametrized by an arbitrary harmonic-analytic vector function, and they admit a manifestly supersymmetric formulation in harmonic superspace. It should be pointed out, however, that the relation between the analytic prepotential and the familiar (harmonic-independent) metric and torsion is much more complicated than in (2,0) or (2,2) case. In the (4,0) case (as in the (4,4) case [14]) one must solve a first-order differential equation on the harmonic 2-sphere, whereas in the (2,0) or (2,2) case only simple differentiation is involved.

The paper is organized as follows. In Section 2 we give the general framework for σ-models with (1,0) supersymmetry and torsion, both in superspace and in components. In Section 3 we study the implications of adding a second supersymmetry and arrive at the known vector prepotential for (2,0) σ-models with torsion [6] .In Section 4 we rederive the same result, this time using the formalism of differential geometry and solving the constraints on the torsion. This is a preparatory step for the much more complicated (4,0) case, where only the systematic study of the torsion constraints can lead to the desired prepotential. In Section 5 we treat the (4,0) constraints by introducing harmonic variables. This enables us to define harmonic analyticity [13] and show its direct link with the torsion

constraints. Then, analyzing the constraints step by step, we find expressions for all the differential geometry objects in terms of one harmonic-analytic vector potential. The latter is shown to determine the (4,0) harmonic superspace action of [15]. Finally we explain the reduction to the torsion-free (hyper-Kähler) case.

We would like to point out that the reader should be familiar with the basics of the harmonic approach [13], in particular with the study of the torsion-free (4,4) σ-models given in [14].

2 (1,0) supersymmetry

We consider a superspace with coordinates $\{x^{(+)}, x^{(-)}, \theta\}$ [6]. The Lorentz group acts on $x^{(\pm)}$ as follows:

$$x^{(\pm)\prime} = e^{\pm\lambda} x^{(\pm)} \tag{1}$$

and on the left-handed fermionic coordinate θ by:

$$\theta' = e^{\frac{\lambda}{2}}\theta. \tag{2}$$

No right-handed coordinate is present, hence the name (1,0) supersymmetry. A chiral superfield $\phi(x,\theta)$ may be expanded as:

$$\phi(x,\theta) = \varphi(x) + \theta\psi(x) \tag{3}$$

where the fermion field $\psi(x)$ has Lorentz charge $-1/2$. One can define a spinor covariant derivative:

$$D = \frac{\partial}{\partial\theta} + i\theta\partial_{(+)} \tag{4}$$

which commutes with supersymmetry and satisfies the relation:

$$D^2 = i\partial_{(+)} \tag{5}$$

The most general σ-model action involving N superfields ϕ_μ, $\mu = 1\cdots N$, has the form:

$$S = \frac{1}{2i}\int d^2x d\theta a_{\mu\nu}(\phi)\partial_{(-)}\phi^\mu D\phi^\nu, \tag{6}$$

or, in terms of component fields:

$$S = \frac{1}{2}\int d^2x [a_{\mu\nu}(\varphi)\partial_{(-)}\varphi^\mu\partial_{(+)}\varphi^\nu - i a_{\mu\nu}(\varphi)\partial_{(-)}\psi^\mu\psi^\nu - i a_{\mu\nu,\lambda}\psi^\lambda\psi^\nu\partial_{(-)}\varphi^\mu]. \tag{7}$$

In the form (6), the action S is manifestly (1,0) supersymmetric for an arbitrary function $a_{\mu\nu}(\phi)$. As we shall see later on, the requirement of having more than one supersymmetry for the action (6) imposes certain restrictions on $a_{\mu\nu}$. Clearly, the action (6) is invariant under diffeomorphisms of the manifold parametrized by ϕ^μ:

$$\delta\phi^\mu = \lambda^\mu(\phi), \tag{8}$$

provided that $a_{\mu\nu}(\phi)$ transforms as a tensor. The symmetric part of $a_{\mu\nu}$,

$$g_{\mu\nu} = \frac{1}{2}(a_{\mu\nu} + a_{\nu\mu}) \tag{9}$$

plays the rôle of the metric on the target manifold. As to the antisymmetric part,

$$b_{\mu\nu} = \frac{1}{2}(a_{\mu\nu} - a_{\nu\mu}), \tag{10}$$

we shall see that it gives rise to torsion on the target manifold. Note that the action (6) is invariant under a gauge transformation of $b_{\mu\nu}$ [3]

$$\delta b_{\mu\nu} = \partial_\mu \xi_\nu - \partial_\nu \xi_\mu. \tag{11}$$

Indeed,

$$\delta_\xi S = \frac{1}{2i}\int d^2x d\theta (\partial_\mu \xi_\nu - \partial_\nu \xi_\mu)\partial_{(-)}\phi^\mu D\phi^\nu$$
$$= \frac{1}{2i}\int d^2x d\theta (\partial_{(-)}\xi_\mu D\phi^\mu - D\xi_\mu \partial_{(-)}\phi^\mu) = 0. \tag{12}$$

The equation of motion for ϕ^μ following from (6) is:

$$\partial_{(-)}D\phi^\mu + \Gamma^\mu_{\nu\lambda}\partial_{(-)}\phi^\nu D\phi^\lambda = 0, \tag{13}$$

where

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2}g^{\mu\rho}(a_{\rho\lambda,\nu} + a_{\nu\rho,\lambda} - a_{\nu\lambda,\rho}) = \gamma^\mu_{\nu\lambda} - \frac{1}{2}T^\mu_{\nu\lambda} \tag{14}$$

is the connection on the manifold. It splits into the standard Christoffel connection:

$$\gamma^\mu_{\nu\lambda} = \frac{1}{2}g^{\mu\rho}(g_{\rho\lambda,\nu} + g_{\nu\rho,\lambda} - g_{\nu\lambda,\rho}), \tag{15}$$

and the torsion:

$$T^\mu_{\nu\lambda} = g^{\mu\rho}(b_{\nu\lambda,\rho} + b_{\rho\nu,\lambda} + b_{\lambda\rho,\nu}). \tag{16}$$

So we see that $b_{\mu\nu}$ serves as a torsion potential [3]. Note that:

$$T_{\mu\nu\lambda} = g_{\mu\rho}T^\rho_{\nu\lambda} = b_{[\mu\nu,\lambda]}, \tag{17}$$

so $T_{\mu\nu\lambda}$ is totally antisymmetric. Obviously

$$\delta_\xi T_{\mu\nu\lambda} = 0. \tag{18}$$

3 (2,0) supersymmetry

3.1 Constraints on the geometry

Now we shall try to make the action (6) invariant under a second supersymmetry,thus extending it to (2,0) supersymmetry. The most general form of such a new supersymmetry, which commutes with the old one, is the following:

$$\delta_2\phi^\mu = \epsilon_2 J^\mu_\nu(\phi) D\phi^\nu. \tag{19}$$

Since in (19) one deals with (1,0) superfields, the relation:

$$[\delta_1, \delta_2]\phi^\mu = 0 \tag{20}$$

is obvious. Further, two transformations of the type (19) should generate a space-time translation:

$$[\delta_2, \delta_2']\phi^\mu = -2i\epsilon_2\epsilon_2'\partial_{(+)}\phi^\mu. \tag{21}$$

As a consequence, the tensor $J^\mu_\nu(\phi)$ in (19) must satisfy the following two conditions [1]:

$$J^\mu_\nu J^\nu_\lambda = -\delta^\mu_\lambda, \tag{22}$$

$$J^\mu_{\nu,\lambda}J^\lambda_\kappa + J^\mu_\lambda J^\lambda_{\kappa,\nu} - (\nu \leftrightarrow \kappa) = 0. \tag{23}$$

Further restrictions on J, as well as on $a_{\mu\nu}$, follow from the invariance of the action. One can easily check that:

$$\delta_2 S = \epsilon_2 \int d^2x d\theta \{(g_{\mu\lambda}J^\lambda_\nu + g_{\nu\lambda}J^\lambda_\mu)(D\phi^\mu\partial_{(-)}D\phi^\nu + D\phi^\nu\partial_{(-)}D\phi^\mu) +$$
$$+[(g_{\mu\lambda}J^\lambda_\nu)_{,\rho} + (a_{\lambda\nu,\rho} + a_{\rho\lambda,\nu} - a_{\rho\nu,\lambda})J^\lambda_\mu](\partial_{(-)}\phi^\rho D\phi^\mu D\phi^\nu)\}. \tag{24}$$

The two structures containing derivatives of ϕ in (24) are independent (one is symmetric in (μ, ν), while the other one is antisymmetric), so one derives two conditions from the requirement $\delta_2 S = 0$:

$$g_{\mu\lambda}J^\lambda_\nu + g_{\nu\lambda}J^\lambda_\mu = 0, \tag{25}$$

$$\nabla_\rho J^\mu_\nu \equiv J^\mu_{\nu,\rho} + \Gamma^\mu_{\rho\lambda}J^\lambda_\nu - \Gamma^\lambda_{\rho\nu}J^\mu_\lambda = 0. \tag{26}$$

Equations (22,23,25,26) are the familiar conditions [2,6] for the existence of a covariantly constant (26), integrable (23) complex structure (22), for which the metric is hermitian (25). Note that in the case of a Riemannian (torsion-free) target manifold, conditions (23,26) are not independent. However, in the presence of torsion they may be combined to get a restriction on the torsion coefficients.

3.2 Solution to the constraints; prepotential

In this subsection we shall find the most general (local) solution to the constraints (22,23,25,26) determining the geometry of a (2,0) σ-model. It will turn out that all the geometric objects can be obtained from a single unconstrained prepotential, a vector field V_m [6].

First, from conditions (22,23) it follows that there exist a set of complex coordinates ϕ^m, $\phi^{\bar{m}}$ in which the complex structure J is constant:

$$J^m_n = i\delta^m_n, J^{\bar{m}}_{\bar{n}} = -i\delta^{\bar{m}}_{\bar{n}}, J^m_{\bar{n}} = J^{\bar{m}}_n = 0. \tag{27}$$

Note that the target manifold must be even dimensional.

[1]The left-hand side of eq. (23) is usually called the Nijenhuis tensor. σ-models with non-vanishing Nijenhuis tensor and an infinite-dimensional supersymmetry algebra have been considered in [16] .

In this new basis, conditions (22,23) are evidently satisfied. Further, condition (25) means that:

$$g_{mn} = g_{\bar{m}\bar{n}} = 0. \tag{28}$$

From (26) one derives two consequences. First,

$$\Gamma^{\bar{l}}_{mn} = 0 \Rightarrow b_{[mn,\bar{l}]} = 0 \tag{29}$$

(see (17)).It has the general solution:

$$b_{mn} = \partial_m \xi_n - \partial_n \xi_m, \tag{30}$$

where $\xi_m(\phi, \bar{\phi})$ is an arbitrary vector. Second,

$$\Gamma^{l}_{m\bar{n}} = 0 \Rightarrow b_{\bar{n}l,m} + a_{m\bar{n},l} - a_{ml,\bar{n}} = 0. \tag{31}$$

Substituting the conjugate of (30) into (31), one finds:

$$(a_{m\bar{n}} - \xi_{\bar{n},m})_{,\bar{l}} - (\bar{n} \leftrightarrow \bar{l}) = 0, \tag{32}$$

thus:

$$a_{m\bar{n}} = \xi_{\bar{n},m} + V_{m,\bar{n}}, \tag{33}$$

where V_m is a new arbitrary vector. Thus, equations (30) and (33) provide us with the general solutions of the constraints following from (2,0) supersymmetry. Recalling the fact that the torsion potential b is defined up to the gauge freedom (11), we see that it is possible to fix a gauge in which, e.g.,

$$a_{mn} = g_{mn} + b_{mn} = 0, \quad a_{m\bar{n}} = g_{m\bar{n}} + b_{m\bar{n}} = V_{m,\bar{n}} \tag{34}$$

We conclude that the geometry of a (2,0) σ-model is entirely determined (locally) by an unconstrained vector prepotential V_m ($V_{\bar{m}} = \overline{V_m}$).

3.3 Superspace action for (2,0) σ-models

In the action (6), only (1,0) supersymmetry is manifest. It is desirable to recast it in a form where the full (2,0) supersymmetry is manifest (provided, of course, that all the conditions are satisfied). To this end we define a (2,0) superspace with Grassmann coordinates θ_1, θ_2. It is convenient to form new, complex θ's:

$$\theta = \theta_1 + i\theta_2, \quad \bar{\theta} = \theta_1 - i\theta_2$$

$$D = \frac{1}{2}(D_1 + iD_2), \ D^2 = \bar{D}^2 = 0, \ \{D, \bar{D}\} = i\partial_{(+)}. \tag{35}$$

Then we can define holomorphic (chiral) superfields:

$$\bar{D}\phi^m = D\phi^{\bar{m}} = 0. \tag{36}$$

This condition is easily solved in a special chiral basis in superspace, e.g.:

$$\bar{D}\phi^m = 0 \Rightarrow \phi^m = \phi^m(x^{(+)} + \frac{i}{2}\theta\bar{\theta}, x^{(-)}, \theta) \tag{37}$$

Given all this, one can write down a manifestly (2,0) supersymmetric action [6]:

$$S^{(2,0)} = i \int d^2x d\theta d\bar{\theta} [V_m(\phi, \bar{\phi}) \partial_{(-)} \phi^m - V_{\bar{m}}(\phi, \bar{\phi}) \partial_{(-)} \phi^{\bar{m}}]. \tag{38}$$

It is not hard to verify that the component form of (38) coincides with the general form of the action (7), but $a_{\mu\nu}$ is restricted according to (34). To do this one uses the θ-expansion:

$$\phi^m = e^{\frac{i}{2}\theta\bar{\theta}\partial_{(+)}}[\varphi^m(x) + \theta\psi^m(x)]. \tag{39}$$

We point out that the action (38) has two gauge invariances. First,

$$\delta V_m = i\partial_m \Lambda(\phi, \bar{\phi}), \quad \Lambda = \bar{\Lambda}, \tag{40}$$

$$\delta S^{(2,0)} = -\int d^2x d\theta d\bar{\theta} (\partial_{(-)}\phi^m \partial_m \Lambda + \partial_{(-)}\phi^{\bar{m}} \partial_{\bar{m}} \Lambda) = -\int d^2x d\theta d\bar{\theta} \partial_{(-)} \Lambda = 0.$$

Second,

$$\delta V_m = \lambda_m(\phi), \quad \partial_{\bar{n}} \lambda_m = 0, \tag{41}$$

$$\delta S^{(2,0)} = i \int d^2x d\theta d\bar{\theta} [\lambda_m(\phi) \partial_{(-)} \phi^m - \lambda_{\bar{m}}(\bar{\phi}) \partial_{(-)} \phi^{\bar{m}}] = 0,$$

since the $d\theta d\bar{\theta}$ integral of holomorphic superfields vanishes. In fact, both those transformations are remnants of the general ξ gauge invariance (11), which are compatible with the ξ gauge (34). The manifestly (2,0) supersymmetric action (38) does not exhibit the ξ invariance in full. The reason is that the action (6) is invariant under (11) up to total deriavative terms,which are not necessarily presentable in a (2,0) supersymmetric form.

The geometry corresponding to the action (38) can be reduced to the more familiar Kähler geometry if one eliminates the torsion. For this one should make the torsion potential b_{mn} in (34) vanish, which leads to:

$$V_m = \partial_m K(\phi, \bar{\phi}). \tag{42}$$

Thus the familiar Kähler potential $K(\phi, \bar{\phi})$ arises. Further, in this case the action (38) takes the form:

$$S^{(2,0)}_{K\ddot{a}hler} = i \int d^2x d\theta d\bar{\theta} (\partial_{(-)}\phi^m \partial_m K - \partial_{(-)}\phi^{\bar{m}} \partial_{\bar{m}} K) \tag{43}$$

One can check that (43) is obtained from the (2,2) Kähler action:

$$S^{(2,2)}_{K\ddot{a}hler} = \int d^2x d^2\theta d^2\bar{\theta} K(\phi, \bar{\phi}) \tag{44}$$

by an expansion in the (0,2) θ's and an appropriate truncation.

4 Differential geometry formalism

So far we have given a complete description of (2,0) σ-models with torsion. Nevertheless, now we shall present an alternative picture in terms of differential geometry constraints [14]. This can be regarded as an exercise for the more complicated case of (4,0) supersymmetry, where only this second method is applicable. The essential information about the geometry of the (2,0) σ-models is contained in the constraints:

$$T^{\bar{l}}_{mn} = 0 \tag{45}$$

(see (29)) and in the fact that the torsion tensor is given in terms of a torsion potential (see (17)). We are going to introduce a vielbein formalism and reformulate the two restrictions in terms of tangent space covariant objects. The vielbein e^A_μ converts a world vector A^μ into a tangent space one:

$$A^A = e^A_\mu A^\mu. \tag{46}$$

Here, the local tangent space group acting on the index A is taken to be $O(2N)$, where $2N$ is the dimension of the manifold. The covariant derivative D_A acts as follows:

$$D_A A^B = e^\mu_A \partial_\mu A^B + \omega_A{}^B{}_C A^C, \tag{47}$$

where:

$$e^\mu_A e^B_\mu = \delta^B_A$$
$$\omega_A{}^B{}_C = \Gamma^\lambda_{\mu\nu} e^\mu_A e^\nu_C e^B_\lambda - e^\mu_A e^\nu_C \partial_\mu e^B_\nu. \tag{48}$$

The second equation in (48) means that the covariant derivative of the vielbein vanishes. The torsion and curvature are defined by the commutator of the two covariant derivatives:

$$[D_A, D_B] = T^C_{AB} D_C + R_{AB} \tag{49}$$

At the same time, they are related to the world-covariant ones, e.g.

$$T^C_{AB} = T^\lambda_{\mu\nu} e^\mu_A e^\nu_B e^C_\lambda. \tag{50}$$

In the case under consideration, we can further specify the tangent space. To this end we replace the index A by a pair $(a,\bar{a})$, such that:

$$J^a_b = e^a_\mu e^\nu_b J^\mu_\nu = i\delta^a_b,\ J^{\bar{a}}_{\bar{b}} = -i\delta^{\bar{a}}_{\bar{b}},\ J^a_{\bar{b}} = J^{\bar{a}}_b = 0. \tag{51}$$

Then, from the covariant constancy of J, eq. (26), one derives:

$$D_A J^C_B = 0 \Rightarrow \omega_{a\bar{c}}{}^b = \omega_{ac}{}^{\bar{b}} = 0, \tag{52}$$

i.e. the connection ω must be a $U(N)$ one. So, we restrict the tangent space group to $U(N)$. Further, the constraint (45) is translated into:

$$T^{\bar{c}}_{ab} \equiv T_{abc} = 0. \tag{53}$$

Thus, the algebra of covariant derivatives takes the form:

$$[D_a, D_b] = T_{ab\bar{c}} D_c + R_{ab}, \tag{54a}$$
$$[D_a, D_{\bar{b}}] = T_{ca\bar{b}} D_{\bar{c}} + T_{\bar{b}\bar{c}a} D_c + R_{a\bar{b}}, \tag{54b}$$
$$[D_{\bar{a}}, D_{\bar{b}}] = T_{\bar{a}\bar{b}c} D_{\bar{c}} + R_{\bar{a}\bar{b}} \tag{54c}$$

The total antisymmetry of T_{ABC} was used in (54b).

Note that the constraints (54) may be interpreted as the integrability conditions for the existence of holomorphic scalar fields defined by:

$$D_a F = 0 \Rightarrow [D_a, D_b] F = 0. \tag{55}$$

Due to the presence of curvature in eq. (54a), the field F cannot have a $U(N)$ index (in contrast to the case without torsion where the curvature term in (54a) also vanishes).

One can proceed to solving the constraints (54a) in the following order. First, from condition (53) on the torsion, one derives the existence of a holomorphic basis $\phi^\mu = (\phi^m, \phi^{\bar{m}})$ in which:

$$e^{\bar{m}}_a = 0 \Rightarrow D_a = e^m_a \partial_m + \omega_a. \tag{56}$$

Thus the diffeomorphism group is restricted to one with holomorphic parameters:

$$\delta\phi^m = \lambda^m(\phi),\ \partial_{\bar{m}} \lambda^m = 0. \tag{57}$$

Second, from (54a) one obtains:

$$T_{ab\bar{c}} = e^m_a \partial_m e^n_b e_{n\bar{c}} + \omega_{ab\bar{c}} - (a \leftrightarrow b). \tag{58}$$

On the other hand, from (54b) one finds:

$$T_{a\bar{b}c} = e^m_a \partial_m e^{\bar{n}}_{\bar{b}} e_{\bar{n}c} + \omega_{a\bar{b}c}. \tag{59}$$

One can use eqs. (58,59) and the total antisymmetry of T_{ABC} to express ω in terms of the vielbeins:

$$\omega_{ab\bar{c}} = -e^m_a \partial_m e^n_b e_{n\bar{c}} + e^m_b \partial_m e^n_a e_{n\bar{c}} + e^m_b \partial_m e^{\bar{n}}_{\bar{c}} e_{\bar{n}a}, \tag{60}$$

Putting this back into (58), one gets:

$$T_{ab\bar{c}} = e^m_a e^n_b e^{\bar{l}}_{\bar{c}} (\partial_m g_{n\bar{l}} - \partial_n g_{m\bar{l}}), \tag{61}$$

where

$$g_{m\bar{n}} = e_{m\bar{a}} e_{\bar{n}a}. \tag{62}$$

Removing the vielbeins from eq. (61), one arrives at the following expression:

$$T_{mn\bar{l}} = \partial_m g_{n\bar{l}} - \partial_n g_{m\bar{l}}\ . \tag{63}$$

The remaining piece of information, which is not encoded in the consraints (54), is that the torsion is the curl of the torsion potential:

$$T_{mn\bar{l}} = \partial_{\bar{l}} b_{mn} + \partial_m b_{n\bar{l}} + \partial_n b_{\bar{l}m}. \tag{64}$$

Similarly,

$$T_{mnl} = \partial_{[m} b_{nl]} = 0, \tag{65}$$

which implies:

$$b_{mn} = \partial_m \xi_n - \partial_n \xi_m, \tag{66}$$

in accordance with eq. (30). Putting (66) into (64), and comparing (64) with (63), one finds:

$$\partial_m (g_{n\bar{l}} - b_{n\bar{l}} - \partial_{\bar{l}} \xi_n) - (m \leftrightarrow n) = 0. \tag{67}$$

This yields the final solution (see eq. (33))

$$a_{\bar{n}m} = g_{\bar{n}m} + b_{\bar{n}m} = \partial_m V_{\bar{n}} + \partial_{\bar{n}} \xi_m. \tag{68}$$

We stress once again that the purpose of the alternative derivation, eqs. (54-68), was to prepare the ground for the (4,0) case. There the only applicable method is to first write down the torsion constraints in tangent space and then find a way of solving them.

5 (4,0) supersymmetry

5.1 New constraints on the geometry

In this subsection we shall study the consequences of adding a third supersymmetry to the existing ones (see eq. (19)) [9,8,11]

$$\delta_3 \phi^\mu = \epsilon_3 K^\mu_\nu(\phi) D\phi^\nu. \tag{69}$$

Once again, repeating the steps after eq. (19), we find that the new tensor K^μ_ν should satisfy the same conditions as J^μ_ν, eqs. (22-26). In addition, from the commutator:

$$[\delta_2, \delta_3]\phi^\mu = 0 \tag{70}$$

one derives:

$$K^\mu_\nu J^\nu_\lambda + J^\mu_\nu K^\nu_\lambda = 0, \tag{71}$$

$$K^\mu_{\nu,\lambda} J^\lambda_\rho + J^\mu_{\nu,\lambda} K^\lambda_\rho - K^\mu_\lambda J^\lambda_{\nu,\rho} - J^\mu_\lambda K^\lambda_{\nu,\rho} - (\nu \leftrightarrow \rho) = 0. \tag{72}$$

Having introduced two complex structures J and K, we automatically obtain a third one, thus increasing the number of supersymmetries to 4:

$$J_1 = J,\ J_2 = K,\ J_3 = JK. \tag{73}$$

The properties (23) of J and K, and eq. (72) can then be rewritten in a combined form:

$$\{[(J_a)^\mu_{\nu,\lambda}(J_b)^\lambda_\rho - (J_a)^\mu_\lambda (J_b)^\lambda_{\nu,\rho}] + (a \leftrightarrow b)\} - (\nu \leftrightarrow \rho) = 0 \tag{74}$$

$J_a = (J_1, J_2, J_3)$. Further, using the covariant constancy of the three complex structures, one can recast eq. (74) in its final form [11]:

$$[\delta_{ab} T^\lambda_{\mu\nu} + (J_a)^\rho_\mu (J_b)^\kappa_\nu T^\lambda_{\kappa\rho} - (J_a)^\lambda_\rho (J_b)^\kappa_\mu T^\rho_{\nu\kappa} + (J_a)^\lambda_\rho (J_b)^\kappa_\nu T^\rho_{\mu\kappa}] + (a \leftrightarrow b) = 0. \tag{75}$$

The next step is similar to the choice of tangent space made after eq. (51). Namely, we split the tangent space indices $A = \alpha i$, where α is a $Sp(n)$ index, and i an $Sp(1)$ index (thus we assume that the manifold is $4N$ dimensional). Then we introduce vielbeins $e^\mu_{\alpha i}$, $e^{\alpha i}_\mu$ such that:

$$(J_a)^{\alpha i}_{\beta j} = e^{\alpha i}_\mu e^\nu_{\beta j} (J_a)^\mu_\nu = i\delta^i_j (\sigma_a)^\alpha_\beta. \tag{76}$$

Then the condition:

$$\mathcal{D}_{\alpha i}(J_a)^{\gamma k}_{\beta j} = 0 \tag{77}$$

yields:

$$\omega_{\alpha i \beta j}{}^{\gamma k} = \omega_{\alpha i \beta}{}^{\gamma} \delta^k_j, \tag{78}$$

i.e. ω is an $Sp(N)$ connection, whereas $Sp(1)$ remains a rigid symmetry. Eq. (75) in tangent space becomes, using (76):

$$[\delta_{ab} T^{\alpha i}_{\beta j\ \gamma k} + (\sigma_a)^l_k (\sigma_b)^p_j T^{\alpha i}_{\beta p\ \gamma l} - (\sigma_a)^i_l (\sigma_b)^p_j T^{\alpha l}_{\beta p\ \gamma k} - (\sigma_a)^i_l (\sigma_b)^p_k T^{\alpha l}_{\beta j\ \gamma p}] + (a \leftrightarrow b) = 0. \tag{79}$$

After some work, one can show that the only consequence of the apparently complicated constraint (79) is:

$$T^{(ijk)}_{\alpha\beta\gamma} = 0 \tag{80}$$

where (ijk) means complete symmetrization. This is still not all: according to our basic assumption, eq. (17), $T_{\alpha i \beta j \gamma k}$ is related to the curl of a torsion potential, which will be taken into account in due course.

5.2 Harmonic space and analytic basis

In the much simpler case of (2,0) supersymmetry, we made heavy use of the existence of a privileged basis in the target manifold, where the complex structure was constant. It is well known that in the presence of three complex structures generally a basis in which all of them are constant simultaneously does not exist. Therefore we have to change the tactics. This time we shall use harmonic variables $u^\pm_i$ [2], which will allow us to find a basis in which a different kind of analyticity (harmonic analyticity [14]) becomes manifest. This, in turn, will help us to solve the torsion constraints.

With the help of $Sp(1)/U(1)$ harmonic variables $u^\pm_i$, we define the projections of the torsion:

$$T^{\pm\pm\pm}_{\alpha\beta\gamma} = u^\pm_i u^\pm_j u^\pm_k T^{ijk}_{\alpha\beta\gamma}. \tag{81}$$

Then, using eq. (80) and the total antisymmetry of $T_{(\alpha i)(\beta j)(\gamma k)}$ in the pairs of indices, we derive the following constraints:

$$T^{+++}_{\alpha\beta\gamma} = 0, \tag{82}$$

$$T^{++-}_{[\alpha\beta\gamma]} = 0, \tag{83}$$

$$T^{++-}_{\alpha\beta\gamma} = -T^{+-+}_{\alpha\gamma\beta} = -T^{++-}_{\beta\alpha\gamma} = T^{-++}_{\gamma\alpha\beta}. \tag{84}$$

[2]The harmonic variables are defined as elements of $Sp(1)$ ($u^{+i} u^-_i = 1$, $u^-_i = \overline{u^{+i}}$) carrying an $Sp(1)$ index i and a $U(1)$ index $\pm$. Further information about the harmonic method may be found in [13,14]

Taking all this into account, we can rewrite the defining relation (49) as follows:

$$[\mathcal{D}^+_\alpha, \mathcal{D}^+_\beta] = T^{++\gamma-}_{\alpha\beta} \mathcal{D}^+_\gamma + R^{++}_{\alpha\beta}. \tag{85}$$

$$[\mathcal{D}^+_\alpha, \mathcal{D}^-_\beta] = -T^{+-\gamma+}_{\alpha\beta} \mathcal{D}^-_\gamma + T^{+-\gamma-}_{\alpha\beta} \mathcal{D}^+_\gamma + R^{+-}_{\alpha\beta}. \tag{86}$$

We shall not need the remaining commutators. The projected covariant derivatives

$$\mathcal{D}^\pm_\alpha = u^\pm_i \mathcal{D}^i_\alpha \tag{87}$$

satisfy the obvious constraints:

$$[D^{++}, \mathcal{D}^+_\alpha] = 0, \tag{88}$$

$$[D^{--}, \mathcal{D}^+_\alpha] = \mathcal{D}^-_\alpha \tag{89}$$

Where D^{++} (95) and D^{--} (100) are the harmonic derivatives. The geometric interpretation of the constraint (85) is analogous to that of the constraint (54a) in the (2,0) case. Namely, the absence of the torsion term $T^{++\gamma+}_{\alpha\beta} \mathcal{D}^-_\gamma$ in (85) guarantees the existence of analytic fields satisfying the condition:

$$\mathcal{D}^+_\alpha F(\phi, u) = 0. \tag{90}$$

Note that this applies only to fields without external tangent space indices, because of the curvature term in eq. (85). One can make this analyticity manifest by going to a new "analytic basis" in the space $(\phi^\mu, u^{\pm i})$. To do this, one replaces the index μ, $\mu = 1 \cdots 4N$, by a pair of indices μi ($\mu = 1 \cdots 2N$, $i = 1, 2$), and then makes two independent coordinate shifts:

$$\phi^{\mu+}_A = \phi^{\mu i} u^+_i + v^{\mu+}(\phi, u), \tag{91a}$$

$$\phi^{\mu-}_A = \phi^{\mu i} u^-_i + v^{\mu-}(\phi, u). \tag{91b}$$

The functions $v^{\mu\pm}$ (called bridges to the analytic basis) can be chosen in such a way that in the new basis $(\phi^\mu_A, u^{\pm i})$ the covariant derivatives $\mathcal{D}^+_\alpha$ will have the form:

$$\mathcal{D}^+_\alpha = E^\mu_\alpha \partial^+_\mu + \omega^+_\alpha, \quad \partial^+_\mu \equiv \frac{\partial}{\partial \phi^{\mu-}_A}. \tag{92}$$

Note the absence of a ∂^-_μ term in (92). The integrability conditions for the existence of such bridges is just the vanishing of the torsion component $T^{+++}_{\alpha\beta\gamma}$. Clearly, in the analytic basis the analyticity condition (90) can be solved explicitly:

$$\mathcal{D}^+_\alpha F = 0 \Rightarrow F = F(\phi^+_A, u), \tag{93}$$

where F is a holomorphic function of ϕ^+_A. Having made the change of variables (91), we have actually changed the diffeomorphism group. The old variables transformed as follows:

$$\delta\phi^{\mu i} = \tau^{\mu i}(\phi), \quad \delta u^{\pm i} = 0. \tag{94}$$

Note that the diffeomorphism parameters $\tau^{\mu i}$ are independent of the harmonic variables, so in that basis the notion of a u-independent function is covariant. In other words, the harmonic derivative

$$D^{++} = u^{+i} \frac{\partial}{\partial u^{-i}} \equiv \partial^{++} \tag{95}$$

needs no covariantization there. On the other hand, the analytic basis coordinates $(\phi^\mu_A, u^{\pm i})$ should transform in such a way that the notion of analytic function is preserved:

$$\delta\phi^{\mu+}_A = \lambda^{\mu+}(\phi^+_A, u)$$

$$\delta\phi^{\mu-}_A = \lambda^{\mu-}(\phi^+_A, \phi^-_A, u), \tag{96}$$

$$\delta u^{\pm i} = 0.$$

This allows the derivative $\mathcal{D}^+_\alpha$ in eq. (92) to be covariant (∂^+_μ transforms homogeneously under the transformations (96)). However, at the same time the harmonic derivative ∂^{++} ceases to be covariant. In other words, by making the change of variables (91), we have generated vielbein terms in the harmonic derivatives:

$$\mathcal{D}^{++} = \partial^{++} + H^{++\mu+} \partial^-_\mu + H^{++\mu-} \partial^+_\mu. \tag{97}$$

Here

$$H^{++\mu+} = \mathcal{D}^{++} v^{\mu+}, \quad H^{++\mu-} = \mathcal{D}^{++} v^{\mu-} - v^{\mu+} + \phi^{\mu+}_A, \tag{98a}$$

$$\delta H^{++\mu\pm} = \mathcal{D}^{++} \lambda^{\mu\pm}. \tag{98b}$$

The parameter $\lambda^{\mu-}$ in eq. (96) is a general function of $\phi^\pm$ and $u^\pm$, so one is able to gauge the vielbein $H^{++\mu-}$ to its flat space limit $\phi^{\mu+}$:

$$H^{++\mu-} = \phi^{\mu+}_A \rightarrow \mathcal{D}^{++} v^{\mu-} = v^{\mu+}, \quad \mathcal{D}^{++} \lambda^{\mu-} = \lambda^{\mu+}. \tag{99}$$

This gauge is always implied henceforth. The other vielbein in (97), $H^{++\mu+}$, as we shall see later, will turn out to be the prepotential of the (4,0) σ-model. In other words, we will be able to express all the other geometric quantities in terms of $H^{++\mu+}$ (and another, pure gauge object). The bridges $v^{\mu\pm}$ will be treated as secondary objects, which can be found as solutions to eq. (98) for a given $H^{++\mu+}$.

The second harmonic derivative,

$$D^{--} = u^{-i} \frac{\partial}{\partial u^{+i}} \equiv \partial^{--}, \tag{100}$$

also acquires vielbeins after the change of variables (91):

$$\mathcal{D}^{--} = \partial^{--} + H^{--\mu+} \partial^-_\mu + H^{--\mu-} \partial^+_\mu$$

$$\delta H^{--\mu\pm} = \mathcal{D}^{--} \lambda^{\mu\pm}. \tag{101}$$

As in (98), the new vielbeins $H^{--\mu\pm}$ can be expressed in terms of the bridges $v^{\mu\pm}$. However, we prefer to find them directly from the relation:

$$[\mathcal{D}^{++}, \mathcal{D}^{--}] = \mathcal{D}^0 \equiv \partial^0 + \phi^{\mu+}_A \partial^-_\mu - \phi^{\mu-}_A \partial^+_\mu. \tag{102}$$

This relation is obviously true in the τ basis (94), so eq. (102) is its covariant version in the λ basis (96). Note that the charge counting derivative D^0 always remains flat. From (102) one finds equations for the vielbeins $H^{--\mu\pm}$:

$$\mathcal{D}^{++} H^{--\mu+} - \mathcal{D}^{--} H^{++\mu+} = \phi^{\mu+}_A,$$

$$\mathcal{D}^{++}H^{--\mu-} = -\phi_A^{\mu-} + H^{--\mu+}. \tag{103}$$

These are linear differential equations for $H^{--\mu\pm}$. Under some very general assumptions they always have unique solutions, which can be found perturbatively. For some special choices of $H^{++\mu+}$ one can even find closed-form expressions for $H^{--\mu\pm}$, although in general this is far from easy. Nevertheless, in what follows we shall assume that eqs. (103) have been solved. A more detailed discussion of this important point can be found in [14].

Let us now return to the constraints (85,86). Before we assumed that the projected covariant derivatives $D_\alpha^\pm$ depended linearly on $u^{\pm i}$, eq. (87), which is guaranteed by the eqs. (88,89). After the change of variables (91) to the λ frame eq. (87) is not valid any more, so it should be replaced by the covariantized version of eqs.(88,89):

$$[\mathcal{D}^{++}, \mathcal{D}_\alpha^+] = 0 \tag{104}$$

$$[\mathcal{D}^{--}, \mathcal{D}_\alpha^+] = \mathcal{D}_\alpha^-. \tag{105}$$

We can now say that eqs. (82-86,102,104,105) are the set of constraints which determine the geometry of the σ-model, independently of the basis. Note, however, that the torsion in (82-86) is further restricted by the requirement that it is obtained from a torsion potential, eq. (17).

5.3 Solution to the constraints. Analytic prepotentials

In this subsection, we shall use all the information encoded in the constraints (82-86,102,104,105), as well as the fact that the torsion is the curl of a torsion prepotential. This will allow us to express the vielbein E_α^μ and the connection ω_α^+ in (92) (and thus the full differential-geometry formalism) in terms of just one field - the analytic vielbein $H^{++\mu+}$.

First of all, we shall assume that eqs. (103) following from the constraint (102), have been solved, so we shall treat the vielbeins $H^{--\mu\pm}$ as quantities determined by $H^{++\mu+}$. Second, we examine the consequences of the constraint (88). Inserting eqs. (92,97) into (88), we find:

$$\partial_\nu^+ H^{++\mu+} = 0, \tag{106}$$

$$\mathcal{D}^{++}E_\alpha^\mu = 0, \tag{107}$$

$$\mathcal{D}^{++}\omega_\alpha^+ = 0. \tag{108}$$

Eq. (106) may be immediately solved by taking $H^{++\mu+}$ to be a holomorphic function

$$H^{++\mu+} = H^{++\mu+}(\phi_A^{\mu+}, u^\pm) \tag{109}$$

(the index μ of $H^{++\mu+}$ is a world one, not a tangent space one, so there is no contradiction with the comment made after eq. (90)). The claim now is that there will be no further restrictions on the analytic field (109), and it will become the unconstrained prepotential of the (4,0) σ-model (the analog of the field (V_m, $V_{\bar{m}}$) in the (2,0) case).

Before discussing the consequences of eqs. (106,107) we need to find an expression for the $Sp(N)$ connection ω_α^+. To do this we shall compare two different expressions for the torsion component $T_{\alpha\beta\gamma}^{++-}$, obtained from eqs. (85,86) and (84). Eq. (85) gives:

$$T_{\alpha\beta\gamma}^{++-} = E_\alpha^\mu(\partial_\mu^+ E_\beta^\nu)E_{\nu\gamma} + \omega_{\alpha\beta\gamma}^+ - (\alpha \leftrightarrow \beta). \tag{110}$$

Further, eq. (105) is nothing but a definition for the covariant derivative $\mathcal{D}_\alpha^-$:

$$\mathcal{D}_\alpha^- = [\mathcal{D}^{--}, \mathcal{D}_\alpha^+] = -E_\alpha^\nu \partial_\nu^+ H^{--\mu+}\partial_\mu^- + (\mathcal{D}^{--}E_\alpha^\mu - E_\alpha^\nu\partial_\nu^+ H^{--\mu-})\partial_\mu^+ + \mathcal{D}^{--}\omega_\alpha^+. \tag{111}$$

Inserting this into (86), it is not hard to find the torsion component:

$$T_{\alpha\beta\gamma}^{+-+} = -E_\alpha^\mu(\partial_\mu^+ E_\beta^\nu)E_{\nu\gamma} - \omega_{\alpha\beta\gamma}^+ - E_\alpha^\mu E_\beta^\nu(\partial_\mu^+\partial_\nu^+ H^{--\rho+})(\partial H)_\rho^{-1\,\lambda}E_{\lambda\gamma}, \tag{112}$$

where

$$(\partial H)_\mu^\nu \equiv \partial_\mu^+ H^{--\nu+}. \tag{113}$$

According to (84) the two expressions (110) and (112) are related, so one obtains an equation for $\omega_{\alpha\beta\gamma}^+$. The solution is:

$$\omega_{\alpha\beta\gamma}^+ = E_\beta^\mu\partial_\mu^+ E_\alpha^\nu E_{\nu\gamma} - E_\alpha^\mu\partial_\mu^+ E_\beta^\nu E_{\nu\gamma} - E_\beta^\mu\partial_\mu^+ E_\gamma^\nu E_{\nu\alpha} - E_\beta^\mu E_\gamma^\nu(\partial_\mu^+\partial_\nu^+ H^{--\rho+})(\partial H)_\rho^{-1\,\lambda}E_{\lambda\alpha}. \tag{114}$$

This expression can now be substituted in eq. (108), and, using (106,107), it is not hard to show that (114) provides the solution to the constraint (108) as well.

By definition, ω_α^+ is a Sp(N) connection, so it must be symmetric in its indices $(\beta\gamma)$. This does not follow from (114), so one should impose it as a further constraint:

$$\omega_{\alpha[\beta\gamma]}^+ = 0 \Rightarrow\ E_\alpha^\mu E_\beta^\nu \partial_{[\mu}^+ E_{\nu]\gamma} + cycle(\alpha\beta\gamma) = 0. \tag{115}$$

In fact, this is nothing but the constraint (84), as can be seen from eqs. (110,114). Eq. (115) can be rewritten in the following form:

$$\partial_{[\mu}^+ H_{\nu\rho]} = 0, \tag{116}$$

where

$$H_{\mu\nu} = E_\mu^\gamma E_{\gamma\nu} = -H_{\nu\mu} \tag{117}$$

is a kind of "symplectic metric" (it has flat limit $\Omega_{\mu\nu}$, the invariant two-form of the symplectic group).

The constraint we have not discussed so far is eq. (107). Multiplying it by $E_{\mu\beta}$ and symmetrizing and antisymmetrizing in $(\alpha\beta)$, one obtains two independent constraints:

$$\mathcal{D}^{++}E_{(\alpha}^\mu E_{\mu\beta)} = 0, \tag{118}$$

$$\mathcal{D}^{++}H_{\mu\nu} = 0. \tag{119}$$

The first of them simply means that part of the vielbein E_α^μ can be gauged away by means of a tangent space u-independent $Sp(N)$ transformation. Indeed, in the linearized limit,

$E^\mu_\alpha = \delta^\mu_\alpha + \epsilon^\mu_\alpha$, so (118) implies $\partial^{++}\epsilon_{(\alpha\mu)} = 0$, but this is exactly the property of the local $Sp(N)$ parameter. The second constraint (119) will be dealt with later on.

It is time now to recall the fact that the torsion is generated by a torsion potential, see eq. (17). This will give us some useful additional information, although the torsion potential will not remain an independent object. In tangent space notation eq. (17) becomes:

$$T_{\alpha i\beta j\gamma k} = \mathcal{D}_{\alpha i} b_{\beta j\gamma k} + T^{\delta l}_{\alpha i\beta j} b_{\gamma k\delta l} + cycle(\alpha i, \beta j, \gamma k), \tag{120}$$

where

$$b_{\alpha i\beta j} = \epsilon_{ij} b_{(\alpha\beta)} + b_{[\alpha\beta](ij)}. \tag{121}$$

Note that b is defined as a u-independent object in the τ-basis, so it satisfies the constraint:

$$\mathcal{D}^{++} b_{\alpha i\beta j} = 0. \tag{122}$$

Since the torsion is restricted by various constraints, b is also a restricted object. To see this, we shall multiply eq. (120) by harmonics. In particular, from (82) we find:

$$\mathcal{D}^{+}_{[\alpha} b^{++}_{\beta\gamma]} + T^{++\delta-}_{[\alpha\beta} b^{++}_{\gamma]\delta} = 0 \tag{123}$$

(note that the term $b_{(\alpha\beta)}$ from eq. (121) is not present in $b^{++}_{\alpha\beta}$). The significance of this constraint is that $b^{++}_{\alpha\beta}$ becomes a pure gauge degree of freedom. This is most easily shown in the linearized approximation. There eq. (123) becomes:

$$\partial^{+}_{[\alpha} b^{++}_{\beta\gamma]} = 0, \tag{124}$$

which has as general solution:

$$b^{++}_{\alpha\beta} = \partial^{+}_{[\alpha} b^{+}_{\beta]}, \ \delta b^{+}_\alpha = \partial^{+}_\alpha b \tag{124}$$

where b^+_α is defined up to δb^+_α. Further, from eq. (122) it follows that:

$$\partial^{++} b^{++}_{\alpha\beta} = 0 \Rightarrow \partial^{+}_{[\alpha}\partial^{++} b^{+}_{\beta]} = 0 \Rightarrow \partial^{++} b^{+}_\alpha = \partial^{+}_\alpha b^{++}, \tag{125}$$

where b^{++} is defined up to the gauge freedom (see eq. (124)):

$$\delta b^{++} = \partial^{++} b + \rho^{++}, \ \partial^{+}_\alpha \rho^{++} = 0. \tag{126}$$

Since $b(\phi, u)$ is a general function, it is always possible to choose a gauge in which $b^{++} = 0$. Then, from eq. (125) one obtains:

$$\partial^{++} b^{+}_\alpha = 0 \Rightarrow \ b^{+}_\alpha = u^{+i} b_{\alpha i}(\phi)$$
$$\Rightarrow \ b^{++}_{\alpha\beta} = u^{+i} u^{+j} (\partial_{\alpha i} b_{\beta j} - \partial_{\beta j} b_{\alpha i}). \tag{127}$$

Now we recall that the torsion potential is defined up to ξ-gauge freedom, eq. (11), so we can choose a gauge in which:

$$b^{++}_{\alpha\beta} = 0. \tag{128}$$

In this gauge the torsion potential is reduced to:

$$b_{\alpha i\beta j} = \epsilon_{ij} b_{(\alpha\beta)}. \tag{129}$$

Then from (92,120) one obtains (already in the non-linear case):

$$T^{++-}_{\alpha\beta\gamma} = \mathcal{D}^{+}_\alpha b_{\beta\gamma} - \mathcal{D}^{+}_\beta b_{\alpha\gamma} - T^{++\delta-}_{\alpha\beta} b_{\delta\gamma} + T^{+-\delta+}_{\alpha\gamma} b_{\beta\delta} - T^{+-\delta+}_{\beta\gamma} b_{\alpha\delta}$$
$$= E^\mu_\alpha \partial^+_\mu b_{\beta\gamma} - E^\mu_\beta \partial^+_\mu b_{\alpha\gamma} + (\omega^{+}_{\alpha\beta}{}^{\delta} - \omega^{+}_{\beta\alpha}{}^{\delta} - T^{++\delta-}_{\alpha\beta}) b_{\delta\gamma} + (\omega^{+}_{\alpha\gamma}{}^{\delta} + T^{+-\delta+}_{\alpha\gamma}) b_{\beta\delta} - (\omega^{+}_{\beta\gamma}{}^{\delta} + T^{+-\delta+}_{\beta\gamma}) b_{\alpha\delta}. \tag{130}$$

Next one defines:

$$b_{\mu\nu} = E^\alpha_\mu E^\beta_\nu b_{\alpha\beta} = b_{\nu\mu}, \tag{131}$$

multiplies eq. (130) by $E^\alpha_\mu E^\beta_\nu E^\gamma_\lambda$ and uses eqs. (110,112) to obtain:

$$T^{++-}_{\mu\nu\lambda} \equiv E^\alpha_\mu E^\beta_\nu E^\gamma_\lambda T^{++-}_{\alpha\beta\gamma} = (\partial H)^\tau_\lambda \partial^+_\mu [(\partial H)^{-1}_\tau{}^\rho b_{\rho\nu}] - (\mu \leftrightarrow \nu). \tag{132}$$

On the other hand, one can find an alternative expression for $T^{++-}_{\mu\nu\lambda}$ by substituting (114) into (110) and using (116):

$$T^{++-}_{\mu\nu\lambda} = (\partial H)^\tau_\lambda \partial^+_\mu [(\partial H)^{-1}_\tau{}^\rho H_{\rho\nu}] - (\mu \leftrightarrow \nu). \tag{133}$$

Comparing (132) with (133) one sees that:

$$\partial^+_\mu [(\partial H)^{-1}_\lambda{}^\rho (H_{\rho\nu} - b_{\rho\nu})] - (\mu \leftrightarrow \nu) = 0. \tag{134}$$

The solution of eq. (134) is given in terms of a new object L^-_λ as:

$$H_{\mu\nu} - b_{\mu\nu} = (\partial H)^\lambda_\mu \partial^+_\nu L^-_\lambda. \tag{135}$$

Since $b_{\mu\nu}$ is symmetric and $H_{\mu\nu}$ is antisymmetric, one can find expressions for both of them from (135). Obviously, the expression for $H_{\mu\nu}$ satisfies the constraint (116). The only constraint which still remains to be imposed is the covariant u-independence of $b_{\mu\nu}$, eq. (122), and $H_{\mu\nu}$, eq. (119). Acting on (135) with $\mathcal{D}^{++}$, and using the fact that

$$[\mathcal{D}^{++}, \partial^+_\mu] = 0 \tag{136}$$

(see eq. (106)), and the relation (103) between $H^{--\mu+}$ and $H^{++\mu+}$, one obtains:

$$\partial^+_\mu (\mathcal{D}^{++} L^-_\nu + \partial^-_\nu H^{++\lambda+} L^-_\lambda) = 0, \tag{137}$$

which leads to:

$$\mathcal{D}^{++} L^-_\nu + \partial^-_\nu H^{++\lambda+} L^-_\lambda = L^+_\nu, \ \partial^+_\nu L^+_\mu = 0. \tag{138}$$

Eq. (138) allows one to express L^-_μ in terms of the newly introduced arbitrary analytic function L^+_μ, and of $H^{++\mu+}$ (in the same way as one solves eq. (103) for $H^{--\mu\pm}$). Thus L^+_μ emerges as a new unconstrained analytic prepotential, in addition to $H^{++\mu+}$. In fact, one can show that L^+_μ is a pure gauge. Indeed, since $H_{\mu\nu}$ has a flat limit $\Omega_{\mu\nu}$, L^-_μ must have flat limit $\phi_A^{\mu-}$ (see eq. (135)) and consequently $L_\mu^+ = \phi_{\mu A}^+ + l_\mu^+(\phi^+, u)$. On the other hand, L_μ^+ transforms as a $\lambda^{\mu+}$- covariant vector:

$$\delta L_\mu^+ = -\partial_\mu^- \lambda^{\nu+} L_\nu^+ \tag{139}$$

under the λ-basis diffeomorphism group, eq. (96). Then in the linearized limit one finds:

$$\delta l_\mu^+ = -\lambda_\mu^+ - \partial_\mu^- \lambda^{\nu+}\phi_\nu^+, \tag{140}$$

so one can use the arbitrary analytic parameter $\lambda^{\mu+}$ to gauge away the analytic field l_μ^+. Thus the conclusion is that $H^{++\mu+}$ is the only true prepotential in the theory, although L_μ^+ should also be kept if one insists on general covariance.

The road to this final solution was rather long and not quite straightforward, so we feel that it is a good idea to summarize the various steps. In the beginning there is the unconstrained analytic prepotential $H^{++\mu+}$. From eq. (103), one can in principle solve for $H^{--\mu\pm}$ (in practice this can be done either pertubatively or exactly for some special choices of $H^{++\mu+}$). Thus both covariant derivatives $\mathcal{D}^{++}$ and $\mathcal{D}^{--}$ are known. Then one solves eq. (138) for L_μ^- (with an arbitrary or gauge fixed L_μ^+), and obtains an expression for $H_{\mu\nu}$, eq. (135) (as well as for the torsion potential $b_{\mu\nu}$, which, however, is not a genuine part of the tangent space differential geometry formalism). This essentially means having the vielbein E_α^μ in eq. (92) (up to tangent space freedom, see eq. (118)). Finally, from eq. (114) one obtains the $Sp(N)$ connection ω_α^+. This completes the construction of the differential geometry formalism, starting from the analytic prepotential $H^{++\mu+}$ (and L_μ^+).

5.4 Superspace action

In the case of (2,0) σ-models we have been able to write down a manifestly supersymmetric action in (2,0) superspace. There the unconstrained prepotential V_m, $V_{\bar m}$ of the geometry determined the superfield action. The natural question now is: can one do the same in the (4,0) case?

The answer has been found some time ago [15]. To this end, one introduces a Grassmann-analytic superfield:

$$\phi^{\mu+}(x, \theta^+, \bar\theta^+, u^\pm),\ \theta^+ = \theta^i u_i^+,\ \bar\theta^+ = \bar\theta^i u_i^+. \tag{141}$$

Here θ^i, $\bar\theta^i$ are the Grassmann coordinates of (4,0) superspace (all of them have Lorentz charge +1/2). Then one writes down the superspace action:

$$S^{(4,0)} = \int d^2x du d\theta^+ d\bar\theta^+ [L_\mu^+(\phi^+, u)\partial_{(-)}^x \phi^{\mu+} + \Lambda_\mu^-(D^{++}\phi^{\mu+} - H^{++\mu+}(\phi^+, u)]. \tag{142}$$

Here,

$$D^{++} = \partial^{++} + i\theta^+\bar\theta^+\partial_{(+)}^x \tag{143}$$

is the supersymmetric-covariant harmonic derivative, and $\Lambda_\mu^-(x, \theta^+, \bar\theta^+, u)$ is another analytic superfield of Lorentz charge +1. It plays the rôle of a Lagrange multiplier, which leads to the equation of motion

$$D^{++}\phi^{\mu+} = H^{++\mu+}(\phi^+, u). \tag{144}$$

This equation can be recast in a different form by making the change of variables:

$$\phi^{\mu+} = \phi^{\mu i}u_i^+ + v^{\mu+}(\phi^{\mu i}, u),\ \phi^{\mu i} = \phi^{\mu i}(x, \theta^i, \bar\theta^i), \tag{145}$$

after which eq. (144) becomes:

$$D^{++}v^{\mu+} = H^{++\mu+}. \tag{146}$$

One sees that eqs. (145) and (146) are the same as the bridge-defining equations (91a) and (98a). So, one concludes that the superspace action corresponding to the most general (4,0) σ-model is obtained by replacing the coordinates of the manifold by Grassmann analytic superfields, and by putting the prepotentials L_μ^+, $H^{++\mu+}$ into the action (142).

After the elimination of the auxiliary fields from the superspace action (142) one obtains a σ-model action with some metric and torsion. One may ask the question: are they the same as the metric and torsion found from the solution of the constraints in subsection 5.3, provided one starts with the same $H^{++\mu+}$? The answer is affirmative [17]. To see this one performs the θ-expansion of the action (142) and substitutes the kinematic equations for some of the component fields.

We would like to point out that the main difference between the case with torsion and the case without is that the prepotential $H^{++\mu+}$ is further restricted in the latter case [14]. Namely, it can be related to the gradient of a scalar prepotential:

$$H^{++\mu+} = L^{\mu\nu}(\partial^{++}L_\nu^+ + \partial_\nu^- L^{++++}), \tag{147}$$

where $L^{\mu\nu}$ is the inverse of the matrix $\partial_\mu^- L_\nu^+ - \partial_\nu^- L_\mu^+$. Correspondingly, the superspace action with (4,4) supersymmetry and no torsion has the form:

$$S^{(4,4)} = \int d^2x du d^2\theta^+ d^2\bar\theta^+ [L_\mu^+(\phi^+, u)D^{++}\phi^{\mu+} - L^{++++}(\phi^+, u)]. \tag{148}$$

It is not hard to show that by expanding the Lagrangian in (148) in the two extra θ's, integrating them out and truncating properly, one obtains an action of the type (142), but with restricted prepotential $H^{++\mu+}$, according to (147).

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