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SOME ASPECTS OF A UNIFIED APPROACH TO GAUGE, DUAL AND GRIBOV THEORIES^{*)}

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ABSTRACT

We consider quantum chromodynamics (QCD) with N_c colours and N_f flavours. Large N expansions for this theory are discussed and their advantages are pointed out, especially in relation to the possibility of unifying gauge, dual and Gribov theories of strong interactions.

We first recall how the $1/N_c$ expansion of 't Hooft can be related to a dual loop expansion with a fixed coupling constant. We point out the necessity for quarkless (purely gluonic) bound states to appear and their importance in maintaining confinement at higher orders in $1/N_c$. We show how non-orientable dual loops are reinterpreted in QCD and how a paradox appears when N_f is such that asymptotic freedom is lost. Some recent results of Cornwall and Tiktopoulos are analyzed in leading order in $1/N_c$.

We then introduce a $1/N$ expansion at $\rho \equiv N_f/N_c$ fixed and show how it is related to the hadronic topological expansion (TE). This allows an unambiguous definition of Reggeon field theory concepts such as the bare Pomeron and diffractive dissociation in QCD. We are able to relate the parameter ρ to the clustering of hadronic final states into resonances. Decreasing ρ corresponds to increasing cluster over gap size. Renormalization of the dual coupling constant as a function of ρ is discussed and an apparent paradox is resolved. We are also able to shed some new light on the problem of f extinction in the TE.

Finally, we compare our approach to other schemes trying to relate different aspects of hadron physics.

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1. - INTRODUCTION AND SUMMARY

Much attention has been recently focused on non-perturbative expansions of the large N variety for strong interactions.

An expansion in $1/N_c$, where N_c is the number of colours in the standard non-Abelian gauge theory of quarks and gluons, known as quantum chromodynamics (QCD), has been proposed by 't Hooft ¹⁾. This expansion is a way to simplify the study of the dynamics of confinement; at the same time, it bears an amazing resemblance to the dual loop expansions. The program can be carried out successfully ²⁾ in two space-time dimensions and there are hopes that it may work as well (if not as simply) in the real four-dimensional case. Analogies with certain dual string models have been pointed out by Bars ³⁾.

On the other hand, an expansion of dual theories in $1/N_f$ (where N_f is the number of flavours) has been proposed ⁴⁾ and extensively studied. This is the so-called topological expansion (TE). A less systematic but basically closely related, approach is the dual unitarization scheme ⁵⁾. These schemes have helped considerably our understanding of strong interaction concepts, such as the dynamics of the bare Reggeon ⁶⁾, the meaning and structure of the bare Pomeron ⁷⁾, and an interpretation ⁸⁾ of the Reggeon calculus of Gribov and co-workers. The TE has also improved our understanding of the Zweig rule and of its violations ⁹⁾ and this has led to a number of interesting phenomenological applications.

What is the relation, if any, between these two expansions? Are they compatible? Is there a framework in which both expansions come out naturally? These are exactly the questions that we are trying to answer in this paper. We will actually show that both expansions can be studied within the framework of QCD with N_c colours and N_f flavours of quarks.

Some of the points we shall make are not entirely new and are enclosed for our sake of being self-contained. We also feel that presenting both expansions in a unified perspective is both pedagogically and conceptually important. There have not been enough interactions so far between $1/N_c$ physicists and $1/N_f$ physicists: if we are able to set up a unified approach and, most important, a unified language, we can produce such an interaction to the advantage of both.

The paper is organized as follows. In Section 2, after establishing our notations in QCD, we give the g, N_c, N_f dependence of an arbitrary vacuum graph with external (colour-singlet) sources attached. In Section 3, we first review briefly the $1/N_c$ expansion of 't Hooft and its relation to a dual loop expansion with a calculable coupling constant. We then extend his analysis to include the purely gluonic sector and show how one can check confinement at higher orders in $1/N_c$. Non-orientable dual loops get re-interpreted as higher order terms in the expansion, while a paradox seems to emerge if N_f is such that asymptotic freedom is lost. We also discuss some recent arguments for confinement by Cornwall and Tiktopoulos in the light of this expansion.

In Section 4 we introduce the topological expansion (TE) of QCD as a $1/N$ expansion at $\rho = N_f/N_c$ fixed and we point out the physical relevance of ρ in hadronic multiparticle production. In particular, an approximate proportionality between ρ and the ratio of average gap to average cluster size is obtained. The renormalization of the dual coupling constant as a function of ρ is discussed and the way out of an apparent paradox indicated. The bare Reggeon and Pomeron of QCD are unambiguously defined and their properties discussed. The question of the extinction of the f in the TE is addressed in this new framework.

Finally, in Section 5, we summarize again the advantages of large N expansions of QCD in the search for a unified approach to strong interaction dynamics and comparison with other approaches is made.

2. - TOPOLOGICAL CONSIDERATIONS ON QCD

We shall work within the standard Lagrangian of QCD :

$$\mathcal{L} = \frac{1}{4} G_{\mu\nu,i}^j G_j^{\mu\nu,i} + \sum_a \bar{q}^{a,i} (\gamma_\mu D_i^{\mu,j} - m_a \delta_i^j) q_j^a + \mathcal{L}_{g.f.} + \mathcal{L}_{ghost} \quad (2.1)$$

where $G_{\mu\nu,i}^j$ and $D_{\mu,i}^j$ are the usual gauge covariant curl and derivative, respectively. In terms of the gauge fields $A_{\mu,i}^j$:

$$G_{\mu\nu,i}^j = \partial_\mu A_{\nu,i}^j - \partial_\nu A_{\mu,i}^j + g [A_\mu, A_\nu]_i^j ;$$

$$D_{\mu,i}^j = \partial_\mu \delta_i^j + g \tilde{A}_{\mu,i}^j ; \tilde{A}_{\mu,i}^j = A_{\mu,i}^j - \frac{1}{N_c} \delta_i^j A_{\mu,k}^k \quad (2.2)$$

The indices $i, j = 1 \dots N_c$ denote colour labels, the index $a = 1 \dots N_f$ denotes flavour. The local gauge group $SU(N_c)$ is exactly conserved, whereas the $SU(N_f)$ global symmetry is broken just by the quark mass term $\bar{q}^{a,i} m_a q_i^a$ (unless $m_a = m$). The traceless field $\tilde{A}_{\mu,i}^j$ is the one physically coupled, but it will be easier to classify topologically the diagrams of QCD by replacing \tilde{A} with A and also by letting $m_a \equiv m$. For actual evaluation of diagrams one will have to keep account of this fact.

Quantization and renormalization of QCD is now well understood. Besides a gauge fixing term $\mathcal{L}_{g.f.}$ one has to add a ghost term \mathcal{L}_{ghost} . For our purposes it is enough to recall the vertices occurring in the Feynman diagrams of QCD in the Feynman gauge. These are shown in Fig. 1, where the following graphical conventions ^{*)} are used [see also Ref. 1]. Gauge vector mesons (ghosts) are denoted by a pair of solid (dotted) lines (1a,b). The two lines have arrows pointing in opposite direction, denoting the fact that the colour structure is that of a $q\bar{q}$ system. Quarks are represented also as a set of two lines (1c) pointing in the same direction. The solid line carries colour, the wavy line carries flavour. Looking in the direction of the arrow, the wavy line will be drawn to the right of the solid one. Finally, in order to define exactly the meaning of concepts such as "planar" or "handle" in QCD, it will be useful to adopt a convention when drawing a graph on a sheet of paper. We call this the clockwise convention. According to it, we draw all vertices (Figs. 1d,e,f,g) in such a way that, following the arrows of the lines going from one particle to another at each vertex, we go around it clockwise. This implies that each vertex is decomposed into various components, one for each cyclic permutation of the legs (one component for the fermion-gluon vertex, two for the three-gluon vertex, six for the four-gluon coupling). It is obvious that, if we draw a graph with the clockwise convention

^{*)} On several occasions, we shall also draw diagrams with ordinary, single line propagators (solid for fermions, wavy for gauge mesons).

(i) all meson propagators are untwisted (the two lines representing them do not cross each other) ; (ii) wavy lines do not cross solid (or dotted) lines because fermion lines emit gluons (or ghosts) on their left.

Consider now an arbitrary but connected vacuum graph G (i.e., with no external lines attached). It is easy to see that, if G is drawn according to our conventions, it can be embedded in a closed two-dimensional orientable surface S (with the clockwise convention satisfied locally on S , with respect to the tangent plane). S is characterized topologically by a certain number of handles h_S . For any G there is a minimal number of handles $\min h_S \equiv h_G$ needed in order to make the embedding. We call this the number of handles of G and denote it simply by h . The (solid and dotted) colour lines of G will make a certain number ℓ of closed loops and the (wavy) flavour lines will make f closed loops of flavour quantum numbers.

It is easy to see that, again within our conventions, the closed loops described above divide the entire surface S_h on which G is embedded into c disjoint "cells" with $c = \ell + f$. We can say that S is divided into ℓ bosonic and f fermionic cells.

It is now straightforward to see how G depends on g , N_c and N_f in the limit of exact $U(N_c) \otimes SU(N_f)$. Using the fact that each trilinear (quartilinear) vertex gives a factor of g (g^2), and that each bosonic (fermionic) cell gives a factor N_c (N_f), one gets

$$G_0(\underline{P}, g, N_c, N_f) = g^{V_3 + 2V_4} (N_c)^\ell (N_f)^f \tilde{G}_0(\underline{P}) \quad (2.3)$$

where V_3 (V_4) is the number of trilinear (quartilinear) vertices and the symbol G_0 denotes a vacuum graph. Using now the relation between the number of vertices and of propagators and the Euler relation among the number of vertices, the number of propagators, c and h , one obtains ^{1),8)} :

$$G_0(\underline{P}, g, N_c, N_f) = (g^2)^{2h-2} (g^2 N_c)^\ell (g^2 N_f)^f \tilde{G}_0(\underline{P}) \quad (2.4)$$

In order to conform to our subsequent notation, we have introduced as variables in Eqs. (2.3), (2.4) a set \underline{P} of momenta. Such set is obviously empty for G_0 .

Next we introduce gauge invariant (colour singlet) currents $J_i(x) = \beta \bar{q}(x)^{j,a} (\lambda_i)_{ab} \Gamma q(x)^{j,b}$, coupled to the quarks through definite combinations of λ matrices in flavour space. This will add the new vertex of Fig. 1h.

Consider now the n current correlation function :

$$\Gamma_n(\underline{p}) \equiv \int d^4x_i e^{i \sum p_i \cdot x_i} \langle 0 | T(J_1(x_1) \dots J_n(x_n)) | 0 \rangle. \quad (2.5)$$

A given graph G_n contributing to Γ_n will be obtained from a vacuum graph G_0 by attaching the n currents on b out of the f fermionic cells. We shall call these b cells boundaries and the remaining $w = f - b$ cells windows. In passing from G_0 to G_n we obviously have to make the replacement :

$$(N_f)^b \Rightarrow \beta^n \prod_{i=1}^b \text{Tr}(\lambda_{j_1} \lambda_{j_2} \dots \lambda_{j_i}) \equiv \beta^n \text{Tr}(\underline{\lambda}^{(b)}) \quad (2.6)$$

where $\sum_{i=1}^b j_i = n$ and the j_i currents coupled to the i^{th} boundary follow the same cyclic order in the trace as they do on the boundary.

The final result is then

$$G_n(\underline{p}, g, N_c, N_f) = \beta^n \text{Tr}(\underline{\lambda}^{(b)}) (g^2)^{b-2} g^{4h} (g^2 N_c)^e (g^2 N_f)^w \tilde{G}_n(\underline{p}) \quad (2.7)$$

This equation will be the starting point of our expansions of QCD.

3. - THE DUAL LOOP EXPANSION OF QCD

3.1. - Definition of the $1/N_c$ expansion

't Hooft has proposed ¹⁾ an expansion of QCD in powers of $1/N_c$ at $g^2 N_c$ and N_f fixed. In such case, the n point Green's function Γ_n of Section 2 is written as

$$\Gamma_n = \sum_{G_n} G_n = \beta^n \sum_{\{b\}} T_r^{(\{b\})}(\lambda)(g^2)^{b-2} \sum_{w,h=0}^{\infty} (g^2 N_c)^w g^{4h} \Gamma_n^{(b,w,h)}(g^2 N_c, \underline{P}),$$

$$\Gamma_n^{(b,w,h)}(g^2 N_c, \underline{P}) \equiv \sum_{G_n(b,w,h)} (g^2 N_c)^{\ell} \tilde{G}_n(\underline{P}) \quad (3.1)$$

where $\tilde{G}_n(\underline{P})$ is defined by Eq. (2.7) and the $\sum_{G_n(b,w,h)}$ denotes a sum over all graphs with fixed $\{b\}$, w , h and with any ℓ .

We notice, with 't Hooft, that each term of the $1/N_c$ expansion is classified in terms of the parameters $\{b\}$, w and h . This is exactly the set of parameters defining orientable dual loops [see Ref. 10) for a review of the dual concepts used in this paper]. One can indeed establish a one-to-one correspondence between orientable dual loops and terms of the $1/N_c$ expansion of QCD. We shall see in Section 3.6 how non-orientable dual loops get reinterpreted in QCD.

The idea behind the $1/N_c$ expansion is that the first term of the expansion (i.e., $\Gamma_n^{(b=1,w=0,h=0)} \equiv \Gamma_n^{(1)}$) is a minimal (though, of course, infinite) set of graphs that can lead to confinement. 't Hooft also argued that confinement in $\Gamma_n^{(1)}$ is not destroyed by higher order corrections in $1/N_c$. We shall see that, actually, a second condition is necessary (and hopefully sufficient) on $\Gamma_n^{(b=2,w=0,h=0)} \equiv \Gamma_n^{(2)}$ in order for colour confinement to persist, order by order, in $1/N_c$.

3.2. - Properties of the leading term of the expansion, $\Gamma_n^{(1)}$

We start by studying the leading term of the expansion, $\Gamma_n^{(1)}$, which is defined in terms of the diagrams shown in Fig. 2. The statement of quark confinement is translated into the mathematical requirement that, although individual graphs contributing to $\Gamma_n^{(1)}$ have quarks and gluons as intermediate states, the sum of all graphs has only mesonic bound states. Therefore one must have :

$$\frac{1}{2i} \text{Disc}_{M^2} \Gamma_n^{(1)} = \sum_R g_{R,n_1} g_{R,n_2} \delta(M^2 - M_R^2) \quad (3.2)$$

where ξ_{R,n_1} (ξ_{R,n_2}) are the couplings of the bound state (later on resonance) R to the n_1 (n_2) external currents defining the planar channel with invariant mass M^2 . The δ functions in Eq. (3.2) indicate the fact that, within the graphs of $\Gamma_n^{(1)}$, our resonances cannot decay¹⁾ (see below). The coupling $\xi_{R,1}$ of a single current to the on-shell resonance R is of order $\beta\sqrt{N_c}$. This is easily seen, noticing that from Eq. (3.1)

$$\Gamma_2^{(1)} = \beta^2 \frac{1}{g^2} T_r(\lambda_1, \lambda_2) F(g^2 N_c) \quad (3.3)$$

and then comparing with Eq. (3.2). It is this factor $\sqrt{N_c}$ that makes e^+e^- annihilation into hadrons proportional to N_c even if no coloured states exist.

Hadronic amplitudes can be constructed from Γ_n through factorization (Fig. 2d) and the n meson amplitude A_n will be of the form :

$$A_n = \left(\frac{1}{\sqrt{N_c}} \right)^{n-2} T_r(\lambda_1, \lambda_2, \dots, \lambda_n) F_n(g^2 N_c, P) \quad (3.4)$$

We will show (Section 3.4) that, using renormalization group arguments, the dependence on $g^2 N_c$ can be traded for a (trivial) dependence on a mass parameter $\sqrt{s_0}$ so that :

$$A_n = \left(\frac{c}{\sqrt{N_c}} \right)^{n-2} T_r(\lambda_1, \lambda_2, \dots, \lambda_n) F_n(P/\sqrt{s_0}) \quad (3.5)$$

where c is a calculable constant and F_n has to obey factorization constraints. The analogy with the dual model¹⁰⁾ is rather striking if we identify :

$$\frac{1}{s_0} \rightarrow \alpha' \quad ; \quad c/\sqrt{N_c} \rightarrow \gamma \quad ; \quad F_n(P/\sqrt{s_0}) \rightarrow B_n(\alpha' s_i) \quad (3.6)$$

where γ is the dual coupling and α' the (universal) dual slope. The analogy is even more remarkable if we notice that F_n should exhibit the following properties :

- (i) planarity and crossing as in planar dual models ;

- (ii) infinitely narrow resonances, i.e., F_n are meromorphic in the Mandelstam variables ; this comes ¹⁾ from the impossibility of finding, within the graphs of $\Gamma_n^{(1)}$, intermediate states of more than one system of colour singlet particles ; factorization at the resonance pole and no-ghost ;
- (iii) if these resonances lie on Regge trajectories, these will be real hence very likely linear.

If the asymptotic behaviour is controlled by these Regge trajectories :

- (iv) duality will hold ¹⁰⁾ in the sense that the sum over s channel resonances is equal to the sum over t channel resonances.

Ghost cancellation in dual models seems to depend a lot on the high degeneracy of the levels. If this holds true here :

- (v) parallel, integer spaced, daughters are expected.

The conclusion of all this is that $\Gamma_n^{(1)}$ should give hadronic amplitudes A_n whose mathematical properties are much simpler than those of each individual term in the sum of graphs defining $\Gamma_n^{(1)}$. Indeed we may expect something of the same order of complexity as a dual Born term. If this should be the case, some magic trick ought to be found in order to sum the leading graphs of the $1/N_c$ expansion in closed form. The importance of solving such a problem (which we may call the problem of planar confinement) can hardly be overestimated.

Notice that the perturbative intermediate states of $\Gamma_n^{(1)}$ (Fig. 2c) have the same colour quantum number structure of the operators

$$\bar{\Psi}_{i_1} A_{\mu_1, i_2}^{i_1} A_{\mu_2, i_3}^{i_2} \dots A_{\mu_n, i_{n+1}}^{i_n} \Psi^{i_{n+1}}$$

which, in turn, remind us of the expansion of the gauge invariant operator

$$\bar{\Psi}_i(x_1) \left[\exp \left(g \int_{x_1}^{x_2} dx_\mu A^\mu \right) \right]_j^i \Psi^j(x_2).$$

Also the flavour quantum numbers do, of course, agree. It is likely that the problem of planar confinement can be simplified by a systematic use of such operators (and similar ones of Section 3.3). These are also the basic operators of the lattice formulation ¹¹⁾ of QCD.

3.3. - Properties of the second term of the expansion

We now turn our attention to $\Gamma_n^{(2)} \equiv \Gamma_n^{(b=2, w=0, h=0)}$ which is of order $1/N_c$ relative to $\Gamma_n^{(1)}$ as is depicted in Fig. 3. We see that a new set of many-gluon intermediate states appears perturbatively in the t channel, defined as the one in which particles on one boundary go into those of the second boundary. If colour is confined order by order in the $1/N_c$ expansion, one should also dispose of such many-gluon states, by replacing them with a new set of mesonic bound states.

In order to see that these states cannot just be neglected, consider the case of external currents coupled directly to gluons. The leading graphs of the $1/N_c$ expansion are now graphs of a pure Yang-Mills theory with a cylindrical shape (unlike the planar shape of the diagrams giving $\Gamma_n^{(1)}$) similar to that of Fig. 3b. The t channel absorptive part of such diagrams, which is given by many gluon states in a colour singlet configuration at each order in g , will have to be produced eventually by mesonic quarkless bound states, if colour is confined.

The colour quantum number structure of the many-gluon intermediate states, appearing perturbatively, is that of a trace $\text{Tr} [A_{\mu_1, i_1}^{i_2} A_{\mu_2, i_2}^{i_3} \dots A_{\mu_n, i_n}^{i_1}]$. Hence these states look similar to the quarkless states of Ref. 11), which are related to the gauge invariant operators $\text{Tr} [\exp(g \int dx_\mu A^\mu)]$.

These states carry no flavour and, like the $q\bar{q}$ states, are expected to be infinitely narrow in this order of the expansion. The analogue of these states in dual theories is well known. It is the so-called Pomeron sector, whose particles are described by the Shapiro-Virasoro non-planar dual model ¹⁰⁾, or by a closed string in the string picture ¹⁰⁾. As in the dual model, particles of this new sector couple to external $q\bar{q}$ states through $q\bar{q}$ bound states. In other words $\Gamma_n^{(2)}$ has poles in t from both types of bound states and there is a direct coupling between quarkless states and the $SU(N_f)$ singlet $q\bar{q}$ states. The t channel discontinuity of $\Gamma_n^{(2)}$ is therefore given by

$$\frac{1}{2i} \text{Disc}_t \Gamma_n^{(2)} = \sum_{\tilde{R}} g_{\tilde{R}n_1} g_{\tilde{R}n_2} \delta(t - m_{\tilde{R}}^2) + (q\bar{q} \text{ bound states}) \quad (3.7)$$

where \tilde{R} denotes a resonance of the new sector.

It is important, for future considerations, to compute the order of magnitude in $1/N_c$, $1/N_f$ of the various couplings of $q\bar{q}$ and of quarkless mesons. For the coupling of three $q\bar{q}$ mesons one has

$$g(R_1, R_2, R_3) \approx \frac{1}{\sqrt{N_c}} \left[T_r(\lambda_1 \lambda_2 \lambda_3) + T_r(\lambda_3 \lambda_2 \lambda_1) \right] \quad (3.8)$$

In particular, if R_3 is the flavour singlet ($\lambda_3 = \lambda_0 = 1/\sqrt{N_f} \cdot 1$)

$$g(R_1, R_2, R_0) \approx \frac{1}{\sqrt{N_c N_f}} \delta_{12} \quad (3.9)$$

For quarkless bound states \tilde{R} we find

$$\begin{aligned} g(R_i, R_j, \tilde{R}) &\approx \frac{1}{N_c} \delta_{ij} \\ g(\tilde{R}, \tilde{R}, \tilde{R}) &= 1/N_c. \end{aligned} \quad (3.10)$$

Finally the direct $R\tilde{R}$ coupling is of order

$$g(R_i, \tilde{R}) \approx \delta_{i0} \sqrt{\frac{N_f}{N_c}} \quad (3.11)$$

Equations (3.8) through (3.11) are easily obtained using Eq. (2.7) and dividing out the couplings to external currents.

3.4. - Non-arbitrariness of the dual coupling constant in QCD

Another interesting property of QCD is that physically observable couplings are in principle calculable (and fixed). This property, known as dimensional transmutation¹²⁾ follows from the renormalization group (RG) equations for $m_a = 0$ (and should be valid also for $m_a \ll \mu =$ subtraction point). For the sake of completeness we shall sketch the argument here.

If $\Gamma(P_i, \mu_1, g_1)$ and $\Gamma(P_i, \mu_2, g_2)$ are Green's functions of massless QCD with renormalization point μ_1 (coupling g_1 at μ_1) and with renormalization point μ_2 (coupling g_2 at μ_2), respectively, the two theories define the same S matrix if

$$\bar{g}(\mu_1/\mu_0, g_1) = \bar{g}(\mu_2/\mu_0, g_2) \quad (3.12)$$

where μ_0 is some fixed reference mass and \bar{g} is the famous running coupling constant satisfying :

$$\begin{aligned} \mu \frac{\partial}{\partial \mu} \bar{g}(\mu/\mu_0, g) &= -\beta(\bar{g}), \quad \bar{g}(1, g) = g \\ \beta(g) \frac{\partial}{\partial g} \bar{g}(\mu/\mu_0, g) &= \beta(\bar{g}) \end{aligned} \quad (3.13)$$

and $\beta(g)$ is the Gell-Mann - Low function.

In the $(\mu/\mu_0, g)$ plane we can draw the curves of constant \bar{g} . We know from asymptotic freedom (which holds a fortiori for the planar theory, since fermion loops are excluded) that $\beta(g) < 0$ for small g , hence $\mu(\partial/\partial \mu) \bar{g} > 0$ for small g . The curves of constant \bar{g} will then look qualitatively like those of Fig. 4, where we have excluded non-trivial zeros of $\beta(g)$. For fixed \bar{g} the relation between μ/μ_0 and f is

$$\log(\mu/\mu_0) = - \int_g^{\bar{g}} dg' / \beta(g') \quad (3.14)$$

and, for $\beta < 0$, g decreases as μ/μ_0 increases. If we consider now a theory with a given value of g and $\mu = \mu_0$, we can always define a new value of μ , say μ_1 , by

$$\mu_1 = \mu_0 \cdot \exp \int_g^{1/\Lambda_c} dg' / \beta(g') \equiv \mu_0 f(g) \quad (3.15)$$

so that

$$\bar{g}(\mu_1/\mu_0, 1/\Lambda_c) = \bar{g}(\mu_0/\mu_0, g) = g \quad (3.16)$$

The original theory (with g, μ_0) will then be equivalent to what we shall call a "standard" theory (with $g_1 = 1/\sqrt{N_c}$, $\mu = \mu_1 = \mu_0 \varphi(g)$). However, all "standard" theories are trivially related, because they only differ by the choice of the renormalization point and a change of it trivially changes the units of mass. The units of mass will be fixed once we fix one dimensional parameter of the theory, e.g., the mass of the first bound state, or the slope of Regge trajectories. From there on, all dimensionless quantities (including mass ratios) will be fixed and are then calculable. Since all Green's functions are given, in the planar approximation, by a known power of N_c and N_f times an unknown (but in principle calculable) function of $g^2 N_c$, and since, in the "standard" theory $g^2 N_c = 1$, all Green's functions will be calculable functions of P/μ (independent of the original g) with a trivial N_c, N_f dependence.

For instance, ratios of masses of bound states will be N independent and so will be Regge intercepts. The couplings among bound states will be given by Eqs. (3.8) to (3.11) with calculable proportionality constants.

The fact that hadronic couplings are fixed and depend, in a well defined way, upon N_c and N_f in leading order in $1/N_c$ will be important for the discussion of Section 4.

3.5. - Comments on a paper by Cornwall and Tiktopoulos

In a recent paper Cornwall and Tiktopoulos ¹³⁾ have made an interesting attempt to evaluate, in the leading logarithm approximation, the infra-red divergences of QCD with the hope of finding some signature of colour confinement. One of their most suggestive results concerns the object Γ_2 considered in this paper, i.e., the two-current correlation function $\langle 0 | J(x) J(0) | 0 \rangle$.

It is argued that, because of infra-red divergences,

$$\frac{1}{2i} \text{Disc}_s \Gamma_2 = \sum_n M_n M_n^* = \sum_n \sigma_n \xrightarrow{\lambda \rightarrow 0} 0 \quad (3.17)$$

Here λ is the gluon mass, introduced by hand to cut-off infra-red singularities, σ_n is the cross-section for producing a $q\bar{q}$ pair and n gluons out of the current $J(x)$ and M_n is the corresponding amplitude. In leading order in $1/N_c$, our Eq. (3.2) should hold at $\lambda = 0$

$$\frac{1}{2i} \text{Disc}_s \Gamma_2 \xrightarrow{\lambda \rightarrow 0} \sum_R g_R^2 \int (s - m_R^2) \quad (3.18)$$

where, averaging over s , one should recover the scaling limit of the naive parton model ($\sigma \approx N_c N_f$ if the current is a flavour singlet).

For λ small but finite, confinement will not be completely effective because the binding potential will be cut-off at large distances by an $e^{-\lambda r}$ factor. In this case we expect Γ_2 to exhibit narrow resonances (rather than bound states) which decay into quarks and gluons with smaller and smaller widths as $\lambda \rightarrow 0$. We then expect Eq. (3.17) to be true except near the resonance mass where actually $\sigma_n \rightarrow \infty$ for $\lambda \rightarrow 0$. Hence we suspect Eq. (3.17) not to be correct for the gluon theory. This result of Ref. 13) has been already criticized¹⁴⁾ on the basis of Kinoshita's theorem (Γ_2 and therefore $\text{disc}_s \Gamma_2$ cannot be infra-red singular). We add to that criticism the observation that, for $N_c \rightarrow \infty$, or just by keeping planar diagrams only, M_n should exhibit a narrow resonance structure as we vary s . Instead the leading log calculations of Ref. 13) give for M_n (e.g., for M_0) a smooth function of s even in the limit $N_c \rightarrow \infty$. We then conclude that leading log calculations are not reliable as long as the confinement problem of QCD is concerned.

3.6. - Higher order contributions

The $1/N_c$ expansion will be an appealing approach if, once colour is confined in the leading order, higher orders in $1/N_c$ do not destroy confinement.

It is instructive to see how this can happen by looking into a few examples.

A) - Addition of planar fermion loops and a paradox

Consider the object $\Gamma^{(b=1, w=1, h=0)}$ which is of order $1/N_c$ when compared to $\Gamma_2^{(1)} \equiv \Gamma_2^{(b=1, w=0, h=0)}$. This object is shown in Fig. 5 with some of its lowest order graphs. It is safe to assume that, once quarks are confined in $\Gamma^{(1)}$, the new graphs with one window will not destroy confinement. We see clearly that the graphs of Fig. 5 just allow one of our bound states to decay into two states of the same family. In other words, the effect of this planar loop is that of starting the process of turning our bound states into resonances of finite width. This is exactly the way it works in dual models, where one never talks about quarks and gluons and writes dual loops by explicit summation over mesonic resonances¹⁰⁾.

All this looks rather trivial ; it is, however, in contrast to what is usually done in perturbation theory. Consider the graph of Fig. 5a : this is a renormalization of the gluon propagator. In the same way, the graph 5b is a correction to the three-gluon vertex. In standard perturbation theory one would add diagrams with more fermion loops attached to the gluon propagator to 5a, and would then construct a renormalized gluon propagator to be inserted in skeleton graphs. In the $1/N_c$ expansion, however, each extra fermion loop means an extra factor $1/N_c$ and one has to sum differently, i.e., adding together (in the same order in $1/N_c$) propagator corrections, vertex corrections and (primitive) skeleton graphs. They are all needed in order to guarantee confinement : in other words, the $1/N_c$ expansion does not like the skeleton expansion. But now we get to a paradox : if $N_f \gg N_c$, we know that asymptotic freedom is lost and that $g=0$ becomes an attractive fixed point for the infra-red limit (as in QED). We would then expect to lose confinement for $N_f \gg N_c$. On the other hand, confinement at the leading order in the $1/N_c$ expansion does not depend at all on N_f and we have just argued that, if one keeps summing according to the $1/N_c$ expansion, nothing dramatic is expected to happen as a result of adding fermion loops (in the same way as nothing should happen at a dual theory if N_f is large). We then end up in a sort of paradox in which two different orders of summation suggest opposite conclusions on the problem of confinement.

B) - Addition of handles

Consider again $\Gamma_2^{(1)}$ and let us add to it a non-planar gluonic insertion (Fig. 6). A single gluon cannot be just non-planar because, as it is easy to check, it would have to be a colour singlet. We can put, however, two or more gluons on the handle and build on that to form the usual cylindrical topology of the handle.

It is now clear that the complete object $\Gamma_2^{(b=1, w=0, h=1)}$ thus obtained will have intermediate states of the type $R\tilde{R}$ where R is a $q\bar{q}$ meson and \tilde{R} a quarkless state. These diagrams will therefore produce $R \rightarrow R + \tilde{R}$ transitions. Seen from a Regge point of view, a Reggeon Pomeron cut is added to the pure pole exchange of $\Gamma^{(1)}$. By the way, this will be the lowest order graph to break ρ, A_2 degeneracy, a problem now under study¹⁵⁾ in the topological expansion (see Section 4). We also remark that it was crucial to confine purely gluonic states (in $\Gamma_n^{(2)}$), otherwise we would have obtained at this order a transition $R \rightarrow R + \text{gluons}$.

C) - Non-orientable dual loops and their reinterpretation in QCD

In Figs. 7a and 7b, we show two typical diagrams of QCD which look like non-orientable dual loops. In dual theory, it has been argued¹⁰⁾ that one can consistently avoid them if $qq, \bar{q}\bar{q}$ states are not present. It is interesting to see what is the possible interpretation of non-orientable dual loops in QCD.

Following our conventions of Section 2, we can redraw the diagrams 7a, 7b as 7c, 7d, respectively, thus reproducing diagrams of higher topological complexity ($h > 0$). Summing order by order in $1/N_c$, qq and $\bar{q}\bar{q}$ states cannot be formed; only $q\bar{q}$ states and quarkless states are found at each order in $1/N_c$. Non orientable dual loops are thus reinterpreted as orientable loops with handles.

Unfortunately something similar happens for qqq states (in general N_c quark states). It seems to be impossible to bind quarks into baryons by just summing diagrams of a given order in $1/N_c$. The binding of quarks into baryons seems to require a non-perturbative effect in the $1/N_c$ expansion.

To conclude this section, we remark that, along the lines of thought presented here, it should be possible to prove, at least formally, that confinement of colour in $\Gamma_n^{(1)}$ and $\Gamma_n^{(2)}$, as discussed in subsections 3.2 and 3.3, guarantees confinement of colour to each order in the $1/N_c$ expansion. Higher orders in $1/N_c$ will just give widths to the bound states and will implement unitarity by introducing cuts, both in the energy and in the angular momentum plane.

We repeat again, however, that baryons are likely to be missed by a $1/N_c$ expansion. If we remember that the consistent introduction of baryons has always been a problem in dual theories, we see that, also in this respect, the $1/N_c$ expansion of QCD and dual models behave in a similar way.

4. - THE TOPOLOGICAL EXPANSION OF QCD

4.1. - Definition of the topological expansion

The $1/N_c$ expansion, although very promising within the problem of confinement, has the phenomenological disadvantage that its higher orders mix together planar-loop corrections and non-planar corrections.

Non-planar corrections can be seen to modify the properties of the leading term in a qualitative way : on one hand they introduce violations of the so-called Zweig rule ($\Gamma_n^{(2)}$ already does that), on the other hand they introduce absorptive corrections (i.e., Regge cuts and long-range correlations) to the pole dominated leading terms of the $1/N_c$ expansion. Unlike non-planar corrections, planar corrections are expected to modify quantitatively, but not qualitatively, the leading term of the expansion. They introduce neither violations of the Zweig rule, nor absorptive corrections : they are simply expected to renormalize the Born term by giving a width to the bound states (i.e., making them decay into light particles) and by enforcing a set of unitarity-like constraints (i.e., planar unitarity) expressing an approximate type of conservation of probability.

Phenomenologically, we have evidence both for an approximate-to-good validity of the Zweig rule and for dominance of short-range correlations in multiparticle production. It seems therefore desirable to treat non-planar corrections (i.e., violations of the Zweig rule and long-range correlations) in a perturbative way. On the other hand, one would like to treat non-perturbatively planar loop corrections in order to correct the Born term for gross violations of unitarity, and in order to predict phenomena like pionization and jets. Also, quantitatively, planar corrections are down by powers of N_f/N_c and are not suppressed enough if $N_f/N_c \approx 1$ 4).

These considerations lead to another expansion of the $1/N$ variety, which we call the topological expansion (TE) of QCD, in which, while expanding in g , both $g^2 N_c$ and $g^2 N_f$ are held fixed. Consequently, the ratio $\rho \equiv N_f/N_c$ is also fixed in the TE. The value of ρ turns out to have important consequences on some phenomenological features of hadronic physics.

From our definition of the TE, we see that, instead of Eq. (3.1), we now have the following expansion of Γ_n :

$$\Gamma_n = \beta^n \sum_{\{b\}} (g^2)^{b-2} T_r(\lambda) \sum_{h=0}^{\infty} g^{2h} \Gamma_n^{(b,h)}(g^2 N_c, \rho, \underline{p}) \quad (4.1)$$

where

$$\Gamma_n^{(b,h)}(g^2 N_c, \rho, \underline{p}) \equiv \sum_w (g^2 N_f)^w \Gamma_n^{(b,w,h)}(g^2 N_c, \underline{p}) = \sum_w (g^2 N_c \rho)^w \Gamma_n^{(b,w,h)}(g^2 N_c, \underline{p}) \quad (4.2)$$

If we consider again the n meson amplitude A_n , and we trade the dependence on $g^2 N_c$ for a mass scale $\sqrt{1/\alpha'} = \mu$, we get

$$A_n = \left(\frac{c}{\sqrt{N_c}}\right)^{n-2} \sum_{\{b\}} T_r(\lambda) \left(\frac{c^2}{N_c}\right)^{b-1} \sum_{h=0}^{\infty} \left(\frac{c^2}{N_c}\right)^{2h} A_n^{(b,h)}(\underline{p}\sqrt{\alpha'}, \rho) \quad (4.3)$$

In this form the TE of QCD looks identical to the TE of dual models 4) except for the fact that the dual coupling is now fixed at $c/\sqrt{N_c}$, with c a calculable constant (see Section 3.4).

The two parameters b and h , defining a generic term of the expansion, have a direct physical meaning. In fact $\{b\}$ gives the $SU(N_f)$ structure of the contribution. In particular $\{b\}$ controls violations of

the Zweig rule, which are automatically suppressed ^{*}) by a factor $(1/N)^{b-1}$. On the other hand the number h denotes the degree of non-planarity of the contribution but does not affect its quantum number structure (a handle brings no flavour). Increasing h will correspond to absorptive corrections of increasing complexity [typically an $(h+1)$ Pomeron cut]. Each extra handle will give an extra $1/N^2$ factor [hence, for instance, $\sigma_{e\bar{e}}/\sigma_{\text{tot}} \approx O(1/N^2)$].

We now proceed to discuss some properties of the TE of QCD.

4.2. - Properties of the bare Reggeon and an apparent paradox

The bare Reggeon is defined as $A_{R_0} \equiv A^{(b=1, h=0)}$ and is the first term of the expansion (4.3). Its topological structure is shown in Fig. 8. This object has been studied extensively in the literature ¹⁶⁾ from the point of view of the dual topological expansion. Here we shall only recall a few properties of the bare Reggeon in the TE and shall spend most of the time illustrating those aspects which are typical of QCD. The standard properties of $A_n^{(b=1, h=0)}$ are those of a planar dual model, i.e. :

- (i) exact exchange degeneracy ;
- (ii) ideal mixing in broken $SU(N_f)$ and exact validity of the Zweig rule ;
- (iii) absence of fixed poles and of Regge cuts ;
- (iv) validity of finite energy and of finite mass sum rules with saturation in terms of resonances of finite width on one hand, and moving Regge poles on the other.

To these linear properties, which are common to the bare Reggeon and to the Born term, we have to add an important non-linear constraint, known as planar unitarity ⁴⁾, which we write schematically as :

$$\frac{1}{2i} \text{Disc}_s A_{n_1+n_2}^{(b=1, h=0)} = \sum_m A_{n_1+m}^{(b=1, h=0)} A_{n_2+m}^{(b=1, h=0)*} \quad (4.4)$$

This equation is illustrated in Figs. 9a,b and discussed in detail in Ref. 4).

^{*}) This depends on the $SU(N_f)$ quantum numbers. It is true for the decay of a pure $q\bar{q}$ state (say, $\lambda\bar{\lambda}$ or $c\bar{c}$) but not for the singlet $1/\sqrt{N_f} (\sum_i q_i \bar{q}_i)$.

Planar unitarity is a very strong constraint on planar amplitudes. As an example it gives a bound on partial waves, $0 \leq \text{Im } A_\ell \leq 1/N_f$, which is N_f times stronger than the usual unitarity bound.

More interesting results emerge if we couple planar unitarity with the multiperipheral (MP) assumption that imaginary parts are dominated, at large s and small t , by the contribution of multiparticle production proceeding via a MP scheme (Figs. 9c,d). Within this planar theory, the same exchange-degenerate set of Reggeons is supposed to be exchanged and produced by the sum over intermediate states. Furthermore, exchange of Regge cuts is safely excluded because of planarity and crossed ladders are out for the same reason. A very tight bootstrap scheme emerges, in which no Regge cut has to appear in the output of the MP integral equation, although it is present for each exclusive cross-section. Amazingly, the requirement of self-consistency on the output pole, combined with the use of finite mass sum rules (valid in the planar theory!) can lead to such an exact cancellation of the cut [see last three papers, Ref. 6]. A general consequence of the scheme ⁶⁾ is the so-called planar bootstrap equation :

$$1 = \frac{\gamma^2 N_f}{16\pi^3} \int d^2 p_\perp \frac{\Gamma^2(t, t_1, t_2)}{(1 + \alpha(t) - \alpha(t_1) - \alpha(t_2))^2} \cos \pi(\alpha(t_1) - \alpha(t_2)) \quad (4.5)$$

where $\gamma\Gamma$ is the triple Regge coupling [γ is our dual coupling, Eq. (3.6), and Γ is normalized to one at some values of t, t_1, t_2]. As a result of Eq. (4.5), one finds typically $(\gamma^2 N_f / 16\pi^2) \approx 0(1)$. The fact that $\gamma^2 N_f$ cannot be arbitrarily small in this theory comes from the fact ¹⁶⁾ that we have taken "almost" linear trajectories (i.e., trajectories whose slope does not depend on γ for small γ) and that slope and cut off in transverse momentum are of the same order. In ordinary field theories (say, $\lambda\phi^3$) this is not what happens : the slope is proportional to g^2 and the whole trajectory moves away from some integer only $0(g^2)$. This allows consistency between multiperipheral behaviour and unitarity even for small values of g .

If QCD is a confining theory, our RG arguments of Section 3.4 have shown that we expect α' to be some constant independent of g . Actually, our discussion in Section 3.4 has shown that, once α' is fixed, γ is a fixed number $c/\sqrt{N_c}$. This would fit perfectly well with Eq. (4.5), except that the latter gives $\gamma^2 \approx 1/N_f$ and not $\gamma^2 \approx 1/N_c$. Hence we seem to be forced into a paradox.

This paradox is not hard to solve. We first realize that the coupling γ^2 appearing in Eq. (4.5) is not the "bare" coupling constant obtained in Section 3 in leading order in $1/N_c$, because it contains all planar fermion loop corrections. Denoting by γ_0 the bare dual coupling ($\gamma_0 = c/\sqrt{N_c}$) we will have

$$\gamma^2 = \gamma_0^2 f(\rho) = \gamma_0^2 \sum_{n=0}^{\infty} f_n \rho^n \quad (4.6)$$

where the \sum_n is the fermion loop expansion. This is enough to solve our paradox for $\rho \gg 1$ ($N_f \gg N_c$). In this case we have no reason to believe $\gamma^2 \approx \gamma_0^2$ and it is perfectly sensible to expect that planar loops will "absorb" the coupling, forcing $\gamma^2/\gamma_0^2 \approx N_c/N_f$ ($f(\rho) \underset{\rho \gg 1}{\sim} \rho^{-1}$). In other words, in the case $\rho \gg 1$, the Born term is too big and violates planar unitarity bounds (which are tighter for large ρ). Planar loops intervene to restore unitarity.

The situation is less trivial for the opposite case $\rho \ll 1$ ($N_f \ll N_c$). In this case (4.6) should give $\gamma^2 \approx \gamma_0^2 + O(\rho) \approx 1/N_c$. This is the case of very small couplings compared to what Eq. (4.5) would like. Since, in deriving Eq. (4.5), the only doubtful assumption we have used is the MP assumption, we are led to reanalyze such an assumption. The point is that, for $N_c \gg N_f$, our resonances have small width (one can easily see that widths are proportional to $g^2 N_f \approx N_f/N_c \approx \rho$ in the planar limit) and, in that situation it is well known that the MP model has to be at least implemented with clustering effects. Qualitatively speaking, this will increase our phase space in Eq. (4.5) because an integration over cluster mass will be added. In this way we can achieve consistency with $\gamma^2 \approx 1/N_c$.

We can make our arguments more quantitative by introducing an average cluster size in rapidity \bar{Y}_c and an average gap size \bar{Y}_g . Equation (4.5) is then easily seen to be modified into

$$\bar{Y}_g (\bar{Y}_c + \bar{Y}_g)^{-1} = N_f \langle \gamma^2 \Gamma^2 \rangle, \quad (4.7)$$

$$\langle \gamma^2 \Gamma^2 \rangle \equiv \gamma^2 \int \frac{d^2 p_\perp}{16\pi^3} \frac{\Gamma^2 \cos \pi(\alpha_1 - \alpha_2)}{(1 + \alpha - \alpha_1 - \alpha_2)^2}$$

Equation (4.7) can be derived in several equivalent ways, the simplest one being a trivial generalization of the method used by Rosenzweig and the author, Ref. 6). From (4.7) we get

$$\frac{\bar{Y}_c}{\bar{Y}_g} = \frac{1 - N_f \langle \gamma^2 \Gamma^2 \rangle}{N_f \langle \gamma^2 \Gamma^2 \rangle} \quad (4.8)$$

Equation (4.8) gives immediately $\langle \gamma^2 \Gamma^2 \rangle \leq 1/N_f$ with the bound saturated for $\bar{Y}_c/\bar{Y}_g \rightarrow 0$. This is the situation we expect to face for large widths ($\rho \gg 1$) in which case an MP model without clusters is expected to work. When ρ is decreased and eventually becomes small ($N_c \gg N_f$) $\langle \gamma^2 \Gamma^2 \rangle \rightarrow 1/bN_c$, with b in principle calculable (like c of Section 2). We now have

$$\bar{Y}_c / \bar{Y}_g \underset{N_c \gg N_f}{\approx} \frac{bN_c - N_f}{N_f} \approx \frac{bN_c}{N_f} = b/\rho$$

and \bar{Y}_c becomes much larger than \bar{Y}_g . At that stage the whole validity of an MP scheme can be put in doubt. It seems preferable to turn to an s channel picture of formation of long-lived compounds, which cascade into pions [see, e.g., Ref. 16]. In this type of picture, unitarity constraints are rather trivial to satisfy, if one takes into account the effect of small widths. This introduces factors $1/\Gamma$ and planar unitarity just gives $\Gamma \sim N_f/N_c$.

The physical situation seems to be midway between the two extreme pictures described above. We have, on one hand, $\rho \approx 1$ *) and, on the other hand, an MP model with production of clusters of mass ~ 1.5 GeV is a rather appealing possibility ¹⁷⁾. From our point of view, we have to investigate the case $\rho \approx 1$. Since we have

$$\begin{aligned} \langle \gamma^2 N_f \Gamma^2 \rangle \xrightarrow{N_f \gg N_c} 1 & \quad ; \quad \langle \gamma^2 \Gamma^2 \rangle \leq 1/N_f \\ \langle \gamma^2 \Gamma^2 \rangle \xrightarrow{N_c \gg N_f} \frac{1}{bN_c} & \end{aligned} \quad (4.9)$$

we are very tempted to use, for all values of ρ

*) In broken $SU(N_f)$ one can see that what counts is some sort of $N_f^{\text{effective}}$ in which degrees of freedom associated with heavy quarks count less.

$$\langle \gamma^2 \Gamma^2 \rangle = (b N_c + N_f)^{-1} \quad (4.10)$$

at least as an approximation. Equation (4.8) then becomes, at all values of ρ ,

$$\bar{Y}_c / \bar{Y}_g = b N_c / N_f \quad (4.11)$$

Equations (4.10) and (4.11) are an appealing set of equations relating quark parameters to phenomenological hadronic quantities. The approximation used by Schaap and the author [Ref. 6] was $\langle \gamma^2 \Gamma^2 \rangle = 1/N_f$ and gave ^{*}

$$\sigma_{\pi^+\pi^-} - \sigma_{\pi^-\pi^-} \approx \frac{100}{N_f} \text{ mb. } (\alpha's)^{\alpha(0)-1} \approx 40 \text{ mb. } (\alpha's)^{\alpha(0)-1};$$

$$\alpha(0) \approx .60.$$

An estimate, made by Stevens et al. [Ref. 7] on the basis of FMSR and low energy data, led to about 26 mb $(\alpha's)^{-\frac{1}{2}}$ for such a quantity. Instead, if we use a ratio 2:1 for ρ coupling to pions and nucleons, and compare with $\sigma_{\pi^-p} - \sigma_{\pi^+p}$ data [for which we expect about 20 mb $(\alpha's)^{-\frac{1}{2}}$] we see that we are likely off by a factor of two. Still we have to remember that our result holds for the unabsorbed ρ exchange and that the inclusion of the first correction term ($b=1$, or torus, or Pomeron, ρ cut) will reduce the cross-section. More important, the value of γ^2 will have to be reduced by a factor $(1 + \bar{Y}_c/\bar{Y}_g)$ which could be close to the factor needed for complete agreement. In any case, because of our approximations, we cannot expect agreement better than within 50%.

We end this subsection by noting that a more precise relation, which, unlike Eq. (4.8), does not introduce the rapidity language, can be derived from the MP integral equation and from the MP expression for the average multiplicity of gaps (hence of clusters). The equation reads :

$$1 - N_f \langle \gamma^2 \Gamma^2 \rangle = \log(\bar{M}_c^2/s_0) \cdot \langle n_c \rangle / \log s \quad (4.12)$$

where $\langle n_c \rangle$ is the average number of clusters and \bar{M}_c^2 the cut-off of the finite mass sum rule over the cluster mass (with $\bar{M}_c^2 = s_0 = 1 \text{ GeV}^2$ for stable particle production). From Ref. 17), we get $\langle n_c \rangle / \log s \approx 0.4$

(notice that only about half of the experimental multiplicity is relevant for the planar theory) and $\log(\bar{M}_c^2/s_0) \approx 1$; hence $N_f \langle \gamma^2 \Gamma^2 \rangle = 0.6$ which looks like a very reasonable number ($\sigma_{\pi^-\pi^+} - \sigma_{\pi^+\pi^+} \approx 24 \text{ mb } (\alpha's)^{-\frac{1}{2}}$).

^{*}) Because of a trivial mistake the couplings quoted there have to be multiplied by a factor of π .

4.3. - The planar approximation to e^+e^- annihilation

We now extend our considerations on the leading term of the TE to include a discussion of e^+e^- annihilation into hadrons in the planar approximation ($b=1, h=0$). There are actually four degrees of satisfaction at which one can describe e^+e^- annihilation. The first one is naive perturbation theory, in which the virtual photon creates a $q\bar{q}$ pair and no gluon is exchanged (Fig. 10a) so that actual quarks are produced. This gives the "right" value of σ but the wrong final states. The second model includes planar gluons (Fig. 10b) and corresponds to our $\Gamma_2^{(1)}$ of Section 3. We still get the right value for σ and no-quarks in the final states : the final states are now $q\bar{q}$ heavy bound states. There is no unitarity and inclusive spectra will diverge. The third degree of sophistication is the one given by the TE of this section. Now we get (Fig. 10c) multiparticle final states possibly with clustering and, most likely, with a jet-type structure. The quantum number structure of the final state is the same as that discussed in several occasions by Feynmann : $q\bar{q}$ pairs are created out of the gluons and each q combines with a neighbouring \bar{q} to form a meson. The final mesons will be ordered very much like in a MP chain with memory of the initial $q\bar{q}$ quantum numbers getting lost as we move towards the centre of the chain. Both here and in the picture of Feynmann interference (crossed) terms are excluded (planarity).

The fourth and last description is the one in which the full $q\bar{q} \rightarrow q\bar{q}$ Green's function is introduced in the black box of Fig. 10d. Now we have a multiparticle final state in which some of the particles may originate from the "middle" of the graph forming a flavour singlet system [in exact $SU(N_F)$]. This can be thought of as the decay of an excited, heavy quarkless state. Even the whole final state can be given by such a system : this is the case for $\Gamma_2^{(b=2, h=0)}$. This case, in which a violation of Zweig's rule originates, will be dealt with in Section 4.4.

We see, in conclusion, that the picture given by the leading term in the TE (Fig. 10c) can be a fairly good approximation of the quantitative description of e^+e^- annihilation away from the $J/\psi, \psi'$ masses where the next term is needed. Inclusive-type sum rules will be satisfied at this stage, as a consequence of planar unitarity. Apparent paradoxes come out as we vary N_F/N_C and their resolution follows the same lines of thought used to overcome the purely hadronic puzzles. We believe that, in this way, a paradox recently found by Schierholz and Schmidt¹⁸⁾ disappears completely.

Our topological considerations can be extended, of course, to other processes with lepton-hadron interaction, e.g., deep inelastic electroproduction. The study of these processes is deferred to some future investigation, but we will just remark that the inclusion of quark loops (i.e., the TE) will be needed if we want to introduce both valence (given by the leading $1/N_c$ term) and sea quarks.

4.4. - The bare Pomeron and the problem of the f extinction

After having discussed the bare Reggeon we turn our attention to the bare Pomeron of QCD which we define as $A_{P_0} \equiv A_{n_1, n_2}^{(b=2, h=0)}$. Its topological structure is shown in Fig. 11. The bare Pomeron of the dual topological expansion has been the subject of much attention in recent years ⁷⁾. Here we recall some of the results obtained so far. Consider for simplicity the case of a four-point function $A_{P_0}(s, t)$

- 1) - $A_{P_0}(s, t)$ has vacuum quantum numbers in the t channel and exotic quantum numbers in s and u provided by non-resonant intermediate states.
- 2) - At fixed t and large s , A_{P_0} is dominated by a Regge pole $\alpha_{P_0}(t)$ whose intercept $\alpha_{P_0}(0)$ is larger than the Reggeon intercept $\alpha_R(0)$ ^{4), 5), 7)}. Arguments can be given ⁷⁾ for $\alpha_{P_0}(0) \approx 1$. Nevertheless the quantity $1 - \alpha_{P_0}(0)$ can have any sign and is of order one in the $1/N$ expansion ⁸⁾.
- 3) - The s channel discontinuity of A_{P_0} is given by a sort of double MP chain ⁷⁾ in which two sets of particles are produced non-diffractively. According to our discussion in Section 4.2, one expects clusters (resonances with $q\bar{q}$ quantum numbers) to be formed if the dual coupling has a moderate value (i.e., for $N_F/N_c \approx 1$). Only particles belonging to the same set can form clusters in the above sense ; this does not mean, however, that particles belonging to different sets will not be able to be close in rapidity. What we expect is that "like" clusters (i.e., clusters made of particles of the same set) will be ordered in rapidity with small overlap among adjacent clusters, whereas "unlike" clusters will interact weakly, will not be ordered in rapidity relative to each other and can actually have a considerable overlap. This point will be taken up again in the question of the f extinction.

- 4) - From the point of view of broken $SU(3)$ [or $SU(4)$] the bare Pomeron is not a pure singlet ⁷⁾ but rather a combination of singlet and octet. In broken $SU(3)$ it will couple to non-strange particles more than to strange particles.
- 5) - The bare Pomeron mixes with the neutral components of the bare Reggeon ⁷⁾ (e.g., with f and f' but not with ρ , A_2). As a result, it breaks ρ , f exchange degeneracy and with it the so-called Zweig rule ⁹⁾. Violations of the Zweig rule are found ⁹⁾ to be strongly t dependent, with large violations occurring at $t \lesssim -0.5 \text{ GeV}^2$ [giving an approximate $SU(3)$ singlet Pomeron] and small violations occurring at $t \gtrsim 0.5 \text{ GeV}^2$ (giving ideal mixing). At $t=0$ (total cross-sections) the situation is sort of midway between $SU(3)$ singlet and ideal mixing. This is what Chew and Rosensweig call asymptotic planarity ⁹⁾.
- 6) - For large t , fixed s A_{P_0} is dominated by the (exotic) Reggeon-Reggeon cut. This helps understanding the smallness of Zweig rule violations ⁹⁾ at $t \rightarrow +\infty$ and also relates it to the experimentally observed suppression of exotic (e.g., double charge) exchanges.
- 7) - Finally, one finds ⁸⁾ that the bare coupling of three-bare Pomerons at $t=0$ is different from zero and of order $1/N$.

We now turn our attention to the problem of the f extinction in the TE, namely the fact that when we add the bare Pomeron amplitude A_{P_0} to the bare Reggeon amplitude A_{R_0} a single leading trajectory ($P_0 = f_0$) with vacuum quantum numbers is usually seen to appear. Theoretically, this is rather unappealing (and still being debated ⁷⁾ phenomenologically), due to the fact that both the dual perturbation theory and the $1/N_c$ expansion of QCD lead us to consider this new sector of bound states (closed strings in dual models, quarkless gluonic states in QCD).

Let us look at the problem again in the context of QCD, as we vary $\rho \equiv N_f/N_c$. If $\rho \ll 1$ we expect the dual perturbation theory to be good and fermion loops to be relatively unimportant. Equations (3.9), (3.10), (3.11) are expected to be accurate and one finds :

$$\begin{aligned} & \gamma_{R_0}^2 \approx 1/N_c N_f \gg \gamma_{P_0}^2 \approx 1/N_c^2 ; \\ \text{and} \quad & \gamma_{R_0 P_0}^2 \approx N_f/N_c = \rho \ll 1 \end{aligned} \quad (4.13)$$

In other words we have two trajectories, the bare Pomeron and the flavour-singlet Reggeon, the first high and weaker, the second lower and stronger. The two are weakly coupled to each other. In this situation the addition of any number of planar loops should not alter qualitatively the situation but the picture for the Pomeron is that of s channel production of two (or more) heavy, overlapping clusters with no possibility of exchanging Reggeons in between. In this case we lose any argument ⁷⁾ for $\alpha_{P_0}(0) \sim 1$ and for any dynamical relation between the Pomeron and the Reggeon.

As we increase ρ , clusters become smaller and we will slowly converge towards an MP of the Reggeon and of the Pomeron. With ρ of order one clusters still have a size comparable to the average gap size, and the probability of having two unlike clusters overlapping is of order $\frac{1}{2}$ or so.

In these circumstances we cannot always exchange twisted Reggeons between unlike clusters and, within the MP picture, we are forced to add background production to resonance production. This background production was neglected in all schemes producing f extinction and we conjecture that this is the explanation of the effect.

As we let $\rho \gg 1$ and the cluster size goes to zero, the renormalization of our couplings replaces Eq. (4.13) by

$$\gamma_{R_0}^2 \approx \gamma_{P_0}^2 \approx 1/N_f^2 \quad ; \quad \gamma_{R_0 P_0} \approx O(1) \quad (4.14)$$

as obtained from the arguments of Section 4.2.

Now the mixing $\gamma_{R_0 P_0}$ is strong and the couplings comparable. Also for $(\bar{Y}_c/\bar{Y}_g) \rightarrow 0$, configurations with overlapping clusters become very rare, and the arguments of Chew and Rosenzweig ⁷⁾ can be used to give decoupling of the non-leading trajectory.

Again we believe the truth lies somewhere in the middle with $\rho \approx 1$ and MP background production appreciable, even if not the dominant mechanism. The introduction of the two mechanisms can be shown to restore the possibility of two leading singularities in the vacuum channel ¹⁹⁾.

There is an exception though. In one-space, one-time dimensions, diagonalization of the MP integral equation again leads to the decoupling of the f . It is amusing to observe that QCD has a similar peculiarity in two-dimensions²⁾. Gauge mesons have no physical degrees of freedom (there are no transverse gluons in two-dimensions!) and quarkless states are not supposed to exist. In such a case our prejudice for another sector falls completely and the Pomeron, even if present, will not be a particle-supporting trajectory.

4.5. - TE and Reggeon field theory

We need not repeat here how the bare Pomeron and Reggeon defined in the TE interact together once diagrams with handles ($h > 0$) are included. As shown in Ref. 8), this leads to a "derivation" of the Reggeon calculus from our diagrammatic viewpoint. The new ingredients given by the TE are :

- 1) - a precise definition of the bare Reggeons and a certain knowledge of their s channel structure ; the relation $\sigma_{el}/\sigma_{tot} \approx 1/N^2$ justifies a perturbative approach à la Gribov ;
- 2) - a Reggeon field theory Lagrangian of the form $\mathcal{L} = \psi^+ \psi F(\psi/N, \psi^+/N)$; i.e., a mass term of order 1, a (non-zero) triple coupling of order $1/N$, etc. ; in particular one does not expect to be at the critical point of the Reggeon field theory ($\alpha_{ren}(0) = 1$) ;
- 3) - one is able to derive a Reggeon field theory with cut and uncut Pomerons and to obtain in this way properties related to s channel unitarity and to multiparticle production ; the cutting rules of Abramovskii, Gribov and Kancheli are rederived and s channel unitarity is built in from the beginning ;
- 4) - One hopes to be able to understand the transition point (in energy) between a perturbative regime and a non-perturbative one ; this is possibly given by $(1/N^2) \log E \approx 1$ and is most likely much above presently available energies.

5. - CONCLUSIONS AND COMPARISON WITH OTHER SCHEMES

In this paper we have related, at least in qualitative way, three of the most promising present approaches to strong interaction physics. The connection has been obtained by means of non-perturbative expansions of the large N variety applied to QCD.

The main outcome of this effort has been, in our opinion, a wide unification of concepts used by people working in different areas of hadronic physics. As a result, new insight has been gained on several aspects of hadronic physics and, at the same time, the most important problems to be solved in each approach have clearly emerged.

Within QCD, the central problem appears to be that of "planar confinement" (i.e., confinement in leading order in the $1/N_c$ expansion). Arguments have been given for adopting an optimistic view towards the possibility of solving this problem by some non-perturbative technique. Existing dual models with their string interpretation could play an important heuristic rôle by showing what type of properties one should expect from QCD in the planar approximation. The deep analogy of QCD in two dimensions and certain dual string models has been already noticed ³⁾ and the challenging problem now is to generalize such analogy to the four-dimensional case.

Another interesting problem in which a unified approach could prove of great value is that of understanding the Zweig rule and its violations. The problem can be approached both from the point of QCD (asymptotic freedom) and from that of dual models and the TE ⁹⁾ (asymptotic planarity). The question of diffractive scattering in QCD is a related problem, the Pomeron and violations of the Zweig rule being just two manifestations of the same phenomenon in different kinematical regions. The TE and continuation from $t \leq 0$ to $t > 0$ clearly show this fact.

We then have the link between dual models and Gribov theories. Here, the problems that emerge as the crucial ones are those of determining the bare Pomeron and its properties, in particular the intercept, the slope, the elastic and triple Pomeron coupling. At the level of the Reggeon field theory itself, our approach gives a simple understanding of s channel unitarity and of the transition between the perturbative regime [small $(1/N^2)$ $\log s$] and the non-perturbative one of extremely high energies.

It is promising to have been able to identify some small parameter in the framework of strong interactions. This parameter is the inverse of the number of degrees of freedom of the hadronic world (either hidden, like colour, or realized, like flavour).

Much work is clearly waiting for us if we want to fill in all the holes still present in the way from QCD to Reggeon field theory which we have outlined here. Besides the obvious question of proving confinement, we just remind the reader of our difficulties in understanding baryons in the $1/N_c$ expansion. If this difficulty cannot be solved, we may be forced with the problem of not being able to make predictions which are directly testable in the laboratory.

Finally, some words about the relation of our approach to that of other authors who have recently tried to relate different concepts of hadronic physics. The approach which is closest to ours is the one of Nussinov²⁰⁾, who tries to develop a semi-perturbative approach to QCD, in which some effects are treated perturbatively and others are not. We do have something similar, except that we make the distinction at the topological level, whereas Nussinov rather distinguishes on kinematical grounds (hard vs. soft gluons). This does not exclude the possibility that the two approaches can be combined to increase the predictive power of both.

Another point of view is the one of Preparata and co-workers²¹⁾, i.e., the massive quark model. The emphasis is somewhat different, in that Preparata aims at immediately deriving experimentally testable quantitative predictions, whereas we look at more general, qualitative and semi-quantitative properties. We also try to be as consistent as possible with current ideas, both in the framework of QCD and in that of dual models, whereas the scheme of Preparata is an independent approach which could turn out to be somewhat inconsistent with (or perhaps just more general than) either one. In any case, we expect $1/N$ expansions of QCD to shed some light on the expansion used by Preparata and co-workers in the massive quark model.

A third approach is the one of Pokorski and Van Hove²²⁾, the so-called quark gluon model of hadronic production and deep inelastic hadron lepton scattering. As we have discussed shortly in Section 4.3, our topological considerations can be used for both hadron-hadron and lepton-hadron scattering; hence, ultimately, our model should be able to connect the two,

something in which the model of Pokorski and Van Hove ²²⁾ is quite successful. At this stage we can only compare the two schemes at the level of purely hadronic scattering and we note that there are qualitative differences between, say, the scheme of Ref. 22) for non-diffractive production and the one of this paper. Since we always interpret purely gluonic states as states of the Pomeron sector, the mechanism for non-diffractive production of Ref. 22) looks superficially like exclusive double Pomeron exchange in our picture. A precise identification is, however, impossible because topological concepts are extraneous to the scheme of Ref. 22).

We conclude the paper by stating again our belief that a joint effort is needed today, more than ever before, towards the building of a realistic theory (or model) of hadrons. Due to the obvious objective difficulties one meets in each single approach, it seems necessary to combine as much as possible ideas, techniques and results known in each different area of study, assuming that gauge theories, dual theories, Regge-Gribov models (and maybe others) are just different ways of looking at the same problematics.

The modest aim of this paper has been just to point out the importance of large N expansions in providing the necessary bridges, or just the necessary dictionary, for such a merging of different ideas.

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REFERENCES

- 1) G. 't Hooft - Nuclear Phys. B72, 461 (1974).
- 2) G. 't Hooft - Nuclear Phys. B75, 461 (1974) ;
C.G. Callan, N. Coote and D.J. Gross - Phys.Rev. D13, 1649 (1976) ;
M.B. Einhorn - Fermilab-Pub-76/22-THY (1976) ;
J. Cardy - UCSB Preprints TH-19 (1975), TH-1 (1976) ;
R.C. Brower and J.H. Weis - Private communication.
- 3) I. Bars - Yale University Preprint (1976).
- 4) G. Veneziano - Phys.Letters 52B, 220 (1974) ; Nuclear Phys. B74, 365 (1974).
- 5) H.M. Chan, J.E. Paton and T.S. Tsun - Nuclear Phys. B86, 479 (1975) ;
H.M. Chan, J.E. Paton, T.S. Tsun and S.W. Ng - Nuclear Phys. B92, 13 (1975).
- 6) G. Veneziano - Ref. 4) ;
H.M. Chan et al. - Ref. 5) ;
C. Rosenzweig and G. Veneziano - Phys.Letters 52B, 335 (1974) ;
M.M. Schaap and G. Veneziano - Nuovo Cimento Letters 12, 204 (1975) ;
N. Sakai - Nuclear Phys. B99, 167 (1975) ;
J. Kwiecinski and N. Sakai - Nuclear Phys. B106, 44 (1976) ;
J. Finkelstein and J. Koplik - Columbia Preprint CO-2271-72 (1975) ;
S. Feinberg and D. Horn - Tel Aviv University Preprint TAUP-511-76 (1976) ;
J.R. Freeman and Y. Zarmi - Weizmann Institute Preprint WIS-76/1 (1976) ;
M. Bishari and G. Veneziano - Phys.Letters 58B, 445 (1975) ;
J.R. Freeman, G. Veneziano and Y. Zarmi - CERN Preprint TH 2211 (1976), to be typed.
- 7) H. Lee - Phys.Rev.Letters 30, 719 (1973) ;
G. Veneziano - Phys.Letters 43B, 413 (1973) ;
H.M. Chan and J.E. Paton - Phys.Letters 46B, 228 (1973) ;
G.F. Chew and C. Rosenzweig - Phys.Letters 58B, 93 (1975) ; Phys.Rev. D12, 3907 (1975) ;
C. Schmid and C. Sorensen - Nuclear Phys. B96, 209 (1975) ;
N. Papadopoulos, C. Schmid, C. Sorensen and D. Webber - Nuclear Phys. B101, 189 (1975) ;
G. Veneziano - Nuclear Phys. B108, 285 (1976) ;
P.R. Stevens, G.F. Chew and C. Rosenzweig - Caltech Preprint, CALT 68-541 (1975) ;
C. Quigg and E. Rabinovici - Phys.Rev. D13, 2525 (1976).
- 8) G. Marchesini and G. Veneziano - Phys.Letters 56B, 271 (1975) ;
M. Ciafaloni, G. Marchesini and G. Veneziano - Nuclear Phys. B98, 472, 493 (1975).

- 9) G.F. Chew and C. Rosenzweig - Nuclear Phys. B104, 290 (1976) ;
M. Bishari - Université P. et M. Curie, Paris, Preprint (1975) ;
H.M. Chan, J. Kwiecinski and R.G. Roberts - Phys.Letters 60B, 367
(1976) ;
H.M. Chan, K. Konishi, J. Kwiecinski and R.G. Roberts - Phys.Letters
60B, 469 (1976) ;
C. Schmid, D. Webber and C. Sorensen - ETH Preprint (1976) ;
E.L. Berger and C. Sorensen - Phys.Letters 62B, 303 (1976) ;
C. Rosenzweig - Phys.Rev. D13, 3080 (1976) ;
T. Inami, K. Kawarabayashi and S. Kitakado - Phys.Letters 61B, 60
(1976) ;
G.F. Chew and C. Rosenzweig - Saclay Preprint (1976).
- 10) Dual Theory, Collection of Physics Reports Reprints (edited by
M. Jacob).
- 11) See : K.G. Wilson - Erice Lecture Notes (1975), and references
therein.
- 12) S. Coleman and E. Weinberg - Phys.Rev. D7, 1888 (1973) ;
D.J. Gross and A. Neveu - Phys.Rev. D10, 3235 (1974).
- 13) J.M. Cornwall and G. Tiktopoulos - Phys.Rev.Letters 35, 338 (1975) ;
Phys.Rev. D13, 3370 (1976).
- 14) T. Appelquist, J. Carazzone, H. Kluberg-Stern and M. Roth - Phys.
Rev.Letters 36, 768 (1976) ;
Y.-P. Yao - Phys.Rev.Letters 36, 653 (1976) ;
E.C. Poggio and H.R. Quinn - Harvard Preprint (1976).
- 15) H.M. Chan, K. Konishi, J. Kwiecinski and R.G. Roberts - Rutherford
Laboratory Preprint RL-76-056 (1976) ;
J. Usherson - Helsinki Preprint (1976) ;
M. Bishari - Helsinki Preprint (1976) ;
Y. Eylon - Berkeley Preprint (1976) ;
- 16) M.B. Green and G. Veneziano - Phys.Letters 36B, 477 (1971).
- 17) L. Caneschi - Nuclear Phys. B108, 417 (1976)
- 18) G. Schierholz and M.G. Schmidt - CERN Preprint TH. 2159 (1976).
- 19) M. Bishari and G. Veneziano - Unpublished.
- 20) S. Nussinov, IAS Preprint COO-2220-57 (1975) ; Phys.Rev.Letters 34,
1286 (1975) ;
F.E. Low - Phys.Rev. D12, 163 (1975).
- 21) G. Preparata - Erice Lecture Notes (1975), and references therein.
- 22) S. Pokorski and L. Van Hove - Acta Phys. Polonica B5, 229 (1974) ;
Nuclear Phys. B86, 243 (1975) ;
L. Van Hove - CERN Preprint TH. 2121 (1976).

FIGURE CAPTIONS

- Figure 1 Graphical conventions used in Section 2 for diagrams of QCD. In the rest of the paper the more common conventions with single line propagators are also used.
- Figure 2 Graphical definition of $\Gamma_n^{(1)}$ and of A_n . Figure 2c uses the graphical rules of Fig. 1, Fig. 2b does not.
- Figure 3 The class of diagrams defining $\Gamma_n^{(2)}$. Figure 3b puts in evidence the purely gluonic intermediate states.
- Figure 4 Curves of constant \bar{g} in the $(g, \mu/\mu_0)$ plane. The two theories with $\mu/\mu_0 = 1$ denoted by a dot and by a cross are equivalent to the two "standard" theories with $g = 1/\sqrt{N_c}$.
- Figure 5 Diagrams contributing to $\Gamma_2^{(b=1, w=1, h=0)}$.
- Figure 6 Diagrams contributing to $\Gamma_2^{(b=1, w=0, h=1)}$. All vertices in the picture are trilinear.
- Figure 7 (a), (b) are diagrams of QCD which look like non-orientable dual loops; (c) and (d) are the same diagrams drawn with the clockwise convention (but with single lines) and can be put on an orientable surface with handles. No quartilinear vertices in Figs. (c) and (d).
- Figure 8 Definition of $A^{(b=1, h=0)}$.
- Figure 9 Planar unitarity for $A^{(b=1, h=0)}$; in general (a), (b), and within the MP assumption (c), (d).
- Figure 10 Four degrees of sophistication in the description of e^+e^- annihilation into hadrons.
- Figure 11 The bare Pomeron of QCD. All vertices in (b) are trilinear.



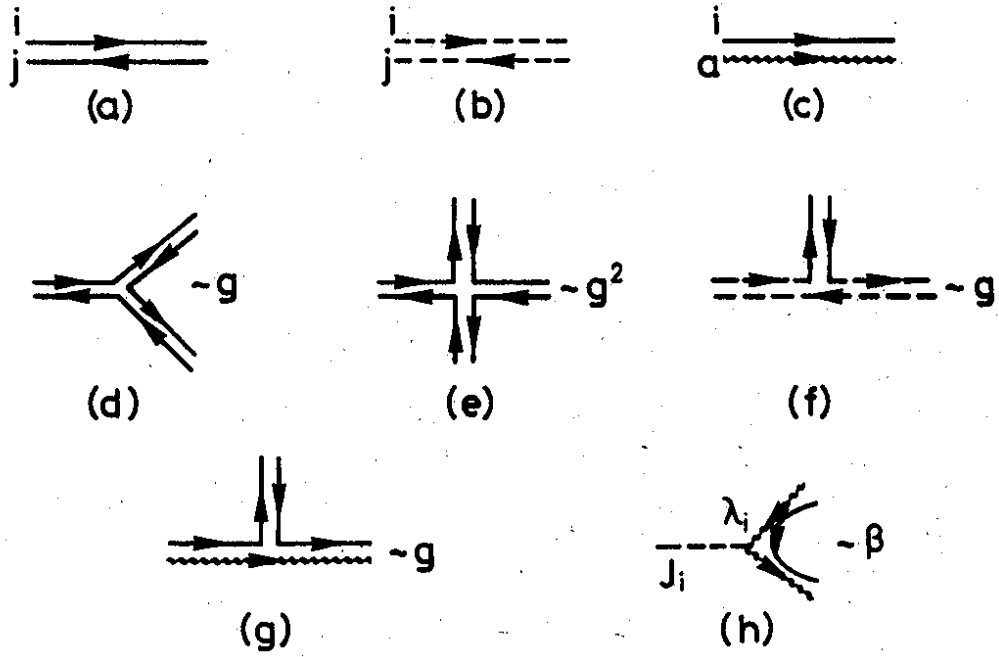


FIG. 1

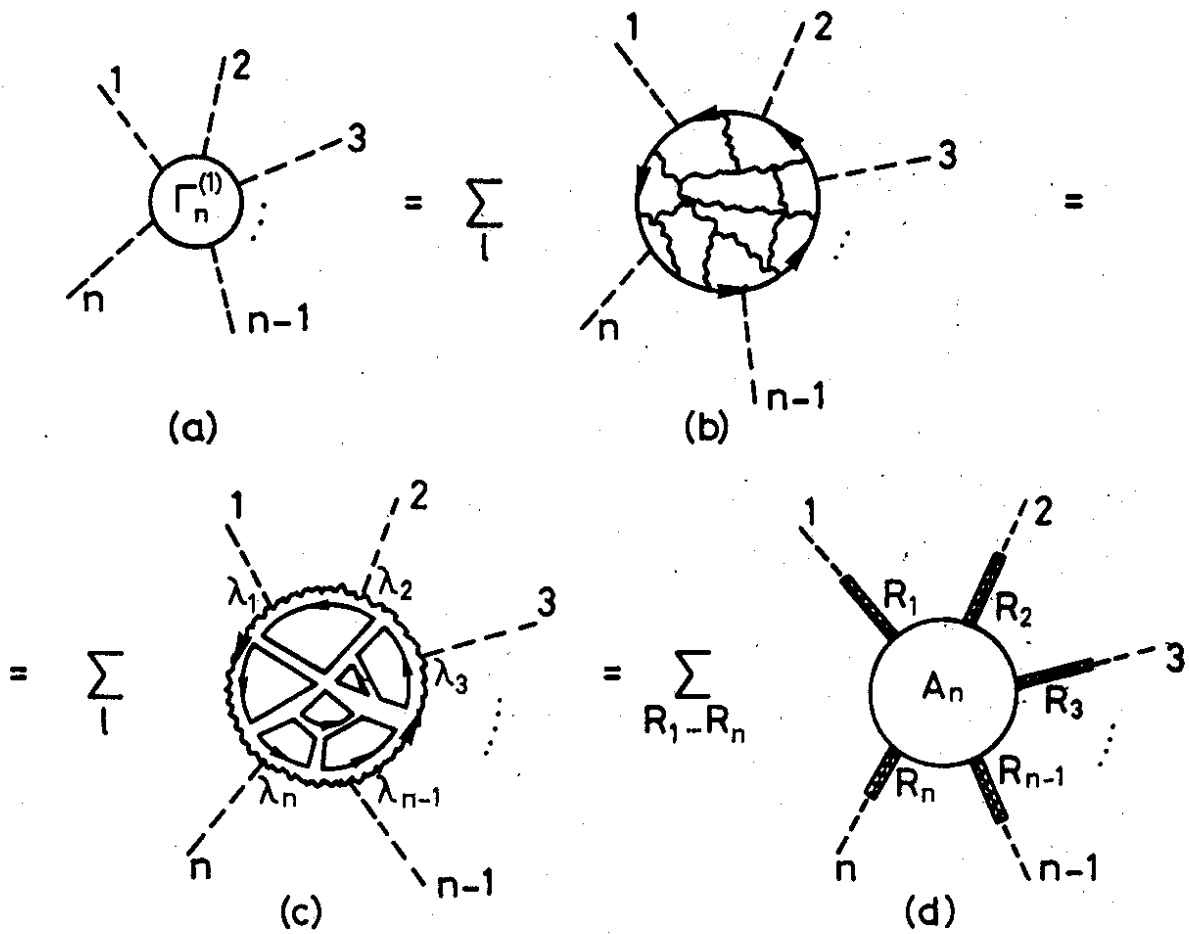


FIG. 2

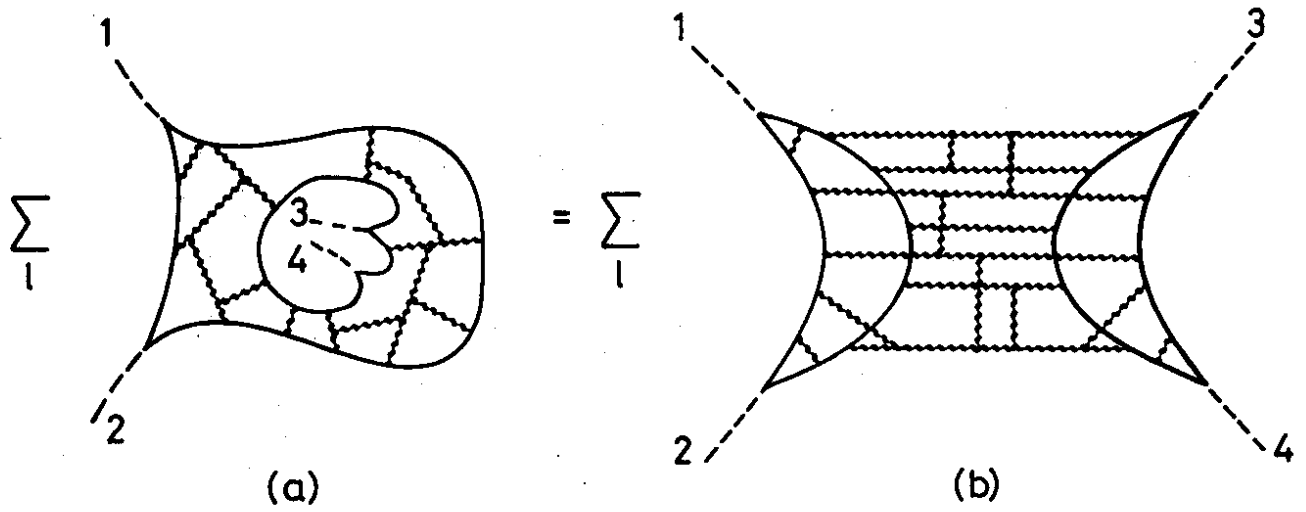


FIG. 3

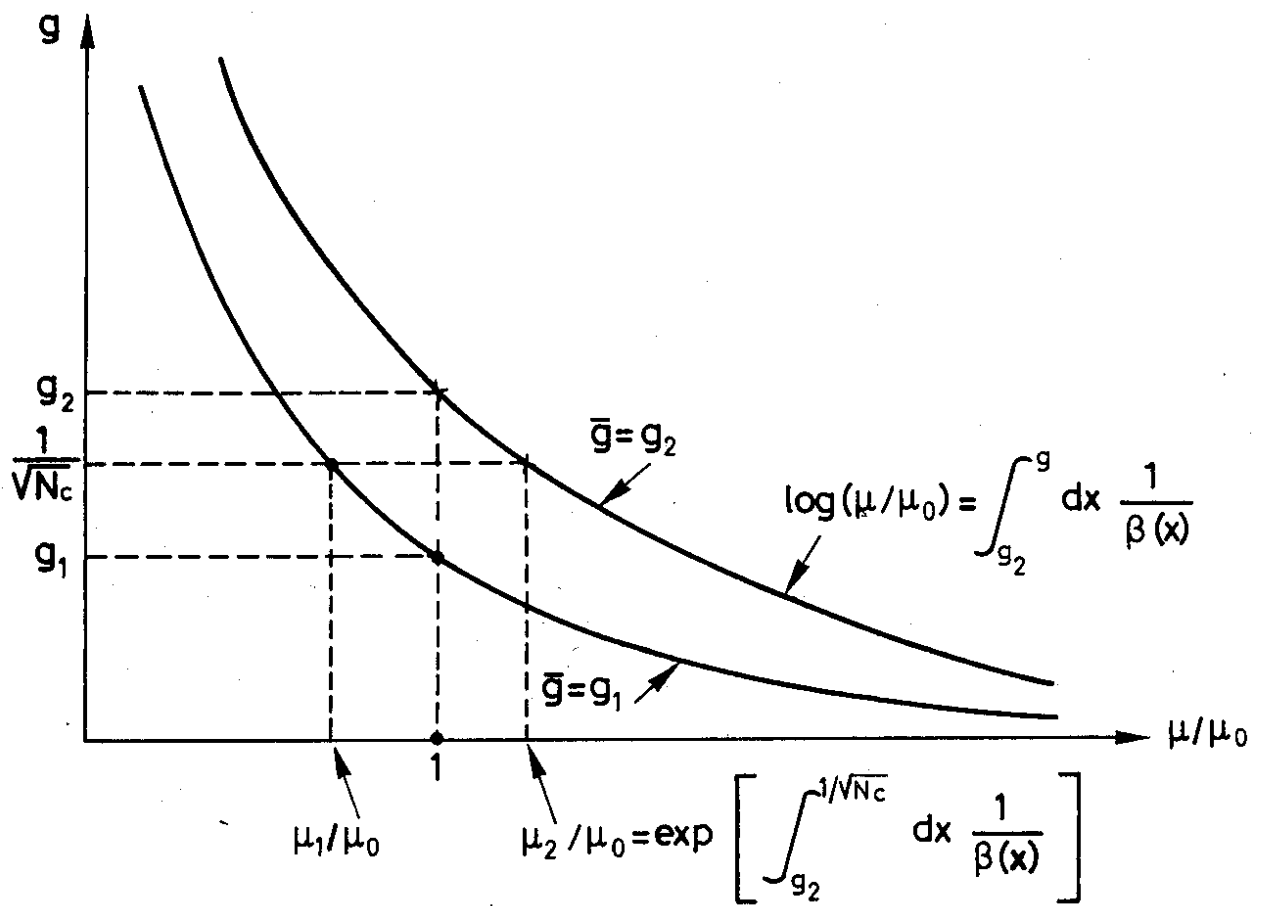


FIG. 4

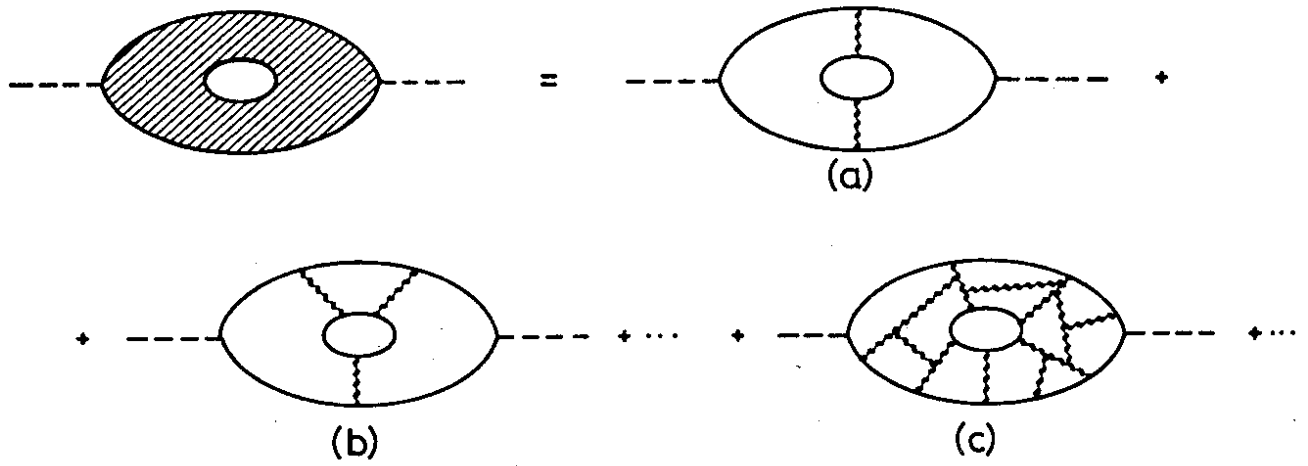


FIG.5

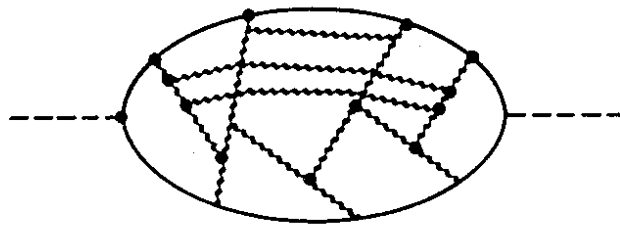


FIG.6

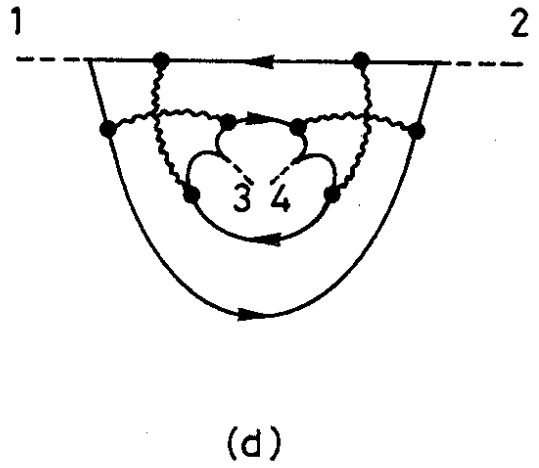
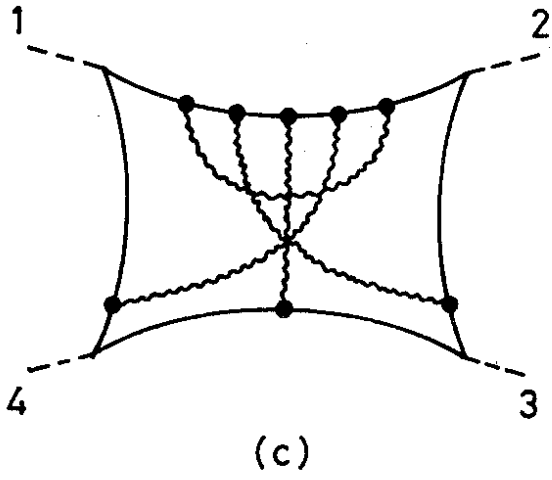
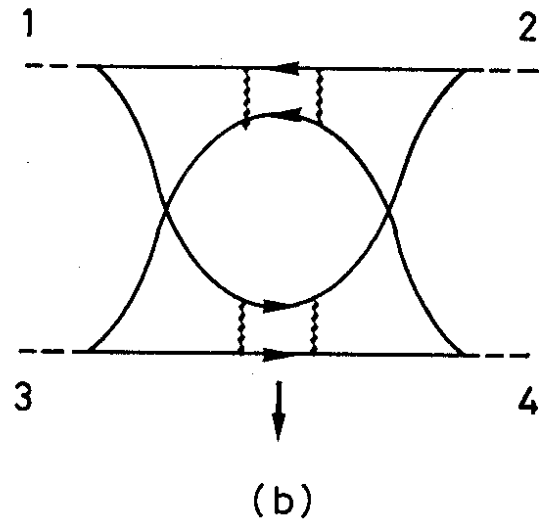
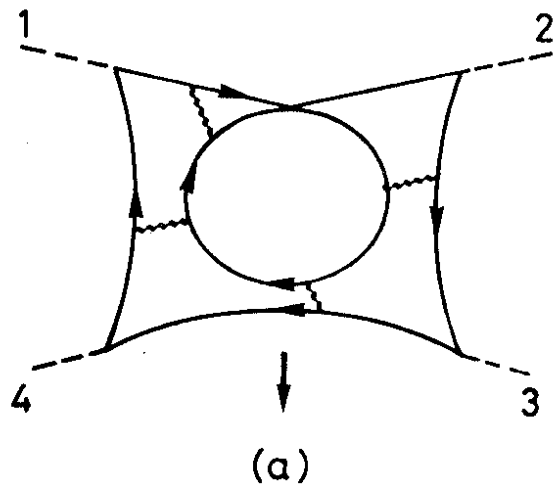


FIG. 7

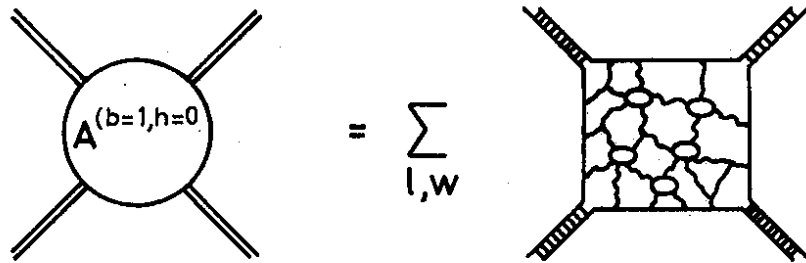


FIG. 8

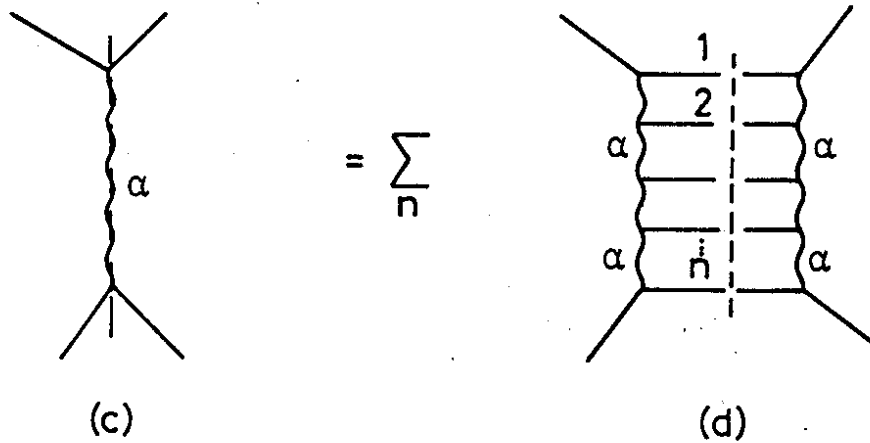
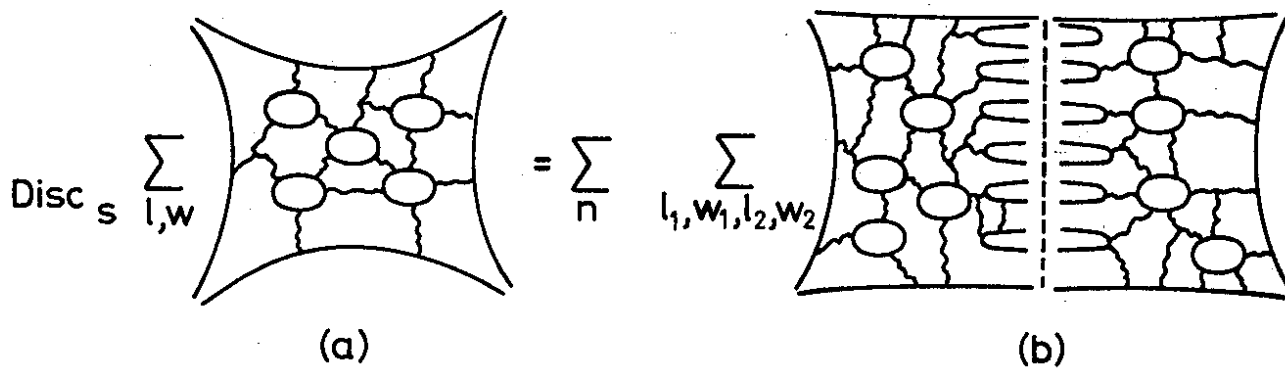


FIG. 9

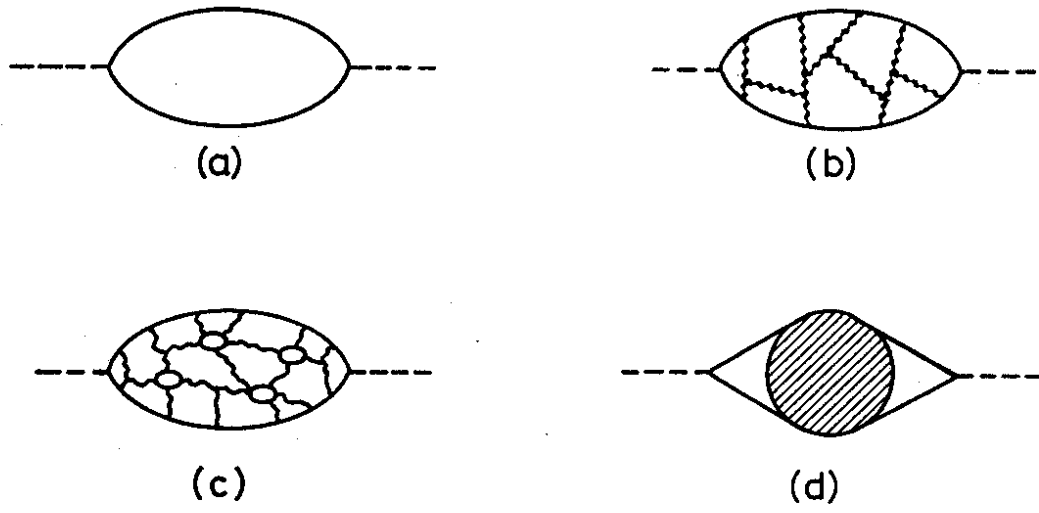


FIG.10

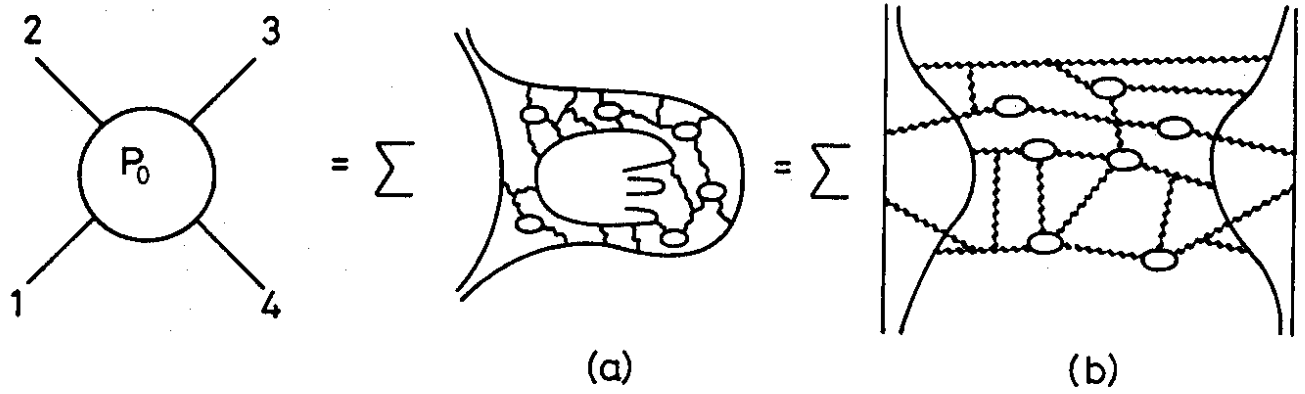


FIG.11