CERN-TH.5508/89

THREE-DIMENSIONAL DUALITY AND HYPERMULTIPLET SELF-INTERACTIONS ON CALABI-YAU VACUA

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ABSTRACT

The structure of quaternionic manifolds that are dual to the special Kähler manifolds occurring in N = 2, 4D supergravity theories is reported. The relevance of these results for the low-energy interactions of superstrings, compactified on general Calabi-Yau spaces, is discussed.

Invited talk given at the
Meeting on Quantum Field Theory and Mathematical Physics,
Liblice, Czechoslovakia, June 1989

CERN-TH.5508/89
August 1989

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In the present lecture I will describe some properties of the field theory limit for ten-dimensional superstrings compactified on complex three-dimensional (Calabi-Yau) manifolds[1].

Although the origin of these properties is motivated by string theory, the study of them leads to a new class of supergravity theories and to new quaternionic manifolds for N = 2 hypermultiplets never encountered before[2,3].

The properties we are going to discuss are, in more general terms, connected with local properties of the moduli space of Calabi-Yau manifolds[4] or, in string language, with the moduli space of (2,2) internal superconformal field theories[5].

The connection with string theory comes from the fact that the target space metric of the scalar field kinetic term

$$-g_{A\bar{B}}\, D_\mu \phi^A D_\mu \bar{\phi}^{\bar{B}} \tag{1}$$

in the effective Lagrangian is related to the correlator of two (truly) marginal operators[6-8]

$$\langle V_\phi^A(1)\, V^{\bar{B}}(0)\rangle = g_{A\bar{B}}(\phi) \tag{2}$$

in the corresponding two-dimensional field theory in which ϕ plays the role of a 'coupling constant' space[6].

We will not discuss in which precise sense Eqs. (1) and (2) are related but we shall point out here what are the local constraints on the metric $g_{A\bar{B}}$ which arise from the fact that Calabi-Yau spaces, or in more general terms (2,2) superconformal systems, can be equally used for type IIA, IIB, or heterotic superstrings[2,7].

In a space-time geometrical language this means that a Calabi-Yau threefold can be used as the 'vacuum' of different ten-dimensional supergravity theories, i.e. N = 1 chiral supergravity coupled to Yang-Mills [with gauge group $E_8 \times E_8$ or SO(32)] or N = 2 non-chiral and chiral supergravity. It is known that these supergravity theories are related to the low-energy limit of heterotic[9], type IIA and type IIB superstrings respectively[10].

When the same Calabi-Yau space is used to compactify different theories one expects there to be a relation among couplings in the effective Lagrangians. The reason for this lies in the fact that all ten-dimensional fields of ten-dimensional supergravity compactified on

$\mathcal{M}_4 \times C_3$ are expanded in terms of the same harmonics[1]. In particular, the massless fields on $\mathcal{M}_4$ are related to closed harmonic (1,1) and (2,1) forms on C_3, which in turn are related to the topological properties of the Calabi-Yau space. As a consequence of this fact the effective interactions of massless fields in four dimensions are expressed by some overlapping integrals on the Calabi-Yau space and these integrals are the same in different theories since they are merely a property of the internal space.

As an illustrative example it is easy to see[11] that the four-dimensional axion coupling

$$d_{abc}\, b^a F^b_{\mu\nu} F^c_{\rho\sigma}\, \varepsilon^{\mu\nu\rho\sigma} \tag{3}$$

in type IIA supergravity, which comes from the ten-dimensional interaction term

$$\int d^{10}_x\, \varepsilon^{\mu_1\cdots\mu_{10}} F_{\mu_1\cdots\mu_4} F_{\mu_5\cdots\mu_8} B_{\mu_9\mu_{10}} \tag{4}$$

is related to the same intersection matrix[12]

$$d_{abc} = \int B_a \wedge B_b \wedge B_c \tag{5}$$

that determines the Yukawa couplings for $\overline{27}$ antifamilies in four-dimensional heterotic strings with gauge group $E_6 \times E_8$.

At the string level the same correspondence arises from the fact that vertex operators for massless particles of heterotic and type II theories are related since they contain the same (2,2) superconformal fields and they only differ from the space-time part and the heterotic gauge fermions[13].

For example, the matter field vertex operators contain the first component of the same N = 2 chiral superfield whose last component is related to the moduli vertex.

This implies a relation between correlators of matter fields (in heterotic strings) and moduli fields which are common to different superstrings. However, since moduli fields are related, by N = 2 space-time supersymmetry, to other bosonic fields in type II theories (Ramond-Ramond scalars and vectors), this also implies that the same couplings will also fix the mutual interactions of moduli fields and Ramond-Ramond fields.

Therefore the rich structure which emerges for (2,2) superconformal systems gives different maps from heterotic, type IIA and type IIB theories. The map from type II to heterotic strings was first discussed by Lerche, Lüst and Schellekens[5] in the lattice construction of four-dimensional superstrings and emphasized by Gepner[5] in his classification of (2,2) superconformal field theories. The map from type IIA to type IIB was first discussed by Seiberg[7], and its general consequences analysed in Refs. 2 and 13.

Let us denote by Z^1 the set of all scalar fields in a given theory. In a string theory compactified on a Calabi-Yau space the scalar fields are members of some supermultiplets, more precisely in heterotic strings they are members of chiral multiplets.

If we neglect the gauge sector the number of massless chiral multiplets is $h_{(1,1)} + h_{(2,1)} + 1$, where $h_{(1,1)}$ and $h_{(2,1)}$ are the only independent Hodge numbers of the Calabi–Yau space[1]. In terms of the metric and antisymmetric tensor the (1,1) scalars are $g_{i\bar{j}}$, $b_{i\bar{j}}$, the (2,1) scalars are g_{ij}, and the remaining chiral multiplet is given by the dilaton ϕ and the space-time axion D (dual to $b_{\mu\nu}$).

From the gauge sector we obtain $h_{(1,1)} + h_{(2,1)}$ chiral multiplets in the $\overline{27}$- and 27-dimensional representation of E_6 and h neutral gauge singlets related to H^1 (end T)[12].

Let us now move to type II strings. In type IIA strings the (1,1) moduli are members of vector mutliplets, while in type IIB strings they are members of hypermultiplets[2,7]. The opposite situation occurs for (2,1) moduli. This is best seen by looking at the space-time vector fields in the two theories[14].

In type IIA they come from the ten-dimensional vector A_μ and ten-dimensional three-form $A_{\hat{\mu}\hat{\nu}\hat{\varrho}}$: A_μ, $A_{\mu i\bar{j}}$ ($\hat{\mu} = 1\ldots10$, $\mu = 1\ldots4$, i, $\bar{j} = 1,2,3$). In type IIB they come from the ten-dimensional four-form $A_{\hat{\mu}\hat{\nu}\hat{\varrho}\hat{\sigma}}$ (with self-dual field strength): $A_{\mu ij\bar{k}}$, $A_{\mu ijk}$.

The total number of vector fields is therefore $h_{(1,1)} + 1$ and $h_{(2,1)} + 1$ in the two different theories. The remaining vector is the graviphoton, the N = 2 partner of the graviton. On the other hand, if one counts the number of hypermultiplets, they are $h_{(2,1)} + 1$ in type IIA theory and $h_{(1,1)} + 1$ in type IIB theory. This is so because g_{ij} pairs with $A_{ij\bar{k}}$ in type IIA while $g_{i\bar{j}}$, $b_{i\bar{j}}$ pair with $b'_{i\bar{j}}$, $A_{\mu i\bar{j}}$ in type IIB to make a total number of degrees of freedom equal to $4h_{(2,1)}$ and $4h_{(1,1)}$ in the two different theories. The extra hypermultiplet corresponds to the universal sector containing the dilaton, the space-time axion, and two extra scalars, which in the IIA and IIB theories are given respectively by A_{ijk}, $b'_{\mu\nu}$, ϕ' [10] (b', ϕ' denote here the imaginary part of the complex dilaton and complex antisymmetric tensor present in the type IIB theory).

The restriction from N = 2 space-time supersymmetry[15–18] implies that the moduli fields should be coordinates of special Kähler manifolds (compatible with N = 2 space-time supersymmetry) when viewed as members of vector multiplets or as coordinates of quaternionic manifolds when viewed as members of hypermultiplets. Of course their roles interchange by going from type IIA to type IIB theory and vice versa. We denote[2] by C-map the mathematical operation which interchanges the effective Lagrangian of type IIA with type IIB theory. In mathematical terms it is a correspondence between two target space manifolds given by

$$\mathcal{M}^A(h_{(1,1)}) \times Q^A(h_{(2,1)}+1)$$

and

$$\mathcal{M}^B(h_{(2,1)}) \times Q^B(h_{(1,1)}+1)$$

where $\mathcal{M}$ and Q refer to Kähler and quaternionic, and in brackets is given their complex and quaternionic dimension.

Since these spaces are factorized a more important operation is the map (called S-map in Ref. 2), which maps a Kähler manifold of (complex) dimension n to a quaternionic manifold

of (quaternionic) dimension n + 1. We note that this map only exists if the original Kähler manifold is of the special type required by N = 2 space-time supersymmetry. The associated Q manifold of a given $\mathcal{M}$ manifold is called dual quaternionic manifold.

This map also exists in the case of global supersymmetry (not related to strings). In that case it is a map from a restricted n-dimensional Kähler manifold to a 2h-dimensional hyper-Kähler manifold (real dimension 4n)[2]. Note that the restricted Kähler spaces of N = 2 rigid supersymmetry are different from the restricted ones of N = 2 local supersymmetry. This corresponds to the fact that the dual manifold is in one case hyper-Kähler and in the other case quaternionic.

An important property of dual quaternionic manifolds of (quaternionic) dimension n + 1 is that they contain, as submanifolds, the Kähler spaces $[SU(1,1)/U(1)] \times \mathcal{M}_n$ of complex dimension n + 1, where $\mathcal{M}_n$ is the original Kähler manifold and SU(1,1)/U(1) is the Kähler space containing the dilaton and the axion (dual to the space-time antisymmetric tensor).

We now give the main (local) properties of Calabi–Yau moduli space and their quaternionic extension in the case of type II superstrings[19]. Let us denote by $\phi_{A(B)}$ the (1,1) and (2,1) moduli.

The moduli space $\mathcal{M}$ of complex dimension $h_{(1,1)} + h_{(2,1)}$ has the product structure $\mathcal{M} = \mathcal{M}_A \times \mathcal{M}_B$, where $\mathcal{M}_A$ and $\mathcal{M}_B$ are Kähler spaces of dimension $h_{(1,1)}$ and $h_{(2,1)}$, respectively, of restricted type.

This means that, in a certain choice of coordinates for the moduli, the Kähler potential K is of the form

$$K_{A(B)} = -\log Y_{A(B)} \tag{6}$$

with $Y_{A(B)}$ given by

$$Y_{A(B)} = f_{A(B)} + \bar{f}_{A(B)} - \frac{1}{2}\left(\frac{\partial f_{A(B)}}{\partial \phi_{A(B)}} - \frac{\partial \bar{f}_{A(B)}}{\partial \bar{\phi}_{A(B)}}\right)\left(\phi_{A(B)} - \bar{\phi}_{A(B)}\right) \tag{7}$$

where $f_{A(B)}$ are holomorphic, i.e. $\partial f_{A(B)}/\partial \bar{\phi}_{A(B)} = 0$.

The functions $f_{A(B)}$ determine all low-energy couplings of heterotic strings as well as type II strings, and their form depends on the particular Calabi–Yau space. In the field theory limit the functions $Y_{A(B)}$ are given by[4,19]

$$Y_A = \int J \wedge J \wedge J = V \tag{8}$$

$$Y_B = i \int \Omega \wedge \bar{\Omega} \tag{9}$$

where V is the volume of the threefold, J is the Kähler form as a function of the (1.1) moduli parameters, and Ω is the holomorphic (3,0) form as a function of the (2,1) moduli. From Eqs. (8) and (9) is manifest the geometrical meaning of the ϕ_A and ϕ_B moduli as deformation parameters of the Kähler class and complex structure respectively.

String-theory arguments and non-renormalization theorems[20] indicate that while Eq. (9) is an exact result and is given by a tree-level σ-model calculation, Eq. (8) is true in σ-model perturbation theory but is not true in a strongly coupled σ-model where non-perturbative corrections, such as world-sheet instantons, become important. Another way of saying this is that the metric of the moduli space of (2,2) superconformal field theories does not coincide with the geometrical metric of the moduli fields of the Calabi–Yau classical threefold [given by Eqs. (6) and (8)].

Recently[21] a Kaluza-Klein argument has been given for the instability of the (1,1) moduli metric against integration over massive Kaluza-Klein modes. In view of the fact that massive Kaluza-Klein and winding stringy modes are related by duality[20] in the moduli fields it is likely that these different arguments are in fact equivalent.

It is our aim to characterize now the properties of the dual quaternionic manifolds $Q(h+1)$, where $h = h_{(2,1)}$ in type IIA and $h = h_{(1,1)}$ in type IIB strings.

An important fact which enables us to compute the Q manifolds is three-dimensional duality[22]. More precisely, if we consider the bosonic sector of N = 2 n-Abelian vector multiplets coupled to N = 2 four-dimensional supergravity and dimensionally reduce the theory to D = 3 dimensions we get an N = 4, D = 3 supergravity theory coupled to a quaternionic σ-model. Note that in D = 3 the holonomy group contains an $SO(4) = SU(2) \times SU(2)$ group and therefore two kinds of quaternionic manifolds. Then two different sets of quaternionic manifolds are nothing but the dimensionally reduced version of vector and hypermultiplet self-couplings.

Note that in D = 3 the Abelian gauge boson coming from the circle compactification $g_{4\mu}$ is just equivalent to a scalar degree of freedom because of three-dimensional duality. This degree of freedom, as we will see shortly, is essential for matching the dimension of the dual quaternionic manifold.

The metric of the dual quaternionic manifold[2] is derived here by performing a dimensional reduction from D = 4 to D = 3 dimensions of N = 2 supergravity coupled to n-vector Abelian multiplets with self coupling specified by a holomorphic function $f(Z^a)$ $(a = 1, \ldots, n)$.

The N = 2 Lagrangian for vector multiplets is (bosonic part)[16-18]

$$e^{-1}\mathcal{L} = \frac{1}{2} R - K_{a\bar{b}} \partial_\mu z^a \partial_\mu \bar{z}^b + \frac{1}{4} \operatorname{Re} \mathcal{N}_{ij} F^i_{\mu\nu} F^j_{\mu\nu} + \frac{1}{4} \operatorname{Im} \mathcal{N}_{ij} F^i_{\mu\nu} \tilde{F}^j_{\mu\nu} \tag{10}$$

with

$$\mathcal{N}_{ij} = \frac{1}{4} \bar{F}_{ij} - \frac{(N\bar{z})_i (N z)_j}{(z N z)} \tag{11}$$

$$N_{ij} = \frac{1}{4} (F_{ij} + \bar{F}_{ij}) \tag{12}$$

$$K = -\log 2 z N \bar{z} = -\log \left[f + f^* - \frac{1}{2} (Z^a - Z^{*a})(f_a - f^*_a) \right] \tag{13}$$

where F is a homogeneous function of degree 2 in n + 1 variables X^i $(Z^i = X^i/X^0)$ and

$$F(\lambda X) = \lambda^2 F(X) \quad , \quad f(Z) = X^{0^{-2}} F(X) \tag{14}$$

Dimensional reduction from D = 4 (N = 2) to D = 3 (N = 4) is obtained using a triangular gauge for the vierbein

$$e^{\hat{a}}_{\hat{\mu}} = \begin{bmatrix} e^a_\mu & 0 \\ \phi^{1/2} B_\mu & \phi^{1/2} \end{bmatrix} \tag{15}$$

and for four-vectors we have

$$A^i_{\hat{\mu}} = (A^i_\mu + B_\mu S^i, S^i) \qquad (S^i = A^i_{\hat{4}}) \tag{16}$$

The Lagrangian (10) reduced to three dimensions, after a Weyl rescaling

$$e^a_\mu \to \phi^{-1/2} e^a_\mu \tag{17}$$

becomes

$$\begin{aligned} e^{-1}\mathcal{L} = {} & \frac{1}{2} R - \frac{1}{4\phi^2}(\partial_\mu \phi)^2 + \frac{1}{4}\phi^2 H_\mu^2 \\ & - K_{a\bar{b}}\, \partial_\mu z^a \partial_\mu \bar{z}^b - \frac{\phi}{2} R_{ij} (F^i_\mu + H_\mu S^i)(F^j_\mu + H_\mu S^j) \\ & + \frac{1}{2\phi} R_{ij} \partial_\mu S^i \partial_\mu S^j - \mathrm{Im}\,\mathcal{N}_{ij} (F^i_\mu + H_\mu S^i) \partial_\mu S^j \end{aligned} \tag{18}$$

where

$$\begin{aligned} & H_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \,, \quad H_\mu = -\tfrac{1}{2} \varepsilon_{\mu\nu\rho} H_{\nu\rho} \\ & F^i_\mu = -\tfrac{1}{2} \varepsilon_{\mu\nu\rho} F^i_{\nu\rho} \,, \quad R_{ij} = \mathrm{Re}\,\mathcal{N}_{ij} \end{aligned} \tag{19}$$

We now use three-dimensional duality to convert the n + 1 vector fields F^i_μ, H_μ into scalars. For this purpose we add the Lagrange multipliers $\tilde{S}_i$, $\tilde{\phi}$

$$-F^i_\mu \partial_\mu \tilde{S}_i + \frac{1}{2} H_\mu \partial_\mu (\tilde{\phi} - S^i \tilde{S}_i) \tag{20}$$

Integration over F^i_μ, H_μ yields

$$\begin{aligned} e^{-1}\mathcal{L} = {} & \frac{1}{2} R - K_{a\bar{b}}\, \partial_\mu z^a \partial_\mu \bar{z}^b + \frac{1}{4\phi^2} (\partial_\mu \phi)^2 \\ & + \frac{1}{4\phi^2} (\partial_\mu \tilde{\phi} + S^i \overleftrightarrow{\partial_\mu} \tilde{S}_i)^2 - \frac{1}{2\phi} R_{ij} \partial_\mu S^i \partial_\mu S^j \\ & - \frac{1}{2\phi} (\partial_\mu \tilde{S}_i + \mathrm{Im}\,\mathcal{N}_{ik} \partial_\mu S^k)(R^{-1})^{ij} (\partial_\mu \tilde{S}_j + \mathrm{Im}\,\mathcal{N}_{j\ell} \partial_\mu S^\ell) \end{aligned} \tag{21}$$

Finally, let us define the fields

$$\begin{aligned} & C_i = \mathcal{N}_{ij} S^j + i \tilde{S}_i \qquad (i = 1 \ldots n+1) \\ & S = \phi + i \tilde{\phi} - \frac{1}{2} (C + \bar{C})_i (R^{-1})^{ij} (C + \bar{C})_j \end{aligned} \tag{22}$$

Then the Lagrangian describing the scalar manifold of (real) dimension 4 (n + 1) is

$$\begin{aligned} e^{-1}\mathcal{L} = {} & - K_{a\bar{b}}\, \partial_\mu z^a \partial_\mu \bar{z}^b - \frac{1}{[S + \bar{S} + \frac{1}{2}(C+\bar{C}) R^{-1} (C+\bar{C})]^2} \,\times \\ & \times \left| \partial_\mu S + (C+\bar{C}) R^{-1} \partial_\mu C - \frac{1}{4} (C+\bar{C}) R^{-1} \partial_\mu \mathcal{N} R^{-1} (C+\bar{C}) \right|^2 \\ & + \frac{1}{[S+\bar{S}+\frac{1}{2}(C+\bar{C}) R^{-1} (C+\bar{C})]} \left[\partial_\mu C - \frac{1}{2} \partial_\mu \mathcal{N} R^{-1} (C+\bar{C})\right] \\ & \times R^{-1} \left(\partial_\mu \bar{C} - \frac{1}{2} \partial_\mu \bar{\mathcal{N}} R^{-1} (C+\bar{C})\right] \end{aligned} \tag{23}$$

Positivity of the kinetic energy requires $K_{a\bar{b}}$ and $-R_{ij}$ to be positive definite matrices {as implied by Eq. (10)}.

Equation (23) defines a manifold for 2(n + 1) complex fields S, Z^a, C which according to the general analysis of Ref. 2 is a dual quaternionic manifold of the original n-dimensional restricted Kähler manifold with coordinates Z^a. The additional n + 1 complex fields which enlarge the moduli space to a quaternionic manifold are the S multiplet, containing the dilaton and axion, and (n + 1) (complex) Ramond fields C_i, which come from Ramond–Ramond scalars.

Let us anticipate some properties shared by all dual quaternionic manifolds[3)].

a) At each point of the moduli space $Z^a = Z^{a0}$ ($\partial_\mu Z^{a0} = 0$) the Ramond–Ramond scalars, the dilaton and the axion, parametrize an $SU(1,n+2)/U(1) \times SU(n+2)$ manifold. For n = 0 this manifold reduces to the universal sector, as obtained in Ref. 2 by general arguments and explicitly constructed elsewhere[23)].
b) If we set the n + 1 Ramond–Ramond scalars $C_i = 0$ ($\partial_\mu C_i = 0$) then the (Z^a,S) fields parametrize the manifold $[SU(1,1)/U(1)] \times M_n$, where M_n is the original (restricted) Kähler manifold.

c) The quaternionic manifold $Q(n+1)$ has at least $2n+4$ isometries acting on all coordinates but the moduli fields Z^a.

d) The dual quaternionic manifolds are Einstein spaces with negative curvature: $R = -8(n+1)(n+3)$.

e) Those moduli which correspond to vanishing Yukawa couplings (in heterotic strings), together with their (Ramond) partners, the dilaton and the axion, span a Kähler quaternionic manifold $SU(2,n'+1)/SU(2) \times SU(n'+1) \times U(1)$. The associated restricted Kähler manifold is in this case $SU(1,n')/U(1) \times SU(n')$.

Properties (a), (b), (c), and (e) can be discussed in a rather straightforward way. It is sufficient to observe that Eq. (23) can be rewritten as

$$e^{-1}\mathcal{L} = -K_{a\bar{b}}\partial_\mu z^a \partial_\mu \bar{z}^b - \tilde{K}_{S\bar{S}} D_\mu S D_\mu \bar{S} - \tilde{K}_{S\bar{C}_i} D_\mu S D_\mu \bar{C}_i - \tilde{K}_{C_i\bar{S}} D_\mu C_i D_\mu \bar{S} - \tilde{K}_{C_i\bar{C}_j} D_\mu C_i D_\mu \bar{C}_j \tag{24}$$

where

$$D_\mu C = \partial_\mu C - \frac{1}{2}\partial_\mu \mathfrak{N} R^{-1}(C+\bar{C}) \tag{25}$$

$$D_\mu S = \partial_\mu S + \frac{1}{4}(C+\bar{C}) R^{-1} \partial_\mu \mathfrak{N} R^{-1}(C+\bar{C}) \tag{26}$$

$$\tilde{K} = -\log\left[S + \bar{S} + \frac{1}{2}(C+\bar{C}) R^{-1}(C+\bar{C})\right] \tag{27}$$

The above equations show that for $\mathrm{Re}\, C_i = 0$ ($\partial_\mu C_i = 0$), the manifold reduces to $SU(1,1)/U(1) \times M_n$, with coordinates (S, Z^a), while for fixed Z^a it contains the submanifold $SU(1,n+2)/U(1) \times SU(n+2)$ with coordinates (S, C^i). The standard metric for this manifold is best seen by making the following field redefinition (holomorphic for fixed Z^a) $S \to S - \frac{1}{2} C R^{-1} C$.

We also remark that if the matrix $\mathfrak{N}(Z,\bar{Z})$ is holomorphic, i.e. does not depend on $\bar{Z}$, then the manifold Q is Kähler with Kähler potential $K + \tilde{K}$. In view of Eq. (11) this is the case if the $f(Z)$ function is a quadratic polynomial and the Kähler quaternionic manifold is therefore given by $SU(2,n+1)/SU(2) \times SU(n+1) \times U(1)$.

Note that a quadratic f implies vanishing Yukawa couplings and this proves statement (e).

To discuss the point (c) we remark that there are $2n+3$ isometries related to the axion and Ramond scalars C^i

$$\begin{aligned} S &\to S + i\alpha - 2C\cdot\gamma - \gamma\cdot\mathfrak{N}\cdot\gamma \\ C &\to C + i\beta + \mathfrak{N}\gamma \end{aligned} \tag{28}$$

where α, β_i, γ_i are $2n+3$ real parameters. The last isometry is the scale transformation

$$S \to \lambda S, \quad C \to \lambda^{1/2} C \tag{29}$$

Therefore the Q manifold has at least $2n+4$ isometries. This is consistent with the statement (a) in which case these isometries are the non-linearly realized part of the non-compact group $SU(1,n+2)$.

It remains to prove that the manifold defined by Eqs. (23) and (24) is a quaternionic manifold of (real) dimension $4(n+1)$.

Let us recall that for quaternionic manifolds of (real) dimension 4d there exist three locally defined tensors $(J^u)^\mu_\nu$ where $u = 1,3$; $\mu,\nu = 1 \ldots 4d$, which satisfy the quaternionic algebra

$$J^u \times J^v = -\delta^{uv} + \varepsilon^{uvw} J^w \tag{30}$$

Moreover, the three two-forms

$$J^u = \frac{1}{2} J^u_{\mu\nu}\, dx^\mu dx^\nu, \quad J^u_{\mu\nu} = g_{\mu\rho}(J^u)^\rho{}_\nu \tag{31}$$

are covariantly constant with respect to an Sp(1) connection ω

$$DJ = dJ + [\omega, J] = 0, \quad J = J^u \sigma^u \tag{32}$$

The Sp(1) curvature is proportional to the J two-forms

$$d\omega + \omega\omega = i\lambda J \tag{33}$$

for some λ. The holonomy group of a quaternionic manifold is contained in $Sp(1) \times Sp(d)$.

In addition, quaternionic manifolds are Einstein spaces with

$$R_{\mu\nu} = 2\lambda(d+2)\, g_{\mu\nu} \tag{34}$$

However, consistent coupling to supergravity requires[15] $\lambda = -1$. This property will, of course, be satisfied for all dual quaternionic spaces irrespective of the choice of the holomorphic function $f(Z)$.

Let us consider the original Kähler manifold $\mathfrak{M}_n$ with Kähler (closed) two-form given by

$$J = i e^A e^{\bar{A}} \qquad (e^A = e^A_a dz^a) \tag{35}$$

The Kähler metric is

$$K_{a\bar{b}} = e^A_a (e^A_b)^* \tag{36}$$

It is convenient to define an $n \times (n+1)$ matrix P^A_i as follows[3]:

$$P^A_a = e^A_a \, , \quad P^A_0 = -e^A_a z^a \tag{37}$$

P satisfies

$$P \cdot Z = 0 \qquad (z^0 = 1)$$

$$P^+ P = -\frac{1}{\bar{Z} N Z}\left[N - \frac{(NZ)(\bar{Z}N)}{\bar{Z} N Z}\right] \tag{38}$$

$$P N^{-1} P^+ = -\frac{1}{\bar{Z} N Z}$$

The vierbein one-forms for the quaternionic manifold are

$$e = P dZ$$

$$E = e^{(\tilde{K}-K)/2} P N^{-1}\left[dC - \tfrac{1}{2} dW R^{-1}(C+\bar{C})\right] \tag{39}$$

$$u = 2 e^{(K+\tilde{K})/2} Z \left[dC - \tfrac{1}{2} dW R^{-1}(C+\bar{C})\right]$$

$$v = e^{\tilde{K}}\left[dS + (C+\bar{C}) R^{-1} dC - \tfrac{1}{4}(C+\bar{C}) R^{-1} dW R^{-1}(C+\bar{C})\right]$$

The Wedge product of forms $dx \wedge dy$ will be denoted by $dx\, dy$. The $\otimes$ symbol denotes the sum of the product of components of two one-forms. We use capital letters for flat indices, small letters for curved indices, initial letters of the alphabet a, A run from 1 up to n while middle letters i, I run from 1 up to n + 1.

The Lagrangian for the quaternionic manifold takes the form

$$-\bar{e}^{1}\mathcal{L} = e \otimes \bar{e} + E \otimes \bar{E} + u \otimes \bar{u} + v \otimes \bar{v}$$

$$= \sum_{\alpha=1,2;\, I=1..n+1} e^{\alpha I} (e^{\alpha I})^* \tag{40}$$

($e \otimes \bar{e}$ is the Lagrangian for the original $\mathfrak{M}_n$ manifold) in terms of the $2(n+1)$ component vierbein:

$$e^{\alpha I} = (e^{+I}, e^{-I}) \, , \quad e^{+I} = \begin{pmatrix} u \\ e^A \end{pmatrix} , \quad e^{-I} = \begin{pmatrix} v \\ E^A \end{pmatrix} \tag{41}$$

To find the connections we compute the exterior derivatives of the vierbein one-forms. For instance,

$$de = -\omega e \tag{42}$$

when ω is the connection of the original Kähler manifold

$$\omega = -\frac{\bar{Z} N dZ - Z N d\bar{Z}}{2 \bar{Z} N Z} + \frac{\bar{Z} N Z}{2}\left\{ dP N^{-1} P^+ - P N^{-1} dP^+ - i P N^{-1} dY N^{-1} P^+ \right\} \tag{43}$$

and

$$N_{ij} + i Y_{ij} = \tfrac{1}{2} F_{ij} \tag{44}$$

The curvature two-form for $\mathfrak{M}_n$

$$d\omega + \omega\omega \tag{45}$$

agrees with known results.

The connections for the Q manifold are given by

$$de^{\alpha I} + p^{\alpha}_{\beta} e^{\beta I} + q^{I}_{J} e^{\alpha J} + t^{I}_{J} \varepsilon^{\alpha\alpha'} \overline{e^{\alpha' J}} = 0 \tag{46}$$

where p is an Sp(1) connection and q,t are entries of an Sp(n + 1) connection.

Explicitly we have

$$p = \begin{bmatrix} \frac{1}{4}(v-\bar{v}) - \frac{1}{4}\frac{\bar{z}Ndz - \bar{z}Nd\bar{z}}{\bar{z}Nz} & -u \\ \bar{u} & -\frac{1}{4}(v-\bar{v}) + \frac{1}{4}\frac{\bar{z}Nd\bar{z} - \bar{z}Nd\bar{z}}{\bar{z}Nz} \end{bmatrix} \tag{47}$$

$$q = \begin{bmatrix} -\frac{3}{4}(v-\bar{v}) - \frac{1}{4}\frac{\bar{z}Ndz - \bar{z}Nd\bar{z}}{\bar{z}Nz} & \bar{E} \\ -E & \omega - \frac{1}{4}(v-\bar{v}) + \frac{1}{4}\frac{\bar{z}Nd\bar{z} - \bar{z}Nd\bar{z}}{\bar{z}Nz} \end{bmatrix} \tag{48}$$

$$t = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{\bar{f}_{ABC}\bar{E}^{C}}{4\bar{z}Nz} \end{bmatrix}$$

The Sp(1) × Sp(n + 1) connection is better seen by defining a 4(n + 1) component vierbein

$$V^{\alpha\Gamma} = \begin{bmatrix} e^{\alpha I} \\ \varepsilon^{\alpha\beta}(e^{\beta I})^{*} \end{bmatrix} \tag{49}$$

The flat metric is $\frac{1}{2}\epsilon_{\alpha\alpha'}\varrho_{\Gamma\Gamma'}$ with

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \tag{50}$$

Then $V^{\alpha\Gamma}$ is covariantly constant

$$(d + \Omega)V = 0 \tag{51}$$

with connection

$$\begin{gathered} \Omega = p \times (\mathbb{1})_{2(n+1)} + (\mathbb{1})_2 \times \begin{pmatrix} q & t \\ -t^{+} & -q^{t} \end{pmatrix} \\ p^{+} = -p,\ q^{+} = -q,\ t^{T} = t,\quad \Omega^{+} = -\Omega \end{gathered} \tag{52}$$

Equation (52) proves that Ω is an Sp(1) × Sp(n + 1) connection.

The Sp(1) curvature is

$$-iJ = dp + pp = \frac{1}{2}\begin{bmatrix} \bar{e}^{+} \\ \bar{e}^{-} \end{bmatrix} \sigma^{u} \begin{bmatrix} e^{+} \\ e^{-} \end{bmatrix} \sigma^{u} \tag{53}$$

or

$$(-iJ)^{\alpha}_{\ \beta} = -V^{\alpha\Gamma} V_{\beta\Gamma} \tag{54}$$

J^{u} defines the three covariantly constant tensors satisfying the quaternionic algebra as given by Eq. (30).

It is of interest to give the Sp(n + 1) curvature as well. This is a 2(n + 1) × 2(n + 1) matrix valued two-form

$$\tilde{R} = \begin{pmatrix} r & r' \\ -r'^{+} & -r^{t} \end{pmatrix} \tag{55}$$

in which r, r′ are (n × 1) × (n + 1) matrix valued two-forms.

Their expression is

$$r^{0}_{0} = -\frac{3}{n}(u\bar{u} + v\bar{v}) - \frac{1}{2}(e\bar{e} + E\bar{E})$$

$$r^{A}_{0} = -(r^{0}_{A})^{*} = \bar{u}e^{A} + \bar{v}E^{A}$$

$$r^{A}_{B} = -\frac{1}{2}\delta^{A}_{B}(e\bar{e} + E\bar{E} + u\bar{u} + v\bar{v}) - e^{A}e^{\bar{B}} - E^{A}\overline{E^{B}}$$
$$- \frac{\bar{f}_{ACE}f_{EDB}}{16(\bar{z}Nz)^{2}}(\bar{e}^{C}e^{D} + \bar{E}^{C}E^{D}) \tag{56}$$

$$r'^{A}_{\bar{B}} = \frac{1}{4\bar{z}Nz}\bar{f}_{ABC}(u\bar{e}^{C} + v\bar{E}^{C})$$
$$+ \frac{1}{16(\bar{z}Nz)^{2}}\bar{f}_{ABC}f_{CDE}e^{D}E^{C} - \frac{1}{4\bar{z}Nz}\frac{1}{4}X^{A}_{\bar{B}\bar{C}\bar{D}}\bar{e}^{C}\bar{E}^{D}$$

where

$$f_{ABC} = f_{abc}e^{a}_{A}e^{b}_{B}e^{c}_{C}, \quad f_{abc} = \frac{\partial}{\partial z^{a}}\frac{\partial}{\partial z^{b}}\frac{\partial}{\partial z^{c}}f$$

$$X_{\bar{a}\bar{b}\bar{c}\bar{d}} = \bar{f}_{abcd} + \frac{(Nz)_{\{a}\bar{f}_{bcd\}}}{\bar{z}Nz}$$
$$- \frac{1}{4}(\bar{f}_{abe}\bar{f}_{fcd} + \bar{f}_{ace}\bar{f}_{fdb} + \bar{f}_{ade}\bar{f}_{fbc})(N^{-1})^{\bar{e}\bar{f}} \tag{57}$$

It can be checked that the quaternionic manifold with curvature given by Eqs. (46) and (48) is an Einstein space with scalar curvature given by

$$R = -8(n+1)(n+3)$$

in agreement with Ref. 15.

It is of interest to remark that for general holomorphic functions f the $Sp(n+1)$ curvature depends both on the third and fourth derivatives of f, unlike the Kähler curvature[18].

Dual quaternionic manifolds which correspond to non-vanishing Yukawa couplings in heterotic strings ($f_{abc} \neq 0$) have a complicated structure unless the Yukawa couplings are independent of the moduli. In the latter case, for trilinear holomorphic functions f, one can recover all symmetric and homogeneous quaternionic manifolds discussed in Refs. 24-26.

For vanishing Yukawa couplings the quaternionic manifold becomes a symmetric Kähler manifold $SU(2,n+1)/SU(2) \times SU(n+1) \times U(1)$.

In this paper we have obtained the dual quaternionic manifolds by using three-dimensional duality in the pure context of four-dimensional supergravity.

An alternative way, which should give the same answer, would be to use the Kaluza-Klein compactification of type II ten-dimensional supergravity on a Calabi-Yau space or to use an S-matrix approach, by computing string amplitudes in type II strings, along lines similar to those recently discussed in Ref. 13.

It would be very interesting to check whether these different approaches give rise to the same answer.

It is worth mentioning that, besides the motivation of describing the low-energy limit of superstrings compactified on (2,2) superconformal systems, the construction of the chiral quaternionic manifolds provides examples of continuous families of quaternionic manifolds which, to our knowledge, were unknown before. Here we have derived the explicit expression for their connection and curvature. Recently the C-map and the construction of dual hyper-Kähler and quaternionic manifolds has also been studied[27] using harmonic superspace[28] which is the best suitable superspace description of hypermultiplet self couplings.

This work was supported in part by the United States Department of Energy under Contract #DE-AT03-88ER40384, Task E.

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