# Three Lectures on Electric – Magnetic Duality

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Abstract

Duality rotations in nonlinear electromagnetism are presented and basic examples reviewed. We then describe the nontrivial example of Born-Infeld theory with $n$ abelian gauge fields and with $Sp(2n, \mathbb{R})$ self-duality. The central role of duality symmetry in four dimensional extended supergravity theories is explained and explicitly illustrated in two examples ($N = 4$ and $N = 8$ supergravities). Duality rotations in $N = 2$ supergravities are also discussed.

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## 1 Introduction

The invariance of Maxwell's equations under rotation of electric field into magnetic field is also shared by other electromagnetic theories, nonlinear (like Born-Infeld electromagnetism), with more than one gauge field, in interaction with scalar fields, with spinor fields and with gravity. The emergence of this duality symmetry in extended supergravity theories [1, 2, 3] led to develop the general theory of duality invariance with abelian gauge fields coupled to fermionic and bosonic matter [4, 5]. Since then, on one hand the duality symmetry of extended supergravity theories has been extensively investigated [6, 7, 8, 9]; on the other hand examples of nonlinear Born-Infeld type lagrangians with electric-magnetic duality have been presented, in the case of one abelian gauge field [10, 11, 12, 13, 14] and in the case of many abelian gauge fields [15, 16, 17, 18]. Their supersymmetric generalizations have been considered in [19, 20] and with different scalar couplings and noncompact duality group in [15, 16, 21, 22, 23].

Duality symmetry can also be generalized to arbitrary even dimensions by using antisymmetric tensor fields such that the rank of their field strengths equals half the dimension of space-time, see [24, 25], and [28, 9, 29, 26, 14, 16, 22, 23].

Duality symmetries arise in many contexts. In superstring theory or $M$ theory electric-magnetic dualities can arise from many sources, namely $S$-duality, $T$-duality or a combination thereof called $U$-duality [27]. From the point of view of a four dimensional observer such dualities manifest as some global symmetries of the lowest order Euler-Lagrange equations of the underlying four dimensional effective theory. Notice that duality rotation symmetries can be further enhanced to local symmetries (gauging of duality groups). The corresponding gauged supergravities appear as string compactifications in the presence of fluxes and as generalized compactifications of (ungauged) higher dimensional supergravities.

Electric-Magnetic duality is also the underlying symmetry which encompasses the physics of extremal black holes and of the "attractor mechanism" [30, 31, 32] (for recent reviews on the attractor mechanism see [33, 34, 35]). Here the Bekenstein-Hawking entropy-area formula $S = A/4$ is directly derived by the evaluation of a certain black hole potential $\mathscr{V}_{BH}$ at its attractive critical points [36]

$$S = \pi\, \mathscr{V}_{BH}|_C$$

where the critical points $C$ satisfy $\partial \mathscr{V}_{BH}|_C = 0$. The potential $\mathscr{V}_{BH}$ is a quadratic invariant of the duality group; it depends on both the matter and the gauge fields configuration. In extended supersymmetries with $N \geq 2$, the entropy $S$ can also be computed via a certain duality invariant combination of the magnetic and electric charges $p, q$ of the fields configuration, see [37, 38] for all the $N > 2$ cases and [39] for the $N = 2$ case.

These three lectures begin with a pedagogical introduction to $U(1)$ duality rotations in nonlinear theories of electromagnetism. The basic aspects of duality symmetry are already present in this simple case with just one abelian gauge field: the hamiltonian is invariant, the lagrangian is not invariant but must transform in a well defined way. The Born-Infeld theory (relevant in describing the low energy effective action of $D$-branes in open string theory) is the main example of duality invariant nonlinear theory.

We next recall the general theory [4, 40] with many abelian gauge fields interacting with bosonic and fermionic matter. The maximal symmetry group in a theory with $n$ abelian gauge fields includes $Sp(2n, \mathbb{R})$. If there are no scalar fields the maximal symmetry group is $U(n)$. The geometry of the symmetry transformations on the scalar fields is that of the coset space $Sp(2n,\mathbb{R})/U(n)$. The kinetic term for the scalar fields is constructed by using this coset space geometry. The Born-Infeld lagrangian with $n$ abelian gauge fields and $Sp(2n,\mathbb{R})$ duality symmetry [16] is then presented. Its duality symmetry is proven by first considering duality rotations with complex field strengths and a Born-Infeld lagrangian with $U(n,n)$ self-duality. This latter theory is per se interesting, the scalar fields span the coset space $U(n,n)/[U(n)\times U(n)]$, and in the case $n=3$ this is the coset space of the scalars of $N=3$ supergravity with 3 vector multiplets. This Born-Infeld lagrangian is then a natural candidate for the nonlinear generalization of $N=3$ supergravity.

In Section 4, following [40], we first apply the general theory of duality rotations to supergravity theories with $N>2$ supersymmetries. In these supersymmetric theories the duality group is always a subgroup $G$ of $Sp(2n,\mathbb{R})$, where $G$ is the isometry group of the sigma model $G/H$ of the scalar fields. Much of the geometry underlying these theories is in the (local) embedding of $G$ in $Sp(2n,\mathbb{R})$. The supersymmetry transformation rules, the structure of the central and matter charges and the duality invariants associated to the entropy and the potential of extremal black holes configurations are all expressed in terms of the embedding of $G$ in $Sp(2n,\mathbb{R})$ [9]. We thus present a unifying formalism. This formalism is based on sections of a symplectic bundle and holds also in the general $N=2$ case where the scalar manifold is no more a coset space but a special Kähler manifold. We explicitly construct the symplectic bundles (vector bundles with a symplectic product on the fibers) associated to the $N>2$ supergravity theories, and prove that they are topologically trivial; this is no more the case for generic $N=2$ supergravities. In this case duality symmetry is needed to globally define the theory. It is not a symmetry of the equation of motions but rather a gauge symmetry with constant $Sp(2n,\mathbb{R})$ gauge transformations.

## 2 $U(1)$ gauge theory and duality symmetry

Maxwell theory is the prototype of electric-magnetic duality invariant theories. In vacuum the equations of motion are

$$\partial_\mu F^{\mu\nu}=0\ ,\qquad \partial_\mu \tilde{F}^{\mu\nu}=0\ , \tag{2.1}$$

where $\tilde{F}^{\mu\nu}\equiv\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. They are invariant under rotations $\binom{F}{\tilde F}\mapsto\left(\begin{smallmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{smallmatrix}\right)\binom{F}{\tilde F}$, or using vector notation under rotations $\binom{\boldsymbol{E}}{\boldsymbol{B}}\mapsto\left(\begin{smallmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{smallmatrix}\right)\binom{\boldsymbol{E}}{\boldsymbol{B}}$. This rotational symmetry, called duality symmetry, and also duality invariance or self-duality, is reflected in the invariance of the hamiltonian $\mathcal{H}=\frac{1}{2}(\boldsymbol{E}^2+\boldsymbol{B}^2)$, notice however that the lagrangian $\mathcal{L}=\frac{1}{2}(\boldsymbol{E}^2-\boldsymbol{B}^2)$ is not invariant. This continuous symmetry is not an internal symmetry because it rotates a tensor into a pseudotensor, however Poincaré Lie algebra transformations and $SO(2)$ Lie algebra ones commute, and this is so also for finite transformations belonging to the connected component $SL(2,\mathbb{C})$ of the Poincaré group and to $SO(2)$.

We study this symmetry for more general electromagnetic theories. In this section and the next one conditions on the lagrangians of (nonlinear) electromagnetic theories will be found that guarantee the duality symmetry (self-duality) of the equations of motion.

The key mathematical point that allows to establish criteria for self-duality, thus avoiding the explicit check of the symmetry at the level of the equation of motions, is that the equations of motion (a system of PDEs) can be conveniently split in a set of equations that is of degree 0 (no derivatives on the field strengths $F$), the so-called constitutive relations (see e.g. (2.5), or (2.8)), and another set of degree 1 (see e.g. (2.2), (2.3) or (2.9), (2.10)). Duality rotations act as an obvious symmetry of the set of equations of degree 1, so all what is left is to check that they act as a symmetry on the set of equations of degree 0. It is therefore plausible that this check can be equivalently formulated as a specific transformation property of the lagrangian under duality rotations (and independent from the spacetime dependence $F_{\mu\nu}(x)$ of the fields), indeed both the lagrangian and the equations of motions of degree 0 are functions of the field strength $F$ and not of its derivatives.

## 21. Duality symmetry in nonlinear electromagnetism

Maxwell equations read

$$\partial_t \boldsymbol{B} = -\nabla \times \boldsymbol{E} \quad , \quad \nabla \cdot \boldsymbol{B} = 0 \tag{2.2}$$

$$\partial_t \boldsymbol{D} = \nabla \times \boldsymbol{H} \quad , \quad \nabla \cdot \boldsymbol{D} = 0 \tag{2.3}$$

they are complemented by the relations between the electric field $\boldsymbol{E}$, the magnetic field $\boldsymbol{H}$, the electric displacement $\boldsymbol{D}$ and the magnetic induction $\boldsymbol{B}$. In vacuum we have

$$\boldsymbol{D} = \boldsymbol{E} \;, \quad \boldsymbol{H} = \boldsymbol{B} \;. \tag{2.4}$$

In a nonlinear theory we still have the equations (2.2), (2.3), but the relations $\boldsymbol{D} = \boldsymbol{E}$, $\boldsymbol{H} = \boldsymbol{B}$ are replaced by the nonlinear constitutive relations

$$\boldsymbol{D} = \boldsymbol{D}(\boldsymbol{E}, \boldsymbol{B}), \qquad \boldsymbol{H} = \boldsymbol{H}(\boldsymbol{E}, \boldsymbol{B}) \tag{2.5}$$

(if we consider a material medium with electric and magnetic properties then these equations are the constitutive relations of the material, and (2.2) and (2.3) are the macroscopic Maxwell equations).

Equations (2.2), (2.3), (2.4) are invariant under the group of general linear transformations

$$\begin{pmatrix} \boldsymbol{B}' \\ \boldsymbol{D}' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \boldsymbol{B} \\ \boldsymbol{D} \end{pmatrix}, \qquad \begin{pmatrix} \boldsymbol{E}' \\ \boldsymbol{H}' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} \quad . \tag{2.6}$$

We study under which conditions also the nonlinear constitutive relations (2.5) are invariant. We find constraints on the relations (2.5) as well as on the transformations (2.6).

We are interested in nonlinear theories that admit a lagrangian formulation so that relativistic covariance of the equations (2.2), (2.3), (2.5) and their inner consistency is automatically ensured. This requirement is fulfilled if the constitutive relations (2.5) are of the form

$$\boldsymbol{D} = \frac{\partial \mathcal{L}(\boldsymbol{E}, \boldsymbol{B})}{\partial \boldsymbol{E}} \quad , \quad \boldsymbol{H} = -\frac{\partial \mathcal{L}(\boldsymbol{E}, \boldsymbol{B})}{\partial \boldsymbol{B}} \quad , \tag{2.7}$$

where $\mathcal{L}(\boldsymbol{E}, \boldsymbol{B})$ is a Poincaré invariant function of $\boldsymbol{E}$ and $\boldsymbol{B}$. Indeed if we consider $\boldsymbol{E}$ and $\boldsymbol{B}$ depending on a gauge potential $A_\mu$ and vary the lagrangian $\mathcal{L}(\boldsymbol{E}, \boldsymbol{B})$ with respect to $A_\mu$, we recover (2.2), (2.3) and (2.7). This property is most easily shown by using four component notation. We group the constitutive relations (2.7) in the constitutive relation[1]

$$\tilde{G}^{\mu\nu} \;=\; 2\frac{\partial\mathcal{L}(F)}{\partial F_{\mu\nu}} \;; \tag{2.8}$$

we also define $G_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\tilde{G}^{\rho\sigma}$, so that $\tilde{G}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}G_{\rho\sigma}$ ($\epsilon^{0123} = -\epsilon_{0123} = 1$). If we consider the field strength $F_{\mu\nu}$ as a function of a (locally defined) gauge potential $A_\mu$, then equations (2.2) and (2.3) are respectively the Bianchi identities for $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the equations of motion for $\mathcal{L}(F(A))$,

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \;, \tag{2.9}$$

$$\partial_\mu \tilde{G}^{\mu\nu} = 0 \;\; . \tag{2.10}$$

In our treatment of duality rotations we study the symmetries of the equations **(2.9), (2.10)** and **(2.8)**. The lagrangian $\mathcal{L}(F)$ is always a function of the field strength $F$; it is not seen as a function of the gauge potential $A_\mu$; accordingly the Bianchi identities for $F$ are considered part of the equations of motions for $F$.

Finally we consider an action $S = \int \mathcal{L}\, d^4x$ with lagrangian density $\mathcal{L} = \mathcal{L}(F)$ that depends on $F$ but not on its partial derivatives; it also depends on a spacetime metric $g_{\mu\nu}$ that we generally omit writing explicitly[2], and on at least one dimensionful constant in order to allow for nonlinearity in the constitutive relations (2.8) (i.e. (2.5)). We set this dimensionful constant to 1.

The duality rotations (2.6) read

$$\begin{pmatrix} F' \\ G' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} \;\; . \tag{2.11}$$

Since by construction equations (2.9) and (2.10) are invariant under (2.11), these duality rotations are a symmetry of the system of equations (2.9), (2.10), (2.8) (or (2.2), (2.3), (2.5)), iff on shell the constitutive relations are invariant in form, i.e., iff the functional dependence of $\tilde{G}'$ from $F'$ is the same as that of $\tilde{G}$ from $F$, i.e. iff

$$\tilde{G}'^{\mu\nu} \;=\; 2\frac{\partial\mathcal{L}(F')}{\partial F'_{\mu\nu}} \;, \tag{2.12}$$

where $F'_{\mu\nu}$ and $G'_{\mu\nu}$ are given in (2.11). This is the condition that constrains the lagrangian $\mathcal{L}(F)$ and the rotation parameters in (2.11). This condition has to hold on shell of (2.8)-(2.10); however (2.12) is not a differential equation and therefore has to hold just using (2.8), i.e., off shell of (2.9) and (2.10) (indeed if it holds for constant field strengths $F$ then it holds for any $F$).

[1] a practical convention is to define $\frac{\partial F_{\rho\sigma}}{\partial F_{\mu\nu}} = \delta^\mu_\rho \delta^\nu_\sigma$ rather than $\frac{\partial F_{\rho\sigma}}{\partial F_{\mu\nu}} = \delta^\mu_\rho \delta^\nu_\sigma - \delta^\nu_\rho \delta^\mu_\sigma$ . This explains the factor 2 in (2.8).

[2] Notice that (2.9), (2.10) are also the equation of motions in the presence of a nontrivial metric. Indeed $S = \int \mathcal{L}\, d^4x = \int L \sqrt{g} d^4x$. The equation of motions are $\partial_\mu(\sqrt{g}\, F^{*\,\mu\nu}) = \partial_\mu \tilde{F}^{\mu\nu} = 0$ , $\partial_\mu(\sqrt{g}\, G^{*\,\mu\nu}) = \partial_\mu \tilde{G}^{\mu\nu} = 0$ , where the Hodge dual of a two form $\Omega_{\mu\nu}$ is defined by $\Omega^*_{\mu\nu} \equiv \frac{1}{2}\sqrt{g}\, \epsilon_{\mu\nu\rho\sigma}\Omega^{\rho\sigma}$ .

In order to study the duality symmetry condition (2.12) let $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) + \epsilon\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) + \ldots$, and consider infinitesimal $GL(2,\mathbb{R})$ rotations $G \to G + \epsilon\Delta G,\ F \to F + \epsilon\Delta F$,

$$\Delta \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} , \tag{2.13}$$

so that the duality condition reads

$$\tilde{G} + \Delta\tilde{G} = 2\frac{\partial \mathcal{L}(F+\Delta F)}{\partial(F+\Delta F)} . \tag{2.14}$$

The right hand side simplifies to[3]

$$\frac{\partial \mathcal{L}(F+\Delta F)}{\partial(F+\Delta F)} = \frac{\partial \mathcal{L}(F+\Delta F)}{\partial F}\frac{\partial F}{\partial(F+\Delta F)} = \frac{\partial \mathcal{L}(F+\Delta F)}{\partial F} - \frac{\partial \mathcal{L}(F)}{\partial F}\frac{\partial(\Delta F)}{\partial F}$$

then, using (2.13) and (2.8), condition (2.14) reads

$$c\tilde{F} + d\tilde{G} = 2\frac{\partial(\mathcal{L}(F+\Delta F) - \mathcal{L}(F))}{\partial F} - 2a\frac{\partial \mathcal{L}(F)}{\partial F} - b\tilde{G}\frac{\delta G}{\partial F} . \tag{2.15}$$

In order to further simplify this expression we write $2\tilde{F} = \frac{\partial}{\partial F}F\tilde{F}$ and we factorize out the partial derivative $\frac{\partial}{\partial F}$. We thus arrive at the equivalent condition

$$\mathcal{L}(F+\Delta F) - \mathcal{L}(F) - \frac{c}{4}F\tilde{F} - \frac{b}{4}G\tilde{G} \;=\; (a+d)(\mathcal{L}(F) - \mathcal{L}_{F=0}) . \tag{2.16}$$

The constant term $(a+d)\mathcal{L}_{F=0}$, nonvanishing for example in D-brane lagrangians, is obtained by observing that when $F=0$ also $G=0$.

Next use $\mathcal{L}(F+\Delta F) - \mathcal{L}(F) = \frac{\partial \mathcal{L}(F)}{\partial F}\Delta F = \frac{1}{2}aF\tilde{G} + \frac{1}{2}bG\tilde{G}$ in order to rewrite expression (2.16) as

$$\frac{b}{4}G\tilde{G} - \frac{c}{4}F\tilde{F} = (a+d)(\mathcal{L}(F) - \mathcal{L}_{F=0}) - \frac{a}{2}F\tilde{G} . \tag{2.17}$$

If we require the nonlinear lagrangian $\mathcal{L}(F)$ to reduce to the usual Maxwell lagrangian in the weak field limit, $F^4 << F^2$, i.e., $\mathcal{L}(F) = \mathcal{L}_{F=0} - 1/4\int FFd^4x + O(F^4)$, then $\tilde{G} = -F + O(F^3)$, and we obtain the constraint (recall that $\tilde{\tilde{G}} = -G$)

$$b = -c \quad , \qquad a = d \ ,$$

the duality group can be at most $SO(2)$ rotations times dilatations. Condition (2.17) becomes

$$\frac{b}{4}\Big(G\tilde{G} + F\tilde{F}\Big) \;=\; 2a\Big(\mathcal{L}(F) - \mathcal{L}_{F=0} - \frac{1}{2}F\frac{\partial \mathcal{L}}{\partial F}\Big) . \tag{2.18}$$

[3]here and in the following we suppress the spacetime indices so that for example $F\tilde{G} = F_{\mu\nu}\tilde{G}^{\mu\nu}$; notice that $F\tilde{G} = \tilde{F}G$, $\tilde{\tilde{F}} = -F$, and $\tilde{F}\tilde{G} = -FG$ where $FG = F^{\mu\nu}G_{\mu\nu}$.

The vanishing of the right hand side holds only if either $\mathcal{L}(F) - \mathcal{L}_{F=0}$ is quadratic in $F$ (usual electromagnetism) or $a = 0$. We are interested in nonlinear theories; by definition in a nonlinear theory $\mathcal{L}(F)$ is not quadratic in $F$. This shows that dilatations alone cannot be a duality symmetry. If we require the duality group to contain at least $SO(2)$ rotations then

$$G\tilde{G} + F\tilde{F} = 0 \; , \tag{2.19}$$

and $SO(2)$ is the maximal duality group. Relation (2.18) is nontrivially satisfied iff

$$a = d = 0 \; ,$$

and (2.19) hold.

In conclusion equation (2.19) is a necessary and sufficient condition for a nonlinear electromagnetic theory to be symmetric under $SO(2)$ duality rotations, and $SO(2) \subset GL(2, \mathbb{R})$ is the maximal connected Lie group of duality rotations of pure nonlinear electromagnetism[4].

This conclusion still holds if we consider a nonlinear lagrangian $\mathcal{L}(F)$ that in the weak field limit $F^4 << F^2$ (up to an overall normalization factor) reduces to the most general linear lagrangian

$$\mathcal{L}(F) = \mathcal{L}_{F=0} - \frac{1}{4}FF + \frac{1}{4}\theta F\tilde{F} + O(F^4) \; .$$

In this case $G = \tilde{F} + \theta F + O(F^3)$. We substitute in (2.17) and obtain the two conditions (the coefficients of the scalar $F^2$ and of the pseudoscalar $F\tilde{F}$ have to vanish separately)

$$c = -b(1+\theta^2) \quad , \qquad d - a = 2\theta b \; . \tag{2.20}$$

The most general infinitesimal duality transformation is therefore

$$\begin{pmatrix} a & b \\ -b(1+\theta^2) & a+2\theta b \end{pmatrix} = \begin{pmatrix} a+\theta b & 0 \\ 0 & a+\theta b \end{pmatrix} + \Theta \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \Theta^{-1} \tag{2.21}$$

where $\Theta = \begin{pmatrix} 1 & 0 \\ \theta & 1 \end{pmatrix}$. We have dilatations and $SO(2)$ rotations, they act on the vector $\begin{pmatrix} F \\ G \end{pmatrix}$ via the conjugate representation given by the matrix $\Theta$. Let's now remove the weak field limit assumtion $F^4 << F^2$. We proceed as before. From (2.12) (or from (2.17)) we immediately obtain that dilatations alone are not a duality symmetry of the nonlinear equations of motion. Then if $SO(2)$ rotations are a duality symmetry we have that they are the maximal duality symmetry group. This happens if

$$G\tilde{G} + (1+\theta)^2 F\tilde{F} = 2\theta F\tilde{G} \; . \tag{2.22}$$

[4]This symmetry cannot even extend to $O(2)$ because already in the case of usual electromagnetism the finite rotation $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ does not satisfy the duality condition (2.12).

Finally we note that the necessary and sufficient conditions for $SO(2)$ duality rotations (2.22) (or (2.19)) can be equivalently expressed as invariance of

$$\mathcal{L}(F) - \frac{1}{4} F\tilde{G} \ . \tag{2.23}$$

Proof: the variation of expression (2.23) under $F \to F + \Delta F$ is given by $\mathcal{L}(F + \Delta F) - \mathcal{L}(F) - \frac{1}{4}\Delta F\,\tilde{G} - \frac{1}{4}F\Delta\tilde{G}$ . Use of (2.16) with $a + d = 0$ (no dilatation) shows that this variation vanishes.

**Invariance of the energy momentum tensor**
The symmetric energy momentum tensor of a nonlinear theory of electromagnetism (obtained via Belinfante procedure or by varying with respect to the metric) is given by[5]

$$T^{\mu}_{\ \nu} = \tilde{G}^{\mu\lambda} F_{\nu\lambda} + \partial^{\mu}_{\ \nu}\mathcal{L} \ . \tag{2.24}$$

The equations of motion (2.10) and (2.9) imply its conservation, $\partial_\mu T^{\mu}_{\ \nu} = 0$. Invariance of the energy momentum tensor under duality rotations is easily proven by observing that for a generic antisymmetric tensor $F_{\mu\nu}$

$$\tilde{F}^{\mu\lambda} F_{\nu\lambda} = -\frac{1}{4}\delta^{\mu}_{\ \lambda}\tilde{F}^{\rho\sigma}F_{\rho\sigma} \ , \tag{2.25}$$

and then by recalling the duality symmetry condition (2.19). In particular the hamiltonian $\mathcal{H} = T^{00} = \boldsymbol{D}\cdot\boldsymbol{E} - \mathcal{L}$ of a theory that has duality rotation symmetry is invariant.

## 22. Born-Infeld lagrangian

A notable example of a lagrangian whose equations of motion are invariant under duality rotations is given by the Born-Infeld one [41]

$$\begin{aligned}
\mathcal{L}_{\rm BI} &= 1 - \sqrt{-\det(\eta + F)} && (2.26)\\
&= 1 - \sqrt{1 + \frac{1}{2}F^{\,2} - \frac{1}{16}(F\tilde{F})^2} && (2.27)\\
&= 1 - \sqrt{1 - \boldsymbol{E}^2 + \boldsymbol{B}^2 - (\boldsymbol{E}\cdot\boldsymbol{B})^2} \ . && (2.28)
\end{aligned}$$

In the second line we have simply expanded the 4x4 determinant and espressed the lagrangian in terms of the only two independent Lorentz invariants associated to the electromagnetic field: $F^{\,2} \equiv F_{\mu\nu}F^{\mu\nu}$ , $F\tilde{F} \equiv F_{\mu\nu}\tilde{F}^{\mu\nu}$.

The explicit expression of $G$ is

$$G_{\mu\nu} = \frac{\tilde{F}_{\mu\nu} + \frac{1}{4}F\tilde{F}\,F_{\mu\nu}}{\sqrt{1 + \frac{1}{2}F^2 - \frac{1}{16}(F\tilde{F})^2}} \ , \tag{2.29}$$

and the duality condition (2.19) is readily seen to hold.

[5] symmetry of $T^{\mu\nu}$ follows immediately by observing that the tensor structure of $\tilde{G}^{\mu\nu}$ implies $\tilde{G}^{\mu\nu} = f_s(F)F^{\mu\nu} + f_p(F)\tilde{F}^{\mu\nu}$ with scalars $f_s(F)$ and $f_p(F)$ depending on $F$, the metric $\eta = diag(-1,1,1,1)$ and the completely antisymmetric tensor density $\epsilon_{\mu\nu\rho\sigma}$. (Actually, if the lagrangian is parity even, $f_s$ is a scalar function while $f_p$ is a pseudoscalar function).

## 23. Legendre Transformations

In the literature on gauge theories of abelian $p$-form potentials, the term duality transformation denotes a different transformation from the one we have introduced, a Legendre transformation, that is not a symmetry transformation. In this section we relate these two different notions, see [13] for further applications and examples.

Consider a theory of nonlinear electrodynamics ($p = 1$) with lagrangian $\mathcal{L}(F)$. The equations of motion and the Bianchi identity for $F$ can be derived from the Lagrangian $\mathcal{L}(F, F_{\rm D})$ defined by

$$\mathcal{L}(F, F_{\rm D}) = \mathcal{L}(F) - \frac{1}{2} F\tilde{F}_{\rm D} \;, \qquad {F_{\rm D}}^{\mu\nu} = \partial^\mu {A_{\rm D}}^\nu - \partial^\nu {A_{\rm D}}^\mu \;, \tag{2.30}$$

where $F$ is now an unconstrained antisymmetric tensor field, $A_{\rm D}$ a Lagrange multiplier field and $F_{\rm D}$ its electromagnetic field. [Hint: varying with respect to $A_{\rm D}$ gives the Bianchi identity for $F$, varying with respect to $F$ gives $G^{\mu\nu} = {F_{\rm D}}^{\mu\nu}$ that is equivalent to the initial equations of motion $\partial_\mu \tilde{G}^{\mu\nu} = 0$ because ${F_{\rm D}}^{\mu\nu} = \partial^\mu {A_{\rm D}}^\nu - \partial^\nu {A_{\rm D}}^\mu$ (Poincaré lemma)].

Given the lagrangian (2.30) one can also first consider the equation of motion for $F$,

$$G(F) \;=\; F_{\rm D} \;, \tag{2.31}$$

that is solved by expressing $F$ as a function of the dual field strength, $F = F(F_{\rm D})$. Then inserting this solution into $\mathcal{L}(F, F_{\rm D})$, one gets the dual model

$$\mathcal{L}_{\rm D}(F_{\rm D}) \equiv \mathcal{L}(F(F_{\rm D})) - \frac{1}{2} F(F_{\rm D}) \cdot \tilde{F}_{\rm D} \;. \tag{2.32}$$

Solutions of the (2.32) equations of motion are, together with (2.31), solutions of the (2.30) equations of motion. Therefore solutions to the (2.32) equations of motion are via (2.31) in 1-1 correspondence with solutions of the $\mathcal{L}(F)$ equations of motion.

One can always perform a Legendre transformation and describe the physical system with the new dynamical variables $A_{\rm D}$ and the new lagrangian $\mathcal{L}_{\rm D}$ rather than $A$ and $\mathcal{L}$.

The relation with the duality rotation symmetry (self-duality) of the previous section is that if the system admits duality rotations then the solution $F_{\rm D}$ of the $\mathcal{L}_{\rm D}$ equations of motion is also a solution of the $\mathcal{L}$ equations of motion, we have a symmetry because the dual field $F_{\rm D}$ is a solution of the original system. This is the case because for any solution $\mathcal{L}$ of the self-duality equation, its Legendre transform $\mathcal{L}_{\rm D}$ satisfies:

$$\mathcal{L}_{\rm D}(F) \;=\; \mathcal{L}(F) \;. \tag{2.33}$$

This follows from considering a finite $SO(2)$ duality rotation with angle $\pi/2$. Then $F \to F' = G(F) = F_{\rm D}$, and invariance of (2.23), i.e. $\mathcal{L}(F') - \frac{1}{4}F'\tilde{G}' = \mathcal{L}(F) - \frac{1}{4}F\tilde{G}$ , implies $\mathcal{L}_{\rm D}(F_{\rm D}) \;=\; \mathcal{L}(F_{\rm D})$, i.e., (2.33).

In summary, a Legendre transformation is a duality rotation only if the symmetry condition (2.8) is met. If the self-duality condition (2.8) does not hold, a Legendre transformation leads to a dual formulation of the theory in terms of a dual Lagrangian $\mathcal{L}_{\rm D}$, not to a symmetry of the theory.

### 24. Extended duality rotations

The duality symmetry of the equations of motion of nonlinear electromagnetism can be extended to $SL(2,\mathbb{R})$. We observe that the definition of duality symmetry we used -symmetry of the system of equations (2.9), (2.10) and (2.8)- can be relaxed by allowing the $F$ dependence of $G$ to change by a linear term: $G = 2\frac{\partial \mathcal{L}}{\partial F}$ and $G = 2\frac{\partial \mathcal{L}}{\partial F} + \vartheta F$ together with the Bianchi identities for $F$ give equivalent equations of motions for $F$. Therefore the transformation

$$\begin{pmatrix} F' \\ G' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \vartheta & 1 \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} \tag{2.34}$$

is a symmetry of any nonlinear electromagnetism. It corresponds to the lagrangian change $\mathcal{L} \to \mathcal{L} + \frac{1}{4}\vartheta F\tilde{F}$. This symmetry alone does not act on $F$, but it is useful if the nonlinear theory has $SO(2)$ duality symmetry. In this case (2.34) extends duality symmetry from $SO(2)$ to $SL(2,\mathbb{R})$ (i.e. $Sp(2,\mathbb{R})$). Notice however that the $SL(2,\mathbb{R})$ transformed solution, contrary to the $SO(2)$ one, has a different energy and energy momentum tensor (recall (2.24)). If the constant $\vartheta$ is promoted to a dynamical field we then have invariance of the energy momentum tensor under $SL(2,\mathbb{R})$ duality.

## 3 General theory of duality rotations

We now briefly describe the most general conditions in order to have theories with duality rotation symmetry, provide three examples and discuss the geometry of the scalar fields, that when present enhance the duality symmetry from a compact group to a noncompact one (like from $SO(2)$ to $SL(2,\mathbb{R})$).

We consider a theory of $n$ abelian gauge fields possibly coupled to other bosonic and fermionic fields that we denote $\varphi^\alpha$, $(\alpha = 1, ...p)$. We assume that the $U(1)$ gauge potentials enter the action $S = S[F, \varphi]$ only trough the field strengths $F^\Lambda_{\mu\nu}$ $(\Lambda = 1, \ldots, n)$, and that the action does not depend on partial derivatives of the field strengths. Define $\tilde{G}_\Lambda{}^{\mu\nu} = 2\frac{\partial \mathcal{L}}{\partial F^\Lambda_{\mu\nu}}$, i.e,

$$\tilde{G}_\Lambda{}^{\mu\nu} = 2\frac{\delta S[F,\varphi]}{\delta F^\Lambda_{\mu\nu}} \ ; \tag{3.1}$$

then the Bianchi identities and the equations of motions for $S[F,\varphi]$ are

$$\partial_\mu \tilde{F}^{\Lambda\,\mu\nu} = 0 \ , \tag{3.2}$$

$$\partial_\mu \tilde{G}_\Lambda{}^{\mu\nu} = 0 \ , \tag{3.3}$$

$$\frac{\delta S[F,\varphi]}{\delta\varphi^\alpha} = 0 \ . \tag{3.4}$$

The field theory is described by the system of equations (3.1)-(3.4). Consider the duality transformations

$$\begin{pmatrix} F' \\ G' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} \tag{3.5}$$

$$\varphi'^\alpha = \Xi^\alpha(\varphi) \tag{3.6}$$

where $(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix})$ is a generic constant $GL(2n,\mathbb{R})$ matrix and the $\varphi^\alpha$ fields transformation in full detail reads $\varphi'^\alpha = \Xi^\alpha(\varphi, (\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}))$, with no partial derivative of $\varphi$ appearing in $\Xi^\alpha$.

These duality rotations are a symmetry of the system of equations (3.1)-(3.4) iff, given $F$, $G$, and $\varphi$ solution of (3.1)-(3.4) then $F'$, $G'$ and $\varphi'$, that by construction satisfy $\partial_\mu \tilde{F}'^{\Lambda\,\mu\nu} = 0$ and $\partial_\mu \tilde{G}'^{\mu\nu}_\Lambda = 0$, satisfy also

$$\tilde{G}'^{\mu\nu}_\Lambda \;=\; 2\frac{\delta S[F',\varphi']}{\delta F'^\Lambda_{\mu\nu}} \;, \tag{3.7}$$

$$\frac{\delta S[F',\varphi']}{\delta \varphi'^\alpha} \;=\; 0 \;. \tag{3.8}$$

The study [4, 40] of these on shell conditions in the case of infinitesimal $GL(2n,\mathbb{R})$ rotations

$$F \to F' = F + \Delta F \quad , \quad G \to G' = G + \Delta G \;,$$

$$\Delta \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} \;, \tag{3.9}$$

$$\Delta \varphi^\alpha = \xi^\alpha(\varphi) \;. \tag{3.10}$$

leads to the condition

$$\mathcal{L}(F',\varphi') - \mathcal{L}(F,\varphi) - \kappa\mathcal{L}(F,\varphi) - \frac{1}{4}\tilde{F}\,c\,F - \frac{1}{4}\tilde{G}\,b\,G \;=\; const_{a,b,c,d} \tag{3.11}$$

If we expand $F'$ in terms of $F$ and $G$, we obtain the equivalent condition

$$\mathcal{L}(F,\varphi') - \mathcal{L}(F,\varphi) = \frac{1}{4}\tilde{F}\,c\,F - \frac{1}{4}\tilde{G}\,b\,G + \kappa\mathcal{L}(F,\varphi) - \frac{1}{2}\tilde{G}\,a\,F + const_{a,b,c,d} \tag{3.12}$$

Moreover the matrices $a, b, c, d$ are constrained by the relations

$$a^t + b = \kappa\mathbb{1} \quad , \quad b^t = b \quad , \quad c^t = c \tag{3.13}$$

so that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an infinitesimal dilatation times an infinitesimal symplectic transformation. When the real parameter $\kappa$ vanishes we just have an infinitesimal symplectic transformation.

Equation (3.12), where $\tilde{G}^{\mu\nu}_\Lambda = 2\partial\mathcal{L}/\partial F^\Lambda_{\mu\nu}$, is a necessary and sufficient condition in order to have duality symmetry. This condition is on shell of the fermions equations of motion, in particular if no fermion is present this condition is off shell. In the presence of fermions, equation (3.12) off shell is a sufficient condition for duality symmetry.

The duality symmetry group is

$$\mathbb{R}^{>0} \times Sp(2n,\mathbb{R}) \;, \tag{3.14}$$

the group of dilatations times symplectic transformations It is also the maximal group of duality rotations as the example (or better, the limiting case) studied in the next section shows.

We note that condition (3.12) in the absence of dilatations ($\kappa = 0$), and for $const_{a,b,c,d} = 0$ (which is almost always the case) is equivalent to the invariance of

$$\mathcal{L} - \frac{1}{4}\tilde{F}G \ . \tag{3.15}$$

**31. The main example and the scalar fields fractional transformations**

Consider the Lagrangian

$$\frac{1}{4}\mathcal{N}_{2\,\Lambda\Sigma}F^{\Lambda}F^{\Sigma} + \frac{1}{4}\mathcal{N}_{1\,\Lambda\Sigma}F^{\Lambda}\tilde{F}^{\Sigma} + \mathscr{L}(\phi) \tag{3.16}$$

where the real symmetric matrices $\mathcal{N}_1(\phi)$ and $\mathcal{N}_2(\phi)$ and the lagrangian $\mathscr{L}(\phi)$ are just functions of the bosonic fields $\phi^i$, $i = 1, \ldots m$, (and their partial derivatives).

Any nonlinear lagrangian in the limit of vanishing fermionic fields and of weak field strengths $F^4 << F^2$ reduces to the one in (3.16). A straightforward calculation shows that this lagrangian has $\mathbb{R}^{>0} \times SL(2n, \mathbb{R})$ duality symmetry if the matrices $\mathcal{N}_1$ and $\mathcal{N}_2$ of the scalar fields transform as

$$\Delta\,\mathcal{N}_1 = c + d\,\mathcal{N}_1 - \mathcal{N}_1 a - \mathcal{N}_1\,b\,\mathcal{N}_1 + \mathcal{N}_2\,b\,\mathcal{N}_2 \ \ , \tag{3.17}$$

$$\Delta\,\mathcal{N}_2 = d\,\mathcal{N}_2 - \mathcal{N}_2 a - \mathcal{N}_1\,b\,\mathcal{N}_2 - \mathcal{N}_2\,b\,\mathcal{N}_1 \ \ , \tag{3.18}$$

and

$$\Delta\,\mathscr{L}(\phi) = \kappa\mathscr{L}(\phi) \ . \tag{3.19}$$

If we define

$$\mathcal{N} = \mathcal{N}_1 + i\mathcal{N}_2 \ ,$$

i.e., $\mathcal{N}_1 = \mathrm{Re}\mathcal{N}$ , $\mathcal{N}_2 = \mathrm{Im}\mathcal{N}$, the transformations (3.17), (3.18) read

$$\Delta\,\mathcal{N} \ = \ c + d\,\mathcal{N} - \mathcal{N}a - \mathcal{N}\,b\,\mathcal{N} \ , \tag{3.20}$$

the finite version is the fractional transformation

$$\mathcal{N}' = (C + D\mathcal{N})\,(A + B\mathcal{N})^{-1} \ . \tag{3.21}$$

Under (3.21) the imaginary part of $\mathcal{N}$ transforms as

$$\mathcal{N}_2' = (A + B\mathcal{N})^{-\dagger}\mathcal{N}_2(A + B\mathcal{N})^{-1} \tag{3.22}$$

where $-\dagger$ is a shorthand notation for the hermitian conjugate of the inverse matrix.

The kinetic term $\frac{1}{4}\mathcal{N}_{2\,\Lambda\Sigma}F^{\Lambda}F^{\Sigma}$ is positive definite if the symmetric matrix $\mathcal{N}_2$ is negative definite. It can be shown (see for example Appendix 7.2 in [40]) that the matrices $\mathcal{N} = \mathcal{N}_1 + i\mathcal{N}_2$ with $\mathcal{N}_1$ and $\mathcal{N}_2$ real and symmetric, and $\mathcal{N}_2$ positive definite, are the coset space $\frac{Sp(2n,\mathbb{R})}{U(n)}$.

A scalar lagrangian that satisfies the variation (3.19) can always be constructed using the geometry of the coset space $\frac{Sp(2n,\mathbb{R})}{U(n)}$, see Section 34..

This example also clarifies the condition (3.13) on the $GL(2n,\mathbb{R})$ generators. It is a straighfoward calculation to check that the equations (3.2), (3.3) and

$$\tilde{G} = \mathcal{N}_2 F + \mathcal{N}_1 \tilde{F} \tag{3.23}$$

have duality symmetry under $GL(2n,\mathbb{R})$ transformations with $\Delta\mathcal{N}$ given in (3.20). However we want the constitutive relations $G = G[F,\varphi]$ to follow from a lagrangian. Those following from the lagrangian (3.16) are (3.23) with $\mathcal{N}_1$ and $\mathcal{N}_2$ necessarily *symmetric* matrices. Only if the transformed matrices $\mathcal{N}_1'$ and $\mathcal{N}_2'$ are again symmetric we can have $\tilde{G}' = \frac{\partial\mathcal{L}(F',\varphi')}{\partial F'}$ as in (3.7), (or more generally $\tilde{G}' = \frac{\partial\mathcal{L}'(F',\varphi')}{\partial F'}$). The constraints $\mathcal{N}_1' = \mathcal{N}_1'^t$, $\mathcal{N}_2' = \mathcal{N}_2'^t$, reduce the duality group to $\mathbb{R}^{>0} \times Sp(2n,\mathbb{R})$.

In conclusion equation (3.12) is a necessary and sufficient condition for a theory of $n$ abelian gauge fields coupled to bosonic matter to be symmetric under $\mathbb{R}^{>0} \times Sp(2n,\mathbb{R})$ duality rotations, and $\mathbb{R}^{>0} \times Sp(2n,\mathbb{R})$ is the maximal connected Lie group of duality rotations.

## 32. An example with fermi fields

Consider the Lagrangian with Pauli coupling

$$\mathcal{L}_0 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\overline{\psi}\partial\!\!\!/\psi - \frac{1}{2}\overline{\xi}\partial\!\!\!/\xi + \frac{1}{2}\lambda F^{\mu\nu}\overline{\psi}\sigma_{\mu\nu}\xi \tag{3.24}$$

where $\sigma^{\mu\nu} = \frac{1}{4}[\gamma^\mu,\gamma^\nu]$ and $\psi,\xi$ are two Majorana spinors. We have

$$\tilde{G}^{\mu\nu} = 2\frac{\partial\mathcal{L}_0}{\partial F_{\mu\nu}} = -F^{\mu\nu} + \lambda\overline{\psi}\sigma^{\mu\nu}\xi \tag{3.25}$$

and the duality condition (3.12) for an infinitesimal $U(1)$ duality rotation $\left(\begin{smallmatrix} 0 & b \\ -b & 0\end{smallmatrix}\right)$ reads

$$\Delta_\psi\mathcal{L}_0 + \Delta_\xi\mathcal{L}_0 = -\frac{b}{4}\lambda\tilde{F}\overline{\psi}\sigma\xi + \frac{b}{4}\lambda^2\,\overline{\psi}\sigma^{\mu\nu}\xi\,\overline{\psi}\tilde{\sigma}_{\mu\nu}\xi\ . \tag{3.26}$$

It is natural to assume that the kinetic terms of the fermion fields are invariant under this duality rotation (this is also the case for the scalar lagrangian $\mathscr{L}(\phi)$ in (3.19)), then using $\gamma_5\sigma^{\mu\nu} = i\tilde{\sigma}^{\mu\nu}$ we see that the coupling of the fermions with the field strength is reproduced if the fermions rotate according to

$$\Delta\psi = \frac{i}{2}b\gamma_5\psi\ , \tag{3.27}$$

$$\Delta\xi = \frac{i}{2}b\gamma_5\xi\ ; \tag{3.28}$$

we also see that we have to add to the lagrangian $\mathcal{L}_0$ a new interaction term quartic in the fermion fields. Its coupling is also fixed by duality symmetry to be $-\lambda^2/8$.

The theory with $U(1)$ duality symmetry is therefore given by the lagrangian [1]

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\overline{\psi}\partial\!\!\!/\psi - \frac{1}{2}\overline{\xi}\partial\!\!\!/\xi + \frac{1}{2}\lambda F^{\mu\nu}\psi\sigma_{\mu\nu}\xi - \frac{1}{8}\lambda^2\,\overline{\psi}\sigma_{\mu\nu}\xi\,\overline{\psi}\sigma^{\mu\nu}\xi\ . \tag{3.29}$$

Notice that fermions transform under the double cover of $U(1)$ indeed under a rotation of angle $b = 2\pi$ we have $\psi \to -\psi$, $\xi \to -\xi$, this is a typical feature of fermions transformations under duality rotations, they transform under the double cover of the maximal compact subgroup of the duality group. This is so because the interaction with the gauge field is via fermions bilinear terms.

## 33. Compact and noncompact duality rotations

The fractional transformation (3.21) is also characteristic of nonlinear theories. The subgroup of $Sp(2n,\mathbb{R})$ that leaves invariant a fixed value of the scalar fields $\mathcal{N}$ is $U(n)$. This is easily seen by setting $\mathcal{N} = -i1\!\!1$. Then infinitesimally we have relations (3.13) with $\kappa = 0$ and $b = -c$, $a = -a^t$, i.e. we have the antisymmetric matrix

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\ ,$$

$a = -a^t$, $b = b^t$. For finite transformations the $Sp(2n,\mathbb{R})$ relations

$$A^tC - C^tA = 0\ \ ,\quad B^tD - D^tB = 0\ \ ,\quad A^tD - C^tB = 1 \tag{3.30}$$

are complemented by

$$A = D\ \ ,\quad B = -C\ . \tag{3.31}$$

Thus $A - iB$ is a unitary matrix. $U(n)$ is the maximal compact subgroup of $Sp(2n,\mathbb{R})$, it is the group of orthogonal and symplectic $2n \times 2n$ matrices.

By freezing the values of the scalar fields $\mathcal{N}$ we have obtained a theory with only gauge fields and with $U(n)$ duality symmetry. Vice versa (see [40], that follows [14] that extends to $U(n)$ the $U(1)$ interacting theory discussed in [12, 13]) given a theory invariant under $U(n)$ duality rotations it is possible to extend it via $n(n+1)$ scalar fields $\mathcal{N}$ to a theory invariant under $Sp(2n,\mathbb{R})$.

## 34. Nonlinear sigma models on $G/H$

In this section we briefly consider the geometry of coset spaces $G/H$. This is the geometry underlying the scalar fields and needed to formulate their dynamics.

We study in particular the case $G = Sp(2n.\mathbb{R})$, $H = U(n)$ [4] and give a kinetic term for the scalar fields $\mathcal{N}$.

The geometry of the coset space $G/H$ is conveniently described in terms of coset representatives, local sections $L$ of the bundle $G \to G/H$. A point $\phi$ in $G/H$ is an equivalence class $gH = \{\tilde{g}\,|\,g^{-1}\tilde{g} \in H\}$. We denote by $\phi^i$ $(i = 1, 2 \ldots m)$ its coordinates (the scalar fields of the theory). The left action of $G$ on $G/H$ is inherited from that of $G$ on $G$, it is given by $gH \mapsto g'gH$, that we rewrite $\phi \mapsto g'\phi = \phi'$. Concerning the coset representatives we then have

$$g'L(\phi) = L(\phi')h\ , \tag{3.32}$$

because both the left and the right hand side are representatives of $\phi'$. The geometry of $G/H$ and the corresponding physics can be constructed in terms of coset representatives. Of course the construction must be insensitive to the particular representative choice, we have a gauge symmetry with gauge group $H$.

When $H$ is compact the Lie algebra of $G$ splits in the direct sum $\mathbb{G} = \mathbb{H} + \mathbb{K}$, where

$$[\mathbb{H}, \mathbb{H}] \subset \mathbb{H} \quad , \quad [\mathbb{K}, \mathbb{K}] \subset \mathbb{H} + \mathbb{K} \quad , \quad [\mathbb{H}, \mathbb{K}] \subset \mathbb{K} \ . \tag{3.33}$$

The last expression defines the coset space representation of $\mathbb{H}$. The representations of the compact Lie algebra $\mathbb{H}$ are equivalent to unitary ones, and therefore there exists a basis $(H_\alpha, K_a)$, where $[H_\alpha, K_a] = C^b_{\alpha a} K_b$ with $C_\alpha = (C^b_{\alpha a})_{a,b=1,\dots m=dimG/H}$ antihermitian matrices. Since the coset representation is a real representation then these matrices $C_\alpha$ belong to the Lie algebra of $SO(m)$.

Given a coset representative $L(\phi)$, the pull back on $G/H$ of the $\mathbb{G}$ Lie algebra left invariant 1-form $\Gamma = L^{-1}dL$ is decomposed as

$$\Gamma = L^{-1}dL = P^a(\phi)K_a + \omega^\alpha(\phi)H_\alpha \ .$$

$\Gamma$ and therefore $P = P^a(\phi)K_a$ and $\omega = \omega^\alpha(\phi)H_\alpha$ are invariant under diffeomorphisms generated by the left $G$ action. Under the local right $H$ action of an element $h(\phi)$ (or under change of coset representative $L'(\phi) = L(\phi)h(\phi)$) we have

$$P \to h^{-1}Ph \quad , \quad \omega \to h^{-1}\omega h + h^{-1}dh \ . \tag{3.34}$$

The 1-forms $P^a(\phi) = P^a(\phi)_i d\phi^i$ are therefore vielbein on $G/H$ transforming in the fundamental of $SO(m)$, while $\omega = \omega(\phi)_i d\phi^i$ is an $\mathbb{H}$-valued connection 1-form on $G/H$. We can then define the covariant derivative $\nabla P^a = [P, \omega]^a = P^b \otimes -C^a_{\alpha b}\omega^\alpha$.

There is a natural metric on $G/H$,

$$g = \delta_{ab} P^a \otimes P^b \ , \tag{3.35}$$

(this definition is well given because we have shown that the coset representation is via infiniesimal $SO(m)$ rotations). It is easy to see that the connection $\nabla$ is metric compatible, $\nabla g = 0$.

If the coset is furthermore a symmetric coset we have

$$[\mathbb{K}, \mathbb{K}] \subset \mathbb{H} \ ,$$

then the identity $d\Gamma + \Gamma \wedge \Gamma = 0$, that is (the pull-back on $G/H$ of) the Maurer-Cartan equation, in terms of $P$ and $\omega$ reads

$$R + P \wedge P = 0 \ , \tag{3.36}$$
$$dP + P \wedge \omega + \omega \wedge P = 0 \ . \tag{3.37}$$

This last relation shows that $\omega$ is torsion free. Since it is metric compatible it is therefore the Riemannian connection on $G/H$. Equation (3.36) then relates the Riemannian curvature to the square of the vielbeins.

By using the connection $\omega$ and the vierbein $P$ we can construct couplings and actions invariant under the rigid $G$ and the local $H$ transformations, i.e. sigma models on the coset space $G/H$.

For example a kinetic term for the scalar fields, which are maps from spacetime to $G/H$, is given by pulling back to spacetime the invariant metric (3.35) and then contracting it with the spacetime metric

$$\mathscr{L}_{\rm kin}(\phi) = \frac{1}{2}P^a_\mu P^\mu_a = \frac{1}{2}P^a{}_i\partial_\mu\phi^i P_{aj}\partial^\mu\phi^j \ . \tag{3.38}$$

By construction the lagrangian $\mathscr{L}_{\rm kin}(\phi)$ is invariant under $G$ and local $H$ transformations; it depends only on the coordinates of the coset space $G/H$.

### 34.1. The case $G = Sp(2n,\mathbb{R})$, $H = U(n)$

A kinetic term for the $\frac{Sp(2n,\mathbb{R})}{U(n)}$ valued scalar fields is given by (3.38). This lagrangian is invariant under $Sp(2n,\mathbb{R})$ and therefore satisfies the duality condition (3.19) with $G = Sp(2n,\mathbb{R})$ and $\kappa = 0$. We can also write

$$\mathscr{L}_{\rm kin}(\phi) = \frac{1}{2}P^a_\mu P^\mu_a = \frac{1}{2}{\rm Tr}(P_\mu P^\mu) \ ; \tag{3.39}$$

where in the last passage we have considered generators $K_a$ so that ${\rm Tr}(K_aK_b) = \delta_{ab}$ (this is doable since $U(n)$ is the maximal compact subgroup of $Sp(2n,\mathbb{R})$).

In order to obtain a more explicit expression for the lagrangian (3.39) we now consider the so-called complex basis representation of the group $Sp(2n.\mathbb{R})$ and of the associated coset $\frac{Sp(2n,\mathbb{R})}{U(n)}$. Rather than using the symplectic matrix $S = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)$ of the fundamental representation of $Sp(2n,\mathbb{R})$, we consider the conjugate matrix

$$\mathcal{A}^{-1}S\mathcal{A} \qquad \text{where} \qquad \mathcal{A} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ -i\mathbb{1} & i\mathbb{1}\end{pmatrix} \ . \tag{3.40}$$

In this complex basis the subgroup $U(n) \subset Sp(2n,\mathbb{R})$ is simply given by the block diagonal matrices $\left(\begin{smallmatrix} u & 0 \\ 0 & \bar{u}\end{smallmatrix}\right)$. We also define the $n \times 2n$ matrix

$$\begin{pmatrix} f \\ h \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} A - iB \\ C - iD\end{pmatrix} \tag{3.41}$$

and the matrix

$$V = \begin{pmatrix} f & \bar{f} \\ h & \bar{h}\end{pmatrix} = \begin{pmatrix} A & B \\ C & D\end{pmatrix}\mathcal{A} \ . \tag{3.42}$$

Then the symplectic and reality conditions of the matrix $S = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)$ read

$$(f^\dagger, h^\dagger)\begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0\end{pmatrix}\begin{pmatrix} f \\ h\end{pmatrix} = i\mathbb{1} \qquad i.e. \quad -f^\dagger h + h^\dagger f = i\mathbb{1} \tag{3.43}$$

and

$$(f^t, h^t)\begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} f \\ h \end{pmatrix} = 0 \qquad i.e. \quad -f^t h + h^t f = 0 \tag{3.44}$$

and we thus have

$$V^{-1}dV = (\mathcal{A}^{-1}S\mathcal{A})^{-1}d(\mathcal{A}^{-1}S\mathcal{A}) = \begin{pmatrix} i(f^\dagger dh - h^\dagger df) & i(f^\dagger d\bar{h} - h^\dagger d\bar{f}) \\ -i(f^t dh - h^t df) & -i(f^t d\bar{h} - h^t d\bar{f}) \end{pmatrix} \equiv \begin{pmatrix} \omega & \bar{\mathcal{P}} \\ \mathcal{P} & \bar{\omega} \end{pmatrix}, \tag{3.45}$$

where in the last passage we have defined the $n \times n$ sub-blocks $\omega$ and $\mathcal{P}$ corresponding to the $U(n)$ connection and the vielbein of $Sp(2n,\mathbb{R})/U(n)$ in the complex basis, (with slight abuse of notation we use the same letter $\omega$ in this basis too).

We further introduce the matrix

$$\mathcal{N} = fh^{-1} \tag{3.46}$$

If we decompose it into real and imaginary parts, $\mathcal{N} = \mathcal{N}_1 + i\mathcal{N}_2 = \mathrm{Re}\mathcal{N} + i\mathrm{Im}N$, then

$$\mathcal{N}_2^{-1} = -2ff^\dagger\ . \tag{3.47}$$

The matrix of scalars $\mathcal{N}$ can be shown to parametrize the coset space $Sp(2n,\mathbb{R})/U(n)$. Under the symplectic rotation $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \to \left(\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right)\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ the matrix $\mathcal{N}$ changes via the fractional transformation $\mathcal{N} \to (C' + D'\mathcal{N})\,(A' + B'\mathcal{N})^{-1}$, (cf. (**??**)). The transformation of the matrix $\mathcal{N}_2$ is given in (3.22).

We now go back to the kinetic term of the $\frac{Sp(2n,\mathbb{R})}{U(n)}$ valued scalar fields and obtain the explicit expression

$$\mathscr{L}_{\mathrm{kin}}(\phi) = \mathrm{Tr}(\bar{\mathcal{P}}_\mu \mathcal{P}^\mu) = \frac{1}{4}\mathrm{Tr}(\mathcal{N}_2^{-1}\partial_\mu\overline{\mathcal{N}}\,\mathcal{N}_2^{-1}\partial^\mu\mathcal{N}) \tag{3.48}$$

where $\mathcal{P} = \mathcal{P}_\mu dx^\mu = \mathcal{P}_i\partial_\mu\phi^i dx^\mu$, $\overline{\mathcal{N}} = \mathcal{N}_1 - i\mathcal{N}_2$ and $\mathcal{N} = \mathcal{N}_1 + i\mathcal{N}_2 = \mathrm{Re}\mathcal{N} + i\mathrm{Im}\mathcal{N}$.

For future reference we present here also another parametrization of the coset space $Sp(2n,\mathbb{R})/U(n)$, it it given by the symmetric matrices $\mathcal{M}$,

$$\begin{aligned}
\mathcal{M}(\mathcal{N}) &= \begin{pmatrix} \mathbb{1} & -\mathcal{N}_1 \\ 0 & \mathbb{1} \end{pmatrix}\begin{pmatrix} \mathcal{N}_2 & 0 \\ 0 & \mathcal{N}_2^{-1} \end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 \\ -\mathcal{N}_1 & \mathbb{1} \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{N}_2 + \mathcal{N}_1\mathcal{N}_2^{-1}\mathcal{N}_1 & -\mathcal{N}_1\mathcal{N}_2^{-1} \\ -\mathcal{N}_2^{-1}\mathcal{N}_1 & \mathcal{N}_2^{-1} \end{pmatrix} \\
&= -i\begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} + \begin{pmatrix} \mathcal{N}\mathcal{N}_2^{-1}\mathcal{N}^\dagger & -\mathcal{N}\mathcal{N}_2^{-1} \\ -\mathcal{N}_2^{-1}\mathcal{N}^\dagger & \mathcal{N}_2^{-1} \end{pmatrix} \qquad (3.49) \\
&= -i\begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} - 2\begin{pmatrix} hh^\dagger & -hf^\dagger \\ -fh^\dagger & ff^\dagger \end{pmatrix} \\
&= -i\begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} - 2\begin{pmatrix} -h \\ f \end{pmatrix}(-h^\dagger\ f^\dagger) \\
&= -2\,\mathrm{Re}\left[\begin{pmatrix} -h \\ f \end{pmatrix}(-h^\dagger\ f^\dagger)\right]\ . \qquad (3.50)
\end{aligned}$$

## 35. The Generalized Born Infeld theory example

In this section we present the Born-Infeld theory with $n$ abelian gauge fields coupled to $n(n+1)/2$ scalar fields $\mathcal{N}$ and show that is has an $Sp(2n,\mathbb{R})$ duality symmetry. If we freeze the scalar fields $\mathcal{N}$ to the value $\mathcal{N} = -i\mathbb{1}$ then the lagrangian has $U(n)$ duality symmetry and reads

$$\mathcal{L} = \mathrm{Tr}[\mathbb{1} - \mathcal{S}_{\alpha,\beta}\sqrt{\mathbb{1} + 2\alpha - \beta^2}] \ , \tag{3.51}$$

where the components of the $n \times n$ matrices $\alpha$ and $\beta$ are

$$\alpha^{\Lambda\Sigma} = \frac{1}{4} F^\Lambda F^\Sigma, \ \beta^{\Lambda\Sigma} = \frac{1}{4} \widetilde{F}^\Lambda F^\Sigma. \tag{3.52}$$

The square root is to be understood in terms of its power series expansion, and the operator $\mathcal{S}_{\alpha,\beta}$ acts by symmetrizing each monomial in the $\alpha$ and $\beta$ matrices. A world (monomial) in the letters $\alpha$ and $\beta$ is symmetrized by averaging over all permutations of its letters. The normalization of $\mathcal{S}_{\alpha,\beta}$ is such that if $\alpha$ and $\beta$ commute then $\mathcal{S}_{\alpha,\beta}$ acts as the identity. Therefore in the case of just one abelian gauge field (3.51) reduces to the usual Born-Infeld lagrangian.

Following [16] we show the duality symmetry of the Born-Infeld theory (3.51) by first showing that a Born-Infeld theory with $n$ *complex* abelian gauge fields written in an auxiliary field formulation has $U(n,n)$ duality symmetry. Thanks to a remarkable property of solutions of matrix equations [17] the auxiliary fields can be eliminated. Then real fields can also be considered.

### 35.1. Duality rotations with complex field strengths

From the general study of duality rotations we know that a theory with $2n$ real fields $F_1^\Lambda$ and $F_2^\Lambda$ ($\Lambda = 1, \dots n$) has at most $Sp(4n,\mathbb{R})$ duality if we consider duality rotations that leave invariant the energy-momentum tensor (and in particular the hamiltonian). We now consider the complex fields

$$F^\Lambda = F_1^\Lambda + iF_2^\Lambda \quad , \qquad \bar{F}^\Lambda = F_1^\Lambda - iF_2^\Lambda \ , \tag{3.53}$$

the corresponding dual fields

$$G = \frac{1}{2}(G_1 + iG_2) \quad , \qquad \bar{G} = \frac{1}{2}(G_1 - iG_2) \ , \tag{3.54}$$

and restrict the $Sp(4n,\mathbb{R})$ duality group to the subgroup of *holomorphic* transformations,

$$\Delta \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} \mathsf{a} & \mathsf{b} \\ \mathsf{c} & \mathsf{d} \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} \tag{3.55}$$

$$\Delta \begin{pmatrix} \bar{F} \\ \bar{G} \end{pmatrix} = \begin{pmatrix} \bar{\mathsf{a}} & \bar{\mathsf{b}} \\ \bar{\mathsf{c}} & \bar{\mathsf{d}} \end{pmatrix} \begin{pmatrix} \bar{F} \\ \bar{G} \end{pmatrix} \ . \tag{3.56}$$

This requirement singles out those matrices, acting on the vector $\begin{pmatrix} F_1 \\ F_2 \\ G_1 \\ G_2 \end{pmatrix}$, that belong to the Lie algebra of $Sp(4n,\mathbb{R})$ and have the form

$$\begin{pmatrix} \mathcal{A}\left(\begin{smallmatrix} \mathsf{a} & 0 \\ 0 & \bar{\mathsf{a}} \end{smallmatrix}\right)\mathcal{A}^{-1} & \frac{1}{2}\mathcal{A}\left(\begin{smallmatrix} \mathsf{b} & 0 \\ 0 & \bar{\mathsf{b}} \end{smallmatrix}\right)\mathcal{A}^{-1} \\ \\ 2\mathcal{A}\left(\begin{smallmatrix} \mathsf{c} & 0 \\ 0 & \bar{\mathsf{c}} \end{smallmatrix}\right)\mathcal{A}^{-1} & \mathcal{A}\left(\begin{smallmatrix} \mathsf{d} & 0 \\ 0 & \bar{\mathsf{d}} \end{smallmatrix}\right)\mathcal{A}^{-1} \end{pmatrix} \tag{3.57}$$

where $\mathcal{A} = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} \mathbb{1} & \mathbb{1} \\ -i\mathbb{1} & i\mathbb{1} \end{smallmatrix}\right)$. The matrix (3.57) belongs to $Sp(4n,\mathbb{R})$ iff the $n \times n$ complex matrices $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}$ satisfy

$$\mathsf{a}^\dagger = -\mathsf{a} \ , \quad \mathsf{b}^\dagger = \mathsf{b} \ , \quad \mathsf{c}^\dagger = \mathsf{c} \ . \tag{3.58}$$

Matrices $\begin{pmatrix} \mathsf{a} & \mathsf{b} \\ \mathsf{c} & \mathsf{d} \end{pmatrix}$, that satisfy (3.58), define the Lie algebra of the real form $U(n,n)$. The group $U(n,n)$ is here the subgroup of $GL(2n,\mathbb{C})$ characterized by the relations

$$M^\dagger \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} M = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \ , \tag{3.59}$$

setting $U = \mathcal{A}^{-1} M \mathcal{A}$ the matrix $U$ satisfies the condition $U^\dagger\left(\begin{smallmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{smallmatrix}\right)U = \left(\begin{smallmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{smallmatrix}\right)$.

One can check that (3.59) implies the following relations for the block components of $M = \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & \mathsf{D} \end{pmatrix}$,

$$\mathsf{C}^\dagger \mathsf{A} = \mathsf{A}^\dagger \mathsf{C} \ , \quad \mathsf{B}^\dagger \mathsf{D} = \mathsf{D}^\dagger \mathsf{B} \ , \quad \mathsf{D}^\dagger \mathsf{A} - \mathsf{B}^\dagger \mathsf{C} = \mathbb{1} \ . \tag{3.60}$$

The Lie algebra relations (3.58) can be obtained from the Lie group relations (3.60) by writing $\left(\begin{smallmatrix} \mathsf{A} & \mathsf{C} \\ \mathsf{B} & \mathsf{D} \end{smallmatrix}\right) = \left(\begin{smallmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{smallmatrix}\right) + \epsilon\left(\begin{smallmatrix} \mathsf{a} & \mathsf{b} \\ \mathsf{c} & \mathsf{d} \end{smallmatrix}\right)$ with $\epsilon$ infinitesimal. Equation (3.57) gives the embedding of $U(n,n)$ in $Sp(4n,\mathbb{R})$.

The theory of holomorphic duality rotations can be seen as a special case of that of real duality rotations, but (as complex geometry versus real geometry) it deserves also an independent formulation based on the holomorphic variables $\begin{pmatrix} F \\ G \end{pmatrix}$ and maps $\begin{pmatrix} \mathsf{a} & \mathsf{b} \\ \mathsf{c} & \mathsf{d} \end{pmatrix}$.

The dual fields in (3.54), or rather the Hodge dual of the dual field strength, $\widetilde{G}_\Lambda^{\ \mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} G_\Lambda^{\ \rho\sigma}$, is equivalently defined via

$$\tilde{G}_\Lambda^{\ \mu\nu} \equiv 2\frac{\partial \mathcal{L}}{\partial \bar{F}^\Lambda_{\ \mu\nu}} \ , \quad \tilde{\bar{G}}_\Lambda^{\ \mu\nu} \equiv 2\frac{\partial \mathcal{L}}{\partial F^\Lambda_{\ \mu\nu}} \ . \tag{3.61}$$

Similarly to the real field strengths case we have that the Bianchi identities and equations of motion $\partial_\mu \tilde{F}^{\Lambda\,\mu\nu} = 0$ , $\partial_\mu \tilde{G}_\Lambda^{\ \mu\nu} = 0$ , $\frac{\delta S[F,\bar{F},\varphi]}{\delta\varphi^\alpha} = 0$ transform covariantly under the holomorphic infinitesimal transformations (3.55) if the lagrangian satisfies the condition (cf. (3.11))

$$\mathcal{L}(F+\Delta F, \bar{F}+\Delta\bar{F}, \varphi+\Delta\varphi) - \mathcal{L}(F,\bar{F},\varphi) - \frac{1}{2}\tilde{F}\,\mathsf{c}\,\bar{F} - \frac{1}{2}\tilde{G}\,\mathsf{b}\,\bar{G} \ = \ const_{\mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d}} \tag{3.62}$$

The maximal compact subgroup of $U(n,n)$ is $U(n)\times U(n)$ and is obtained by requiring (3.60) and

$$\mathsf{A}=\mathsf{D}\ ,\quad \mathsf{B}=-\mathsf{C}\ .$$

The corresponding infinitesimal relations are (3.58) and $\mathsf{a}=\mathsf{d}\ ,\ \mathsf{b}=-\mathsf{c}$ .

The coset space $\frac{U(n,n)}{U(n)\times U(n)}$ is the space of all negative definite hermitian matrices $\mathcal{M}$ of $U(n,n)$ (see for example [16, 40]) or equivalently of matrices

$$\mathcal{N}\equiv\mathcal{N}_1+i\mathcal{N}_2 \tag{3.63}$$

where $\mathcal{N}_1$ is hermitian and $\mathcal{N}_2$ is hermitian and negative definite.

Since any complex matrix can always be decomposed into hermitian matrices as in (3.63), the only requirement on $\mathcal{N}$ is that $\mathcal{N}_2$ is negative definite. The relation between $\mathcal{M}$ and $\mathcal{N}$ is given in the line (3.49). Under the left action of $U(n,n)$ on itself $g\rightarrow\left(\begin{smallmatrix}\mathsf{A}&\mathsf{B}\\ \mathsf{C}&\mathsf{D}\end{smallmatrix}\right)g$, we have the transformation $\mathcal{M}\rightarrow\left(\begin{smallmatrix}\mathsf{D}&-\mathsf{C}\\ -\mathsf{B}&\mathsf{A}\end{smallmatrix}\right)\mathcal{M}\left(\begin{smallmatrix}\mathsf{D}&-\mathsf{C}\\ -\mathsf{B}&\mathsf{A}\end{smallmatrix}\right)^\dagger$ (that follows from $\mathcal{M}=-{g^\dagger}^{-1}g^{-1}$) and

$$\mathcal{N}\rightarrow\mathcal{N}'=(\mathsf{C}+\mathsf{D}\mathcal{N})\,(\mathsf{A}+\mathsf{B}\mathcal{N})^{-1}\ , \tag{3.64}$$

$$\mathcal{N}_2\rightarrow\mathcal{N}_2'=(\mathsf{A}+\mathsf{B}\mathcal{N})^{-\dagger}\mathcal{N}_2(\mathsf{A}+\mathsf{B}\mathcal{N})^{-1}\ . \tag{3.65}$$

### 35.2. Born-Infeld with auxiliary fields

A lagrangian that satisfies condition (3.62) is

$$\mathcal{L}=\mathrm{Re}\,\mathrm{Tr}\,[\,i(\mathcal{N}-\lambda)\chi-\frac{i}{2}\lambda\chi^\dagger\mathcal{N}_2\chi-i\lambda(\alpha+i\beta)\,]\ , \tag{3.66}$$

where now the $\alpha$ and $\beta$ matrices are the Lorentz invariant combinations

$$\alpha^{ab}\equiv\frac{1}{2}F^a\bar{F}^b,\quad \beta^{ab}\equiv\frac{1}{2}\widetilde{F}^a\bar{F}^b. \tag{3.67}$$

The auxiliary fields $\chi$ and $\lambda$ and the scalar field $\mathcal{N}$ are $n$ dimensional complex matrices. We can also add to the lagrangian a duality invariant kinetic term for the scalar field $\mathcal{N}$, (cf (3.48))

$$\mathrm{Tr}(\mathcal{N}_2^{-1}\partial_\mu\mathcal{N}^\dagger\,\mathcal{N}_2^{-1}\partial^\mu\mathcal{N})\ . \tag{3.68}$$

In order to prove the duality of (3.66) we first note that the last term in the Lagrangian can be written as

$$-\mathrm{Re}\,\mathrm{Tr}\,[\,i\lambda(\alpha+i\beta)\,]=-\mathrm{Tr}(\lambda_2\alpha+\lambda_1\beta)\ .$$

If the field $\lambda$ transforms by fractional transformation and $\lambda_1$, $\lambda_2$ and the gauge fields are real this is the $U(1)^n$ Maxwell action (3.16), with the gauge fields interacting with the scalar field $\lambda$. This term by itself has the correct transformation properties under the duality group. Similarly for hermitian $\alpha$, $\beta$, $\lambda_1$ and $\lambda_2$ this term by itself satisfies equation (3.62). It follows that the rest of the Lagrangian must be duality invariant. The duality transformations of the scalar and auxiliary fields are[6]

$$\lambda' = (\mathsf{C} + \mathsf{D}\lambda)\,(\mathsf{A} + \mathsf{B}\lambda)^{-1}\ , \tag{3.69}$$

$$\chi' = (\mathsf{A} + \mathsf{B}\mathcal{N})\chi(\mathsf{A} + \mathsf{B}\lambda^\dagger)^\dagger\ , \tag{3.70}$$

and (3.64). Invariance of $\mathrm{Tr}[i(\mathcal{N} - \lambda)\chi]$ is easily proven by using (3.60) and by rewriting (3.69) as

$$\lambda' = (\mathsf{A} + \mathsf{B}\lambda^\dagger)^{-\dagger}\,(\mathsf{C} + \mathsf{D}\lambda^\dagger)^\dagger\ . \tag{3.71}$$

Invariance of the remaining term which we write as $\mathrm{Re\,Tr}\,[-\frac{i}{2}\lambda\chi^\dagger\mathcal{N}_2\chi] = \mathrm{Tr}\,[\frac{1}{2}\lambda_2\chi^\dagger\mathcal{N}_2\chi]$ , is straightforward by using (3.65) and the following transformation obtained from (3.71),

$$\lambda_2' = (\mathsf{A} + \mathsf{B}\lambda^\dagger)^{-\dagger}\lambda_2(\mathsf{A} + \mathsf{B}\lambda^\dagger)^{-1}\ . \tag{3.72}$$

### 35.3. Elimination of the Auxiliary Fields

The equation of motion obtained by varying $\lambda$ gives an equation for $\chi$,

$$\chi + \frac{1}{2}\chi^\dagger\mathcal{N}_2\chi + \alpha + i\beta = 0\ , \tag{3.73}$$

using this equation in the Lagrangian (3.66) we obtain

$$\mathcal{L} = \mathrm{Re\,Tr}\ (i\mathcal{N}\chi) \tag{3.74}$$

$$= \mathrm{Re\,Tr}\ (-\mathcal{N}_2\chi) + \mathrm{Tr}\,(\mathcal{N}_1\beta)\ , \tag{3.75}$$

where $\chi$ is now a function of $\alpha$, $\beta$ and $\mathcal{N}_2$ that solves (3.73). In the second line we observed that the anti-hermitian part of (3.73) implies $\chi_2 = -\beta$.

In order to obtain an explicit expression of $\mathcal{L}$ in terms of $\alpha$, $\beta$ and $\mathcal{N}$ we set

$$\widehat{\chi} = R\chi R^\dagger\ , \qquad \widehat{\alpha} = R\alpha R^\dagger\ , \qquad \widehat{\beta} = R\beta R^\dagger\ , \tag{3.76}$$

where, $R^\dagger R = -\mathcal{N}_2$. The equation of motion for $\chi$ is then equivalent to

$$\widehat{\chi} - \frac{1}{2}\widehat{\chi}^\dagger\widehat{\chi} + \widehat{\alpha} - i\widehat{\beta} = 0\ . \tag{3.77}$$

The anti-hermitian part of (3.77) implies $\widehat{\chi}_2 = -\widehat{\beta}$, thus $\widehat{\chi}^\dagger = \widehat{\chi} - 2i\widehat{\beta}$. This can be used to eliminate $\widehat{\chi}^\dagger$ from (3.77) and obtain the quadratic equation for $\widehat{\chi}$,

$$\widehat{\chi} = -\widehat{\alpha} + i\widehat{\beta} - i\widehat{\beta}\widehat{\chi} + \frac{1}{2}\widehat{\chi}^2\ . \tag{3.78}$$

A key point is that this equation is a unilateral matrix equation, i.e. the coefficients, that are matrices, appear for example only on the left hand side, like $a_0 + a_1\widehat{\chi} +$

[6] In [16] we use different notations: $\mathcal{N} \to S^\dagger, \lambda \to \lambda^\dagger, \chi \to \chi^\dagger, \left(\begin{smallmatrix}\mathsf{A} & \mathsf{B}\\ \mathsf{C} & \mathsf{D}\end{smallmatrix}\right) \to \left(\begin{smallmatrix}\mathsf{D} & \mathsf{C}\\ \mathsf{B} & \mathsf{A}\end{smallmatrix}\right)$.

$\ldots a_n \widehat{\chi}^n = 0$. The trace $Tr(\widehat{\chi})$ of the solution of unilateral matrix equations is an expression symmetric in the coefficients $a_0, a_1, \ldots a_n$ [17], see also [42] and [43].

The trace of the solution of equation (3.78) gives the explicit expression of the lagrangian $\mathcal{L} = \mathrm{Re}\,\mathrm{Tr}\,(-\widehat{\chi}) + \mathrm{Tr}\,(\mathcal{N}_1\beta)$, in terms of $\alpha$, $\beta$ and $\mathcal{N}$,

$$\mathcal{L} = \mathrm{Tr}[\mathbb{1} - \mathcal{S}_{\alpha,\beta}\sqrt{\mathbb{1} + 2\widehat{\alpha} - \widehat{\beta}^2} + \mathcal{N}_1\beta]\ . \tag{3.79}$$

The right hand side formula is understood this way: first expand the square root as a power series in $\widehat{\alpha}$ and $\widehat{\beta}$ assuming that $\widehat{\alpha}$ and $\widehat{\beta}$ commute. Then solve the ordering ambiguities arising from the noncommutativity of $\widehat{\alpha}$ and $\widehat{\beta}$ by symmetrizing, with the operator $\mathcal{S}_{\widehat{\alpha},\widehat{\beta}}$, each monomial in the $\widehat{\alpha}$ and $\widehat{\beta}$ matrices. A world (monomial) in the letters $\widehat{\alpha}$ and $\widehat{\beta}$ is symmetrized by considering the sum of all the permutations of its letters, then normalize the sum by dividing by the number of permutations. This normalization of $\mathcal{S}_{\widehat{\alpha},\widehat{\beta}}$ is such that if $\widehat{\alpha}$ and $\widehat{\beta}$ commute then $\mathcal{S}_{\widehat{\alpha},\widehat{\beta}}$ acts as the identity. Therefore in the case of just one abelian gauge field (3.51) reduces to the usual Born-Infeld lagrangian.

Let's now come back to the case of real fields strengths. It can be shown that if we set $\widehat{\alpha} = R\alpha R^t$, $\widehat{\beta} = R\beta R^t$, $\mathcal{N}_2 = -R^tR$, where now $R$ is a real matrix, and $\alpha^{\Lambda\Sigma} = \frac{1}{4}F^\Lambda F^\Sigma$, $\beta^{\Lambda\Sigma} = \frac{1}{4}\tilde{F}^\Lambda F^\Sigma$ as in (3.52), then the lagrangian (3.79) that depends on $n$ real field strengths $F^\Lambda$ and the $n(n+1)$ real scalar fields $\mathcal{N}$ is self dual with $Sp(2n,\mathbb{R})$ duality group.

### 35.4. Supersymmetric Theory

In this section we briefly discuss supersymmetric versions of some of the Lagrangians introduced. First we discuss the supersymmetric form of the Lagrangian (3.66). Consider the superfields $V^\Lambda = \frac{1}{\sqrt{2}}(V_1^\Lambda + iV_2^\Lambda)$ and $\check{V}^\Lambda = \frac{1}{\sqrt{2}}(V_1^\Lambda - iV_2^\Lambda)$ where $V_1^\Lambda$ and $V_2^\Lambda$ are real vector superfields, and define

$$W_\alpha^\Lambda = -\frac{1}{4}\bar{D}^2 D_\alpha V^\Lambda\ , \quad \check{W}_\alpha^\Lambda = -\frac{1}{4}\bar{D}^2 D_\alpha \check{V}^\Lambda\ .$$

Both $W^\Lambda$ and $\check{W}^\Lambda$ are chiral superfields and can be used to construct a matrix of chiral superfields

$$M^{\Lambda\Sigma} \equiv W^\Lambda \check{W}^\Sigma\ .$$

The supersymmetric version of the Lagrangian (3.66) is then given by

$$\mathcal{L} = \mathrm{Re}\int d^2\theta \left[\mathrm{Tr}\,(i(\mathcal{N}-\lambda)\chi - \frac{i}{2}\lambda\bar{D}^2(\chi^\dagger\mathcal{N}_2\chi) + i\lambda M)\right]\ ,$$

where $\mathcal{N}$, $\lambda$ and $\chi$ denote chiral superfields with the same symmetry properties as their corresponding bosonic fields. While the bosonic fields $\mathcal{N}$ and $\lambda$ appearing in (3.66) are the lowest component of the superfields denoted by the same letter, the field $\chi$ in the action (3.66) is the highest component of the superfield $\chi$. A supersymmetric kinetic term for the scalar field $\mathcal{N}$ can also be written [44].

Just as in the bosonic Born-Infeld theory, one would like to eliminate the auxiliary fields. This is an open problem if $n \neq 1$. For $n = 1$ just as in the bosonic case the theory with auxiliary fields also admits both a real and a complex version, i.e. one can also consider a Lagrangian with a single real superfield. Then by integrating out the auxiliary superfields the supersymmetric version of the Born-Infeld lagrangian is obtained

$$\mathcal{L} = \int d^4\theta \frac{\mathcal{N}_2^2 W^2 \bar{W}^2}{1 + A + \sqrt{1 + 2A + B^2}} + \mathrm{Re}\left[\int d^2\theta (\frac{i}{2}\mathcal{N} W^2)\right] \,, \tag{3.80}$$

where

$$A = \frac{1}{4}(D^2(\mathcal{N}_2 W^2) + \bar{D}^2(\mathcal{N}_2 \bar{W}^2)) \,, \quad B = \frac{1}{4}(D^2(\mathcal{N}_2 W^2) - \bar{D}^2(\mathcal{N}_2 \bar{W}^2)) \,.$$

If we only want a $U(1)$ duality invariance we can set $\mathcal{N} = -i$ and then the lagrangian (3.80) reduces to the supersymmetric Born-Infeld lagrangian described in [45, 46, 47].

In the case of weak fields the first term of (3.80) can be neglected and the Lagrangian is quadratic in the field strengths. Under these conditions the combined requirements of supersymmetry and self duality can be used [48] to constrain the form of the weak coupling limit of the effective Lagrangian from string theory. Self-duality of Born-Infeld theories with $N = 2$ supersymmetries is discussed in [22].

## 4 Dualities in extended Supergravities

Four dimensional $N$-extended supergravities contain in the bosonic sector, besides the metric, a number $n$ of vectors and $m$ of (real) scalar fields. The relevant bosonic action is known to have the following general form:

$$\begin{aligned} \mathcal{S} = \frac{1}{4}\int \sqrt{-g}\, d^4x \Bigg( & -\frac{1}{2}\, R + \mathrm{Im}\, \mathcal{N}_{\Lambda\Gamma} F^{\Lambda}_{\mu\nu} F^{\Gamma\, \mu\nu} + \frac{1}{2\sqrt{-g}}\, \mathrm{Re}\, \mathcal{N}_{\Lambda\Gamma} \epsilon^{\mu\nu\rho\sigma}\, F^{\Lambda}_{\mu\nu} F^{\Gamma}_{\rho\sigma} + \\ & + \frac{1}{2}\, g_{ij}(\phi) \partial_\mu \phi^i \partial^\mu \phi^j \Bigg) \,, \end{aligned} \tag{4.1}$$

where $g_{ij}(\phi)$ $(i, j, \cdots = 1, \cdots, m)$ is the scalar metric on the $\sigma$-model described by the scalar manifold $M_{scalar}$ of real dimension $m$ and the vectors kinetic matrix $\mathcal{N}_{\Lambda\Sigma}(\phi)$ is a complex, symmetric, $n \times n$ matrix depending on the scalar fields. The number of vectors and scalars, namely $n$ and $m$, and the geometric properties of the scalar manifold $M_{scalar}$ depend on the number $N$ of supersymmetries and for $N > 2$ are summarized in Table 1.

The duality group of these theories is in general not the maximal one $Sp(2n, \mathbb{R})$ because the requirement of supersymmetry constraints the number and the geometry of the scalar fields in the theory. We first study the case where the scalar fields manifold is a coset space $G/H$, and we see that the duality group in this case is $G$. We then briefly describe the general $N = 2$ case where the target space is a special Kähler manifold $M$ and thus in general we do not have a coset space. In both cases the symplectic geometry dictates the structure of the scalar kinetic term, of the supersymmetry transformations and of the charges and their invariant combinations. In the general $N = 2$ case however the $Sp(2n, \mathbb{R})$ transformations are needed in order to globally define the supergravity theory. We do not have a duality symmetry of the theory; $Sp(2n, \mathbb{R})$ is rather a gauge symmetry of the theory, in the sense that only $Sp(2n, \mathbb{R})$ invariant expressions are physical ones.

The case of duality rotations in $N = 1$ supergravity is considered in [7], [49], see also [23]. In this case there is no vector potential in the graviton multiplet hence no scalar central charge in the supersymmetry algebra. Duality symmetry is due to the number of matter vector multiplets in the theory, the coupling to eventual chiral multiplets must be via a kinetic matrix $\mathcal{N}$ holomorphic in the chiral fields. We see that the structure of duality rotations is similar to that of $N = 1$ rigid supersymmetry. For duality rotations in $N = 1$ and $N = 2$ rigid supersymmetry using superfields see the review [22].

## 41. Extended supergravities with target space $G/H$

In $N \geq 2$ supergravity theories where the scalars target space is a coset $G/H$, the scalar sector has a Lagrangian invariant under the global $G$ rotations. Since the scalars appear in supersymmetry multiplets the symmetry $G$ should be a symmetry of the whole theory. This is indeed the case and the symmetry on the vector potentials is duality symmetry.

Let's examine the gauge sector of the theory. We recall from Section 3.1 that we have an $Sp(2n, \mathbb{R})$ duality group if the vector $\binom{F}{G}$ transforms in the fundamental of $Sp(2n, \mathbb{R})$, and the gauge kinetic term $\mathcal{N}$ transforms via fractional transformations, if $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n, \mathbb{R})$,

$$\mathcal{N} \to \mathcal{N}' = (C + D\mathcal{N})\,(A + B\mathcal{N})^{-1} \ . \tag{4.2}$$

Thus in order to have $G$ duality symmetry, $G$ needs to act on the vector $\binom{F}{G}$ via symplectic transformations, i.e. via matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in the fundamental of $Sp(2n, \mathbb{R})$. This requires a homomorphism

$$S : G \to Sp(2n, \mathbb{R}) \ . \tag{4.3}$$

Different infinitesimal $G$ transformations should correspond to different infinitesimal symplectic rotations so that the induced map $\mathrm{Lie}(G) \to \mathrm{Lie}(Sp(2n, \mathbb{R}))$ is injective, and equivalently the homomorphism $S$ is a local embedding (in general $S$ it is not globally injective, the kernel of $S$ may contain some discrete subgroups of $G$).

Since $U(n)$ is the maximal compact subgroup of $Sp(2n, \mathbb{R})$ and since $H$ is compact, we have that the image of $H$ under this local embedding is in $U(n)$. It follows that we have a $G$-equivariant map

$$\mathcal{N} : G/H \to Sp(2n, \mathbb{R})/U(n) \ , \tag{4.4}$$

explicitly, for all $g \in G$,

$$\mathcal{N}(g\phi) = (C + D\mathcal{N}(\phi))\,(A + B\mathcal{N}(\phi))^{-1} \ , \tag{4.5}$$

where with $g\phi$ we denote the action of $G$ on $G/H$, while the action of $G$ on $Sp(2n, \mathbb{R})/U(n)$ is given by fractional transformations via the matrix $S(g) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Notice that we have identified $Sp(2n,\mathbb{R})/U(n)$ with the space of complex symmetric matrices $\mathcal{N}$ that have imaginary part $\mathrm{Im}\mathcal{N} = -i(\mathcal{N} - \overline{\mathcal{N}})/2$ negative definite.

The $D = 4$ supergravity theories with $N > 2$ have all target space $G/H$, they are characterized by the number $n$ of total vectors, the number $N$ of supersymmetries, and the coset space $G/H$, see Table 1 [7].

Table 1: *Scalar Manifolds of $N > 2$ Extended Supergravities*

| N | Duality group $G$ | isotropy $H$ | $M_{scalar}$ | $n$ | $m$ |
|---|---|---|---|---|---|
| 3 | $SU(3,n')$ | $S(U(3)\times U(n'))$ | $\frac{SU(3,n')}{S(U(3)\times U(n'))}$ | $3+n'$ | $6n'$ |
| 4 | $SU(1,1)\times SO(6,n')$ | $U(1)\times S(O(6)\times O(n'))$ | $\frac{SU(1,1)}{U(1)}\times\frac{SO(6,n')}{S(O(6)\times O(n'))}$ | $6+n'$ | $6n'+2$ |
| 5 | $SU(5,1)$ | $S(U(5)\times U(1))$ | $\frac{SU(5,1)}{S(U(5)\times U(1))}$ | 10 | 10 |
| 6 | $SO^\star(12)$ | $U(6)$ | $\frac{SO^\star(12)}{U(6)}$ | 16 | 30 |
| $7,8$ | $E_{7(7)}$ | $SU(8)/\mathbb{Z}_2$ | $\frac{E_{7(7)}}{SU(8)/\mathbb{Z}_2}$ | 28 | 70 |

*In the table, $n$ stands for the number of vectors and $m = \dim M_{scalar}$ for the number of real scalar fields. In all the cases the duality group $G$ is (locally) embedded in $Sp(2\,n,\mathbb{R})$. The number $n$ of vector potentials of the theory is given by $n = n_g + n'$ where $n'$ is the number of vectors potentials in the matter multiplet while $n_g$ is the number of graviphotons (i.e. of vector potentials that belong to the graviton multiplet). We recall that $n_g = \frac{N(N-1)}{2}$ if $N \neq 6$ ; and $n_g = \frac{N(N-1)}{2} + 1 = 16$ if $N = 6$ ; we also have $n' = 0$ if $N > 4$. The scalar manifold of the $N = 4$ case is usually written as $SO_o(6,n')/SO(6)\times SO(n')$ where $SO_o(6,n')$ is the component of $SO(6,n')$ connected to the identity. The duality group of the $N = 6$ theory is more precisely the double cover of $SO^*(12)$. Spinors fields transform according to $H$ or its double cover.*

In general the isotropy group $H$ is the product

$$H = H_{\mathrm{Aut}} \times H_{\mathrm{matter}} \tag{4.6}$$

where $H_{\mathrm{Aut}}$ is the automorphism group of the supersymmetry algebra, while $H_{\mathrm{matter}}$ depends on the matter vector multiplets, that are not present in $N > 4$ supergravities.

In Section 3.5 we have described the geometry of the coset space $G/H$ in terms of coset representatives, local sections $L$ of the bundle $G \to G/H$. Under a left action of $G$ they transform as $gL(\phi) = L(\phi')h$, where the $g$ action on $\phi \in G/H$ gives the point $\phi' \in G/H$.

We now recall that duality symmetry is implemented by the symplectic embeddings (4.3) and (4.4) and conclude that the embeddings of the coset representatives $L$ in

[7] In Table 1 the group $S(U(p)\times U(q))$ is the group of block diagonal matrices $\left(\begin{smallmatrix} P & 0 \\ 0 & Q\end{smallmatrix}\right)$ with $P \in U(p)$, $Q \in U(q)$ and $\det P \det Q = 1$. There is a local isomorphism between $S(U(p)\times U(q))$ and the direct product group $U(1)\times SU(p)\times SU(q)$, in particular the corresponding Lie algebras coincide. Globally these groups are not the same, for example $S(U(5)\times U(1)) = U(5) = U(1)\times PSU(5) \neq U(1)\times SU(5)$.

$Sp(2n,\mathbb{R})$ will play a central role. Recalling (3.41) these embeddings are determined by defining

$$L \to f(L) \quad \text{and} \quad L \to h(L)\ . \tag{4.7}$$

In the following we see that the matrices $f(L)$ and $h(L)$ determine the scalar kinetic term $\mathcal{N}$, the supersymmetry transformation rules and the structure of the central and matter charges of the theory. We also derive the differential equations that these charges satisfy and consider their positive definite and duality invariant quadratic expression $\mathscr{V}_{BH}$. These relations are similar to the Special Geometry ones of $N=2$ supergravity.

From the equation of motion

$$dF^\Lambda = 4\pi j_m^\Lambda \tag{4.8}$$

$$dG_\Lambda = 4\pi j_{e\Lambda} \tag{4.9}$$

We associate with a field strength 2-form $F$ a magnetic charge $p^\Lambda$ and an electric charge $q_\Lambda$ given respectively by:

$$p^\Lambda = \frac{1}{4\pi}\int_{S^2} F^\Lambda \quad , \qquad q_\Lambda = \frac{1}{4\pi}\int_{S^2} G_\Lambda \tag{4.10}$$

where $S^2$ is a spatial two-sphere containing these electric and magnetic charges. These are not the only charges of the theory, in particular we are interested in the central charges of the supersymmetry algebra and other charges related to the vector multiplets. These latter charges result to be the electric and magnetic charges $p^\Lambda$ and $q_\Lambda$ dressed with the scalar fields of the theory. In particular these dressed charges are invariant under the duality group $G$ and transform under the isotropy subgroup $H = H_{Aut} \times H_{matter}$.

According to the transformation of the coset representative $gL(\phi) = L(\phi')h$, under the action of $g \in G$ on $G/H$ we have

$$S(\phi)\mathcal{A} \to S(\phi')\mathcal{A} = S(g)S(\phi)S(h^{-1})\mathcal{A} = S(g)S(\phi)\mathcal{A}U^{-1} \tag{4.11}$$

where $\mathcal{A} = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} \mathbb{1} & \mathbb{1} \\ -i\mathbb{1} & i\mathbb{1}\end{smallmatrix}\right)$ is unitary and symplectic, $S(g) = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)$ and $S(h)$ are the embeddings of $g$ and $h$ in the fundamental of $Sp(2n,\mathbb{R})$, while $U = \mathcal{A}^{-1}S(h)\mathcal{A}$ is the embedding of $h$ in the complex basis of $Sp(2n,\mathbb{R})$. Explicitly

$$U = \begin{pmatrix} \bar{u} & 0 \\ 0 & u \end{pmatrix}$$

where $u$ is in the fundamental of $U(n)$. Therefore the symplectic matrix

$$V = S\mathcal{A} = \begin{pmatrix} f & \bar{f} \\ h & \bar{h} \end{pmatrix} \tag{4.12}$$

transforms according to

$$V(\phi) \to V(\phi') = S(g)V(\phi)\begin{pmatrix} \bar{u}^{-1} & 0 \\ 0 & u^{-1} \end{pmatrix}\ . \tag{4.13}$$

The dressed field strengths transform only under a unitary representation of $H$ and, in accordance with (4.13), are given by [9]

$$\begin{pmatrix} T \\ -\bar{T} \end{pmatrix} = -i\,\overline{V(\phi)}^{-1} \begin{pmatrix} F \\ G \end{pmatrix} \; ; \tag{4.14}$$

$$T \to uT \; . \tag{4.15}$$

Explicitly, since

$$-i\,\bar{V}^{-1} = \begin{pmatrix} h^t & -f^t \\ -h^\dagger & f^\dagger \end{pmatrix} \tag{4.16}$$

we have

$$\begin{aligned} T_{AB} &= h_{\Lambda\,AB} F^\Lambda - f^\Lambda{}_{AB} G_\Lambda \\ \bar{T}_{\bar{I}} &= \bar{h}_{\Lambda\bar{I}} F^\Lambda - \bar{f}^\Lambda_{\bar{I}} G_\Lambda \end{aligned} \tag{4.17}$$

where we used the notation $T = (T^{\bar{M}}) = (T_M) = (T_{AB}, \bar{T}_{\bar{I}})$,

$$\begin{aligned} f &= (f^\Lambda{}_M) = (f^\Lambda{}_{AB}, \bar{f}^\Lambda{}_{\bar{I}}) \; , \\ h &= (h_{\Lambda M}) = (h_{\Lambda AB}, \bar{h}_{\Lambda\bar{I}}) \; . \end{aligned} \tag{4.18}$$

While the index $\Lambda$ is used for the fundamental representation of $Sp(2n;R)$ the index $M$ is used for that of $U(n)$. According to the local embedding

$$H = H_{Aut} \times H_{matter} \to U(n) \tag{4.19}$$

the index $M$ is further divided as $M = (AB, \bar{I})$ where $\bar{I}$ refers to $H_{matter}$ and $AB = -BA \quad (A = 1, \ldots, N)$ labels the two-times antisymmetric representation of the $R$-symmetry group $H_{Aut}$. We can understand the appearence of this representation of $H_{Aut}$ because this is a typical representation acting on the central charges.

From (4.17) the central charges are

$$Z_{AB} = -\frac{1}{4\pi}\int_{S^2_\infty} T_{AB} = f^\Lambda{}_{AB} q_\Lambda - h_{\Lambda\,AB} p^\Lambda \tag{4.20}$$

$$\bar{Z}_{\bar{I}} = -\frac{1}{4\pi}\int_{S^2_\infty} \bar{T}_{\bar{I}} = \bar{f}^\Lambda_{\bar{I}} q_\Lambda - \bar{h}_{\Lambda\bar{I}} p^\Lambda \tag{4.21}$$

where the integral is considered at spatial infinity and, for spherically symmetric configurations, $f$ and $h$ in (4.20), (4.21) are $f(\phi_\infty)$ and $h(\phi_\infty)$ with $\phi_\infty$ the constant value assumed by the scalar fields at spatial infinity.

The integral of the graviphotons $T_{AB\,\mu\nu}$ gives the value of the central charges $Z_{AB}$ of the supersymmetry algebra, while by integrating the matter field strengths $T_{I\,\mu\nu}$ one obtains the so called matter charges $Z_I$.

The dressed graviphotons field strength 2-forms $T_{AB}$ enter the supersymmetry transformation law of the gravitino field in the interacting theory, namely:

$$\delta\psi_A = \nabla\varepsilon_A + \alpha T_{AB\,\mu\nu}\gamma^a\gamma^{\mu\nu}\varepsilon^B V_a + \ldots \tag{4.22}$$

Here $\nabla$ is the covariant derivative in terms of the space-time spin connection and the composite connection of the automorphism group $H_{Aut}$, $\alpha$ is a coefficient fixed by supersymmetry, $V^a$ is the space-time vielbein. Here and in the following the dots denote trilinear fermion terms which are characteristic of any supersymmetric theory but do not play any role in the following discussion. The dressed fields $T_{AB}$ enter also the dilatino transformation law,

$$\delta\chi_{ABC} = \mathcal{P}_{ABCD\,\ell}\partial_\mu\phi^\ell\gamma^\mu\varepsilon^D + \beta T_{[AB\,\mu\nu}\gamma^{\mu\nu}\varepsilon_{C]} + \ldots \tag{4.23}$$

Analogously, when vector multiplets are present, the matter vector field strengths $T_I$ appearing in the transformation laws of the gaugino fields, are linear combinations of the field strengths dressed with a different combination of the scalars:

$$\delta\lambda_{IA} = i\mathcal{P}_{I\,AB\,r}\partial_\mu\phi^r\gamma^\mu\varepsilon^B + \gamma T_{I\,\mu\nu}\gamma^{\mu\nu}\varepsilon_A + \ldots \tag{4.24}$$

Here $\mathcal{P}_{ABCD} = \mathcal{P}_{ABCD\,\ell}\,d\phi^\ell$ and $\mathcal{P}^I_{AB} = \mathcal{P}^I_{AB\,r}\,d\phi^r$ are the vielbein of the scalar manifolds spanned by the scalar fields $\phi^i = (\phi^\ell, \phi^r)$ of the gravitational and vector multiplets respectively (more precise definitions are given below), and $\beta$ and $\gamma$ are constants fixed by supersymmetry.

The charges $Z_{AB}, Z_I$ of these dressed field strength have a profound meaning and play a key role in the physics of extremal black holes. In particular, recalling (4.13) the quadratic combination (black hole potential)

$$\mathscr{V}_{BH} := \frac{1}{2}\bar{Z}^{AB}Z_{AB} + \bar{Z}^I Z_I \tag{4.25}$$

(the factor 1/2 is due to our summation convention that treats the $AB$ indices as independent) is invariant under the symmetry group $G$. In terms of the charge vector

$$Q = \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} , \tag{4.26}$$

we have the formula for the potential (also called charges sum rule)

$$\mathscr{V}_{BH} = \frac{1}{2}\bar{Z}^{AB}Z_{AB} + \bar{Z}^I Z_I = -\frac{1}{2}Q^t\mathcal{M}(\mathcal{N})Q \tag{4.27}$$

where

$$\mathcal{M}(\mathcal{N}) = -(i\bar{V}^{-1})^\dagger i\bar{V}^{-1} = -(S^{-1})^t S^{-1} \tag{4.28}$$

is a negative definite matrix, here depending on $\phi_\infty$. The relation between $\mathcal{M}(\mathcal{N})$ and $\mathcal{N}$ was given in (3.49).

We now derive some differential relations among the central and matter charges. We recall the symmetric coset space geometry $G/H$ studied in Section 3.5, and in particular relations (3.36), (3.37) that express the Maurer-Cartan equation $d\Gamma + \Gamma\wedge\Gamma = 0$ in terms of the vielbein $P$ and of the Riemannian connection $\omega$. Using the (local) embedding of $G$ in $Sp(2n,\mathbb{R})$ we consider the pull back on $G/H$ of the $Sp(2n,\mathbb{R})$ Lie algebra left invariant one form $V^{-1}dV$ given in (3.45), we have

$$V^{-1}dV = \begin{pmatrix} i(f^\dagger dh - h^\dagger df) & i(f^\dagger d\bar{h} - h^\dagger d\bar{f}) \\ -i(f^t dh - h^t df) & -i(f^t d\bar{h} - h^t d\bar{f}) \end{pmatrix} = \begin{pmatrix} \omega & \bar{\mathcal{P}} \\ \mathcal{P} & \bar{\omega} \end{pmatrix} , \tag{4.29}$$

where with slight abuse of notation we use the same letters $V$, $\mathcal{P}$ and $\omega$ for the pulled back forms (we also recall that $\mathcal{P}$ denotes $P$ in the complex basis). Relation (4.29) equivalently reads

$$dV = V \begin{pmatrix} \omega & \bar{\mathcal{P}} \\ \mathcal{P} & \bar{\omega} \end{pmatrix} , \tag{4.30}$$

that is equivalent to the $n \times n$ matrix equations:

$$\nabla f = \bar{f}\,\mathcal{P} , \tag{4.31}$$

$$\nabla h = \bar{h}\,\mathcal{P} , \tag{4.32}$$

where

$$\nabla f = df - f\omega \quad , \qquad \nabla h = dh - h\omega \; . \tag{4.33}$$

Using the definition of the charges we then get the differential relations among charges:

$$\nabla Z_M = \bar{Z}_{\bar{N}} \mathcal{P}^{\bar{N}}{}_M , \tag{4.34}$$

where $\nabla Z_M = \frac{\partial Z_M}{\partial \phi^i_\infty} d\phi^i_\infty - Z_N \omega^N{}_M$, with $\phi^i_\infty$ the value of the $i$-th coordinate of $\phi_\infty \in G/H$ and $\phi_\infty = \phi(r = \infty)$.

It is useful to rewrite (4.31), (4.32), (4.34) with $AB$ and $\bar{I}$ indices. The embedded connection $\omega$ and vielbein $\mathcal{P}$ are decomposed as follows:

$$\omega = (\omega^N{}_M) = \begin{pmatrix} \omega^{AB}{}_{CD} & 0 \\ 0 & \omega^{\bar{I}}{}_{\bar{J}} \end{pmatrix} , \tag{4.35}$$

$$\mathcal{P} = (\mathcal{P}^{\bar{N}}{}_M) = (\mathcal{P}_{NM}) = \begin{pmatrix} \mathcal{P}^{\bar{A}\bar{B}}{}_{CD} & \mathcal{P}^{\bar{A}\bar{B}}{}_{\bar{J}} \\ \mathcal{P}^{\bar{I}}{}_{CD} & \mathcal{P}^{\bar{I}}{}_{\bar{J}} \end{pmatrix} = \begin{pmatrix} \mathcal{P}_{ABCD} & \mathcal{P}_{AB\bar{J}} \\ \mathcal{P}_{ICD} & \mathcal{P}_{I\bar{J}} \end{pmatrix} , \tag{4.36}$$

the subblocks being related to the vielbein of $G/H$, written in terms of the indices of $H_{Aut} \times H_{matter}$. We used the following indices conventions:

$$f = (f^\Lambda{}_M) \quad , \quad f^{-1} = (f^M{}_\Lambda) = (f_{\bar{M}\Lambda}) \quad \text{etc.} \tag{4.37}$$

where in the last passage, since we are in $U(n)$, we have lowered the index $M$ with the $U(n)$ hermitian form $\eta = (\eta_{M\bar{N}})_{M,N=1,\dots n} = diag(1, 1, \dots .1)$. Similar conventions hold for the $AB$ and $I$ indices, for example $\overline{f^\Lambda{}_I} = \bar{f}^\Lambda{}_{\bar{I}} = \bar{f}^{\Lambda I}$.

Using further the index decomposition $M = (AB, \bar{I})$, relations (4.31), (4.32) read (the factor 1/2 is due to our summation convention that treats the $AB$ indices as independent):

$$\nabla f^{\Lambda}_{AB} = \frac{1}{2}\bar{f}^{\Lambda CD}\mathcal{P}_{CDAB} + f^{\Lambda}_{\ I}\mathcal{P}^{I}_{\ AB} \ , \tag{4.38}$$

$$\nabla h^{\Lambda}_{AB} = \frac{1}{2}\bar{h}^{\Lambda CD}\mathcal{P}_{CDAB} + h^{\Lambda}_{\ I}\mathcal{P}^{I}_{\ AB} \ , \tag{4.39}$$

$$\nabla f^{\Lambda}_{\ \bar{I}} = \frac{1}{2}\bar{f}^{\Lambda CD}\mathcal{P}_{CD\bar{I}} + f^{\Lambda\bar{J}}\mathcal{P}_{\bar{J}\bar{I}} \ , \tag{4.40}$$

$$\nabla h^{\Lambda}_{\ \bar{I}} = \frac{1}{2}\bar{h}^{\Lambda CD}\mathcal{P}_{CD\bar{I}} + h^{\Lambda\bar{J}}\mathcal{P}_{\bar{J}\bar{I}} \ . \tag{4.41}$$

As we will see, depending on the coset manifold, some of the sub-blocks of (4.36) can be actually zero. For $N > 4$ (no matter indices) we have that $\mathcal{P}$ coincides with the vielbein $\mathcal{P}_{ABCD}$ of the relevant $G/H$.

Using the $AB$ and $I$ indices, the central charges relations (4.34) read

$$\nabla Z_{AB} = Z_I\mathcal{P}^{I}_{\ AB} + \frac{1}{2}\bar{Z}^{CD}\mathcal{P}_{CDAB} \ , \tag{4.42}$$

$$\nabla \bar{Z}_{\ \bar{I}} = \frac{1}{2}\bar{Z}^{AB}\mathcal{P}_{AB\bar{I}} + Z^{\bar{J}}\mathcal{P}_{\bar{J}\bar{I}} \ . \tag{4.43}$$

## 42. Two examples

We now describe in more detail the $N = 3$ and the $N = 8$ supergravities, and refer to [40] for the $N = 4, 5, 6$ cases. The aim is to write down the group theoretical structure of these theories, their symplectic (local) embeddings $S : G \to Sp(2n, \mathbb{R})$ and $\mathcal{N} : G/H \to Sp(2n, \mathbb{R})/U(n)$, the vector kinetic matrix $\mathcal{N}$, the supersymmetric transformation laws, the structure of the central and matter charges, their differential relations originating from the Maurer-Cartan equations (3.36),(3.37), and the invariants $\mathscr{V}_{BH}$ and $\mathscr{S}$.

### 42.1. The $N = 3$ theory

In the $N = 3$ case [50] the coset space is:

$$G/H = \frac{SU(3, n')}{S(U(3) \times U(n'))} \tag{4.44}$$

and the field content is given by:

− Gravitational multiplet:

$$(V^a_\mu, \psi_{A\mu}, A^{AB}_\mu, \chi_{(L)}) \qquad A = 1, 2, 3 \tag{4.45}$$

− Vector multiplets:

$$(A_\mu, \lambda_A, \lambda_{(R)}, 3\, z)^I \qquad I = 1, \ldots, n' \tag{4.46}$$

Here $A, B, \ldots$ are indices of $SU(3)$, $SU(3) \times U(1)$, being $H_{aut}$ the automorphism group of the $N=3$–extended supersymmetry algebra. The fermions $\lambda_A$ are left handed while the singlet $\lambda_{(R)}$ is right handed.

The transformation properties of the fields are given in Table 2.

Table 2: Transformation properties of fields in $D=4$, $N=3$

| | $V^a_\mu$ | $\psi_{A\mu}$ | $A^\Lambda_\mu$ | $\chi_{(L)}$ | $\lambda^I_A$ | $\lambda^I_{(L)}$ | $L^\Lambda_{AB}$ | $L^\Lambda_{\bar{I}}$ | $R_H$ |
|---|---|---|---|---|---|---|---|---|---|
| $SU(3,n')$ | 1 | 1 | $3+n'$ | 1 | 1 | 1 | $3+n'$ | $3+n'$ | - |
| $SU(3)$ | 1 | 3 | 1 | 1 | 3 | 1 | 3 | 1 | 3 |
| $SU(n')$ | 1 | 1 | 1 | 1 | $n'$ | $n'$ | 1 | $n'$ | $n'$ |
| $U(1)$ | 0 | $\frac{n'}{2}$ | 0 | $3\frac{n'}{2}$ | $3+\frac{n'}{2}$ | $-3(1+\frac{n'}{2})$ | $n'$ | $-3$ | $3+n'$ |

*In this and in the following table, $R_H$ is the representation under which the scalar fields of the linearized theory, or the vielbein $\mathcal{P}$ of $G/H$ of the full theory transform (recall text after (3.33) and that $\mathcal{P}$ is $P$ in the complex basis). Only the left–handed fermions are quoted, right handed fermions transform in the complex conjugate representation of $H$.*

We consider the (local) embedding of $SU(3,n')$ in $Sp(3+n', \mathbb{R})$ defined by the following dependence of the matrices $f$ and $h$ in terms of the $G/H$ coset representative $L$,

$$f^\Lambda_{\ \Sigma} = \frac{1}{\sqrt{2}}(L^\Lambda{}_{AB}, \bar{L}^\Lambda{}_{\bar{I}}) \tag{4.47}$$

$$h_{\Lambda\Sigma} = -i(\eta f \eta)_{\Lambda\Sigma} \qquad\qquad \eta = \begin{pmatrix} \mathbb{1}_{3\times 3} & 0 \\ 0 & -\mathbb{1}_{n'\times n'} \end{pmatrix} \tag{4.48}$$

where $AB$ are antisymmetric $SU(3)$ indices, $I$ is an index of $SU(n')$ and $\bar{L}^\Lambda{}_{\bar{I}}$ denotes the complex conjugate of the coset representative. We have:

$$\mathcal{N}_{\Lambda\Sigma} = (hf^{-1})_{\Lambda\Sigma} = -i(\eta f \eta f^{-1})_{\Lambda\Sigma} \tag{4.49}$$

The supercovariant field strengths and the supercovariant scalar vielbein are:

$$\begin{aligned} \hat{F}^\Lambda &= dA^\Lambda + [\,\frac{i}{2}\bar{f}^\Lambda_{\bar{I}}\bar{\lambda}^{\bar{I}}_A\gamma_a\psi^A V^a - \frac{1}{2}f^\Lambda_{AB}\bar{\psi}^A\psi^B + if^\Lambda_{AB}\bar{\chi}_{(R)}\gamma_a\psi_C\epsilon^{ABC}V^a + h.c.] \\ \hat{\mathcal{P}}_I{}^A &= \mathcal{P}_I{}^A - \bar{\lambda}_{IB}\psi_C\epsilon^{ABC} - \bar{\lambda}_{I(R)}\psi^A \end{aligned} \tag{4.50}$$

where the only nonvanishing entries of the vierbein $\mathcal{P}$ are

$$\mathcal{P}^A_I = \frac{1}{2}\epsilon^{ABC}\mathcal{P}_{IBC} = \mathcal{P}^A_{I\,i}\,dz^i \tag{4.51}$$

$z^i$ being the (complex) coordinates of $G/H$. In the first expression we see the presence of the symplectic sections $(f^\Lambda{}_{AB}, \bar{f}^\Lambda{}_{\bar{I}}, \bar{f}^\Lambda{}_{AB}, f^\Lambda{}_I)$. From the expressions for $\hat{F}^\Lambda$ and $\hat{\mathcal{P}}_I{}^A$, the supersymmetry transformation laws of the vector and scalar fields are retrieved. The dressed field strengths from which the central and matter charges are constructed appear instead in the susy transformation laws of the fermions, up to trilinear fermions terms:

$$\delta\psi_A = D\epsilon_A + 2iT_{AB\,\mu\nu}\gamma^a\gamma^{\mu\nu}V_a\epsilon^B + \cdots \tag{4.52}$$

$$\delta\chi_{(L)} = 1/2\,T_{AB\,\mu\nu}\gamma^{\mu\nu}\epsilon_C\epsilon^{ABC} + \cdots \tag{4.53}$$

$$\delta\lambda_{IA} = -i\mathcal{P}_I{}^B{}_i\partial_\mu z^i\gamma^\mu\epsilon^C\epsilon_{ABC} + T_{I\,\mu\nu}\gamma^{\mu\nu}\epsilon_A + \cdots \tag{4.54}$$

$$\delta\lambda^I_{(L)} = i\mathcal{P}_I{}^A{}_i\partial_\mu z^i\gamma^\mu\epsilon_A + \cdots \tag{4.55}$$

where $T_{AB}$ and $T_I$ have the general form given in equation (4.17).

From the general form of the equations (4.31), (4.32) for $f$ and $h$ we find:

$$\nabla f^\Lambda_{AB} = f^\Lambda{}_I\mathcal{P}^I{}_{AB}\ , \tag{4.56}$$

$$\nabla h^\Lambda_{AB} = h^\Lambda{}_I\mathcal{P}^I{}_{AB}\ , \tag{4.57}$$

$$\nabla f^\Lambda{}_{\bar I} = \frac{1}{2}\bar f^{\Lambda CD}\mathcal{P}_{CD\bar I}\ , \tag{4.58}$$

$$\nabla h^\Lambda{}_{\bar I} = \frac{1}{2}\bar h^{\Lambda CD}\mathcal{P}_{CD\bar I}\ . \tag{4.59}$$

According to the general study of Section 4.1, using (4.20), (4.21) one finds

$$\nabla^{(H)}Z_{AB} = \bar Z^I\mathcal{P}_I{}^C\epsilon_{ABC} \tag{4.60}$$

$$\nabla^{(H)}Z_I = \frac{1}{2}\bar Z^{AB}\mathcal{P}_I{}^C\epsilon_{ABC} \tag{4.61}$$

and the formula for the potential, cf. (4.27),

$$\mathscr{V}_{BH} = \frac{1}{2}Z^{AB}\bar Z_{AB} + Z^I\bar Z_I = -\frac{1}{2}Q^t\mathcal{M}(\mathcal{N})Q \tag{4.62}$$

where the matrix $\mathcal{M}(\mathcal{N})$ has the same form as in equation (3.49) in terms of the kinetic matrix $\mathcal{N}$ of equation (4.49), and $Q$ is the charge vector $Q = \binom{g}{e}$.

For each of the supergravities with target space $G/H$ there is another $G$ invariant expression $\mathscr{S}$ quadratic in the charges $Z_{AB}, Z_I$[51]; the invariant $\mathscr{S}$ is independent from the scalar fields of the theory and thus depends only on the electric and magnetic charges $p^\Lambda$ and $q_\Lambda$. In extremal black hole configurations $\pi\mathscr{S}$ is the entropy of the black hole. It turns out that $\mathscr{S}$ coincides with the potential $\mathscr{V}_{BH}$ computed at its critical point (attractor point) [36, 38, 51]. The invariants $\mathscr{S}$ are obtained by considering among the $H$ invariant combination of the charges those that are also $G$ invariant, i.e. those that do not depend on the scalar fields. This is equivalent to require invariance of $\mathscr{S}$ under the coset space covariant derivative $\nabla$ defined in Section 3.5, see also (4.33).

The $G = SU(3,n')$ invariant is $\frac{1}{2}Z_{AB}\bar Z^{AB} - Z_I\bar Z^I$ (one can check that $\partial_i(\frac{1}{2}Z_{AB}\bar Z^{AB} - Z_I\bar Z^I) = \nabla_i^{(H)}(\frac{1}{2}Z_{AB}\bar Z^{AB} - Z_I\bar Z^I) = 0$) so that

$$\mathscr{S} = |\frac{1}{2}Z_{AB}\bar Z^{AB} - Z_I\bar Z^I|\ . \tag{4.63}$$

### 42.2. The $N = 8$ theory

In the $N = 8$ case [3] the coset manifold is:

$$G/H = \frac{E_{7(7)}}{SU(8)/\mathbb{Z}_2}. \tag{4.64}$$

The field content and group assignments are given in Table 3.

Table 3: Field content and group assignments in $D = 4$, $N = 8$ supergravity

| | $V^a$ | $\psi_A$ | $A^{\Lambda\Sigma}$ | $\chi_{ABC}$ | $S_r^\alpha$ | $R_H$ |
|---|---|---|---|---|---|---|
| $E_{7(7)}$ | 1 | 1 | - | 1 | 56 | - |
| $SU(8)$ | 1 | 8 | 1 | 56 | $28 + 28$ | 70 |

The embedding in $Sp(56, \mathbb{R})$ is automatically realized because the 56 defining representation of $E_{7(7)}$ is a real symplectic representation. The components of the $f$ and $h$ matrices and their complex conjugates are

$$f^{\Lambda\Sigma}_{\ \ AB} \ , \quad h_{\Lambda\Sigma AB} \ , \quad \bar{f}_{\Lambda\Sigma}^{\ \ AB} \ , \quad \bar{h}^{\Lambda\Sigma\, AB} \ , \tag{4.65}$$

here $\Lambda\Sigma, AB$ are couples of antisymmetric indices, with $\Lambda, \Sigma, A, B$ running from 1 to 8. The 70 under which the vielbein of $G/H$ transform is obtained from the four times antisymmetric of $SU(8)$ by imposing the self duality condition

$$\bar{t}^{\bar{A}\bar{B}\bar{C}\bar{D}} = \frac{1}{4!}\epsilon^{\bar{A}\bar{B}\bar{C}\bar{D}}_{\ \ \ \ \ \ A'B'C'D'} t^{A'B'C'D'} \tag{4.66}$$

The supercovariant field strengths and coset manifold vielbein are:

$$\hat{F}^{\Lambda\Sigma} = dA^{\Lambda\Sigma} + [f^{\Lambda\Sigma}_{\ \ AB}(a_1\bar{\psi}^A\psi^B + a_2\bar{\chi}^{ABC}\gamma_a\psi_C V^a) + h.c.] \tag{4.67}$$

$$\hat{\mathcal{P}}_{ABCD} = \mathcal{P}_{ABCD} - \bar{\chi}_{[ABC}\psi_{D]} + h.c. \tag{4.68}$$

where $\mathcal{P}_{ABCD} = \frac{1}{4!}\epsilon_{ABCDEFGH}\bar{\mathcal{P}}^{EFGH} \equiv (L^{-1}\nabla^{SU(8)}L)_{AB\,CD} = \mathcal{P}_{ABCD\,i}d\phi^i$ ($\phi^i$ coordinates of $G/H$). The coefficients $a_1, a_2$ (as well as the later ones $a_3$ and $a_4$) can in principle be determined, we here are mainly interested in the structure of the supersymmetry transformations and its dependence on the embedding of $E_{7(7)}$ into $Sp(56, \mathbb{R})$.

In the complex basis the vielbein $\mathcal{P}_{ABCD}$ of $G/H$ are $28 \times 28$ matrices completely antisymmetric and self dual as in (4.66). The fermion transformation laws are given by:

$$\delta\psi_A = D\epsilon_A + a_3 T_{AB\,\mu\nu}\gamma^a\gamma^{\mu\nu}\epsilon^B V_a + \cdots \tag{4.69}$$

$$\delta\chi_{ABC} = a_4\mathcal{P}_{ABCD\,i}\partial_a\phi^i\gamma^a\epsilon^D + a_5 T_{[AB\,\mu\nu}\gamma^{\mu\nu}\epsilon_{C]} + \cdots \tag{4.70}$$

where:

$$T_{AB} = \frac{1}{2}(h_{\Lambda\Sigma AB}F^{\Lambda\Sigma} - f^{\Lambda\Sigma}_{\ \ AB}G_{\Lambda\Sigma}) \tag{4.71}$$

with:

$$\mathcal{N}_{\Lambda\Sigma\Gamma\Delta} = \frac{1}{2}\, h_{\Lambda\Sigma AB}(f^{-1})^{AB}{}_{\Gamma\Delta}\ . \tag{4.72}$$

With the usual manipulations we obtain the central charges:

$$Z_{AB} = \frac{1}{2}(h_{\Lambda\Sigma AB}p^{\Lambda\Sigma} - f^{\Lambda\Sigma}{}_{AB}q_{\Lambda\Sigma}), \tag{4.73}$$

the differential relations:

$$\nabla^{SU(8)}Z_{\ AB} = \frac{1}{2}\bar{Z}^{\ CD}\mathcal{P}_{ABCD} \tag{4.74}$$

and the formula for the potential, cf. (4.27),

$$\mathscr{V}_{BH} = \frac{1}{2}\bar{Z}^{AB}Z_{AB} = -\frac{1}{2}Q^t\mathcal{M}(\mathcal{N})Q \tag{4.75}$$

where the matrix $\mathcal{M}(\mathcal{N})$ is given in equation (3.49), and $\mathcal{N}$ in (4.72).
For $N = 8$ the $SU(8)$ invariants are

$$I_1 = (TrA)^2 \tag{4.76}$$

$$I_2 = Tr(A^2) \tag{4.77}$$

$$I_3 = Pf\ Z = \frac{1}{2^4 4!}\epsilon^{ABCDEFGH}Z_{AB}Z_{CD}Z_{EF}Z_{GH} \tag{4.78}$$

where $Pf Z$ denotes the Pfaffian of the antisymmetric matrix $(Z_{AB})_{A,B=1,\dots 8}$, and where $A_A^{\ B} = Z_{AC}\bar{Z}^{CB}$. One finds the following $E_{7(7)}$ invariant [37]:

$$\mathscr{S} = \frac{1}{2}\sqrt{|4\mathrm{Tr}(A^2) - (TrA)^2 + 32\mathrm{Re}\,(Pf\ Z)|} \tag{4.79}$$

For a recent study of $E_{7(7)}$ duality rotations and of the corresponding conserved charges see [52].

### 42.3. Electric subgroups and the $D = 4$ and $N = 8$ theory

A duality rotation is really a strong-weak duality if there is a rotation between electric and magnetic fields, more precisely if some of the rotated field strengths $F'^{\Lambda}$ depend on the initial dual fields $G^{\Sigma}$, i.e. if the submatrix $B \neq 0$ in the symplectic matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. Only in this case the gauge kinetic term may transform nonlinearly, via a fractional transformation. On the other hand, under infinitesimal duality rotations $\left(\begin{smallmatrix} 1\!\!1 & 0 \\ 0 & 1\!\!1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right)$, with $b = 0$, the lagrangian changes by a total derivative so that (in the absence of instantons) these transformations are symmetries of the action, not just of the equation of motion. Furthermore if $c = 0$ the lagrangian itself is invariant.

We call electric any subgroup $G_e$ of the duality group $G$ with the property that it (locally) embeds in the symplectic group via matrices $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ with $B = 0$. The parameter space of true strong-weak duality rotations is $G/G_e$.

The electric subgroup of $Sp(2n, \mathbb{R})$ is the subgroup of all matrices of the kind

$$\begin{pmatrix} A & 0 \\ C & A^{t-1} \end{pmatrix} ; \tag{4.80}$$

we denote it by $Sp_e(2n, \mathbb{R})$. It is *the* electric subgroup because any other electric subgroup is included in $Sp_e(2n, \mathbb{R})$. This subgroup is maximal in $Sp(2n, \mathbb{R})$ (see for example the appendices in[53, 54]). In particular if an action is invariant under infinitesimal $Sp_e(2n, \mathbb{R})$ transformations, and if the equations of motion admit also a $\pi/2$ duality rotation symmetry $F^\Lambda \to G^\Lambda$, $G^\Lambda \to -F^\Lambda$ for one or more indices $\Lambda$ (no transformation on the other indices) then the theory has $Sp(2n, \mathbb{R})$ duality.

It is easy to generalize the results of Section 2.2 and prove that duality symmetry under these $\pi/2$ rotations is equivalent to the following invariance property of the lagrangian under the Legendre transformation associated to $F^\Lambda$,

$$\mathcal{L}_D(F, \mathcal{N}') = \mathcal{L}(F, \mathcal{N}) \ , \tag{4.81}$$

where $\mathcal{N}' = (C + D\mathcal{N})(A + B\mathcal{N})^{-1}$ are the transformed scalar fields, the matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ implementing the $\pi/2$ rotation $F^\Lambda \to G^\Lambda$, $G^\Lambda \to -F^\Lambda$. We conclude that $Sp(2n, \mathbb{R})$ duality symmetry holds if there is $Sp_e(2n, \mathbb{R})$ symmetry and if the lagrangian satisfies (4.81).

When the duality group $G$ is not $Sp(2n, \mathbb{R})$ then there may exist different maximal electric subgroups of $G$, say $G_e$ and $G'_e$. Consider now a theory with $G$ duality symmetry, the electric subgroup $G_e$ hints at the existence of an action $S = \int \mathcal{L}$ invariant under the Lie algebra Lie$(G_e)$ and under Legendre transformation that are $\pi/2$ duality rotation in $G$. Similarly $G'_e$ leads to a different action $S' = \int \mathcal{L}'$ that is invariant under Lie$(G'_e)$ and under Legendre transformations that are $\pi/2$ duality rotation in $G$. The equations of motion of both actions have $G$ duality symmetry. They are equivalent if $\mathcal{L}$ and $\mathcal{L}'$ are related by a Legendre transformation. Since $\mathcal{L}'(F, \mathcal{N}') \neq \mathcal{L}(F, \mathcal{N})$, this Legendre transformation cannot be a duality symmetry, it is a $\pi/2$ rotation $F^\Lambda \to G^\Lambda$, $G^\Lambda \to -F^\Lambda$ that is not in $G$, this is possible since $G \neq Sp(2n, \mathbb{R})$.

As an example consider the $G_e = SL(8, \mathbb{R})$ symmetry of the $N = 8$, $D = 4$ supergravity lagrangian whose duality group is $G = E_{7,(7)}$ this is the formulation of Cremmer-Julia. An alternative formulation, obtained from dimensional reduction of the $D = 5$ supergravity, exhibits an electric group $G'_e = [E_{6,(6)} \times SO(1,1)] \ltimes T_{27}$ where the non semisimple group $G'_e$ is realized as a lower triangular subgroup of $E_{7,(7)}$ in its fundamental (symplectic) 56 dimensional representation. $G_e$ and $G'_e$ are both maximal subgroups of $E_{7(7)}$. The corrseponding lagrangians can be related only after a proper duality rotation of electric and magnetic fields which involves a suitable Legendre transformation.

A way to construct new supergravity theories is to promote a compact rigid electric subgroup symmetry to a local symmetry, thus constructing gauged supergravity models (see for a recent review [55], and references therein). Inequivalent choices of electric subgroups give different gauged supergravities. Consider again $D=4$, $N=8$ supergravity. The maximal compact subgroups of $G_e = SL(8,\mathbb{R})$ and of $G'_e = [E_{6,(6)} \times SO(1,1)] \ltimes T_{27}$ are $SO(8)$ and $Sp(8) = U(16) \cap Sp(16,\mathbb{C})$ respectively. The gauging of $SO(8)$ corresponds to the gauged $N=8$ supergravity of De Witt and Nicolai [56]. As shown in [57] the gauging of the non semisimple group $U(1) \ltimes T_{27} \subset G'_e$ corresponds to the gauging of a flat group in the sense of Scherk and Schwarz dimensional reduction [58], and gives the massive deformation of the $N=8$ supergravity as obtained by Cremmer, Scherk and Schwarz [59].

## 43. The flat symplectic bundle of extended supergravities

The formalism we have developed so far for the $D=4$, $N>2$ theories is completely determined by the (local) embedding of the coset representative of the scalar manifold $M = G/H$ in $Sp(2n,\mathbb{R})$. It leads to a flat -actually a trivial- symplectic vector bundle of rank $2n$ that we now describe. In order to show that we have a flat (zero curvature) bundle we observe that if we are able to find $2n$ linearly independent row vectors $V^\xi = (V^\xi{}_\zeta)_{\zeta=1,\ldots 2n}$ then the matrix $V$ in (4.30) is invertible and therefore the connection $\left(\begin{smallmatrix}\omega & \bar{\mathcal{P}}\\ \mathcal{P} & \bar{\omega}\end{smallmatrix}\right)$ is flat. If these vectors are mutually symplectic then we have a symplectic frame, the transition functions are constant symplectic matrices, the connection is symplectic.

The flat symplectic bundle explicitly is,

$$G \times_H \mathbb{R}^{2n} \to G/H\,;$$

this bundle is the space of all equivalence classes $[g,v] = \{(gh, S(h)^{-1}v)\,,\ g \in G, v \in \mathbb{R}^{2n}, h \in H\}$. The symplectic structure on $\mathbb{R}^{2n}$ immediately extends to a well defined symplectic structure on the fibers of the bundle. Using the local sections of $G/H$ and the usual basis $\{e_\xi\} = \{e_M, e^M\}$ of $\mathbb{R}^{2n}$ ($e_1$ is the column vector with with 1 as first and only nonvanishing entry, etc.) we obtain immediately the local sections $s_\xi = [L(\phi), e_\xi]$ of $G \times_H \mathbb{R}^{2n} \to G/H$. Since the action of $H$ on $\mathbb{R}^{2n}$ extends to the action of $G$ on $\mathbb{R}^{2n}$, we can consider the new sections $\mathsf{e}_\xi = s_\zeta S^{-1}(L(\phi))^\zeta{}_\xi = [L(\phi), S^{-1}(L(\phi))e_\xi]$ , that are determined by the column vectors $S^{-1}(L(\phi))\ {}_\xi = (S^{-1}(L(\phi))^\zeta{}_\xi)_{\zeta=1,\ldots 2n}$. These sections are globally defined and linearly independent. Therefore this bundle is not only flat, it is trivial. If we use the complex local frame $\mathcal{V}_\xi = \{s_\zeta \mathcal{A}^\zeta{}_\xi\}$ rather than the $\{s_\xi\}$ one (we recall that $\mathcal{A} = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix}\mathbb{1} & \mathbb{1}\\ -i\mathbb{1} & i\mathbb{1}\end{smallmatrix}\right)$), then the global sections $\mathsf{e}_\xi$ are determined by the column vectors $V^{-1}(L(\phi))\ {}_\xi = (V^{-1}(L(\phi))^\zeta{}_\xi)_{\zeta=1,\ldots 2n}$,

$$\mathsf{e}_\xi = \mathcal{V}_\eta\, V^{-1\eta}{}_\xi\ . \tag{4.82}$$

The sections $\mathcal{V}_\xi$ too form a symplectic frame (a symplectonormal basis, indeed $V^\rho_\xi \Omega_{\rho\sigma} V^\sigma_\zeta = \Omega_{\xi\zeta}$, where $\Omega = \left(\begin{smallmatrix}0 & -\mathbb{1}\\ \mathbb{1} & 0\end{smallmatrix}\right)$), and the last $n$ sections are the complex conjugate of the first $n$ ones, $\{\mathcal{V}_\xi\} = \{\mathcal{V}_M, \bar{\mathcal{V}}_{\bar{M}}\}$. Of course the column vectors $V\,{}_\eta = (V^\xi_\eta)_{\xi=1,\ldots 2n}$, are the coefficients of the sections $\mathcal{V}_\eta$ with respect to the flat basis $\{\mathsf{e}_\xi\}$.

Also the rows of the $V$ matrix define global flat sections. Let's consider the dual bundle of the vector bundle $G \times_H \mathbb{R}^{2n} \to G/H$, i.e. the bundle with fiber the dual vector space. If $\{s_\zeta\}$ is a frame of local sections of $G \times_H \mathbb{R}^{2n} \to G/H$, then $\{s^\zeta\}$, with $\langle s^\zeta, s_\xi\rangle = \delta^\zeta_\xi$, is the dual frame of local sections of the dual bundle. Concerning the transition functions, if $s'_\zeta = s_\eta S^\eta{}_\zeta$ then $s'^\xi = S^{-1}{}^\xi{}_\lambda s^\lambda$. This dual bundle is also a trivial bundle and a trivialization is given by the global symplectic sections $\mathsf{e}^\xi = V^\xi{}_\eta \mathcal{V}^\eta$, whose coefficients are the row vectors $V^\xi = (V^\xi{}_\zeta)_{\zeta=1,\dots 2n}$ i.e., the rows of the symplectic matrix $V$ defined in (4.12),

$$\begin{aligned}\left(V^\Lambda{}_\zeta\right)_{\zeta=1,\dots 2n} &= \left(f^\Lambda{}_M, \bar{f}^\Lambda{}_{\bar{M}}\right)_{M=1,\dots n} ,\\ \left(V_{\Lambda\zeta}\right)_{\zeta=1,\dots 2n} &= \left(h_{\Lambda M}, \bar{h}_{\Lambda\bar{M}}\right)_{M=1,\dots n} .\end{aligned} \tag{4.83}$$

### 44. Special Geometry and $N = 2$ Supergravity

In the case of $N = 2$ supergravity the requirements imposed by supersymmetry on the scalar manifold $M_{scalar}$ of the theory dictate that it should be the following direct product: $M_{scalar} = M \times M^Q$ where $M$ is a special Kähler manifold of complex dimension $n$ and $M^Q$ a quaternionic manifold of real dimension $4n_H$, here $n$ and $n_H$ are respectively the number of vector multiplets and hypermultiplets contained in the theory. The direct product structure imposed by supersymmetry precisely reflects the fact that the quaternionic and special Kähler scalars belong to different supermultiplets. We do not discuss the hypermultiplets any further and refer to [60] for the full structure of $N = 2$ supergravity. Since we are concerned with duality rotations we here concentrate our attention to an $N = 2$ supergravity where the graviton multiplet, containing besides the graviton $g_{\mu\nu}$ also a graviphoton $A^0_\mu$, is coupled to $n'$ vector multiplets. Such a theory has a bosonic action of type (4.1) where the number of (real) gauge fields is $n = 1 + n'$ and the number of (real) scalar fields is $2n'$. Compatibility of their couplings with local $N = 2$ supersymmetry led to the formulation of special Kähler geometry [61],[62].

The formalism we have developed so far for the $D = 4$, $N > 2$ theories is completely determined by the (local) embedding of the coset representative of the scalar manifold $M = G/H$ in $Sp(2n, \mathbb{R})$. It leads to a flat -actually a trivial- symplectic bundle with local symplectic sections $\mathcal{V}_\eta$, determined by the symplectic matrix $V$, or equivalently by the matrices $f$ and $h$. We want now to show that these matrices, the differential relations among charges and their quadratic invariant $\mathscr{V}_{BH}$ (4.27) are also central for the description of $N = 2$ matter-coupled supergravity. This follows essentially from the fact that, though the scalar manifold $M$ of the $N = 2$ theory is not in general a coset manifold, nevertheless, as for the $N > 2$ theories, we have a flat symplectic bundle associated to $M$, with symplectic sections $\mathcal{V}_\eta$. While the formalism is very similar there is a difference, the bundle is not a trivial bundle anymore, and it is in virtue of duality rotations that the theory can be globally defined on $M$.

The local symplectic sections $\mathcal{V}_\eta$ are determined (with respect to a local flat symplectic frame as in (4.82)) by the matrices

$$\begin{aligned}f &= (f^\Lambda{}_M) = (f^\Lambda{}_{AB}, \bar{f}^\Lambda{}_{\bar{I}}) ,\\ h &= (h_{\Lambda M}) = (h_{\Lambda AB}, \bar{h}_{\Lambda\bar{I}}) ,\end{aligned} \tag{4.84}$$

where $A, B = 1, 2$ are in the fundamental of $U(2)$, the automorphism group of the $N = 2$ supersymmetry algebra (and therefore the antisymmetric $AB$ indices in (4.84) can assume just one value).

Since the gauge kinetic term $\mathcal{N}_{\Lambda\Sigma} = h_{\Lambda M} f^{-1}{}^{M}_{\Sigma}$ depends on the choice of the flat symplectic frame $\{\mathsf{e}_\xi\} = \{\mathsf{e}_\Lambda, \mathsf{f}^\Lambda\}$ and this is only locally defined, in another region we have a different frame $\{\mathsf{e}'_\xi\} = \{\mathsf{e}'_\Lambda, \mathsf{f}'^\Lambda\}$ and therefore a different gauge kinetic term $\mathcal{N}'_{\Lambda\Sigma}$. In the common overlapping region the two formulations should give the same theory, this is indeed the case because the corresponding equations of motion are related by a duality rotation. As a consequence the notion of electric or magnetic charge depends on the flat frame chosen. In this sense the notion of electric and magnetic charge is not a fundamental one. The symplectic group is a gauge group (where just constant gauge transformations are allowed) and only gauge invariant quantities are physical. It is in this sense that duality rotations in $N = 2$ theories are not a symmetry of the theory, they are rather a gauge symmetry.

To complete the analogy between the $N = 2$ theory with $n'$ vector multiplets and the higher $N$ theories in $D = 4$, we also give the supersymmetry transformation laws, the central and matter charges, the differential relations among them and the formula for the potential $\mathscr{V}_{BH}$.

In analogy with the the higher $N$ theories in $D = 4$, the supercovariant electric field strength $\hat{F}^\Lambda$ is

$$\hat{F}^\Lambda = F^\Lambda + f^\Lambda \bar{\psi}^A \psi^B \epsilon_{AB} - \mathrm{i} \bar{f}^\Lambda_{\bar{I}} \bar{\lambda}^{\bar{I}}_A \gamma_a \psi_B \epsilon^{AB} V^a + h.c. \tag{4.85}$$

where $f^\Lambda = f^\Lambda_{AB}$. The transformation laws for the chiral gravitino $\psi_A$ and gaugino $\lambda^{IA}$ fields are:

$$\delta\psi_{A\mu} = \nabla_\mu \epsilon_A + \epsilon_{AB} T_{\mu\nu} \gamma^\nu \epsilon^B + \cdots , \tag{4.86}$$

$$\delta\lambda^{IA} = i\mathcal{P}^I_i \partial_\mu z^i \gamma^\mu \epsilon^A + \frac{i}{2} \bar{T}^I_{\mu\nu} \gamma^{\mu\nu} \epsilon^{AB} \epsilon_B + \cdots , \tag{4.87}$$

where:

$$T = h_\Lambda F^\Lambda - f^\Lambda G_\Lambda , \tag{4.88}$$

$$\bar{T}_{\bar{I}} = \bar{h}_{\Lambda \bar{I}} F^\Lambda - \bar{f}^\Lambda_{\bar{I}} G_\Lambda , \tag{4.89}$$

are respectively the graviphoton and the matter vectors, and $\mathcal{P}^I{}_i$ is the vielbein associated to the holomorphic coordinates $z^i$ of $M$. In (4.85)-(4.87) the position of the $SU(2)$ automorphism index $A$ ($A, B = 1, 2$) is related to chirality, namely $(\psi_A, \lambda^{IA})$ are chiral, $(\psi^A, \bar{\lambda}^{\bar{I}}_A)$ antichiral.

In order to define the symplectic invariant charges let us recall the definition of the magnetic and electric charges (the moduli independent charges) in (4.10). The central charges and the matter charges are then defined as the integrals over a sphere at spatial infinity of the dressed graviphoton and matter vectors (4.17), they are given in (4.20), (4.21):

$$(Z_M) = \big(Z_{AB}, \bar{Z}_{\bar{I}}\big) = i\overline{V(\phi_\infty)}^{-1} Q \tag{4.90}$$

where $\phi_\infty$ is the value of the scalar fields at spatial infinity. The special Kähler geometry of the manifold describing the scalars of $N = 2$ supegravity implies

$$\nabla Z_{AB} = Z_I \mathcal{P}^I \epsilon_{AB} \ . \tag{4.91}$$

where $\mathcal{P}^I$ is the vielbein 1-form on $M$.

The positive definite quadratic invariant $\mathscr{V}_{BH}$ in terms of the charges $Z$ and $Z_I$ reads

$$\mathscr{V}_{BH} = \frac{1}{2} Z\bar{Z} + Z_I \bar{Z}^I = -\frac{1}{2} Q^t \mathcal{M}(\mathcal{N}) Q \ . \tag{4.92}$$

Equation (4.92) is obtained by using exactly the same procedure as in (4.27). Invariance of $\mathscr{V}_{BH}$ implies that it is a well defined positive function on $M$.

For an arbitrary special Kähler manifolds it is in general hard to compute the invariant combination $\mathscr{S}$ of the charges $Z$, $Z_I$. The general formula for special Kähler manifolds that are symmetric coset spaces is given in [39].

**Acknowledgments**
P.A. thanks the organizers of the Summer School and Conference on Modern Mathematical Physics, 6-17 July 2008, Belgrade, Serbia, for the warm and fruitful atmosphere of the meeting.

The work of P.A. was supported in part by the Italian MIUR under contract PRIN-2007 Algebre di Operatori e Applicazioni. The work of S. F. has been supported in part by INFN - Frascati National Laboratories, and by D.O.E. grant DE-FG03-91ER40662, Task C. The work of B.Z. was supported in part by the Director, Oce of Science, Oce of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract No. DE-AC02-05CH11231, and in part by NSF grant 10996-13607-44 PHHXM.

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