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THE RIEMANN MONODROMY PROBLEM AND CONFORMAL FIELD THEORIES ON THE TORUS

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ABSTRACT

We generalize the Riemann monodromy problem approach to the investigation of conformal field theories on a higher genus Riemann surface. In particular, we discuss the associated differential equations for the correlators. This approach reveals the connection between the s-point correlation function on a genus g Riemann surface and the s-2 point function on a genus g+1 Riemann surface. For the case of the torus we show how this approach gives the Samir, Mukhi and Sen differential equations.

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It is a well-established fact that conformal field theories (CFT) play a major role in the study of 2-dimensional critical phenomena as well as string theory. A systematic investigation of CFT's was initiated in the work of Polyakov, Belavin and Zamolodchikov [1] . Recently the bootstrap approach to the study of CFT's has been further developed by several groups [2] , [3] , [4] , [5] . One key element in their study is the characterisation of conformal blocks by their monodromy properties.

Given a conformal field theory we would like to be able to calculate its correlation functions. In a previous paper [6] we have used the bootstrap approach together with the Riemann monodromy problem to investigate the correlation functions on the sphere.

In the present letter we would like to generalize our approach to the study of the correlation functions on the torus. In particular we show that the differential equations for the correlation functions which were found recently [7] arise naturally from the study of the appropriate Riemann monodromy problem. Although we concentrate on the torus, our approach is quite general and can be applied to the higher genus Riemann surfaces. Let us also note that the correlation functions on the torus were calculated recently in a number of conformal field theories using different methods [8] , [9] , [10] . A similar approach can be used to investigate characters (see e.g. refs. [7] , [11]). We begin our investigation by formulating the Riemann monodromy problem on a Riemann surface Γ . For a given set of s points $a_1, ... a_s$ on Γ and a set of constant m× m matrices $M_1, ..., M_s$ find a system of m functions $y_1, ..., y_m$ such that for any closed path in $\Gamma - \{a_1, ..., a_s\}$ the monodromy is given by a specified constant matrix M^γ. Under an analytic continuation along the path γ

$$y_i \to y_j M^\gamma_{ji} \tag{1}$$

where the dependence on γ is only through its homotopy class $[\gamma]$. The monodromy data is therefore specified by giving s+2g matrices M_k that correspond to the generators of G=$\pi_1(\Gamma - \{a_1, ... a_s\})$. There are s generators associated with closed loops around each a_i and 2g generators associated with the 2g non-trivial cycles on Γ.

As in the case of the sphere, it is possible to prove [12] that for every such system of matrices M_k (k=1,....,s+2g) there exists a unique (up to multiplication by a matrix of rational functions) set of m rationally independent solutions of the problem. Note that the fundamental solutions $y_i^{(j)}$form a m×m matrix Y. The matrix Y satisfies the differential equations

$$\frac{dY}{dz} = QY \tag{2}$$

where Qdz is a m×m matrix of meromorphic one-forms with the first -order poles as their only singularities. In the case of conformal field theories the monodromy data can be constructed in terms of the crossing (braid) matrices. Note that this construction is the same as for the sphere. We can follow the same arguments that were presented in the case of the sphere [6] to show that the solution of Riemann problem is completely determined once we prescribe the leading behaviour at all singular points $a_1, ...a_s$, as well as the behaviour at the so-called apparent singularities (i.e. points where the conformal blocks have zeroes). The leading behaviour at the singular points $a_1, ...a_s$ is usually controlled by the conformal dimensions of the primary fields.

Next we compare the constraints on the monodromy matrices that we have to impose in order to provide the solution of the Riemann problem. Recall that on the sphere we have [6]

$$M_1....M_s M_\infty = 1 \tag{3}$$

which for the case of the 4-point function reads

$$M_s M_t M_u = 1 \tag{4}$$

where M_s, M_t, M_u are the monodromy matrices associated with the singularities at the points 0,1,∞ respectively. The torus can be described in terms of a parallelogram (fig. 1) with opposite sides identified. Consider the 2-point function $< \Phi_j(z)\Phi_j(0) >$ on the torus. The analytic continuation around the singularity at z=0 can be represented by the action of the matrix $M_\sigma M_\tau M_\sigma^{-1} M_\tau^{-1}$ where M_σ and M_τ are the monodromy matrices associated with the non-trivial cycles on the torus (see fig.1). Hence we obtain

$$M_\sigma M_\tau M_\sigma^{-1} M_\tau^{-1} M_t = 1 \tag{5}$$

Here the matrix M_t is the same one as in eq. (4) since the analytic continuation around the point z=0 on the torus is the same operation as the analytic continuation around the point 1 (corresponding to the t-channel) for the corresponding 4-point correlator on the sphere.

This operation is depicted graphically in fig. 2. The sphere is obtained from the torus by taking the $Im\tau \to \infty$ limit. In order to convince ourselves that equations (4) and (5) are indeed the same we can use the explicit expressions for the monodromy matrices in terms of the crossing (braid) matrices and the T matrices of conformal dimensions given in ref. [6]. Moreover we also need the explicit expressions of M_σ and M_τ associated with the 2-point function $< \Phi_j(z)\Phi_j(0) >$ which are given in ref.[3]

$$\begin{aligned} M_\sigma &= exp(2\pi i(\Delta_q - \Delta_i)) \\ M_\tau &= B_q \begin{pmatrix} j & j \\ i & i \end{pmatrix} P_{qi} \end{aligned} \tag{6}$$

where B is the braid matrix and P_{qi} is the transposition operator (see fig. 3 for precise notations). A straightforward substitution of these expressions into eqs. (4) and (5) allows us to conclude that the consistency condition for the monodromy problem associated with the 4-point correlation function on the sphere is the same as the one obtained for the 2-point function on the torus. This is just one example of the general result that the monodromy problem for the p-point function on a genus g surface is equivalent to the monodromy problem for the p-2 point function on a genus g+1 surface. For each monodromy problem we formulate on a genus g surface there is a corresponding Riemann problem on a genus g+1 surface which gives the same consistency conditions. Going to higher genus surfaces does not involve any new monodromy data beyond the one given on the sphere. We need, therefore, only the orders of the leading singularities on the torus in order to specify completely the solution to the Riemann monodromy problem.

On any Riemann surface we can start from the system of differential equations (2). We can replace this system of m first-order differential equations by an m-th order differential equation of the Fuchs type. In order for this equation to have m independent solutions with only regular singularities the Fuchs condition must be satisfied. We have already discussed this condition for the case of the sphere in our previous paper [6]. For a genus g surface this condition reads [13]

$$\sum_{j=1}^{m}\sum_{i=1}^{s}\rho_j^{(i)} = \frac{m(m-1)}{2}(s+2(g-1)) \tag{7}$$

where $\rho_j^{(i)}$ is the order of the singularity of the j-th component of the solution of the Riemann monodromy problem near the point a_i. (Let us recall [6] that the unknown function in the differential equation corresponds to one of the rationally independent solutions of the Riemann monodromy problem, while the linear independent solutions of this m-th order differential equation correspond to different components of this solution). Let us stress again that one should include also apparent singularities (i.e. zeroes) among the singular points.

Note that if we go from the s point function on a genus g surface to the s-2 point function on a genus g+1 surface, then the r.h.s. of eq.(7) remains the same, i.e. the corresponding sums over the order of the singularities of those conformal blocks are the same. This is yet another manifestation of the close connection between the s-point function on a genus g surface and the s-2 point function on a genus g+1 surface. This kind of connection can be explained also as a result of factorization [14] .

Following our investigation of the Riemann problem on the sphere [6] we can define two indices associated with the problem. The first index, ind_1 is defined to be equal to the r.h.s. of eq. (7)

$$Ind_1 = \frac{m(m-1)}{2}(s + 2(g-1)) \tag{8}$$

while the second index , Ind_2 is defined as the sum over the orders of singularities computed naively in terms of the conformal dimensions by using the appropriate operator product expansions (see [6]):

$$Ind_2 = \sum_{j=1}^{m}\sum_{i=1}^{s}\tilde{\rho}_j^{(i)} \tag{9}$$

The orders of singularities $\tilde{\rho}$ are determined from the O.P.E.

$$\Phi_\alpha(z)\Phi_\beta(0) \sim \sum_\gamma \frac{a_{\alpha\beta}^{\gamma}\Phi_\gamma}{z^{\Delta_\gamma-\Delta_\alpha-\Delta_\beta}} \tag{10}$$

assuming that the coefficients $a_{\alpha\beta}^{\gamma}$ differ from zero. The same proof we gave [6] for the case of the sphere holds also in the general case and we conclude that if

$$Ind_1 = Ind_2 \tag{11}$$

then there are no apparent singularities (zeroes of the conformal blocks outside the set of branching points) and the knowledge of conformal dimensions (including their integer parts) uniquely determines the solution to the Riemann problem, i.e. the z dependence of the conformal blocks. In the case of the torus, the the condition that the Wronskian W(z) defined in ref. [7] is a constant follows from eq. (11). If eq. (11) holds there are no apparent singularities and z=0 is the only singular point of the 2-point function. Considering the Wronskian defined in ref. [7] one readily establishes that for $z \approx 0$

$$W(z) \sim z^{Ind_1 - m(m-1)/2} \tag{12}$$

Using eq. (8) we conclude that W(z), which is a meromorphic function does not have any singularities i.e. it is a constant. Note that eq. (11) is only a sufficient condition for having no apparent singularities. If eq. (11) does not hold, but no apparent singularities exist, we can still use the monodromy data and the knowledge of the leading behaviour near the branch points to determine uniquely the correlators from the solutions of the Riemann monodromy problem. The fact that ind_1 differs from ind_2 is due to the vanishing of some of the leading coefficients in the relevant operator product expansions. In this case the leading behaviour near the branch points is not determined by the conformal dimensions. However, eq. (12) still holds and W(z) remains constant. Recall that the method of ref. [7] is most succesful in the case when W(z) is a constant. Otherwise a free undetermined parameter always appears in the differential equation for the correlation functions.

Next we would like to consider the question of modular invariance. For the sake of simplicity, from now on we shall concentrate on the case of the torus. We would , however, like to point out that the whole discussion can be generalized in a straightforward way to higher genus surfaces.

So far we have cocentrated on the z-dependence of the correlators (conformal blocks) for a given Riemann surface Γ characterized by definite values of parameters in the moduli space of genus g Riemann surfaces. (For the case of the torus we have one modular parameter τ). We would like now to consider the τ dependence. Since $\frac{dY}{dz}$ has the same monodromy properties in z as Y we can express it in terms of the fundamental solution Y of eq. (2):

$$\frac{dY}{d\tau} = AY \tag{13}$$

where A=A(z,τ) is a m×m matrix whose entries are rational functions on the Riemann surface. The selfconsistency of eqs. (2) and (13) gives

$$\frac{dQ}{d\tau} - \frac{dA}{dz} + [Q, A] = 0 \tag{14}$$

Equation (14) detemines A in terms of Q and therefore we can find from eq. (13) the τ dependence of Y (eq. (13)) provided that Q is known. Equations (2) , (13) , and (14) contain all the information which is provided by the Riemann monodromy problem.

Let us now investigate the constraints that are imposed on the solution of the Riemann monodromy problem on the torus by modular invariance.

Modular covariance of the conformal blocks implies

$$\begin{aligned} y_j(z, \tau + 1) &= B_{jk} y_k(z, \tau) \\ y_j(z/\tau, -1/\tau) &= C_{jk} y_k(z, \tau) \end{aligned} \tag{15}$$

where B and C are constant (i.e. independent of z and τ) matrices. Equation (15) can be used to obtain the following constraints on the matrix Q:

$$\begin{aligned} Q(z/\tau, -1/\tau) &= \tau Q(z, \tau) \\ Q(z, \tau + 1) &= Q(z, \tau) \end{aligned} \tag{16}$$

A straightforward consequence of these equations is

$$Q(-z, \tau) = -Q(z, \tau) \tag{17}$$

Thus modular invariance implies that we have to solve the differential equations associated with the Riemann problem (eq. (2)) with Q that is determined through the monodromy data and which satisfies eq. (16). Now let us discuss whether equations (2) ,(13) , (14) , and (16) uniquely determine the τ and z dependence of solutions. To answer this question let us note that, as was mentioned above, the knowledge of the monodromy data and of the behaviour of conformal blocks at the singular points uniquely determines the z dependence of solutions. The only remaining freedom is to multiply the fundamental system of solutions by a matrix whose entries are z independent constants. In the case of the sphere this arbitrariness caused no trouble since the conformal blocks depended only on one variable z. In the present case the entries of the matrix can be arbitrary functions of τ. Clearly, multiplication by such a matrix changes, neither the monodromy properties in z nor the leading z-behaviour near any singularity. Since eq. (2) is linear such multiplication does not change Q. Hence Q is completely defined by the monodromy data, leading z-behaviour near the singularities and by the constraints (16). In order to fix completely the τ dependence of the conformal blocks, let us note that in the limit when all branching points approach each other the conformal blocks factorize into the product of the character and a τ-independent function of z which corresponds to the appropriate correlator on the sphere. This can be easily verified for the case of the 2-point correlation function using the explicit expressions of the conformal blocks on the torus (see e.g. ref. [15]). For the general case it follows from the arguments given in ref. [14]. On the other hand the characters are uniquely defined once we know the conformal dimensions and the modular S and T matrices. The latter can be easily expressed through the crossing (braid) matrices [2] ,[3] . From all these arguments it follows that the monodromy data and the leading behaviour of the conformal blocks near the singular points, together with the constraints of the modular invariance of the characters, uniquely define both the z and τ dependence of the conformal blocks. Finally let us note that to the extent that eq. (14) determines A we can find the τ-dependence of Y by using eq.(13) without imposing the constraints connected with modular invariance. Physical solutions must, of course, satisfy the modular invariance constraints (16). It is still an open question whether every solution of the Riemann monodromy problem associated with the monodromy data defined through the braid matrices given in refs. [2] ,[3] ,[4] and [5] satisfies automatically the modular constraints (16). In practice it is usually easier to determine the matrix Q by imposing those constraints from the start .

The most general form of Q in the case of the sphere is given in terms of a sum of rational functions with first order poles in the positions of the zeroes and the branching points (of the conformal blocks) as their only singularities. The most general expression for a 1-form on a Riemann surface with first- order poles as its only singularities is given in refs. [16] , [17] :

$$Q_{ij}(z,\tau) = \frac{d}{dz}\sum_k h_k^{ij} Log(\theta(z-a_k)) \tag{18}$$

where θ is a theta-function with zero characteristics. The positions of the singularities are denoted by a_k. In particular for the case of the torus

$$Q_{ij}(z,\tau) = \sum_k h_k^{ij}\zeta(z-a_k,\tau) \tag{19}$$

where $\zeta(z,\tau)$ is the Weierstrass ζ-function:

$$\frac{d}{dz}\zeta(z) = -\wp(z) \tag{20}$$

and $\wp(z)$ is the Weierstrass function.

It is a well- known result [18] that the residua of 1-forms on a Riemann surface satisfy the constraint

$$\sum_k h_k = 0 \tag{21}$$

Moreover, using the known transformation properties of the θ-function under modular transformation

$$\begin{aligned} \theta(z/\tau,-1/\tau) &= \frac{1}{\sqrt{\tau}} exp(i\pi z^2/\tau)\theta(z,\tau) \\ \theta(z,\tau+1) &= exp(i\pi/4)\theta(z,\tau) \end{aligned} \tag{22}$$

together with eqs. (16) and (18) we get

$$\sum_k h_k a_k = 0 \tag{23}$$

Equations (21) and (23) summarize the constraints that are imposed on the matrix Q by eq. (16) . The monodromy data can be alternatively described in terms of the positions of the singular points a_k and the matrices of residua h_k.

We would like to conclude this letter by giving as an example the explicit construction of the differential equations of the k=1 SU(2) WZW 2-point correlator on the torus. In this case we have two conformal blocks. It is easy to check that the leading singularities in z as determined by the O.P.E. are -1/2 and 3/2 respectively. The difference in the leading behaviour is due to the vanishing of the first two leading coefficients in the O.P.E. which controls the behaviour of the second block [7]. However, since the sum of these two numbers is equal to ind_1 there are no apparent singularities and we can use our formalism to determine the two-point correlation function on the torus.

It is easy to construct the matrix elements of Q:

$$\begin{aligned}
Q_{12} =& \zeta(z+\omega_2)+\zeta(z-\omega_2)-2\zeta(z) \\
Q_{11} =& \frac{1}{2}(\zeta(z+\omega_3)+\zeta(z-\omega_3)-2\zeta(z))-xQ_{12} \\
Q_{22} =& xQ_{12}-\frac{1}{2}(\zeta(z-\omega_1)+\zeta(z+\omega_2-\omega_3)-2\zeta(z+\omega_2)) \\
& -(\zeta(z+\omega_3)+\zeta(z-\omega_3)-2\zeta(z)) \\
Q_{21} =& -yQ_{12}-u(\zeta(z+\omega_3)+\zeta(z-\omega_3)-2\zeta(z))
\end{aligned} \tag{24}$$

where

$$\begin{aligned}
x &= \frac{e_1-4e_2}{8(e_1-e_3)} \\
y &= 4x^2+2x \\
u &= \frac{x(2e_2-4e_1)}{4e_1}+1+\frac{e_2}{e_1}
\end{aligned} \tag{25}$$

and e_i is $\wp(\omega_i)$; ω_1 and ω_3 are half periods on the torus and $\omega_2=-(\omega_1+\omega_3)$.

It is straightforward to check that all constraint equations are indeed satisfied by using the well-known identities (see e.g. refs.[16] , [17])

$$\begin{aligned}
\zeta(z+\omega_\alpha)+\zeta(z-\omega_\alpha)-2\zeta(z) &= \frac{\wp'(z)}{\wp(z)-e_\alpha} \\
\wp(z)-e_\alpha &= \frac{1}{4}(\zeta(z+\omega_\beta)+\zeta(z-\omega_\beta)-2\zeta(z))(\zeta(z+\omega_\gamma)+\zeta(z-\omega_\gamma)-2\zeta(z))
\end{aligned} \tag{26}$$

and the differential equation for the Weierstrass function.

We can replace the system of two first-order equations by a single second-order differential equation using the second equation of this system to eliminate the second component. We end with the following equation

$$y''-\frac{3}{4}\wp(z)y=0 \tag{27}$$

This is precisely the equation which was derived in ref.[7] by using a different approach. From our investigation it is clear that this equation results from the Riemann monodromy problem. Since there are no apparent singularities, we indeed expect that the solution to the Riemann monodromy problem is determined uniquely. It is also easy to check that the solution of this equation has the right monodromy properties.

We summarize this letter by repeating our main results:

1. The differential equations for the correlators on the Riemann surfaces (e.g. those given in ref. [7]) follow from the Riemann monodromy problem.

2. When no apparent singularities are present the Riemann monodromy problem uniquely determines the correlators once the orders of the leading singularities are given (including the integer parts). When $ind_1 = ind_2$ the orders of the singularities are determined by the conformal diminsions (including the integer part).

3. The monodromy data on any given Riemann surface is identical to the one on the sphere i.e. given in terms of the crossing (braid) matrices and conformal dimensions.

4. The solution of the Riemann monodromy problem reflects the intimate relationship between the s-point function on a genus g surface and the s-2 point function on a genus g+1 Riemann surface.

After finishing this paper we received a preprint by E. Kiritsis [19] which addresses related issues .

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FIGURE CAPTIONS

Fig.1: Analytical continuation around z=0 in terms of $M_\sigma M_\tau M_\sigma^{-1} M_\tau^{-1}$.
Fig.2: Relation between the analytic continuation around z=0 for the 2-point correlation function on the torus and M_t.
Fig.3: Pictorial representation of the transformation associated with M_τ.

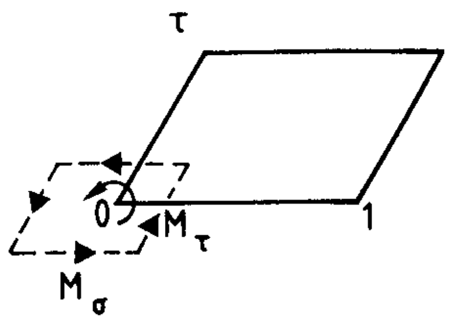

Fig. 1

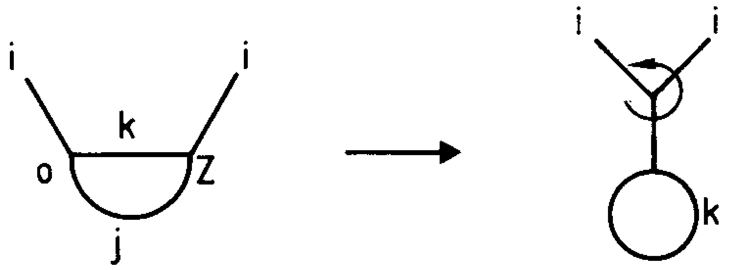

Fig. 2

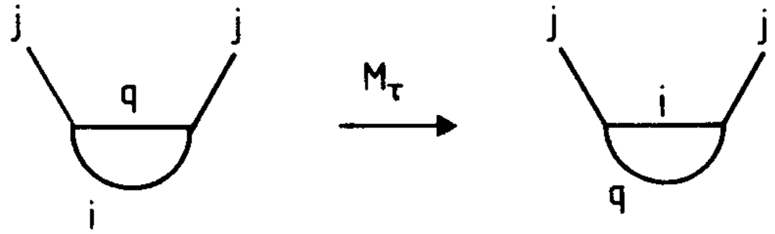

Fig. 3