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Modular Invariance in Supersymmetric Field Theories*

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ABSTRACT

We show how duality invariance of the supergravity action restricts the Kähler-potential and the superpotential and connects them to the theory of modular forms. This has relevance in string-induced supergravity for those scalar fields which are moduli of the underlying string compactification. We also discuss the restrictions imposed on globally supersymmetric theories.

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It is well known [1, 2] that the spectrum of a closed string, when compactified on a circle of radius R, is invariant under the discrete duality transformation $R \to \frac{1}{2R}$. In fact, this duality transformation is an exact symmetry to all orders in string perturbation theory, and the moduli space of such compactifications can be taken to be $R \in [\frac{1}{\sqrt{2}}, \infty)$ instead of $R \in \mathbf{R}^+$. More generally, the moduli space of the heterotic string compactified on a D-dimensional torus has locally the structure of the coset space $\frac{SO(16+D,D)}{SO(16+D)\times SO(D)}$ [3, 4]. However, taking into account the invariance of the spectrum under generalized duality transformations $SO(16+D, D; \mathbf{Z})$ [5, 6, 7] the moduli space is not a coset manifold but actually a fundamental region where points connected by the discrete $SO(16+D, D; \mathbf{Z})$ transformations are identified. This space is in general not a manifold but has orbifold singularities at points fixed by finite subgroups of $SO(16+D, D; \mathbf{Z})$. It seems to be a general feature that these special points correspond to compactifications with enhanced gauge symmetry; in addition, the vacuum energy is extremized at isolated fixed points [5]. For $D = 6$ the resulting effective low energy field theory possesses $N = 4$ space-time supersymmetry. The couplings of the 22 massless $N = 4$ vector multiplets (which contain the 132 moduli of the torus compactification) to supergravity are uniquely described by a non-linear σ-model with underlying coset $\frac{SO(22,6)}{SO(22)\times SO(6)}$ [8]. Recently it was also shown [9, 10] that duality symmetry is preserved for orbifold compactifications of the heterotic string. Aside from this specific context of superstring theories on which we mainly focus in this note, one may investigate the role of duality also in a broader class of field theories and their supersymmetric extensions, as discussed in reference [5].

To study the implications of the modular invariance for the supergravity action of massless matter fields, let us consider the simplified model of one chiral multiplet ϕ coupled to $N = 1$ supergravity. We denote the complex scalar component of ϕ by $t = 2(R^2 + ib)$ [11, 12], where b and R are real. In a string theory context the parameter t could be, for example, the complex modulus describing two-dimensional torus compactifications [5, 7] with background metric $G_{ij} = R^2\delta_{ij}$ $(i, j = 1, 2)$ and internal axion $B_{12} = b$ [11, 12]. More generally, one may think of t as being the modulus whose real part describes the overall scale of a compact six-manifold the string is compactified on, and whose imaginary part is the internal axion. The duality transformations are now simply $SL(2;\mathbf{Z})$ transformations of t:

$$t \to \frac{at - ib}{ict + d}, \qquad ad - bc = 1. \tag{1}$$

In terms of the field $\tau = it$ these are the usual $SL(2;\mathbf{Z})$ transformations. The corresponding non-linear σ-model is based on the coset space $\frac{SU(1,1)}{U(1)} = \frac{SL(2;\mathbf{R})}{U(1)}$ which is isomorphic to the complex upper half-plane $\mathrm{Im}\,\tau \geq 0$. Dividing the upper half-plane by the action of the modular group restricts the modular space of the τ-field to the fundamental domain $\{|\tau| \geq 1,\ 0 \leq \mathrm{Re}\,\tau \leq \frac{1}{2},\ \mathrm{Im}\,\tau > 0\} \cup \{|\tau| > 1,\ -\frac{1}{2} < \mathrm{Re}\,\tau < 0,\ \mathrm{Im}\,\tau > 0\}$.

The standard supergravity action [13] of the t-field is completely specified by the Kähler potential $K(t,\bar{t}) = -n\log(t+\bar{t})$ where the integer n is related to the curvature of $\frac{SU(1,1)}{U(1)}$ ($n = 3$ for compactification on a six-dimensional manifold). This Kähler potential leads to the correct Kähler metric $K_{t\bar{t}} = \partial_t\partial_{\bar{t}}K(t,\bar{t})$ of the $\frac{SU(1,1)}{U(1)}$ non-linear σ-model with bosonic action

$$S = K_{t\bar{t}}\partial_\mu t\partial^\mu \bar{t} = \frac{n}{(t+\bar{t})^2}\partial_\mu t\partial^\mu \bar{t}. \tag{2}$$

Here we have assumed that the superpotential of the t-field vanishes. In string theory this is true at least perturbatively, reflecting the fact that t is a modulus of the underlying compact six-manifold. (For $(2,2)$ compactifications the t-field superpotential vanishes even after taking into account non-perturbative σ-model corrections. In $(0,2)$ compactifications the superpotential may receive non-vanishing contributions due to world-sheet instantons [12, 14].) The action eq.(2) is trivially invariant under $SL(2;\mathbf{Z})$ duality transformations, since it is invariant under $SL(2,\mathbf{R})$ due to is geometrical interpretation as coset non-linear σ-model. If we now add a superpotential $W(t)$, the question of $SL(2;\mathbf{Z})$ invariance becomes non-trivial. In a string theory context we might think that the origin of the superpotential is due to non-perturbative string effects which lift the vacuum degeneracy of the background fields. It is easy to show that a non-vanishing superpotential explicitly breaks $SL(2;\mathbf{R})$ invariance. However, we still want to demand invariance under the duality group $SL(2;\mathbf{Z})$ since we restrict the parameter domain of integration of t to the fundamental region. This requirement gives severe restrictions on the form of the superpotential $W(t)$ and establishes a connection to the theory of modular forms [15]. In the following we give some examples.

For our first example, let us consider how to implement modular invariance in field theories with global supersymmetry. Here the Kähler potential $K(\phi, \bar{\phi})$ and the superpotential $W(\phi)$ are unconnected and the non-linear σ-model action has the form [16]

$$S = \int d^4x d^4\theta K(\phi, \bar{\phi}) + \int d^4x d^2\theta W(\phi) + \text{h.c.} \tag{3}$$

The chiral superfield ϕ transforms under duality transformations like its scalar component t. Then the transformation of its fermionic component χ is $\chi \to (ict+d)^{-2}\chi$. In order for the globally supersymmetric σ-model to be $SL(2;\mathbf{Z})$ invariant we have to demand that the Kähler potential be invariant up to a Kähler transformation. The superpotential, being holomorphic, must be modular invariant; i.e.

$$\begin{aligned} K(t,\bar{t}) &\to K(t,\bar{t}) + f(t) + \bar{f}(\bar{t}) \\ W(t) &\to W(t). \end{aligned} \tag{4}$$

For the superpotential we can take any polynomial of the modular function $j(q)$ which is given by

$$j(q) = \frac{3^6 5^3}{\pi^{12}} \frac{G_4^3(q)}{\eta(q)^{12}} = \frac{1}{q} + 744 + 196884q + \ldots \tag{5}$$

(A definition of the Eisenstein function $G_4(q)$ will be given below. $\eta(q)$ is the Dedekind eta-function.) q is related to t via $q = e^{2\pi i\tau} = e^{-2\pi t}$. $j(q)$ has a triple zero at $\tau = i$ and a pole at $\tau = i\infty$.

Let us now turn to the more interesting case of local supersymmetry [13]. The Kähler potential and the superpotential are now connected and the matter part of the supergravity Lagrangian is now described by a single function

$$G(t,\bar{t}) = K(t,\bar{t}) + \log W(t) + \log \bar{W}(\bar{t}). \tag{6}$$

The component form of the action is

$$\begin{aligned} e^{-1}\mathcal{L} = e^{G}\left\{3 - G_t(G_{t\bar{t}})^{-1}G_{\bar{t}}\right\} + \Big\{ & e^{G/2}\left[-G_{tt} - (G_t)^2 + G_t(G_{t\bar{t}})^{-1}G_{\bar{t}tt}\right]\bar{\chi}_L\chi_L \\ & + e^{G/2}\bar{\psi}_{\mu R}\sigma^{\mu\nu}\psi_{\nu R} - e^{G/2}G_t\bar{\psi}_R\cdot\gamma\chi_L + \text{h.c.}\Big\} + (\text{terms not involving } e^{G}). \end{aligned} \tag{7}$$

Here we have only written down the terms which arise after adding the superpotential (ψ_μ is the gravitino). The first two terms correspond to the scalar potential and the Yukawa couplings. Because of the appearance of e^G in the above Lagrangian we have to demand that G is modular invariant. It is then easy to check that the action eq.(7) is invariant under $SL(2;\mathbf{Z})$ transformations.

Modular invariance of G can now be implemented in two different ways. The first possibility is that $K(t,\bar{t})$ is invariant up to a Kähler transformation which now has to be absorbed by the transformation of the superpotential $W(t)$. To be specific, let us choose again $K(t,\bar{t}) = -n\log(t+\bar{t})$. Then all terms in the Lagrangian which are not proportional to e^G, that is all those we have not displayed in eq.(7), are automatically $SL(2;\mathbf{R})$ invariant due to the geometric construction of the non-linear σ-model. On the other hand, the terms in eq.(7) can never be $SL(2;\mathbf{R})$ invariant. However $SL(2;\mathbf{Z})$ invariance can be maintained if the superpotential transforms under modular transformations like a modular function of weight $-n$, up to a t-independent phase; i.e. if

$$W(t) \to e^{i\alpha(a,b,c,d)}(ict+d)^{-n}W(t). \tag{8}$$

Let us study this situation more carefully. G_t is given by

$$G_t(t,\bar{t}) = -\frac{n}{t+\bar{t}} + \frac{\partial \log W}{\partial t} \tag{9}$$

Since G is modular invariant, G_t, which is non-holomorphic, must transform with weight 2, i.e. $G_t \to (ict+d)^2 G_t$. $\partial_t \log W$ on the other hand is holomorphic but transforms non-covariantly under modular transformations:

$$\partial_t \log W(t) \to (ict+d)^2 \partial_t \log W(t) - inc(ict+d). \tag{10}$$

Functions with exactly these transformation properties are known from the theory of modular forms. Consider the Eisenstein functions $G_{2k}(\tau)$ [15, 17]:

$$G_{2k}(\tau) = \sum_{m,n\in\mathbf{Z}}{}'(m\tau+n)^{-2k}. \tag{11}$$

For $k>1$ these are holomorphic functions of modular weight $2k$. For $k=1$ however, the sum does not converge and a necessary regularization procedure leads to two alternative definitions of G_2:

$$\begin{aligned}\hat{G}_2(\tau) &= \sum_{m,n}{}' \lim_{s\to 0}(m\tau+n)^{-2}|m\tau+n|^{-s} \\ G_2(\tau) &= 2\zeta(2) + 2\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}(m\tau+n)^{-2}\end{aligned} \tag{12}$$

The regularization destroys either holomorphicity or modular covariance. $\hat{G}_2$ is of weight two but not holomorphic whereas G_2 is holomorphic and transforms under $SL(2;\mathbf{Z})$ as

$$G_2(t) \to (ict+d)^2 G_2(t) - 2\pi ic(ict+d). \tag{13}$$

G_2 and $\hat{G}_2$ are related by

$$\hat{G}_2(t) = G_2(t) - \frac{\pi}{\mathrm{Re}\, t} \tag{14}$$

This is exactly what we need for the construction of the supergravity action. Namely, suppose we make the identifications

$$\begin{aligned} \hat{G}_2(t) &= \frac{2\pi}{n} G_t(t) \\ G_2(t) &= \frac{2\pi}{n} \partial_t \log W(t) \end{aligned} \tag{15}$$

Then we obtain the following expression for the superpotential $W(t)$:

$$\begin{aligned} W(t) &= \exp\left\{\frac{n}{2\pi}\int^t \mathrm{d}t' G_2(t')\right\} = \left[\eta(t)\right]^{-2n} \\ &= e^{n\pi t/6}\left(1 + 2ne^{-2\pi nt} + 2n(2+n)e^{-4\pi nt} + \ldots\right) \end{aligned} \tag{16}$$

which is of the type one expects from non-perturbative string effects. Supersymmetry is unbroken for $\partial_t(e^{G/2}) = 0$. This occurs at the two orbifold points $t = 1$ and $t = e^{-\pi i/6}$ of the fundamental region for t which are the zeros of G_t. At these points the superpotential is finite (since the only zero of $\eta(t)$ is at $t = \infty$) and the cosmological constant is minimized. In fact, the superpotential will always be extremized at these two points [5]. The gravitino mass is given by

$$e^{G(t)} = \frac{1}{(t+\bar{t})^n}|\eta(t)|^{-4n}. \tag{17}$$

Note that this is the one-loop partition function of the bosonic string in $2n$ transverse dimensions.

Let us now turn to the second possibility to obtain a modular invariant $G(t,\bar{t})$. This is given by the choice of separately modular invariant expressions for $K(t,\bar{t})$ and $W(t)$. One possibility along these lines is to replace in $K = -\log(t+\bar{t})^n \sim -\log V$ ($V \sim R^{2n}$ is the volume of the $2n$-dimensional internal space) the volume V by a modular invariant expression which behaves in the limit $R \to \infty$ as R^{2n} and in the limit $R \to 0$ as R^{-2n} [6]. For the simple case considered here, the ansatz of ref. [6] gives

$$K(t,\bar{t}) = -2n \log\Big(\sum_{p,q \in \mathbf{Z}} \exp(-\frac{\pi}{\mathrm{Re} t}|p + iqt|^2)\Big) \tag{18}$$

Other interesting examples of this kind of scenario are the no-scale supergravity theories [18] with vanishing scalar potential $e^G(-3 + G_t G_{t\bar{t}}^{-1} G_{\bar{t}}) \equiv 0$. This leads to broken supersymmetry with vanishing cosmological constant if the superpotential is a constant and the Kähler potential is chosen as $K(t,\bar{t}) = -3\log(F(j(t)) + \bar{F}(\bar{j}(\bar{t})))$ where F is such that $\mathrm{Re} F > 0$ for t in the fundamental domain.

In closing, we would like to mention further extensions of this analysis. It is clearly trivial to extend it to theories with N chiral superfields t_i with Kähler potential $G = \sum_{i=1}^{N} G_i(t_i, \bar{t}_i)$. Then duality invariance extends to an invariance under $\big(SL(2;\mathbf{Z})\big)^N$. This, e.g., applies to toroidal compactification of two dimensions with $N = 2$ [5, 7]. It is, however, more interesting to consider cases where the moduli space does not have this simple product structure. As discussed in refs. [6, 5], for background fields

$$G = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \qquad\qquad B = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \tag{19}$$

where g and b are symmetric $n \times n$ matrices, the relevant duality transformations are generated by elements of the symplectic modular group $Sp(2n;\mathbf{Z})$ acting on the complex matrix $b + ig$. Therefore, the constraint of modular invariance of the supergravity action is related to the theory of modular functions on Riemann surfaces of genus n.

We can also imagine several other, less straightforward extensions. One interesting problem is the inclusion of gauge fields and the dilaton multiplet and the relation of gauge and modular invariance in the corresponding supergravity theories. Another is to understand the role played by duality in theories based on Calabi-Yau or K_3 compactifications.

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