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Schrödinger-picture field theory in Robertson-Walker flat spacetimes

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A Schrödinger formulation of a free scalar field propagating in Robertson-Walker flat spacetimes is presented. Gaussian wave functionals which describe homogeneous, isotropic quantum states are considered. Several identities relating the real and imaginary parts of the Gaussian width are derived. Covariant conservation of the stress tensor is demonstrated using these relations. The formalism is applied in detail to reproduce known results in a de Sitter background.

INTRODUCTION

The Schrödinger picture affords a relatively clear viewpoint from which to understand many aspects of quantum field theory,[1] and has been developed in a curved-space background with some success in recent years.[2] This formulation of the theory is particularly appealing in Robertson-Walker spacetimes because of the natural time slicing which appears in these metrics and because of the maximal symmetry of the associated spatial sections. Many standard calculations in curved space[3] can be reproduced in a straightforward manner, and in particular, the de Sitter-space vacuum expectation value of $T_{\mu\nu}$ is easily obtained.

We begin with a general discussion of the Schrödinger formulation of the quantum theory of a free scalar field in a Robertson-Walker flat background. The work proceeds in terms of the Fourier transform of the field configuration. The wave functional is assumed to have a general Gaussian form, which is then simplified by enforcing homogeneity and isotropy. We include a thorough discussion of the Schrödinger equation as it applies to this class of states. Specifically, we obtain in Appendix A several identities obeyed by the Gaussian width in momentum-space field variables, and using these, verify covariant conservation of the stress tensor. This is important since it verifies the consistency of the use of this stress tensor as the right-hand side of the Einstein equations in a semiclassical treatment of general relativity. We argue for the naturalness of the Bunch-Davies vacuum entirely within the Schrödinger formalism, and discover a one-parameter family of vacua for which $\langle \varphi^2 \rangle$ is constant. In Appendix B the Schrödinger propagator is calculated. We conclude the paper with a derivation of the renormalized vacuum stress tensor in the Bunch-Davies vacuum, employing dimensional regularization and DeWitt-Schwinger subtractions.

Our conventions are

$$\eta_{\mu\nu}=\text{diag}(-1,1,1,1),\quad R^{\mu}_{\ \nu\alpha\beta}\equiv\Gamma^{\mu}_{\nu\alpha\beta}-\Gamma^{\mu}_{\nu\beta,\alpha},\ \ldots,$$

$$\Gamma^{\mu}_{\alpha\beta}\equiv\tfrac{1}{2}g^{\mu\rho}(g_{\rho\alpha,\beta}+g_{\rho\beta,\alpha}-g_{\alpha\beta,\rho}),\quad \hbar=c=1\ .$$

Furthermore, since we will be working on d-dimensional, flat, spacelike hypersurfaces with coordinates x_i, we will adopt the notation $x=(x_1,\ldots,x_d)$. Similarly, for the space of Fourier transforms with coordinates k_i, we will write $k=(k_1,\ldots,k_d)$, and for general d-vectors a and b, $a\cdot b=a_1b_1+\cdots+a_db_d$.

I. FUNCTIONAL SCHRÖDINGER PICTURE OF QUANTUM FIELD THEORY ON A ROBERTSON-WALKER FLAT SPACETIME

In this discussion, the spacetime is assumed to admit a description by Robertson-Walker flat coordinates, with the line element $ds^2=-dt^2+a^2(t)d\mathbf{x}^2$. The timelike vector ∂_t which appears in these coordinates is unit normal to the family of flat, spacelike hypersurfaces, $t=$const. In this case, the Ricci tensor $R_{\mu\nu}$ can be expressed in terms of a and its time derivatives:

$$R_{00}=-d(\dot{H}+H^2),\quad R_{ij}=a^2(\dot{H}+dH^2)\delta_{ij},\quad R_{0i}=0\ ,$$

where $H\equiv(\dot{a}/a)$ and $\dot{f}(x;t)\equiv\partial_t f(x;t)$. Contracting $R_{\mu\nu}$ yields the Ricci scalar $R=d[2\dot{H}+(d+1)H^2]$.

A free scalar field of mass m propagating in a background $(d+1)$-dimensional spacetime with metric $g_{\mu\nu}$ is described classically by the action

$$S=\int d^{d+1}x\sqrt{-g}\,\mathcal{L}\ ,$$

where (1.1)

$$\mathcal{L}=-\tfrac{1}{2}[g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi+(m^2+\xi R)\varphi^2]\ .$$

The number of space dimensions d is analytically continued in order to regularize the theory and will be set equal to 3 when the renormalization is complete.

It is now straightforward to define a Hamiltonian on a hypersurface of fixed t. For this purpose, it is convenient to work with a Lagrangian density $L\equiv\sqrt{-g}\,\mathcal{L}$ rather than the more usual scalar $\mathcal{L}$. In Robertson-Walker coordinates

$$L=\tfrac{1}{2}\sqrt{-g}\,[(\dot{\varphi})^2-a^{-2}|\nabla\varphi|^2-(m^2+\xi R)\varphi^2]\ ,$$

where $\sqrt{-g}=a^d(t)$. The momentum density Π may now be defined canonically from L, $\Pi\equiv\partial L/\partial\dot{\varphi}=\sqrt{-g}\,\dot{\varphi}$, in terms of which a time-dependent Hamiltonian $H[\varphi,\Pi;t]$ may be constructed through a functional Legendre transformation:

$$H[\varphi,\Pi;t]\equiv\int d^dx[\Pi\dot{\varphi}-L(\varphi,\dot{\varphi})]$$
$$=\frac{1}{2}\int d^dx\sqrt{-g}\left[\frac{\Pi^2}{|g|}+a^{-2}|\nabla\varphi|^2+(m^2+\xi R)\varphi^2\right]. \tag{1.2}$$

To quantize this theory, equal-time commutation relations are introduced among the field operators φ and Π: i.e.,

$$[\varphi(x,t),\Pi(x',t)]=i\delta^d(x-x'),$$
$$[\varphi(x,t),\varphi(x',t)]=0, \tag{1.3}$$

and

$$[\Pi(x,t),\Pi(x',t)]=0.$$

If we were to proceed in the Heisenberg formulation of quantum field theory, the time dependence of an arbitrary operator $O[\varphi,\Pi,t]$ would be defined by the equation of motion

$$i\dot{O}=[O,H]+i\partial_t O.$$

In particular, the evolution equations for φ and Π,

$$\dot{\varphi}=\frac{1}{\sqrt{-g}}\Pi$$

and

$$\dot{\Pi}=\sqrt{-g}\,[a^{-2}\nabla^2\varphi-(m^2+\xi R)\varphi],$$

would imply the operator field equation

$$(-\Box+m^2+\xi R)\varphi=0.$$

We choose, however, to work within the functional Schrödinger formulation.[4] At any fixed time t_0, the eigenstates of the field operator $\varphi(x)$ are assumed to form a complete orthonormal set $\{|\phi(x)\rangle\}$; i.e., $|\phi(x)\rangle$ satisfies

$$\varphi(x)|\phi(x)\rangle=\phi(x)|\phi(x)\rangle,$$
$$\langle\phi(x)|\tilde{\phi}(x)\rangle=\delta(\phi(x)-\tilde{\phi}(x)),$$

and

$$\int D\phi(x)|\phi(x)\rangle\langle\phi(x)|=1.$$

An arbitrary quantum-mechanical state $|\Psi(t)\rangle$ evolves according to the Schrödinger equation $H_{\rm op}|\Psi(t)\rangle=i\partial_t|\Psi(t)\rangle$ and may be expanded as

$$|\Psi(t)\rangle=\int\mathcal{D}\phi(x)|\phi(x)\rangle\langle\phi(x)|\Psi(t)\rangle.$$

Here $H_{\rm op}$ is defined canonically as in Eq. (1.2). The wave functional $\Psi[\phi(x),t]\equiv\langle\phi(x)|\Psi(t)\rangle$ which represents $|\Psi(t)\rangle$ is a c-number functional of the field $\phi(x)$. In this representation, the action of $\varphi(x)$ on the state is realized as multiplication by the c-number function $\phi(x)$,

$$\langle\phi(x)|\varphi(x)|\Psi(t)\rangle=\phi(x)\langle\phi(x)|\Psi(t)\rangle,$$

and that of $\Pi(x)$ by functional differentiation,

$$\langle\phi(x)|\Pi(x)|\Psi(t)\rangle=-i\frac{\delta}{\delta\phi(x)}\langle\phi(x)|\Psi(t)\rangle.$$

The Schrödinger equation for a free scalar field theory on a fixed Robertson-Walker flat background is thus expressed as a second-order functional partial differential equation

$$\frac{1}{2}\sqrt{-g}\int d^dx\left[-\frac{1}{|g|}\frac{\delta^2}{\delta\phi(x)^2}+a^{-2}|\nabla\phi|^2+(m^2+\xi R)\phi^2\right]\Psi[\phi(x),t]=i\partial_t\Psi[\phi(x),t]. \tag{1.4}$$

[The propagator $\langle\tilde{\phi}(x),t|\phi(x),0\rangle$ will be calculated for de Sitter space in Appendix B. In this notation the state $|\phi(x),t\rangle$ is the Heisenberg eigenket of the Heisenberg operator $\varphi(x,t)$ with functional eigenvalue $\phi(x)$.]

Because the spatial sections are flat, it is natural to perform a Fourier transformation on the space dependence of the field configuration: $\phi(x)=\int d\tilde{k}\,e^{ik\cdot x}\alpha(k)$, where $d\tilde{k}\equiv d^dk/(2\pi)^d$. Similarly, one can implicitly define $\delta/\delta\alpha(k)$ by

$$\frac{\delta}{\delta\phi(x)}=\int d\tilde{k}\,e^{ik\cdot x}\frac{\delta}{\delta\alpha(k)}.$$

The reality of ϕ implies $\alpha(k)^*=\alpha(-k)$ and since

$$\frac{\delta\phi(x)}{\delta\phi(x')}=\delta^d(x-x')$$

it follows that

$$\frac{\delta\alpha(k)}{\delta\alpha(k')}=(2\pi)^d\delta^d(k+k').$$

Functional integrals in the k-space variables require extra care because of the constraint $\alpha(k)^*=\alpha(-k)$. The path-integral measure $\int\mathcal{D}\phi(x)$ is rewritten as

$$\int\mathcal{D}\alpha(k)\mathcal{D}\alpha^*(k)\delta(\alpha(-k)-\alpha^*(k)).$$

In the expression which results, it is simplest to perform all partial functional derivatives before carrying out the functional integrations. If this is done, the $\alpha^*(k)$ integration will simply kill the δ function and replace $\alpha^*(k)$ by $\alpha(-k)$ throughout. [In the expressions that follow, we assume the $\alpha^*(k)$ integration has been performed and the constraint has been enforced.]

In k-space, multiplication by k replaces the gradient operator, so that

$$H_{\rm op}=\frac{1}{2}\int d\tilde{k}\left[-\frac{1}{\sqrt{-g}}\frac{\delta^2}{\delta\alpha(k)\delta\alpha(-k)}+\sqrt{-g}\,(a^{-2}k^2+m^2+\xi R)\alpha(k)\alpha(-k)\right]. \tag{1.5}$$

As expected in a free theory, a complete decoupling of modes has been achieved by the Fourier transformation. In fact, for each k, the integrand in Eq. (1.5) represents a harmonic oscillator with the time-dependent mass $\sqrt{-g}$ and frequency $(a^{-2}k^2+m^2+\xi R)^{1/2}$. It is also possible to express $H_{\rm op}$ in the form

$$H_{\rm op}=\tfrac{1}{2}\int d\tilde{k}[\mathcal{A}^{\dagger}(k)\mathcal{A}(k)+V_d](a^{-2}k^2+m^2+\xi R)\ ,$$

where

$$\mathcal{A}(k;t)=-i\frac{1}{[2\sqrt{-g}\,(a^{-2}k^2+m^2+\xi R)]^{1/2}}\times\left[\frac{\delta}{\delta\alpha(k)}+\sqrt{-g}\,(a^{-2}k^2+m^2+\xi R)\alpha(k)\right]$$

and $V_d=(2\pi)^d\delta_k(0)$.

In general, we do not expect the physical states to eigenstates of the (time-dependent) Hamiltonian defined in Eq. (1.5) (Ref. 5). However, the wave functional describing a Fock vacuum can still be written as a Gaussian in the field variables:[6]

$$\Psi=\mathcal{N}_0(t)\exp\left[-\tfrac{1}{2}\int d\tilde{k}\,d\tilde{k}'\tilde{\mathcal{A}}(k,k';t)\alpha(k)\alpha(k')-i\Omega(t)\right]. \tag{1.6}$$

Furthermore, the requirements of spatial homogeneity and isotropy restrict $\tilde{A}$ to take the form

$$\tilde{A}(k,k';t)\sim(2\pi)^dA(\,|\,k\,|\,,t)\delta(k+k')\ ,$$

so that

$$\Psi=\mathcal{N}_0(t)\exp\left[-\tfrac{1}{2}\int d\tilde{k}\;A(\,|\,k\,|\,,t)\alpha(k)\alpha(-k)-i\Omega(t)\right]. \tag{1.7}$$

That this Ψ describes a homogeneous and isotropic state is obvious. That this is the most general allowed (Gaussian) form can be shown as follows. Homogeneity implies that $\Psi[\phi(x)]=\Psi[\phi(x+s)]$, where s represents an arbitrary spatial displacement. The α's are now expressed in terms of the ϕ's, so that

$$\Psi[\phi(x+s)]=\exp\left[-\tfrac{1}{2}\int d\tilde{k}\,\tilde{k}'\tilde{\mathcal{A}}(k,k';t)\int d^dx\,e^{-ikx}\phi(x+s)\int d^dx' e^{-ik'x'}\phi(x'+s)\right].$$

Then

$$\left.\frac{\partial}{\partial s_i}\Psi[\phi(x+s)]\right|_{s=0}=0$$

implies

$$\int d\tilde{k}\,d\tilde{k}'\tilde{A}(k,k';t)(k_i+k_i')\alpha(k)\alpha(k')=0$$

for all functions $\alpha(k)$ and thus

$$\tilde{A}(k,k';t)=(2\pi)^dA(k;t)\delta(k+k')\ .$$

Isotropy then implies that $A(k)=A(\,|\,k\,|\,)$. In momentum space, one can say that the zero-point fluctuations must be generated "back to back" and "equally in all directions" in order to preserve the symmetries of the space sections.

If the metric had contained any space dependence, the modes would not have decoupled and, as a consequence, the Gaussian form in Eq. (1.6) would not have simplified.

The equation of motion for $A(k,t)$ is obtained by substituting Eq. (1.7) into the Schrödinger equation and equating coefficients of $\alpha(k)\alpha(-k)$:

$$i\dot{A}(k,t)=\frac{A^2(k,t)}{\sqrt{-g(t)}}-\sqrt{-g}\,[a^{-2}(t)k^2+m^2+\xi R]\ . \tag{1.8}$$

By specifying the initial conditions, we obtain a unique solution to Eq. (1.8). This in turn completely determines our vacuum wave functional. In other words, the choice of $A(k,t_{\rm init})$ is the Schrödinger picture analog to the definition of positive frequency in the canonical formalism. The following definitions will facilitate the solution of Eq. (1.8) later.

We introduce an intermediate variable Γ defined by $A=\sqrt{-g}[\Gamma+(id/2)H]$, where $H=\dot{a}/a$, so that

$$i\dot{\Gamma}=\Gamma^2+\frac{d^2}{4}H^2+\frac{d}{2}\dot{H}-(a^{-2}k^2+m^2+\xi R)\ . \tag{1.9}$$

Now $u(k,t)$ defined by

$$u(k,t)\equiv u(k,0)\exp\left[i\int_0^t\Gamma(k,t)dt\right], \tag{1.10}$$

satisfies

$$\ddot{u}-\left[\frac{d^2}{4}H^2+\frac{d}{2}\dot{H}-(a^{-2}k^2+m^2+\xi R)\right]u=0\ . \tag{1.11}$$

An equation of the same form as the Fourier-transformed wave equation (including the Robertson-Walker "damping" factor) is obtained by defining $w=(-g)^{-1/4}u$, which then satisfies

$$\ddot{w}+dH\,\dot{w}+(a^{-2}k^2+m^2+\xi R)w=0\ . \tag{1.12}$$

The phase Ω which appears in Eq. (1.7) is determined by equating the α-independent terms in the Schrödinger equation

$$\dot{\Omega}(t)=-\frac{1}{2\sqrt{-g}}(2\pi)^d\delta_k(0)\int d\tilde{k}\;A(k,t)\ .$$

Ω will not in general be real, but it can always be absorbed into a time-dependent normalization $\mathcal{N}(t)=\mathcal{N}_0(t)\exp[-i\Omega(t)]$ and thus will not affect expectation values.

II. THE STRESS TENSOR

A. Evaluation of some fundamental two-point functions in terms of A

In the Schrödinger picture, the probability density in configuration space is defined as

$$|\Psi|^2=|\mathcal{N}(t)|^2\exp\left[-\tfrac{1}{2}\int d\tilde{q}\,2\,\mathrm{Re}\,A(q,t)\alpha(q)\alpha(-q)\right] .$$

This requires us to choose $\mathcal{N}(t)$, such that

$$\int \mathcal{D}\alpha(k)\,|\Psi[\alpha(k)]|^2=1 .$$

In addition, the vacuum expectation value of an arbitrary operator $\mathcal{O}$ is defined by

$$\langle \mathcal{O}\rangle \equiv \langle \Psi(t)|\mathcal{O}|\Psi(t)\rangle=\int \mathcal{D}\alpha(k)\Psi^*\mathcal{O}\Psi .$$

We proceed to calculate the two-point functions which appear in $\langle T_{\mu\nu}\rangle$.

1. $\langle \alpha(k)\alpha(k')\rangle$

$\langle \alpha(k)\alpha(k')\rangle$ will be evaluated by introducing a source term in the vacuum probability density as follows:

$$|\Psi|^2[j]\equiv |\Psi|^2\exp\left[\int d\tilde{k}\,\alpha(k)j(k)\right] .$$

By functionally integrating this quantity over all $\alpha(k)$, functionally differentiating twice with respect to the source j, and then setting $j=0$, we obtain

$$\langle \alpha(k)\alpha(k')\rangle=\frac{\delta^2}{\delta j(k)\delta j(k')}|\mathcal{N}(t)|^2\int \mathcal{D}\alpha(q)\exp\left[-\tfrac{1}{2}\int d\tilde{q}\{2\,\mathrm{Re}[A(q,t)]\alpha(q)\alpha(-q)-2\alpha(q)j(q)\}\right]\Bigg|_{j=0}$$

$$=\frac{\delta^2}{\delta j(k)\delta j(k')}\exp\left[\tfrac{1}{2}\int d\tilde{q}\frac{(2\pi)^{2d}j(q)j(-q)}{2\,\mathrm{Re}\,A(q,t)}\right]\Bigg|_{j=0}=(2\pi)^d\frac{\delta(k+k')}{2\,\mathrm{Re}\,A(k,t)} . \tag{2.1}$$

This result amounts to the statement that the two-point function for a Gaussian distribution is just the squared width $\sim 1/(2\,\mathrm{Re}A)$.

2. $\langle \alpha(k)[\delta/\delta\alpha(k')]+[\delta/\delta\alpha(k)]\alpha(k')\rangle$

Using the fact that $[\delta/\delta\alpha(k)]\Psi=-A(k,t)\alpha(k)\Psi$, integrating by parts once in the second term, and dropping the surface term, we find

$$\left\langle \alpha(k)\frac{\delta}{\delta\alpha(k')}+\frac{\delta}{\delta\alpha(k)}\alpha(k')\right\rangle$$
$$=[-A(k',t)+A^*(k,t)]\langle\alpha(k)\alpha(k')\rangle$$
$$=-i\frac{\mathrm{Im}\,A}{\mathrm{Re}\,A}(2\pi)^d\delta(k+k') . \tag{2.2}$$

3. $\langle \delta^2/\delta\alpha(k)\delta\alpha(k')\rangle$

In order to evaluate $\langle \delta^2/\delta\alpha(k)\delta\alpha(k')\rangle$, it is again convenient to perform an integration by parts on the functional integral. Neglecting the surface term, we find

$$\left\langle\frac{\delta^2}{\delta\alpha(k)\delta\alpha(k')}\right\rangle=-\left\langle\frac{\overleftarrow{\delta}}{\delta\alpha(k)}\frac{\overrightarrow{\delta}}{\delta\alpha(k')}\right\rangle$$
$$=-A^*(k,t)A(k',t)\langle\alpha(k)\alpha(k')\rangle$$
$$=-\frac{|A(k,t)|^2}{2\,\mathrm{Re}\,A(k,t)}(2\pi)^d\delta(k+k') . \tag{2.3}$$

B. Mode sums

Having obtained expressions (2.1)–(2.3), it is a simple matter to express the relevant expectation values $\langle\varphi^2(t)\rangle$ and $\langle T_{\mu\nu}(t)\rangle$ in terms of A. $\langle\varphi^2\rangle$ takes a particularly simple form

$$\langle\varphi^2(t)\rangle=\int d\tilde{k}\,d\tilde{k}'\langle\alpha(k)\alpha(k')\rangle e^{i(k+k')\cdot x}$$
$$=\int\frac{d\tilde{k}}{2\,\mathrm{Re}\,A(k,t)} . \tag{2.4}$$

The space dependence in φ has been suppressed since the state considered is spatially homogeneous. The canonical stress tensor is given by

$$T^{\mathrm{can}}_{\mu\nu}=\partial_\mu\varphi\partial_\nu\varphi+g_{\mu\nu}\mathcal{L} .$$

For nonminimal coupling to curvature ($\sim\xi R\varphi^2$), this does not coincide with the gravitational stress tensor [which must appear on the right-hand side (RHS) of the Einstein equations]

$$T^{\mathrm{grav}}_{\mu\nu}\equiv\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}}$$
$$=T^{\mathrm{can}}_{\mu\nu}+\xi R_{\mu\nu}\varphi^2+\xi(g_{\mu\nu}\Box-D_\mu D_\nu)\varphi^2 . \tag{2.5}$$

In what follows, we will work only with $T^{\mathrm{grav}}_{\mu\nu}$ and will henceforth suppress the superscript. In Robertson-Walker flat coordinates, this definition reduces to

$$T_{00}=\tfrac{1}{2}\dot\varphi^2+\tfrac{1}{2}(a^{-2}|\nabla\varphi|^2+m^2\varphi^2)$$
$$+\xi G_{00}\varphi^2+\xi\,dH(\varphi\dot\varphi+\dot\varphi\varphi) ,$$
$$T_{ij}=\partial_i\varphi\partial_j\varphi-\tfrac{1}{2}\delta_{ij}a^2(-\dot\varphi^2+a^{-2}|\nabla\varphi|^2+m^2\varphi^2)$$
$$+\xi G_{ij}\varphi^2+\xi a\dot a\delta_{ij}(\varphi\dot\varphi+\dot\varphi\varphi)-2\xi a^2\delta_{ij}\dot\varphi^2$$
$$+2\xi\delta_{ij}|\nabla\varphi|^2+2\xi a^2\delta_{ij}(m^2+\xi R)\varphi^2 ,$$
$$T_{0i}=\partial_i\varphi\dot\varphi+2\xi H\varphi\partial_i\varphi .$$

Total spatial derivatives have been discarded, and use has been made of the operator equations of motion in order to express $T_{\mu\nu}$ entirely in terms of operators which have direct representations in the space of wave functionals. Hence, in the Schrödinger picture,

$$T_{00}=\tfrac{1}{2}\int d\tilde{k}\, d\tilde{k}\,'e^{i(k+k')\cdot x}\left[\frac{1}{|g|}\frac{-\delta^2}{\delta\alpha(k)\delta\alpha(k')}+(-a^{-2}k\cdot k'+m^2)\alpha(k)\alpha(k')\right.$$
$$\left.+2\xi G_{00}\alpha(k)\alpha(k')-i2\xi\frac{dH}{\sqrt{-g}}\left[\alpha(k)\frac{\delta}{\delta\alpha(k')}+\frac{\delta}{\delta\alpha(k)}\alpha(k')\right]\right],$$

$$T_{ij}=\frac{a^2}{2}\int d\tilde{k}\, d\tilde{k}\,'e^{i(k+k')\cdot x}\left[\frac{\delta_{ij}}{|g|}\frac{-\delta^2}{\delta\alpha(k)\delta\alpha(k')}+[a^{-2}(2k_ik_j-\delta_{ij}k^2)-\delta_{ij}m^2]\alpha(k)\alpha(k')\right.$$
$$+2\xi a^{-2}G_{ij}\alpha(k)\alpha(k')-i2\xi\delta_{ij}\frac{dH}{\sqrt{-g}}\left[\alpha(k)\frac{\delta}{\delta\alpha(k')}+\frac{\delta}{\delta\alpha(k)}\alpha(k')\right]$$
$$\left.+4\xi\frac{\delta_{ij}}{|g|}\frac{\delta^2}{\delta\alpha(k)\delta\alpha(k')}+4\xi\delta_{ij}(-a^{-2}k\cdot k'+m^2+\xi R)\alpha(k)\alpha(k')\right],$$

$$T_{0i}=\int d\tilde{k}\, d\tilde{k}\,'e^{i(k+k')\cdot x}\left[k_i\alpha(k)\frac{\delta}{\delta\alpha(k')}+i2\xi Hk_i\alpha(k)\alpha(k')\right].$$

The apparent ordering ambiguity in T_{0i} is unimportant since $[\alpha(k),\delta/\delta\alpha(k')]=-(2\pi)^d\delta(k+k')$, and the possible commutator term integrates to zero. The expectation value of the stress tensor can now be expressed entirely in terms of the inverse-squared Gaussian width A and the Robertson-Walker scale factor $a(t)$:

$$\langle T_{00}\rangle=\frac{1}{2}\int\frac{d\tilde{k}}{2\,\mathrm{Re}\,A(k,t)}\left[\frac{|A(k,t)|^2}{|g|}+(a^{-2}k^2+m^2)+2\xi G_{00}-2\xi\frac{dH}{\sqrt{-g}}2\,\mathrm{Im}\,A(k,t)\right],$$

$$\langle T_{ij}\rangle=\tfrac{1}{2}\delta_{ij}a^2\int\frac{d\tilde{k}}{2\,\mathrm{Re}\,A(k,t)}\left[\frac{|A(k,t)|^2}{|g|}+\left[\frac{2}{d}-1\right]a^{-2}k^2-m^2+\frac{2}{d}\xi a^{-2}\sum_l G_{ll}\right.$$
$$\left.-2\xi\frac{H}{\sqrt{-g}}\mathrm{Im}\,A(k,t)-4\xi\frac{1}{|g|}|A(k,t)|^2+4\xi(a^{-2}k^2+m^2+\xi R)\right], \tag{2.6}$$

$$\langle T_{0i}\rangle=0\ .$$

In establishing Eq. (2.6), we have made use of Eqs. (2.1), (2.2), and (2.3), and the following relations:

$$\int d\tilde{k}\, k^i f(|k|)=0\ ,$$

$$\int d\tilde{k}\, k^ik^jf(|k|)=\frac{1}{d}\delta_{ij}\int d\tilde{k}\, k^2f(|k|)\ .$$

The energy density and pressure can be defined by

$$\langle\rho\rangle\equiv\langle T_{00}\rangle$$

and

$$\langle P\rangle\equiv\frac{1}{d}g^{ij}\langle T_{ij}\rangle\ .$$

With respect to these quantities, the stress tensor takes the diagonal form

$$\langle T^\mu{}_\nu\rangle=\begin{bmatrix}-\langle\rho\rangle & 0 & 0 & 0\\ 0 & \langle P\rangle & 0 & 0\\ 0 & 0 & \langle P\rangle & 0\\ 0 & 0 & 0 & \langle P\rangle\end{bmatrix}, \tag{2.7}$$

which is characteristic of a perfect fluid; it is also the most general form consistent with the assumptions of spatial isotropy and homogeneity.

C. Conservation of the stress tensor

If the renormalized vacuum stress tensor $\langle T_{\mu\nu}\rangle_{\rm ren}$ is to be used as a source for the Einstein equations, it must be covariantly conserved, $D_\mu\langle T^{\mu\nu}\rangle_{\rm ren}=0$. Since the renormalization scheme we adopt later involves the subtraction only of terms which are themselves individually conserved, it is sufficient to check that the regularized $\langle T_{\mu\nu}\rangle$ is conserved.

In a Robertson-Walker flat background spacetime, the conservation equations reduce to the single equation

$$\langle\dot\rho\rangle+d\frac{\dot a}{a}\langle\rho+P\rangle=0\ .$$

Now treating $\langle\rho\rangle$ and $\langle P\rangle$ mode by mode, and omitting terms which cancel arithmetically, we can write

$$\langle \dot\rho_k \rangle + d\frac{\dot a}{a}\langle \rho_k + P_k \rangle = \left[\frac{1}{2|g|}d_t\left[\frac{|A|^2}{2\,\mathrm{Re}A}\right] + d_t\left[\frac{1}{2\,\mathrm{Re}A}\right]\frac{f}{2} - d_t\left[\frac{1}{2\,\mathrm{Re}A}\right]\right]\left[\xi\frac{R}{2} + \xi G_{00}\right]$$
$$ - \frac{\xi d}{\sqrt{-g}}\left[\frac{\mathrm{Im}A}{\mathrm{Re}A}\right](H^2+\dot H) - \frac{\xi d}{\sqrt{-g}}d_t\left[\frac{\mathrm{Im}A}{\mathrm{Re}A}\right] - \frac{\xi d}{|g|}\left[\frac{|A|^2}{\mathrm{Re}A}\right]H + \xi d\left[\frac{f}{\mathrm{Re}A}\right]H\Bigg] .$$

In obtaining this expression, use has been made of the relations $d_t(-g)^r = 2r\,dH(-g)^r$ and $D^\mu G_{\mu\nu}=0$ (the contracted Bianchi identity), and we have introduced $f \equiv (a^{-2}+m^2+\xi R)$. As for the remainder, the first two terms vanish together [Eq. (A7)], as do the third and fourth terms [Eq. (A4)]. Similarly, the fifth, sixth, and seventh terms sum to zero [Eq. (A5)], so that conservation of the vacuum stress tensor is established before summing over modes.[7]

III. DE SITTER SPACE

In de Sitter space, $a_0(t)=e^{H_0 t}$, and with the change of variable $H_0\tau \equiv e^{-H_0 t}$, Eq. (1.8) becomes

$$d_{k\tau}^2 u_0 + \frac{1}{k\tau}d_{k\tau}u_0 + \left[1-\frac{\nu^2}{(k\tau)^2}\right]u_0 = 0 , \tag{3.1}$$

where $\nu^2 = d^2/4 - \tilde m^2/H_0^2$ and $\tilde m^2 \equiv m^2 + d(d+1)\xi H_0^2$. This is Bessels' equation with the solution

$$u_0(k,t) = \mathcal{B}_1(k)H_\nu^1(k\tau) + \mathcal{B}_2(k)H_\nu^2(k\tau) , \tag{3.2}$$

where $H_\nu^{(1)}$ and $H_\nu^{(2)}$ are the Hankel functions of the first and second kinds, respectively, related by complex conjugation. The choice of $\mathcal{B}$'s will complete our vacuum definition.

Since A depends on u only through its logarithmic derivative, the overall (complex) normalization of u is irrelevant. We motivate our choice of $\mathcal{B}_1$ and $\mathcal{B}_2$ by considering the asymptotic form of $\Psi(\alpha(k),t)$ as $t \to -\infty$ (Refs. 8 and 9). In this region, the metric is essentially static, so it is reasonable to choose a $\Psi(\alpha(k),t)$ which tends asymptotically to the Minkowski vacuum wave functional. In Minkowski space, $a(t)=1$ and the equation for A reads, $i\dot A = A^d - (k^2+m^2)$. Since the space is static, we can choose $\dot A = 0$, and we discover $A = \pm(k^2+m^2)^{1/2}$ (Ref. 10). By analogy with the quantum mechanics of a static oscillator, we choose $A(k) = +(k^2+m^2)^{1/2}$ which corresponds to a normalizable ground state in a one-dimensional system. Notice that had we defined $\Psi_{\mathrm{Minkowski}}$ by $\mathcal{A}(k)\Psi_{\mathrm{Minkowski}}=0$, we would not have encountered a sign ambiguity. To relate $\Psi(\alpha(k),t)$ to the Minkowski vacuum, we must consider the asymptotic behavior of the Hankel functions,

$$H_\nu^{1,2}(z) \approx \left[\frac{2}{\pi}\right]^{1/2}\exp\left[\pm i\left[z-\frac{\pi\nu}{2}-\frac{\pi}{4}\right]\right] \quad \text{for } z \gg 1 . \tag{3.3}$$

Thus, $d_t H_\nu^{1,2}/H_\nu^{1,2} \approx \mp iH_0\tau k$, and by choosing $\mathcal{B}_2=1$ and $\mathcal{B}_1=0$, we obtain $A(k,t\to-\infty)\approx\sqrt{-g}\,a_0^{-1}k$. The comparison with ordinary Minkowski space is correct, as can most easily be seen by reexpressing the de Sitter and Minkowski wave functionals in terms of $\phi(x)$. In order to identify our de Sitter space with Minkowski space, we must define a new spatial coordinate $y \equiv a_0 x$. The $\sqrt{-g}$ which appears in A survives the inverse Fourier transform and is reabsorbed into the measure of the integral over the new spatial coordinate. Similarly, the a_0^{-2} multiplies ∇^2, and by reexpressing ∇ in terms of the new coordinate, we find that our result exactly agrees with the wave functional in Minkowski space. The mass term has been lost since it is dominated by $a_0^{-2}k^2$ at early times. The vacuum which we have selected is known in the literature as the Bunch-Davies[11] or Hawking[12] vacuum.

IV. STRESS TENSOR IN THE BUNCH-DAVIES VACUUM

If the vacuum state is required to be de Sitter invariant, then $\langle\varphi^2\rangle$ must be independent of time. This requirement is in fact trivially satisfied by a family of solutions to Eq. (1.8) specified by one complex parameter. This can be seen as follows. Begin with

$$\langle\varphi^2(t)\rangle = \int\frac{d\tilde k}{2\,\mathrm{Re}A_0(k,t)} , \tag{4.1}$$

where

$$2\,\mathrm{Re}A_0(k,t) = (H_0\tau)^{-d}2\,\mathrm{Re}\Gamma_0(k,t) .$$

Since

$$\Gamma_0(k,t) = -id_t \ln u_0(k,t) ,$$
$$u_0(k,t) = \beta H_\nu^{(1)}(k\tau) + H_\nu^{(2)}(k\tau)$$

(up to an irrelevant constant),

$$\mathrm{Im}\,d_t\ln u_0 = \frac{1}{2i}\left[\frac{\beta\dot H_\nu^{(1)}+\dot H_\nu^{(2)}}{\beta H_\nu^{(1)}+H_\nu^{(2)}} - \frac{\beta^*\dot H_\nu^{(2)}+\dot H_\nu^{(1)}}{\beta^* H_\nu^{(2)}+H_\nu^{(1)}}\right]$$
$$= \frac{1}{2i}\left[\frac{(1-|\beta|^2)W_t(H_\nu^{(1)},H_\nu^{(2)})}{|\beta H_\nu^{(1)}+H_\nu^{(2)}|^2}\right] ,$$

where $W_t(u,v)\equiv u\dot v - v\dot u$ denotes the Wronskian with respect to t. Using the fact that $W_t(H_\nu^{(1)},H_\nu^{(2)}) = 4iH_0/\pi$, we discover that

$$\frac{1}{2\,\mathrm{Re}A_0} = a_0^{-d}\frac{\pi}{4H_0}\frac{|\beta H_\nu^{(1)}+H_\nu^{(2)}|^2}{1-|\beta|^2} . \tag{4.2}$$

By introducing a dimensionless variable of integration $y \equiv k\tau$, one can show that $\int d\tilde k/(2\,\mathrm{Re}A)$ is constant as long as $\beta(k)$ is independent of k. Notice that different choices of β within this one-parameter family can give different values of $\langle\varphi^2\rangle$, and since these expectation values are constant, all such vacua remain distinct for all times. This is to be expected, since the vacua are related by k-independent Bogoliubov transformations and relative to a given vacuum, the others will contain excitations at arbitrarily high frequencies which can never be redshifted away.

In what follows, we restrict ourselves to the Bunch-Davies vacuum, i.e., the case in which $\beta(k)=0$ and

$$\langle \varphi^2(x)\rangle=\frac{\pi}{4}H_0^{d-1}\int d\tilde{y}\,|H_\nu^{(1)}(y)|^2 \,. \tag{4.3}$$

Upon integrating Eq. (A5) over k, we obtain

$$\int\frac{d\tilde{k}}{|g|}\frac{|A_0|^2}{2\,\mathrm{Re}A_0}=\int d\tilde{k}\frac{f}{2\,\mathrm{Re}A_0}\,,$$

since the remaining term can be expressed as

$$\frac{1}{\sqrt{-g}}d_t(\sqrt{-g}\,d_t\langle\varphi^2\rangle)\,.$$

(In de Sitter space, the kinetic and potential energies contribute equally to the total vacuum energy.) Furthermore, since

$$\{\mathrm{Im}A/[(-g)^{1/2}\mathrm{Re}A]\}=-d_t(1/2\,\mathrm{Re}A)\,,$$

it will integrate to zero. As a result, the vacuum stress tensor as given in Eq. (2.6) simplifies to

$$\langle T_{00}\rangle=\int\frac{d\tilde{k}}{2\,\mathrm{Re}A_0}(a^{-2}k^2+m^2+\xi R+\xi R_{00})$$

and

$$\langle T_{ij}\rangle=\frac{1}{d}a^2\int\frac{d\tilde{k}}{2\,\mathrm{Re}A_0}(\delta_{ij}a^{-2}k^2+d\xi\,a^{-2}R_{ij})\,,$$

or

$$\langle T_{00}\rangle=\frac{\pi}{4}H_0^{d+1}\int d\tilde{y}\,|H_\nu^{(1)}(y)|^2\left[y^2+\frac{\tilde{m}^2}{H_0^2}-d\xi\right]$$

and

$$\langle T_{ij}\rangle=\frac{1}{d}\delta_{ij}a_0^2\frac{\pi}{4}H_0^{d+1}\int d\tilde{y}\,|H_\nu^{(1)}(y)|^2(y^2+d^2\xi)\,, \tag{4.4}$$

where we recall that $\tilde{m}^2\equiv m^2+d(d+1)\xi H_0^2$. The integrations appearing here are of the Weber-Schaeftlin type[13] and can be performed analytically:

$$J_l\equiv\frac{\pi}{4}H_0^{d-1}\int d\tilde{y}\,y^l\,|H_\nu^{(1)}(y)|^2$$

$$=H_0^{d-1}2^{l-1}\pi^{-d/2}\frac{\Gamma(1-l-d)\Gamma\left[\frac{l+d}{2}\right]}{\Gamma\left[\frac{d}{2}\right]\Gamma\left[1-\frac{l+d}{2}\right]}\times\left[\frac{\Gamma\left[\frac{l+d}{2}+\nu\right]\Gamma\left[\frac{l+d}{2}-\nu\right]}{\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{1}{2}-\nu)}\right]. \tag{4.5}$$

In particular, $\langle\varphi^2\rangle=J_0$. Using the relation $\Gamma(z+1)=z\Gamma(z)$ and the definition $\nu^2=d^2/4-\tilde{m}^2/H_0^2$, it is easily established that

$$J_2=-\frac{d}{1+d}\frac{\tilde{m}^2}{H_0^2}J_0\,, \tag{4.6}$$

and therefore that

$$\langle T_{ij}\rangle=-\delta_{ij}a^2\langle T_{00}\rangle\,.$$

Thus, the vacuum stress tensor is manifestly de Sitter invariant and $\langle T_{\mu\nu}\rangle$ is determined completely by its trace:

$$\langle T_{\mu\nu}\rangle=-\frac{m^2}{d+1}\langle\varphi^2\rangle g_{\mu\nu}\,. \tag{4.7}$$

If we had assumed our state to be de Sitter invariant, we would have written

$$\langle T_{\mu\nu}\rangle=\frac{\langle \mathrm{tr}T\rangle}{d+1}g_{\mu\nu}$$

immediately, where $\mathrm{tr}T$ is given by Eq. (2.2):

$$\mathrm{tr}T=-\frac{d+1}{2}(\partial\varphi)^2-\frac{d+1}{2}m^2\varphi^2-\xi\frac{d+1}{2}R\varphi^2+d\xi\Box\varphi^2$$

$$=d\left[\xi-\frac{1}{4}\frac{d-1}{d}\right]\Box\varphi^2-m^2\varphi^2\,.$$

The second equality was established by using the identity $\Box\varphi^2=2\varphi\Box\varphi+2(\partial\varphi)^2$, and then the field equation to eliminate $\Box\varphi$.

It is a simple matter to confirm that $\langle\Box\varphi^2\rangle$ vanishes as expected in a de Sitter-invariant state, so that $\langle \mathrm{tr}T\rangle=-m^2\langle\varphi^2\rangle$ as calculated above.

In what follows, we work with $n=d+1$ to establish contact with standard conventions in dimensional regularization. The expression for $\langle\varphi^2\rangle$,

$$\langle\varphi^2\rangle=\frac{\pi^{-(n-1)/2}}{2}H_0^{n-2}\Gamma(2-n)\frac{1}{\Gamma\left[\frac{3-n}{2}\right]}I_n\,, \tag{4.8}$$

where

$$I_n=\frac{\Gamma\left[\frac{n-1}{2}+\nu\right]\Gamma\left[\frac{n-1}{2}-\nu\right]}{\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{1}{2}-\nu)}\,,$$

diverges in the limit of physical interest, $n\to4$. The divergence appears as a pole in $\Gamma(2-n)$. To isolate it, notice that the behavior of $\Gamma(z)$ in the vicinity of its poles at the negative integers $z=-1,-2,\ldots$ is completely determined by its behavior at $z=0$:

$$\Gamma(z)=\frac{1}{z}-\gamma+O(z)\,, \tag{4.9}$$

where γ is Euler's constant. $\Gamma(2-n)$ may now be expanded about the pole at $n=4$:

$$\Gamma(2-n)=\frac{\Gamma(4-n)}{(2-n)(3-n)}$$
$$=-\frac{1}{(2-n)(3-n)}\left[\frac{1}{n-4}+\gamma+O(n-4)\right] .$$

$[\Gamma((3-n)/2)]^{-1}$ and I_n are both finite as $n \to 4$. It is necessary, however, to expand them to $O(n-4)$ because when multiplied by the pole term in $\Gamma(2-n)$, the $O(n-4)$ terms survive as finite contributions to $\langle \varphi^2 \rangle$:

$$\frac{1}{\Gamma\left[\frac{3-n}{2}\right]}=\tfrac{1}{2}(3-n)\frac{1}{\Gamma(\frac{1}{2})}\left[1-(n-4)\frac{d_n\Gamma}{\Gamma}\left[\frac{3-n}{2}\right]\Bigg|_{n=4}+O((n-4)^2)\right]$$
$$=\frac{1}{2\sqrt{\pi}}(3-n)[1+\tfrac{1}{2}(n-4)\psi(\tfrac{1}{2})+O((n-4)^2)] ,$$

where $\psi(\frac{1}{2})=-\gamma-2\ln 2$,

$$I_n=I_4+(n-4)\frac{d}{dn}I_n\Bigg|_{n=4}+O((n-4)^2) ,$$

where $I_4=(\frac{1}{2}-\nu)(\frac{1}{2}+\nu)$ and

$$\frac{d}{dn}I_n\Bigg|_{n=4}=\frac{I_4}{2}\left[\frac{\Gamma'(\frac{3}{2}+\nu)}{\Gamma(\frac{3}{2}+\nu)}+\frac{\Gamma'(\frac{3}{2}-\nu)}{\Gamma(\frac{3}{2}-\nu)}\right]$$
$$=\frac{I_4}{2}[\psi(\tfrac{3}{2}+\nu)+\psi(\tfrac{3}{2}-\nu)] .$$

If the action is to maintain the same dimension for all n, an arbitrary mass parameter should be introduced by replacing $\int d^nx$ by $\int d^nx\,\mu^{(n-4)}$ in the original action. This modifies $\langle \varphi^2 \rangle$ simply by a factor of $\mu^{(4-n)}$. We include this factor by in turn replacing H_0^{n-2} with $(H_0/\mu)^{n-4}H_0^2$ and again expanding about $n=4$.

By assembling the various terms, the following expression for $\langle \varphi^2 \rangle$ is obtained:

$$\langle \varphi^2 \rangle=\frac{H_0^2}{8\pi^2}(\tfrac{1}{4}-\nu^2)\left[\frac{1}{n-4}+\left[\ln\frac{H_0^2}{4\pi\mu^2}+\gamma+\psi(\tfrac{3}{2}+\nu)+\psi(\tfrac{3}{2}-\nu)\right]\right]+O(n-4) . \tag{4.10}$$

The divergences of quantum field theory are associated with its short-distance behavior, and as such, should only depend on the local background geometry so long as we only consider states with finite energy relative to one another. A means of isolating these divergences is provided by the DeWitt-Schwinger series for the dimensionally regularized effective action $W=-i\ln\langle 0,\text{out}\,|\,0,\text{in}\rangle$, where $|0,\text{in}\rangle$ and $|0,\text{out}\rangle$ refer to the in and out vacua, respectively. This expansion is particularly nice because it identifies the infinities as coefficients to geometric quantities of a form which also appear in or can be added to the classical gravitational action. Hence the divergences can be absorbed into a redefinition of the gravitational parameters such that all physical (measurable) quantities are finite.

Since in curved space well-defined in and out regions do not in general exist, the effective action is taken to be defined by the logarithm of the Feynman path integral over the exponential of the action. To make sense of the resulting expression, it will in general be necessary to specify some kind of boundary conditions on the fields. However, we will be concerned only with an asymptotic expansion of the effective action at a given spacetime point, and the inclusion of a small negative imaginary mass is sufficient to uniquely determine this expansion to arbitrary order in powers of inverse mass squared. By functionally differentiating our expression for the effective action with respect to the metric, we obtain the asymptotic inverse mass expansion for

$$\frac{\langle 0,\text{out}\,|\,T_{\mu\nu}\,|\,0,\text{in}\rangle}{\langle 0,\text{out}\,|\,0,\text{in}\rangle} .$$

It is reasonable to use the low-order terms of this expansion in the renormalization of $\langle T_{\mu\nu}\rangle$, since these terms represent contributions which correspond to the ultraviolet behavior of the field, and since we do not expect to depend upon the global properties of the states involved. (As long as $|0,\text{out}\rangle$, $|0,\text{in}\rangle$, and $|\text{vac}\rangle$ can be expressed relative to one another, as linear combinations of excited states of finite energy, it is obvious that

$$\langle 0,\text{out}\,|\,T_{\mu\nu}\,|\,0,\text{in}\rangle/\langle 0,\text{out}\,|\,0,\text{in}\rangle$$

differs from $\langle \text{vac}\,|\,T_{\mu\nu}\,|\,\text{vac}\rangle$ by a finite amount.)[14]

In the present example, while our chosen $|\text{vac}\rangle$ is the de Sitter equivalent to $|0,\text{in}\rangle$, (Ref. 15) the state $|0,\text{out}\rangle$ cannot be defined even approximately. Nevertheless, in this case, we will discover that the identification of $|0,\text{in}\rangle$ with $|\text{vac}\rangle$ is sufficient to allow us to employ the DeWitt-Schwinger expansion of

$$\langle 0,\text{out}\,|\,T_{\mu\nu}\,|\,0,\text{in}\rangle/\langle 0,\text{out}\,|\,0,\text{in}\rangle$$

in the renormalization of $\langle \text{vac}\,|\,T_{\mu\nu}\,|\,\text{vac}\rangle$ $(=\langle 0,\text{in}\,|\,T_{\mu\nu}\,|\,0,\text{in}\rangle)$. In contrast, we note that the vacua we would have obtained by choosing other k-independent linear combinations of the Hankel functions in the solution for $A(k;t)$ would not produce formal expectation values for $T_{\mu\nu}$ which could be renormalized through these same DeWitt-Schwinger subtractions. This corresponds to the fact that these other vacuum states contain nonvanishing excitations (of k-independent magnitude) in all modes, relative to the in vacuum. With these preliminaries, we proceed with the renormalization.

If we write $W=\int d^4x\sqrt{-g}\,\mathcal{L}_{\rm eff}$, the DeWitt-Schwinger series for $\mathcal{L}_{\rm eff}$ is

$$\mathcal{L}_{\rm eff}=\frac{1}{(4\pi)^2}\left[\frac{1}{n-4}+\frac{1}{2}\left[\gamma+\ln\frac{m^2}{4\pi\mu^2}\right]\right]\times\left[\frac{4m^4}{n(n-2)}-\frac{2m^2}{(n-2)}[\Omega_1]+[\Omega_2]\right]+\Delta\ . \qquad (4.11)$$

In this expression,

$$[\Omega_1]=(\tfrac{1}{6}-\xi)R$$

and

$$[\Omega_2]=\tfrac{1}{180}(R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}-R^{\mu\nu}R_{\mu\nu})+\tfrac{1}{2}(\tfrac{1}{6}-\xi)^2R^2+\tfrac{1}{6}(\tfrac{1}{5}-\xi)\Box R\ .$$

The Δ appearing in Eq. (4.11) represents terms which are finite as $n\to 4$, and which vanish in the high-mass limit. Specifically,

$$\Delta\sim\sum_{k=3}^{\infty}\frac{(k-3)!}{2(4\pi)^2}\int dx\sqrt{-g}\,\frac{[\Omega_k]}{(m^2)^{k-2}}$$

is an asymptotic expansion in inverse powers of the mass m. The $[\Omega_k]$'s are local polynomials of degree k in the curvature, so that the ratio of successive terms is of order $(\lambda/R_0)^2$, where $\lambda\sim 1/m$ is the Compton wavelength of particles, and R_0 is some characteristic curvature radius. Only those terms in the asymptotic inverse mass expansion which do not vanish in the high-mass limit will figure in the subtraction process, and thus the exact form of Δ is not necessary.

By functionally differentiating $W=-i\ln\langle 0,\text{out}\,|\,0,\text{in}\rangle$ with respect to $g_{\mu\nu}$, we obtain the expectation value of $T_{\mu\nu}$ between in and out vacua:

$$\frac{\langle 0,\text{out}\,|\,T_{\mu\nu}\,|\,0,\text{in}\rangle}{\langle 0,\text{out}\,|\,0,\text{in}\rangle}=\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}W\ .$$

An asymptotic inverse mass expansion can be obtained for this quantity by making use of the expansion for W:

$$\frac{\langle 0,\text{out}\,|\,T_{\mu\nu}\,|\,0,\text{in}\rangle}{\langle 0,\text{out}\,|\,0,\text{in}\rangle}=-\frac{1}{(4\pi)^2}\left[\frac{1}{n-4}+\frac{1}{2}\left[\gamma+\ln\frac{m^2}{4\pi\mu^2}\right]\right]\times\left[\frac{4m^4}{n(n-2)}g_{\mu\nu}+\frac{4m^2}{n-2}(\tfrac{1}{6}-\xi)G_{\mu\nu}+\tfrac{1}{90}(H_{\mu\nu}-{}^{(2)}H_{\mu\nu})+(\tfrac{1}{6}-\xi)^2\,{}^{(1)}H_{\mu\nu}\right]+O\left[\frac{1}{m^2}\right]\ . \qquad (4.12)$$

Here $O(1/m^2)$ represents terms which result from the differentiation of Δ and

$$G_{\mu\nu}=R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R,\quad H_{\mu\nu}=\frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int dx\sqrt{-g}\,R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu},\quad {}^{(1)}H_{\mu\nu}=\frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int dx\sqrt{-g}\,R^2\ ,$$

and

$${}^{(2)}H_{\mu\nu}=\frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int dx\sqrt{-g}\,R^{\mu\nu}R_{\mu\nu}$$

are covariantly conserved. Hence, $\langle 0,\text{out}\,|\,T_{\mu\nu}\,|\,0,\text{in}\rangle/\langle 0,\text{out}\,|\,0,\text{in}\rangle$ is manifestly conserved to $O(1/m^2)$. That it is conserved to all orders follows as a consequence of the invariance of W under infinitesimal coordinate transformations, $\delta g_{\mu\nu}=2D_{(\mu}\xi_{\nu)}$:

$$\begin{aligned}0=\delta\langle 0,\text{out}\,|\,0,\text{in}\rangle=\delta\int\mathcal{D}\varphi e^{iS}&=\int d^4x\sqrt{-g}\int\mathcal{D}\varphi\frac{i}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}e^{iS}\\&=\int d^4x\sqrt{-g}\int\mathcal{D}\varphi\frac{2i}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}D^{\mu}\xi^{\nu}e^{iS}\\&=-\int d^4x\sqrt{-g}\int\mathcal{D}\varphi iD^{\mu}\left[\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}\right]\xi^{\nu}e^{iS}\ .\end{aligned}$$

However, ξ is arbitrary, so that

$$D^{\mu}\langle 0,\text{out}\,|\,T_{\mu\nu}\,|\,0,\text{in}\rangle=0\ .$$

It is easily checked that the expression given in Eq. (4.12), determined entirely by local, conserved geometric tensors and the particle mass, agrees with the series obtained by directly expanding our regularized expression for $\langle T_{\mu\nu}\rangle$ in powers of the inverse mass squared. In the limit where $m\to\infty$, the radius of curvature of spacetime will greatly exceed the Compton wavelength of particles and the Minkowski-space limit for local operators should be recovered. In particular the vacuum stress tensor should vanish. This observation serves to motivate the renormalization prescription which we will adopt from now on, namely, to subtract from $\langle T_{\mu\nu}\rangle$ all terms in the DeWitt-Schwinger expansion for

$$\frac{\langle 0,\mathrm{out}\,|\,T_{\mu\nu}\,|\,0,\mathrm{in}\rangle}{\langle 0,\mathrm{out}\,|\,0,\mathrm{in}\rangle}$$

which do not vanish in the high-mass limit. They are precisely those terms which are given explicitly in Eq. (4.13).

This subtraction corresponds to a renormalization of the bare gravitational parameters which appear in the Einstein equations. In four dimensions, our prescription requires the inclusion of a curvature-squared term in the bare gravitational action whose coefficient can be chosen to exactly cancel the contribution to the total action (and subsequently to $T_{\mu\nu}$) coming from the $[\Omega_2]$ term. As constructed, this scheme is covariant.

In the present case, the vacuum is de Sitter invariant and the stress tensor is proportional to the metric tensor. Thus to renormalize $\langle T_{\mu\nu}\rangle$, only the trace of Eq. (4.12) need be considered further:

$$\frac{\langle 0,\mathrm{out}\,|\,\mathrm{tr}T\,|\,0,\mathrm{in}\rangle}{\langle 0,\mathrm{out}\,|\,0,\mathrm{in}\rangle}=-\frac{1}{(4\pi)^2}\left[\frac{1}{n-4}+\frac{1}{2}\left[\gamma+\ln\frac{m^2}{4\pi\mu^2}\right]\right]\times\left[\frac{4m^4}{n-2}+\frac{4m^2}{n-2}(\tfrac{1}{6}-\xi)G^\mu_\mu+\tfrac{1}{90}(H^\mu_\mu-{}^{(2)}H^\mu_\mu)+(\tfrac{1}{6}-\xi)^2\,{}^{(1)}H^\mu_\mu\right]+O\left[\frac{1}{m^2}\right].$$

The tensors H^μ_ν, ${}^{(1)}H^\mu_\nu$, and ${}^{(2)}H^\mu_\nu$ are traceless when $n=4$:

$$H^\mu_\mu\sim\tfrac{1}{2}(n-4)R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu},\quad {}^{(1)}H^\mu_\mu\sim\tfrac{1}{2}(n-4)R^2,\quad {}^{(2)}H^\mu_\mu\sim\tfrac{1}{2}(n-4)R^{\mu\nu}R_{\mu\nu}\,,$$

so that

$$\mathrm{tr}\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int dx\sqrt{-g}\,[\Omega_2]=(n-4)[\Omega_2]\,,$$

and the pole term multiplying $[\Omega_2]$ is exactly canceled to give a finite contribution. Rewriting

$$\frac{\langle 0,\mathrm{out}\,|\,\mathrm{tr}T\,|\,0,\mathrm{in}\rangle}{\langle 0,\mathrm{out}\,|\,0,\mathrm{in}\rangle}=-\frac{m^2}{(4\pi)^2}\left[\frac{1}{n-4}+\frac{1}{2}\left[\gamma+\ln\frac{m^2}{4\pi\mu^2}\right]\right]\left[\frac{4m^2}{n-2}-2(\tfrac{1}{6}-\xi)R\right]-\frac{1}{(4\pi)^2}\frac{[\Omega_2]}{2}+O\left[\frac{1}{m^2}\right].\qquad(4.13)$$

The renormalized vacuum stress tensor $\langle T_{\mu\nu}\rangle_{\mathrm{ren}}$ is now defined by

$$\langle T_{\mu\nu}\rangle_{\mathrm{ren}}\equiv\frac{g_{\mu\nu}}{4}\langle \mathrm{tr}T\rangle_{\mathrm{ren}}=\frac{g_{\mu\nu}}{4}\left[\langle\mathrm{tr}T\rangle-\left[\frac{\langle 0,\mathrm{out}\,|\,\mathrm{tr}T\,|\,0,\mathrm{in}\rangle}{\langle 0,\mathrm{out}\,|\,0,\mathrm{in}\rangle}\right]_{\text{first three dS terms}}\right]$$
$$=\frac{g_{\mu\nu}}{64\pi^2}\left[m^2[m^2+(\xi-\tfrac{1}{6})R\,]\left[\psi(\tfrac{3}{2}+\nu_4)+\psi(\tfrac{3}{2}-\nu_4)+\ln\frac{12m^2}{R}\right]-m^2(\xi-\tfrac{1}{6})m^2R-\tfrac{1}{18}m^2R-\tfrac{1}{2}(\xi-\tfrac{1}{6})^2R^2+\tfrac{1}{2160}R^2\right],\qquad(4.14)$$

where $R=12H_0^2$ and

$$\nu_4^2\equiv[\tfrac{9}{4}-(m^2+12\xi H_0^2)/H_0^2]\,.$$

In the limit of a massless, conformally coupled ($\xi=\frac{1}{6}$) theory,

$$\langle\mathrm{tr}T\rangle_{\mathrm{ren}}=\frac{1}{250\pi^2}H_0^4=-\frac{1}{16\pi^2}[\Omega_2]_{(\xi=1/6)}\,.$$

This is the well-known trace anomaly.

Up to this point, the de Sitter background has played no dynamical role. In order to understand the back reaction of matter on the geometry of spacetime, we consider the semiclassical Einstein equations

$$G^\mu_\nu+\Lambda\delta^\mu_\nu=-8\pi G\langle T^\mu_\nu\rangle\mathrm{ren}\,.$$

Either by including the appropriate classical matter distribution, or by explicitly inserting a positive cosmological constant, we can consistently solve these equations for a de Sitter metric with any Hubble constant H_0, which will be shifted from its classical value by the inclusion of the quantum-mechanical vacuum stress.

CONCLUSION

The functional Schrödinger picture has been developed for a scalar field theory in a Robertson-Walker flat spacetime. The Schrödinger equation for a Gaussian wave functional was shown to imply useful identities for manipulating the formal expressions obtained in calculating the vacuum stress tensor. In particular, conservation of the vacuum stress has been established using three of these identities. Another identity allowed $\langle T_{\mu\nu}\rangle$ to be written entirely in terms of the real part of the Gaussian width and its time derivatives. This simplification facilitates its explicit evaluation in nearly de Sitter spacetimes.[16] In de Sitter space, the vacuum wave functional has been obtained explicitly, and the vacuum expectation value of $T_{\mu\nu}$ has been shown to agree with the well-known Bunch-Davies result.

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APPENDIX A: SOME USEFUL IDENTITIES

From the equation of motion for $A(k,T)$,

$$i\dot{A}=\frac{A^2}{\sqrt{-g}}-\sqrt{-g}\,(a^{-2}k^2+m^2+\xi R)\ , \tag{A1}$$

it is possible to derive several useful identities relating quantities which appear in the expression for the stress tensor.

Equating real and imaginary parts separately, we obtain the following two equations:

$$-\mathrm{Im}\,\dot{A}=\frac{\mathrm{Re}[A^2]}{\sqrt{-g}}-\sqrt{-g}\,f\ , \tag{A2a}$$

$$\mathrm{Re}\,\dot{A}=\frac{\mathrm{Im}[A^2]}{\sqrt{-g}}\ , \tag{A2b}$$

where $f\equiv a^{-2}k^2+m^2+\xi R$. By now making use of the identity

$$A^2=|A|^2-2(\mathrm{Im}A)^2+2i\,\mathrm{Re}A\,\mathrm{Im}A\ ,$$

Eq. (A2a) may be recast as

$$-\mathrm{Im}\,\dot{A}=\frac{1}{\sqrt{-g}}[\,|A|^2-2(\mathrm{Im}A)^2]-\sqrt{-g}\,f\ , \tag{A3a}$$

$$\frac{\mathrm{Re}\,\dot{A}}{\mathrm{Re}A}=\frac{2}{\sqrt{-g}}\mathrm{Im}A\ , \tag{A3b}$$

the latter of which is equivalent to

$$-d_t\left[\frac{1}{2\,\mathrm{Re}A}\right]=\frac{1}{\sqrt{-g}}\left[\frac{\mathrm{Im}A}{\mathrm{Re}A}\right]. \tag{A4}$$

If Eq. (A3a) is now divided through by $\mathrm{Re}A$, and Eq. (A3b) is used to eliminate one factor of $\mathrm{Im}A$ on the right-hand side in favor of $\mathrm{Re}\dot{A}/\mathrm{Re}A$, we obtain

$$-d_t\left[\frac{\mathrm{Im}A}{\mathrm{Re}A}\right]=\frac{1}{\sqrt{-g}}\frac{|A|^2}{\mathrm{Re}A}-\frac{\sqrt{-g}\,f}{\mathrm{Re}A}\ . \tag{A5}$$

Again applying Eq. (A3b) to the left-hand side of Eq. (A5), the following useful relation is obtained:

$$\frac{1}{|g|}\frac{|A|^2}{2\,\mathrm{Re}A}=(a^{-2}k^2+m^2+\xi R)\frac{1}{2\,\mathrm{Re}A}+\frac{1}{2\sqrt{-g}}d_t\left[\sqrt{-g}\,d_t\left[\frac{1}{2\,\mathrm{Re}A}\right]\right]. \tag{A6}$$

$|A|^2/(|g|\,2\,\mathrm{Re}A)$ and $f/2\,\mathrm{Re}A$ are, respectively, the zero-point kinetic energy T_k and potential energy V_k found in the oscillator of frequency k and thus the second term on the right-hand side of Eq. (A6) represents the difference T_k-V_k.

In the proof of stress energy conservation, we also made use of the equation

$$d_t\left[\frac{|A|^2}{2\,\mathrm{Re}A}\right]=-|g|\,f d_t\left[\frac{1}{2\,\mathrm{Re}A}\right]. \tag{A7}$$

To prove this, Eq. (A1) is multiplied by A^* and the complex-conjugated equation subtracted to obtain

$$d_t(|A|^2)=\frac{|A|^2}{\sqrt{-g}}2\,\mathrm{Im}A+\sqrt{-g}\,2\,\mathrm{Im}A\ .$$

Equation (A7) follows immediately upon dividing across by $2\,\mathrm{Re}A$ and making use of Eq. (A3b) to eliminate $\mathrm{Im}A$ in favor of $\mathrm{Re}A$.

APPENDIX B: THE SCHRÖDINGER KERNEL FOR DE SITTER SPACE

To calculate $\langle\tilde{\varphi}(x),t\,|\,\varphi(x)\rangle$, it is useful to introduce k-space Heisenberg operators $a(k,t)$ and $P(k,t)$ defined by the Fourier transformation of the coordinate-space operators $\varphi(x,t)$ and $\Pi(x,t)$,

$$\varphi(x,t)=\int d\tilde{k}\,e^{ik\cdot x}\alpha(k,t)\ ,$$

$$\Pi(x,t)=\int d\tilde{k}\,e^{ik\cdot x}P(k,t)\ .$$

For each k, $\alpha(k,t)$ will satisfy the operator field equation

$$\ddot{\alpha}+dH_0\,\dot{\alpha}+(e^{-2H_0t}k^2+m^2)\alpha=0\ , \tag{B1}$$

with the Hankel functions $H_\nu^{(1)}(k\tau)$ and $H_\nu^{(2)}(k\tau)$ as solutions. The operator-valued coefficients of the Hankel functions will be determined by the initial conditions. After a little algebra, the operators $\alpha(k,t)$ and $P(k,t)=(H_0\tau)^d\dot{\alpha}(k,t)$ at an arbitrary time t may be expressed in terms of their $t=0$ values, say, $\alpha(k)$ and $P(k)$, as

$$\alpha(k,t)=\frac{\pi}{4iH_0}(H_0\tau)^{d/2}[A(k,t)\alpha(k)+B(k,t)P(k)]\ ,$$
$$P(k,t)=\frac{\pi}{4iH_0}(H_0\tau)^{-d/2}[C(k,t)\alpha(k)+D(k,t)P(k)]\ , \tag{B2}$$

where

$$A(k,t)=\mathcal{M}^*\left[\frac{k}{H_0}\right]H_\nu^{(1)}(k\tau)-\mathcal{M}\left[\frac{k}{H_0}\right]H_\nu^{2}(k\tau)\ ,$$

$$B(k,t)=H_\nu^{(1)}\left[\frac{k}{H_0}\right]H_\nu^{(2)}(k\tau)-H_\nu^{(2)}\left[\frac{k}{H_0}\right]H_\nu^{(1)}(k\tau)\ ,$$

$$C(k,t)=\mathcal{M}^*\left[\frac{k}{H_0}\right]\mathcal{M}(k\tau)-\mathcal{M}\left[\frac{k}{H_0}\right]\mathcal{M}^*(k\tau)\ ,$$

$$D(k,t)=H_\nu^{(1)}\left[\frac{k}{H_0}\right]\mathcal{M}^*(k\tau)-H_\nu^{(2)}\left[\frac{k}{H_0}\right]\mathcal{M}(k\tau)\ ,$$

and

$$\mathcal{M}(y) = -H_0\left[\frac{d}{2}H_\nu^{(1)}(y) + yH_\nu^{1'}\right].$$

Define $K[\alpha,\tilde{\alpha},t] = \langle\tilde{\alpha},t\mid\alpha\rangle$. By operating in turn with $\alpha(k,t)$, $P(k,t)$, and the Schrödinger equation, it is possible to completely determine K up to a constant factor. Although the parameters appearing are time dependent, this is the same technique used to determine the propagator $\langle x',t\mid x\rangle$ for a regular harmonic oscillator in quantum mechanics.

Consider first $\langle\tilde{\alpha},t\mid\alpha(k,t)\mid\alpha\rangle$. Equating the result of operating to the left on the bra $(\tilde{\alpha}\langle\tilde{\alpha}\mid\alpha\rangle)$, with that of operating to the right on the ket $[A\alpha + iB(\delta/\delta\alpha)]K$, gives rise to the functional partial differential equation

$$i\frac{\delta}{\delta\alpha}K = \frac{\tilde{\alpha} - A\alpha}{B}, \tag{B3}$$

which is easily integrated:

$$K = \mathcal{C}[t,\tilde{\alpha}]\exp\left[\frac{1}{i}\int d\tilde{k}\left[\frac{\tilde{\alpha}(k)\alpha(-k) - A(k,t)/2\mid\alpha(k)\mid^2}{B(k,t)}\right]\right]. \tag{B4}$$

Next consider $\langle\tilde{\alpha},t\mid P(k,t)\mid\alpha\rangle$. Equating again the result of operating to the left on the bra $[-i(\delta/\delta\tilde{\alpha})K]$, with that of operating to the right on the ket $[C\alpha + Di(\delta/\delta\alpha)]K$, allows the $\tilde{\alpha}$ dependence of $\mathcal{C}[t,\tilde{\alpha}]$ to be determined:

$$-\frac{i}{C}\frac{\delta C}{\delta\tilde{\alpha}} = -\alpha\left[\frac{AD - BC - 1}{B}\right] + \frac{D}{B}\tilde{\alpha}. \tag{B5}$$

$AD - BC = 1$ because the equations of motion preserve the equal-time commutation relations. Equation (B5) is easily integrated:

$$\mathcal{C}[t,\tilde{\alpha}] = \mathcal{C}(t)\exp\left[-\frac{1}{i}\left[\int d\tilde{k}\frac{D}{2B}\mid\tilde{\alpha}(k)\mid^2\right]\right], \tag{B6}$$

so that

$$K = \mathcal{C}(t)\exp\left[\frac{1}{i}\int d\tilde{k}\frac{1}{B}\{-\tfrac{1}{2}[A\mid\alpha(k)\mid^2 + D\mid\tilde{\alpha}(k)\mid^2] + \alpha(k)\tilde{\alpha}(-k)\}\right]. \tag{B7}$$

The remaining unknown $\mathcal{C}(t)$ is determined by means of the Schrödinger equation:

$$i\partial_t K = \tfrac{1}{2}\int d\tilde{k}\left[-(H_0\tau)^d\frac{\delta^2}{\delta\tilde{\alpha}(k)\tilde{\alpha}(-k)} + (H_0\tau)^{-d}[(H_0\tau)^2k^2 + m^2]\mid\tilde{\alpha}(k)\mid^2\right].$$

Operating on Eq. (B7) with $i\partial_t$ gives

$$i\frac{\partial_t K}{K} = i\frac{\dot{\mathcal{C}}}{\mathcal{C}} + \int d\tilde{k}\left\{\left[\frac{1}{B}\right]^{\cdot}\alpha(k)\tilde{\alpha}(-k) - \frac{1}{2}\left[\left[\frac{A}{B}\right]^{\cdot}\mid\alpha\mid^2 + \left[\frac{D}{B}\right]^{\cdot}\mid\tilde{\alpha}\mid^2\right]\right\} \tag{B8a}$$

and

$$\frac{1}{K}H_{\rm op}K = \tfrac{1}{2}(H_0\tau)^d\left[\frac{1}{i}(2\pi)^d\delta_k(0)\int d\tilde{k}\frac{Dd}{B} + \left[\frac{1}{B}\right]^2[\alpha(k) - D\tilde{\alpha}(k)][\alpha(-k) - D\tilde{\alpha}(-k)] + \tfrac{1}{2}(H_0\tau)^{-d}[(H_0\tau)^2k^2 + m^2]\mid\tilde{\alpha}(k)\mid^2\right]. \tag{B8b}$$

The following identities hold among the coefficients A, B, C, and D as a consequence of the equations of motion:

$$\left[\frac{D}{B}\right]^{\cdot} = \tfrac{1}{2}(H_0\tau)^2\left[\frac{D}{B}\right]^2 + \tfrac{1}{2}(H_0\tau)^{-d}[(H_0\tau)^2k^2 + m^2], \tag{B9a}$$

$$\left[\frac{A}{B}\right]^{\cdot} = -(H_0\tau)^d\frac{1}{B^2}, \tag{B9b}$$

$$\left[\frac{1}{B}\right]^{\cdot} = -(H_0\tau)^d\frac{D}{B^2}. \tag{B9c}$$

As a result, Eq. (B8) reduces to

$$\frac{\dot{\mathcal{C}}}{\mathcal{C}} = \tfrac{1}{2}(2\pi)^d\delta_k(0)(H_0\tau)^d\int d\tilde{k}\frac{D}{B}, \tag{B10}$$

which can be solved for $\mathcal{C}(t)$. The solution is facilitated by making use of Eq. (B9c) in the form

$$\frac{D}{B} = (H_0\tau)^{-d}(\ln B)^{\cdot},$$

with which Eq. (B10) easily integrates to give the following particularly simple form for $\mathcal{C}(t)$:

$$\mathcal{C}(t)=\mathcal{C}_0\exp\left[-\left[\tfrac{1}{2}V_d\int d\tilde{k}\,\ln B(k,t)\right]\right],$$

up to a constant determined by the normalization $\langle\tilde{\alpha},t\,|\,\alpha\rangle=\delta(\tilde{\alpha}-\alpha)$. $V_d=(2\pi)^d\delta_k(0)$ is the volume of space.

Clearly this calculation can be easily generalized to any Robertson-Walker flat background (assuming the mode functions are known).

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[1]See, for example, R. Jackiw, in *Current Algebra and Anomalies,* edited by S. Treiman, R. Jackiw, B. Zumino, and E. Witten (World Scientific, Singapore/Princeton University Press, Princeton, NJ, 1985); R. Jackiw, in *Theories and Experiments in High Energy Physics,* edited by B. Kursunoglu, A. Perlmutter, and S. Widmayer (Plenum, New York, 1975).

[2]K. Freese, C. Hill, and M. Muller, Nucl. Phys. **B255**, 693 (1985).

[3]The following references contain summaries of much of the existing work on field theories in curved-space backgrounds. The second also includes an extensive bibliography. B. DeWitt, Phys. Rep. **19C**, 297 (1975); N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).

[4]C. Hill, Fermilab Report No. Pub-85/37-T, 1985 (unpublished).

[5]Although the time dependence of the oscillator frequencies makes it impossible to define a physical state $|0\rangle$ which is annihilated by the $\mathcal{A}(k;t)$ for all times, we do expect the high-k modes to behave as in Minkowski space. More precisely, we should be able to define our vacuum state such that $\mathcal{A}(k;t)\,|\,0\rangle\approx 0$ for $a^{-1}k\gg\dot{a}/a$. This condition on k ensures that the oscillator's frequency is large compared to the expansion rate, or equivalently, that the frequency varies adiabatically. For this range of k, we can sensibly define a state containing n particles of momentum k by operating on a vacuum state Ψ n times with $\mathcal{A}^\dagger(k)$: $\Psi_n\sim H_n[\alpha(k)]\Psi$, where $H_n[\alpha]$ denotes the Hermite polynomial of order n in $\alpha(k)$.

[6]R. Floreanini, C. T. Hill, and R. Jackiw, Ann. Phys. (N.Y.) **175**, 354 (1987).

[7]In the canonical formalism, conservation is established by the following argument. The quantum field φ is decomposed into creation and annihilation parts, $\varphi=\sum_i(\phi_i a_i+\phi_i^* a_i^\dagger)$, and the vacuum state is defined by $a_i\,|\,0\rangle=0$. With these definitions, the vacuum stress tensor is expressed as the sum over modes $\langle T_{\mu\nu}\rangle=\sum_i T_{\mu\nu}(\phi_i,\phi_i^*)$, where

$$\begin{aligned}T_{\mu\nu}(\phi_i,\phi_i^*)\equiv&\,\partial_\mu\phi_i\partial_\nu\phi_i^*\\&-\tfrac{1}{2}g_{\mu\nu}[\partial_\rho\phi_i\partial^\rho\phi_i^*+(m^2+\xi R)\phi_i\phi_i^*]\\&-\xi R_{\mu\nu}\phi_i\phi_i^*+\xi(g_{\mu\nu}\Box-D_\mu D_\nu)\phi_i\phi_i^*\,.\end{aligned}$$

It is easily checked that $D_\mu[T^{\mu\nu}(\phi_i,\phi_i^*)+T^{\mu\nu}(\phi_i^*,\phi_i)]$ vanishes identically when ϕ_i satisfies the wave equation $[\Box-(m^2+\xi R)]\phi_i\equiv 0$, thereby establishing conservation.

[8]The choice of vacuum as implemented canonically (following Ref. 9) is presented here. The field modes satisfy the differential equation $(-\Box+\tilde{m}^2)\phi=0$. Fourier transforming on the space sections and defining $u=(-g)^{-1/4}\phi$, we obtain Bessel's equation, Eq. (3.1), with the solution given in Eq. (3.2). $u(k,t)$ describes a field excitation of physical wavelength $(2\pi/|\,k\,|)e^{H_0 t}$ which at sufficiently early times will be small compared to the de Sitter horizon distance H_0^{-1}. On such short length scales, it is expected that de Sitter space will be indistinguishable from Minkowski space. In particular, the solutions to Eq. (3.1) should resemble complex exponentials. A more precise statement can be made by examining the asymptotic behavior of the Hankel functions given in Eq. (3.3). Suppose that t_0 is fixed so that

$$z_0=k\tau_0=\frac{1}{2\pi}\frac{\text{horizon length}}{\text{physical wavelengths}}\gg 1\,.$$

Now, for z such that $|\,z/z_0-1\,|\ll 1$ $(H_0\,|\,t-t_0\,|\ll 1)$,

$$H_\nu^{1,2}(z)\sim\left[\frac{2}{\pi z_0}\right]^{1/2}\times\exp\left\{\pm i\left[\left[z_0-\frac{\pi\nu}{2}-\frac{\pi}{4}\right]+(d_t z)_0(t-t_0)\right]\right\}.$$

But $(d_t z)_0=|\,k\,|\,e^{-H_0 t_0}$, which is just the physical frequency $\omega(t_0)$ at time t_0, so that, up to a phase $H_\nu^{1,2}\sim\exp[\mp i\omega(t_0)(t-t_0)]$. Since $\dot{\omega}/\omega\sim H_0\ll\omega$, the changes in frequency are adiabatic and the modes indeed behave like complex exponentials. It is $H_\nu^{(1)}$ which corresponds to the positive-frequency Minkowski-space solution at early times; thus the solution $\mathcal{B}_2=0$ is singled out by the physical requirement that de Sitter space be indistinguishable from Minkowski space on length scales well within the horizon.

[9]A. Guth and S.-Y. Pi, Phys. Rev. D **32**, 1899 (1985).

[10]If $A(t)\,|_{t=0}=\epsilon(k^2+m^2)^{1/2}$, then

$$A(t)=\sqrt{k^2+m^2}\,\frac{1+Ce^{-2i(k^2+m^2t)^{1/2}}}{1-Ce^{-2i(k^2+m^2t)^{1/2}}}\,,$$

where $C\equiv(1-\epsilon)/(1+\epsilon)$. In words, if the kth mode begins in a Gaussian with "the wrong width," it will oscillate about the width of the ground-state solution with a frequency determined by the associated quadratic potential.

[11]T. Bunch and P.C.W. Davies, Proc. R. Soc. London **A360**, 117 (1978).

[12]G. Gibbons and S. Hawking, Phys. Rev. D **15**, 2738 (1977).

[13]*Higher Transcendental Functions* (Bateman Manuscript Project), edited by A. Erdélyi (McGraw-Hill, New York, 1954).

[14]It is not enough to insist that all the states we consider simply belong to the same Fock space. It is easy to construct states with infinite energy density relative to a given vacuum which can be expanded in terms of a Fock basis with coefficients whose squares sum to one; e.g., the state

$$|\,\Psi\rangle\equiv\mathcal{N}\left[\,|\,k,x_0,t,a\,\rangle+\frac{1}{2^2}\,|\,(2^4)k,x_0,t,a\,\rangle+\frac{1}{3^2}\,|\,(3^4)k,x_0,t,a\,\rangle+\cdots\right]$$

where $|p,x_0,t,a\rangle$ is defined as a Gaussian wave packet centered about x_0 at time t with width a and momentum p, and in which $\mathcal{N}$ is chosen so that $\langle\Psi|\Psi\rangle=1$, evidently belongs to the Fock space built upon $|\text{vac}\rangle$ and yet it will yield an infinite energy density relative to this vacuum at spacetime points near (x_0,t).

[15]Strictly speaking, de Sitter space belongs to that class of spacetimes with no static in region, and hence without a well-defined $|0,\text{in}\rangle$. However, as was shown in Sec. III, when working in Robertson-Walker flat coordinates, for fixed k, the space becomes quasi-Minkowski in the limit $t\to-\infty$. Hence, a unique choice of positive and negative frequencies can be made, mode by mode, in the asymptotic past, thus defining an effective in a vacuum.

[16]J. Guven and B. Lieberman, MIT Report No. 1521 (unpublished).