CANONICAL DESCRIPTION OF A SECOND-ORDER ACHROMAT[*]

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ABSTRACT
Charged particle motion in a second-order magnetic optical achromat is described using a canonical perturbation theory. Necessary and sufficient conditions for the existence of such a device are presented. Given these conditions, the second-order matrix elements at the end of the achromat are found explicitly. It is shown that all geometric matrix elements are equal to zero and all chromatic matrix elements are either also equal to zero or proportional to the corresponding chromaticity. Thus all second-order matrix elements vanish simultaneously when the two chromaticities are made to be equal to zero.

1. INTRODUCTION

A fascinating property of the second-order achromat invented by K. L. Brown [1] is that all the matrix elements describing both *geometric* and *chromatic* aberrations (with the exception of the elements connected to the path length of a particle trajectory) of the second-order matrix *simultaneously* vanish at the achromat exit provided that certain conditions are satisfied.

The proof of such a property for a second-order achromat with respect to the geometric aberrations was given by K. L. Brown in Ref. [1]. This paper also contains a practical recipe for constructing a second-order achromat.

On the other hand, a proof pertaining to *chromatic aberrations* is much more difficult. The first attempt of such a proof belongs to D. Carey [2], who showed that the recipe works.

Here we use a canonical perturbation theory to describe the particle motion in a beam line [3]. This approach brings us in a natural way to the formulation of the *necessary and sufficient* conditions for the existence of the second-order achromat. We show *why* the second-order achromat works and provide the reader with a simple physical explanation of the underlying reasons for its existence. By calculating the second-order matrix elements directly we show why they vanish when the achromatic conditions are fulfilled.

This paper will proceed as follows. In Section 2 the canonical formalism [4] for a particle motion in a magnetic field is described. The particle motion is separated into betatron and synchrotron parts. In Section 3 the betatron motion is described using canonical (angle-action) variables. We obtain formulae for horizontal and vertical chromaticities which agree with results obtained by M. Bassetti [5]. We then proceed to give the solution of the Hamilton's equations up to the second-order terms. Section 4 contains calculation of the second-order matrix elements. In particular, the chromatic matrix elements are found explicitly. The necessary and sufficient conditions for a second-order achromat are derived in Section 5. A special case of an achromat built from identical cells is considered.

This paper is in essence a short version of the paper [6], where more detailed calculations can be found.

2. THE HAMILTONIAN FOR THE MOTION OF A CHARGED PARTICLE IN A MAGNETIC FIELD

In the vicinity of a planar reference curve which follows the path of a particle with the reference momentum P_0 in a guiding magnetic field, the motion of an arbitrary particle is governed by a Hamiltonian [4] which we represent here with all the terms up to and including those which are cubic in coordinates and momenta:

$$H_1(x, p_x, y, p_y; s) = -eA_s/cP_0 - (1 + xh)(1 + \delta) + (1 + xh)(p_x^2 + p_y^2)/2(1 + \delta) \quad . \tag{1}$$

Here we use the notation $h = 1/\rho$, where ρ is local radius of curvature of the reference trajectory. The coordinate x is the horizontal (in the plane of the reference trajectory) deviation from the reference trajectory while y is the vertical deviation. The canonically conjugate momenta are p_x and p_y, respectively. The fractional difference of the particle momentum P from that of the reference momentum P_0 defines δ: $P = P_0(1 + \delta)$. The distance s along the reference curve measured from an arbitrary point serves as an independent variable.

The quantity A_s which appears in Eq. (1) is the *canonical* vector potential [4] representing the guide field. Again retaining the terms up to and including those which are cubic in the coordinates, A_s is given by the following expression [7]:

$$\frac{-e}{cP_0}A_s(x, y, s) = xh(1 + hx/2) + K_1(s)(x^2 - y^2)/2$$
$$+ K_1(s)(hx^3/3 - hxy^2/2 - h^2y^4/24) + K_2(s)(x^3 - 3xy^2)/6 \quad , \tag{2}$$

where $K_1 = \frac{h}{B_0}\frac{\partial B_y}{\partial x}$ and $K_2 = \frac{h}{B_0}\frac{\partial^2 B_y}{\partial x^2}$ are local (at the location s) strengths of the quadrupole and sextupole components of the magnetic field B_y, respectively.

The term linear in x in the Hamiltonian (1) makes the equation of motion for the x–plane inhomogeneous. The particular solution of this inhomogeneous equation gives *the equilibrium orbit* of the off-momentum particle. The reference orbit is but a special case of the more general equilibrium orbit for which $\delta = 0$. The solution of the homogeneous equation describes (in case of stable motion) *betatron oscillations*. An off-momentum particle performs betatron oscillations about its corresponding equilibrium orbit.

To describe the horizontal betatron motion of the off-momentum particle one can use a canonical transformation from x, p_x, y, p_y to x_β, p_β, y, p_y to eliminate linear terms in the Hamiltonian. New coordinates and momenta are connected to the old ones by $x_\beta = x - D(s)\delta$, $p_x = p_\beta + \varsigma(s)\delta$.

As shown in Ref. [8], functions $D(s)$ and $\varsigma = D'(1 + \delta)/(1 + hD\delta)$ can be found from a solution of the following second-order differential equation:

$$D'' = (1 + hD\delta)(h - h^2D - K_1D - K_1hD^2\delta - K_2D^2\delta/2)/(1 + \delta) + h(D')^2/2(1 + hD\delta) \quad . \tag{3}$$

Equation (3), together with corresponding initial conditions, defines the *dispersion* function which describes the equilibrium orbit for an off-momentum particle. For δ small, a solution of Eq. (3) can be found as an expansion in δ

$$D(s) = D_0(s) + \delta D_1(s) + \cdots \quad , \tag{4}$$

where the *zero-th* order function D_0 is a solution of the equation

$$D_0'' + (K_1 + h^2)D_0 = h \quad ; \tag{5}$$

the *first*-order function D_1 is a solution of the equation

$$D_1'' + (K_1 + h^2)D_1 = (K_1 + 2h^2)D_0 - h - 2K_1hD_0^2 - K_2D_0^2/2 - h^3D_0^2 + h(D_0')^2/2 \quad , \tag{6}$$

and so on.

The particular *zero-th* order dispersion function $d(s)$ which has the special initial conditions $d(0) = d'(0) = 0$ defines the first-order matrix elements $R_1^6(s)$ and $R_2^6(s)$ [9] which couple x and x' to δ [10].

On the other hand, for periodic systems with period l it may be convenient to define a periodic dispersion function. Such a dispersion function is called *the matched dispersion*. One should distinguish the matched dispersion from functions $D(s)$ with other possible initial conditions. We will denote the *zero-th* order matched dispersion as $\eta(s)$: $\eta(s+l) = \eta(s), \eta'(s+l) = \eta'(s)$. It can be found as solution of Eq. (5) with the following boundary conditions: $\eta(l) = \eta(0), \eta'(l) = \eta'(0)$.

Any *zero-th* order dispersion function $D_o(s)$ with arbitrary initial conditions $D_o(0)$ and $D_o'(0)$ (including the matched dispersion) is connected to function $d(s)$ by the following matrix expression:

$$
\begin{pmatrix} D_o(s) \\ D_o'(s) \\ 1 \end{pmatrix} = \begin{pmatrix} R_1^1(s) & R_1^2(s) & d(s) \\ R_2^1(s) & R_2^2(s) & d'(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D_o(0) \\ D_o'(0) \\ 1 \end{pmatrix} \quad , \tag{7}
$$

where R_i^k are the first-order matrix elements for transition from an arbitrary point 0 to an arbitrary point s of the system. This feature will be used below for calculation of the second-order chromatic matrix elements.

3. BETATRON MOTION

In the new variables x_β, p_β, y, p_y the Hamiltonian has no linear terms and describes betatron motion of a particle with respect to its corresponding equilibrium orbit:

$$
\begin{aligned}
H_2 &= (p_\beta^2 + p_y^2)/2(1+\delta) + K_x x_\beta^2/2 + K_y y^2/2 \\
&\quad + K_{2x} D\delta x_\beta^2/2 - K_{2y} D\delta y^2/2 + (K_{2x} x_\beta^3 - 3K_{2y} x_\beta y^2)/6 \\
&\quad + h x_\beta(p_\beta^2 + p_y^2)/2(1+\delta) + h D\delta(p_\beta^2 + p_y^2)/2(1+\delta) + h\zeta\delta p_\beta x_\beta/(1+\delta) \quad ,
\end{aligned} \tag{8}
$$

where the following notations are used: $K_x = K_1 + h^2$, $K_y = -K_1$, $K_{2x} = K_2 + 2hK_1$, $K_{2y} = K_2 + hK_1$.

As in Ref. [4] we now perform a canonical transformation to the action-angle variables J_x, ϕ_x, J_y, ϕ_y. The old coordinates and momenta are expressed in terms of the new ones as

$$
x_\beta = \sqrt{\frac{2J_x \beta_x}{(1+\delta)}} \cos\phi_x, \quad p_\beta = -\sqrt{\frac{2J_x(1+\delta)}{\beta_x}}(\sin\phi_x + \alpha_x \cos\phi_x) \tag{9}
$$

and similar expressions for y, p_y. The new Hamiltonian is

$$
H_3(J_x, \phi_x, J_y, \phi_y; s) = \frac{J_x}{\beta_x} + \frac{J_y}{\beta_y} + C_{0,0}^{1,0} J_x + C_{0,0}^{0,1} J_y + V(J_x, \phi_x, J_y, \phi_y; s) \quad , \tag{10}
$$

where

$$
V(J_x, \phi_x, J_y, \phi_y; s) = \sum_{m,n,k,l} J_x^{k/2} J_y^{l/2} [C_{m,n}^{k,l} \cos(m\phi_x + n\phi_y) + S_{m,n}^{k,l} \sin(m\phi_x + n\phi_y)] \quad . \tag{11}
$$

(k, l, m and n are all integers.) All non-zero coefficients $C_{m,n}^{k,l}$ and $S_{m,n}^{k,l}$ are given in Table 1. The functions $\beta(s)$ and $\alpha(s) = -\beta'/2$, used to describe the modulated betatron oscillations are *the matched functions* for the system under consideration. For a periodic system the matched β-function is also periodic as required by Floquet's theorem [11]. For a nonperiodic system the matched function is defined by the condition that the final values for the function and its derivative are the same as the initial values. Of course, it would be possible to parametrize the motion in terms of unmatched β-functions. In this case, the following results for the second-order achromat are unchanged; however, the proofs are somewhat

Table 1.
Non-Zero Coefficients of the Second-Order Hamiltonian

Notations:

$$H = \frac{J_x}{\beta_x} + \frac{J_y}{\beta_y} + \sum_{m,n,k,l} J_x^{k/2} J_y^{l/2} [C_{m,n}^{k,l} \cos(m\phi_x + n\phi_y) + S_{m,n}^{k,l} \sin(m\phi_x + n\phi_y)], \quad \Delta = 1 + \delta.$$

N	Coefficient	
1	$C_{0,0}^{1,0}$	$(K_{2x}D - K_x)\delta\beta_x/2\Delta + hD\delta\gamma_x/2 - h\varsigma\delta\alpha_x/\Delta$
2	$C_{0,0}^{0,1}$	$-(K_{2y}D + K_y)\delta\beta_y/2\Delta + hD\delta\gamma_y/2$
3	$C_{2,0}^{1,0}$	$(K_{2x}D - K_x)\delta\beta_x/2\Delta + hD\delta(1 - \alpha_x^2)/2\beta_x - h\varsigma\delta\alpha_x/\Delta$
4	$S_{2,0}^{1,0}$	$hD\delta\alpha_x/\beta_x - h\varsigma\delta/\Delta$
5	$C_{0,2}^{0,1}$	$-(K_{2y}D + K_y)\delta\beta_y/2\Delta - hD\delta(1 - \alpha_y^2)/2\beta_y$
6	$S_{0,2}^{0,1}$	$hD\delta\alpha_y/\beta_y$
7	$C_{3,0}^{3,0}$	$K_{2x}\beta_x^{3/2}/6\sqrt{2}\Delta^{3/2} - h(1 - \alpha_x^2)/\sqrt{2\beta_x}\Delta$
8	$C_{1,0}^{3,0}$	$K_{2x}\beta_x^{3/2}/2\sqrt{2}\Delta^{3/2} + \sqrt{2}h(1 + \alpha_x^2)/\sqrt{\beta_x}\Delta - h(1 - \alpha_x^2)/\sqrt{2\beta_x}\Delta$
9	$C_{1,0}^{1,2}$	$-K_{2y}\beta_x^{1/2}\beta_y/2\sqrt{2}\Delta^{3/2} + h\gamma_y\sqrt{2\beta_x}/\Delta$
10	$C_{1,2}^{1,2}$	$-K_{2y}\beta_x^{1/2}\beta_y/2\sqrt{2}\Delta^{3/2} - h(1 - \alpha_y^2)\sqrt{\beta_x/2\Delta}/\beta_y$
11	$C_{1,-2}^{1,2}$	$-K_{2y}\beta_x^{1/2}\beta_y/2\sqrt{2}\Delta^{3/2} - h(1 - \alpha_y^2)\sqrt{\beta_x/2\Delta}/\beta_y$
12	$S_{3,0}^{3,0}$	$h\alpha_x\sqrt{2/\beta_x}\Delta$
13	$S_{1,0}^{3,0}$	$h\alpha_x\sqrt{2/\beta_x}\Delta$
14	$S_{1,2}^{1,2}$	$h\alpha_y\sqrt{2\beta_x/\Delta}/\beta_y$
15	$S_{1,-2}^{1,2}$	$-h\alpha_y\sqrt{2\beta_x/\Delta}/\beta_y$

more complicated. Finally, we choose the β-function to be independent of momentum, treating all the terms in the Hamiltonian dependent on δ as perturbations.

Due to the explicit dependence of the Hamiltonian (10) on the angles ϕ_x and ϕ_y, J_x and J_y are not constants of motion as would be the case for the unperturbed Hamiltonian. Note, however, that the perturbation term V is zero when averaged over all values of the angle variables. This allows us to evaluate the effects due to the leading order terms (e.g., changes in phase advance of the betatron oscillations) by doing such averaging.

For a system of total length L we define a symbol $< \; >$ for the averaging over the angle variables. From the equation of motion the horizontal phase advance is:

$$\phi_{xo}(s) \approx \phi_{xo} + \psi_x(s) + \Delta\psi_x(s) + \ldots \quad , \tag{12}$$

where ψ_{xo} is an initial phase of the horizontal betatron oscillation,

$$\psi_x(s) = \int_0^s \frac{ds'}{\beta_x(s')} \tag{13}$$

is the linear (unperturbed) horizontal betatron phase, and

$$\Delta\psi_x(s) = \frac{\delta}{2(1 + \delta)} \int_0^s ds' [(K_{2x}D - K_x)\beta_x + \frac{hD(1 + \delta)(1 + \alpha_x^2)}{\beta_x} - 2h\varsigma\alpha_x] \tag{14}$$

is the horizontal tune shift due to the nonlinear terms in the Hamiltonian. Similarly for the vertical tune, these quantities are given, respectively by:

$$\phi_{yo} \approx \phi_{yo} + \psi_y(s) + \Delta\psi_y(s) + \dots, \tag{15}$$

$$\psi_y(s) = \int_0^s \frac{ds'}{\beta_y(s')} \quad, \tag{16}$$

$$\Delta\psi_y(s) = -\frac{\delta}{2(1+\delta)} \int_0^s ds' [(K_{2y}D + K_y)\beta_y - \frac{hD(1+\delta)(1+\alpha_y^2)}{\beta_y}] \quad. \tag{17}$$

The *chromaticity* $\xi = (1/2\pi)(\partial\psi(L)/\partial\delta)_{\delta=0} \equiv (\partial\nu/\partial\delta)_{\delta=0}$ is a measure of the linear dependence of the tune on δ. From Eqs. (14) and (17) one finds:

$$\xi_x = \frac{1}{4\pi} \int_0^L ds' [(K_{2x}D_o - K_x)\beta_x + hD_o\gamma_x - 2hD_o'\alpha_x] \tag{18}$$

and

$$\xi_y = -\frac{1}{4\pi} \int_0^L ds' [(K_{2y}D_o + K_y)\beta_y - hD_o\gamma_y] \quad. \tag{19}$$

Up to now the initial conditions for the dispersion function $D_o(s)$ which enters the above formulae has not been specified. Sometimes special initial conditions are appropriate. For example, for a system built of several identical cells the *periodic matched dispersion* η is convenient. Since β-functions are also periodic, the integrands in Eqs. (18) and (19) become periodic as well. In this case, it is sufficient to perform the integration over one period l:

$$\xi_x = \frac{1}{4\pi} \int_0^l ds' [(K_{2x}\eta - K_x)\beta_x + h\eta\gamma_x - 2h\eta'\alpha_x] \quad, \tag{20}$$

$$\xi_y = -\frac{1}{4\pi} \int_0^l ds' [(K_{2y}\eta + K_y)\beta_y - h\eta\gamma_y] \quad. \tag{21}$$

Equations (20) and (21) were obtained by M. Bassetti [5] following a simple intuitive approach.

It is both interesting and important to note that it follows, from Eq. (7) and from the conditions (to be shown) required for a system to become a second-order achromat, that the values for chromaticities given by Eqs. (18) and (19) are independent of the initial conditions for the dispersion function.

Now we consider the perturbation V [Eq. (11)] in the Hamiltonian [Eq. (10)]. To describe the perturbed betatron motion of a particle we apply the canonical perturbation theory [3] and use the results obtained in Ref. [6]. The generating function which eliminates all the second-order terms from the Hamiltonian in Eq. (10) is given by the expression (see Appendix):

$$G(I_x, \phi_x, I_y, \phi_y, s) = -\int_0^s ds' \sum_{m,n,k,l} I_x^{k/2} I_y^{l/2} \Big(C_{m,n}^{k,l} \cos[m(\phi_x + \tilde{\psi}_x' - \tilde{\psi}_x) + n(\phi_y + \tilde{\psi}_y' - \tilde{\psi}_y)]$$
$$+ S_{m,n}^{k,l} \sin[m(\phi_x + \tilde{\psi}_x' - \tilde{\psi}_x) + n(\phi_y + \tilde{\psi}_y' - \tilde{\psi}_y)] \Big) \quad, \tag{22}$$

where the *perturbed* phases are $\tilde{\psi}_{x,y} = \psi_{x,y}(s) + \Delta\psi_{x,y}(s)$, $\tilde{\psi}_{x,y}' = \psi_{x,y}(s') + \Delta\psi_{x,y}(s')$. The new invariants of the motion are $I_x = J_x - G_{\phi_x}$ and $I_y = J_x - G_{\phi_y}$, where a *subscript* in G indicates the partial differentiation with respect to the subscript.

The solution of the Hamilton's equations up to the second-order terms now is:

$$x_\beta(s) = \sqrt{\frac{2(I_x + G_{\phi_x})\beta_x}{(1+\delta)}} \cos(\phi_{xo} + \psi_x + \Delta\psi_x - G_{J_x}) \quad , \tag{23}$$

$$x_\beta'(s) = -\sqrt{\frac{2(I_x + G_{\phi_x})}{\beta_x(1+\delta)}}[\sin(\phi_{xo} + \psi_x + \Delta\psi_x - G_{J_x}) + \alpha_x\cos(\phi_{xo} + \psi_x + \Delta\psi_x - G_{J_x})] \quad , \tag{24}$$

and similar expressions for the vertical plane.

4. SECOND-ORDER MATRIX ELEMENTS

We will now apply the general results obtained in the previous section to the second order achromat as defined in Ref. [1] — a magnetic system for which all first-order terms represent the identity transformation and all second-order aberrations vanish at its end.

Let a single component of the vector in the phase space $(x_\beta, x_\beta', y, y', \Delta l, \delta)$ be expressed by $X_i(s)$. Then, at each point s, component $X_i(s)$ can be represented as a power series in initial values $X_k(0)$ by expanding the following expressions:

$$x(s) = x_\beta(s) + D(s)\delta, \quad x'(s) = x_\beta'(s) + D'(s)\delta \quad , \tag{25}$$

$$y(s) = y_\beta, \quad y'(s) = y_\beta' \quad . \tag{26}$$

This gives, in general,

$$X_i(s) = R_i^k(s)X_k(0) + T_i^{kj}(s)X_k(0)X_j(0) \quad . \tag{27}$$

Here the summation from 1 to 6 over repeated indices (one upper and one lower) is to be understood. From definition (27) it follows that matrix T is symmetric in the upper indices: $T_i^{kj} = T_i^{jk}$. In order for the first order matrix $R_i^k(L)$ to be the unity matrix for a whole achromat the tunes ν_x and ν_y should be integers and $\sin 2\pi\nu_x = 0$, $\cos 2\pi\nu_x = 1$. Similar expressions hold for y-plane.

There are three categories of the second-order matrix elements $T_i^{kj}(s)$: a) *geometric* matrix elements, i.e., all T_i^{kj} with $i, k, j = 1, 2, 3, 4$; b) *single-chromatic* matrix elements, i.e., elements with k or $j = 6$ and c) *double-chromatic* matrix elements, i.e., with $k = j = 6$.

Equations in the form of Eq. (27) are obtained by expansions of Eqs. (23) and (24) for small G_{ϕ_x}, G_{J_x} and $\Delta\psi_x$ and of similar equations for the vertical plane. Introduce the amplitudes $a_x = \sqrt{2I_x/(1+\delta)}$, $a_y = \sqrt{2I_y/(1+\delta)}$ and express $x_i(L)$ in terms of $x_i(0)$ by excluding a_x, ϕ_x, a_y, ϕ_y .

a) Geometric matrix elements

All the second-order *geometric* matrix elements are proportional to one of the derivatives $G_{I_x}, G_{I_y}, G_{\phi_x}$ or G_{ϕ_y}. Therefore, if these derivatives vanish simultaneously at the point $s = L$, so will the second-order geometric matrix elements. Below we will find the conditions under which these derivatives vanish. For now suppose that these conditions are met.

b) Single-chromatic matrix elements

Consider now the terms which are proportional to $\Delta\psi_x$. These we expand in δ and keep terms linear in δ. That gives:

$$x(L) = x(0) + 2\pi\xi_x\delta[\beta_x(0)x'(0) + \alpha_x(0)x(0)] \quad , \tag{28}$$

$$x'(L) = x'(0) - 2\pi\xi_x\delta[\alpha_x(0)x'(0) + \gamma_x(0)x(0)] \quad . \tag{29}$$

Comparing Eqs. (28) and (29) with Eq. (27) we find:

$$\begin{pmatrix} T_1^{16} & T_1^{26} \\ T_2^{16} & T_2^{26} \end{pmatrix}_L = 2\pi\xi_x \begin{pmatrix} \alpha_x & \beta_x \\ -\gamma_x & -\alpha_x \end{pmatrix}_0 \quad , \tag{30}$$

$$\begin{pmatrix} T_3^{36} & T_3^{46} \\ T_4^{36} & T_4^{46} \end{pmatrix}_L = -2\pi\xi_y \begin{pmatrix} \alpha_y & \beta_y \\ -\gamma_y & -\alpha_y \end{pmatrix}_0 \quad . \tag{31}$$

Recall that for any lattice the initial values for the matched β, α and γ are uniquely defined. When the conditions $\xi_x = 0$ and $\xi_y = 0$ are met all the single-chromatic second-order matrix elements become zero simultaneously.

c) Double-chromatic matrix elements

From Eqs. (4) and (25) it follows that the double-chromatic matrix elements are: $T_1^{66}(L) = D_1(L)$, $T_2^{66}(L) = D_1'(L)$.

The solution of Eq. (6) for the zero initial conditions $D_1(0) = D_1'(0) = 0$ gives the following expressions for the second-order dispersion and its derivative at the exit of the achromat $s = L$:

$$D_1(L) = \sqrt{\beta_x(L)} \left(\sin\psi_x(L) \int_0^L ds' P_1(s')\sqrt{\beta_x(s')} \cos\psi_x' - \cos\psi_x(L) \int_0^L ds' P_1(s')\sqrt{\beta_x(s')} \sin\psi_x' \right) \quad , \tag{32}$$

$$D_1'(L) = \frac{1}{\sqrt{\beta_x(L)}} \Big([\cos\psi_x(L) - \alpha_x(L)\sin\psi_x(L)] \int_0^L ds' P_1(s')\sqrt{\beta_x(s')} \cos\psi_x'$$

$$+ [\sin\psi_x(L) + \alpha_x(L)\cos\psi_x(L)] \int_0^L ds' P_1(s')\sqrt{\beta_x(s')} \sin\psi_x' \Big) \quad , \tag{33}$$

where $P_1(s)$ is the right-hand side of Eq. (6):

$$P_1(s) = K_x(1 - hD_o)D_o - h - K_{2x}D_o^2/2 + h^2 D_o + h(D_o')^2/2 \quad . \tag{34}$$

First consider the matched dispersion $D_o = \eta$. Then $P_1(s)$ is a periodic function and as is shown below for the second-order achromat, it follows that both $D_1(L)$ and $D_1'(L)$ and, consequently, $T_1^{66}(L)$ and $T_2^{66}(L)$ vanish.

Next consider a dispersion function D_0 with arbitrary initial conditions. Then, using Eq. (7), D_o can be represented as a sum of an η -function and a free oscillation function. Again it will be shown below that contribution to $D_1(L)$ and $D_1(L)$ due to this oscillation also vanishes. Hence, the second-order double-chromatic matrix elements become zeros at the end of an achromat independently of the initial conditions for the dispersion function.

5. SECOND–ORDER ACHROMAT

We have shown that in order for the geometric matrix elements to vanish the derivatives of the generating function G must be equal to zero at the end of an achromat. Now we will find the conditions under which that is true.

The generating function $G(s)$ as well as its derivatives over canonical variables I_x, I_y, ϕ_x and ϕ_y are sums of the terms which are linearly independent. In order for the sum to be zero for all possible values

of these variables, each term separately must be zero. Considering a typical term of those sums the following conditions should be satisfied:

$$\int_0^L ds' F_{mn}(s') \exp(i[m\tilde{\psi}_x' + n\tilde{\psi}_y']) = 0 \quad , \tag{35}$$

where F_{mn} are the complex amplitudes obtained from Eq. (22) and Table 1.

Equations $\xi_x = 0, \xi_y = 0$ *, Eq. (35) and the conditions that the tunes* ν_x *and* ν_y *for a whole achromat should be integers constitute the most general set of the necessary and sufficient conditions for a system to be a second-order achromat.*

Consider now a particular case of an achromat built out of N identical cells of length l and tunes per cell ν_x^c and ν_y^c. As for any second-order achromat the total tunes $N\nu_x^c = integer$, $N\nu_y^c = integer$. Assume that values for K_{2x} and K_{2y} have been found such that chromaticities ξ_x and ξ_y in Eqs. (20) and (21) are equal to zero. We will show that in this case Eq. (35) is satisfied, provided that the unperturbed tunes ν_x^c and ν_y^c both avoid the following resonance values:

$$m\nu_x^c + n\nu_y^c \neq integer \quad , \tag{36}$$

where $m = 1, 2$ or 3 for $n = 0$, and $m = 0$ or 1 for $n = 2$ or -2.

Indeed, let us assume first that functions F_{mn} contain the matched dispersion function η and consequently they all are periodic functions with the period l. Then each integral can be evaluated in the following way

$$\int_0^L ds' F_{mn}(s') \exp(im\tilde{\psi}_x' + in\tilde{\psi}_y')$$

$$= \int_0^l ds' F_{mn}(s') \exp(im\tilde{\psi}_x' + in\tilde{\psi}_y') \frac{1 - \exp[2\pi i N(m\nu_x^c + n\nu_y^c)]}{1 - \exp[2\pi i(m\nu_x^c + n\nu_y^c)]} \quad . \tag{37}$$

The unperturbed tunes are used here since the chromaticities are equal to zero. Restrictions in Eq. (36) follow from here.

Now consider the case where the dispersion function D is not periodic. As we discussed earlier it can be expressed as a sum of a periodic function and a free oscillation function [see Eq. (7)]. Examining Table 1 we find that the coefficients containing D give rise to terms already made to vanish by the conditions in Eq. (36).

If, as is usual for an optical magnetic line, $\nu_x^c = \nu_y^c = \nu^c$, all the essential conditions in Eq. (36) are reduced to only one requirement $3\nu^c \neq integer$. In this case, condition (36) is equivalent to the statement that the number of cells in the second-order achromat should be more than 3 ($N \geq 4$). In this particular form the conditions for the second-order achromat were originally formulated by K. Brown [1].

CONCLUSIONS

Using a canonical perturbation theory we have developed a complete theory of the second-order achromat. The necessary and sufficient conditions for the existence of such a device are given. We have shown the reason why all the second-order geometric matrix elements are zeros simultaneously when these conditions are met. The second-order chromatic matrix elements were found explicitly. It was shown that the double-chromatic elements are zeros and that the single chromatic elements are proportional to corresponding chromaticities and consequently vanish when the chromaticities were made equal to zero. We have also shown that for a second-order achromat all the formulae for the second-order matrix elements which explicitly contain the dispersion function do not depend on the initial conditions for that function.

ACKNOWLEDGMENTS

REFERENCES

1. K. L. Brown, *A Second-Order Magnetic Optical Achromat*, Report SLAC-PUB-2257, SLAC, Stanford University, February 1979; IEEE Trans. Nucl. Sci. NS-26 (3), p. 3490 (1979).

2. D. C. Carey, *Why a Second-Order Magnetic Optical Achromat Works*, Fermilab Pub. 79/61-Exp 2042; Nucl. Instr. Meth. **189**, 365 (1981).

3. A. N. Kolmogorov, *On Conservation of Conditionally-Periodic Motions of a Small Change in Hamilton's Function*, Dokl. Acad. Nauk SSSR **98**, 4, 525 (1954).

4. R. D. Ruth, Single Particle Dynamics and Nonlinear Resonances in Circular Accelerators in the book *Linear Dynamics Aspects of Particle Accelerators*, Proceed. of the Joint US-CERN School on Particle Accelerators, Sardinia, 1985, Springer Verlag, p. 37.

5. M. Bassetti, *A Simple Derivation of Chromaticity Formulae*, LEP Note 504, Report CERN, Geneva, Switzerland, 18 June 1985.

6. S. A. Kheifets, T. H. Fieguth and R. D. Ruth, *Canonical Description of a Second-Order Achromat*, SLAC Note AP-56, SLAC, Stanford University, October 1986.

7. F. Ch. Iselin, *The MAD Program (Methodical Accelerator Design) Reference Manual*, Report CERN-LEP-TH/85-15, Geneva, Switzerland, 1985.

8. J. M. Jowett, *Non-Linear Dissipative Phenomena in Electron Storage Rings*, ibid. 4, p. 343.

9. K. L. Brown et al., *TRANSPORT—a Computer Program for Designing Charged Particle Beam Transport Systems*, Report SLAC-91, Rev. 2, UC-28, SLAC, Stanford University, May 1977.

10. K. L. Brown, *A First- and Second-Order Matrix Theory for Design of Beam Transport Systems and Charged Particle Spectrometers*, Report SLAC-75, SLAC, Stanford University, June 1982.

11. J. Stoker, *Non-Linear Vibrations*, Interscience, 1950.

12. A. N. Kolmogorov, *On Conservation of Conditionally-Periodic Motions of a Small Change in Hamilton's Function*, Dokl. Acad. Nauk SSSR **98**, 4, 525 (1954).

13. T. O. Raubenheimer and R. D. Ruth, *Superconvergent Tracking*, SLAC-PUB-4370, Stanford University, July 1987, to be published.

APPENDIX I

CANONICAL PERTURBATION THEORY

In this Appendix we derive the generating function (22) used to solve the Hamilton's equations with an accuracy up to the second order in perturbation. The method was developed by Kolmogorov and is known as *canonical perturbation theory* [12].

We start with the Hamiltonian Eq. (10):

$$H_3(\vec{J}, \vec{\phi}, s) = H_o(\vec{J}, s) + V(\vec{J}, \vec{\phi}, s) \quad . \tag{A.1}$$

All vectors here and in what follows are two-dimensional, for example, $\vec{J} \equiv (J_x, J_y)$. The nonlinear perturbation $V(\vec{J}, \vec{\phi}, s)$ is periodic in angles $\vec{\phi}$ and has zero average over these angles. Our goal is to find canonically conjugate new variables such that the new Hamiltonian is a function of the new generalized momenta alone plus terms of order V^2.

Consider a canonical transformation from the variables $(\vec{J}, \vec{\phi})$ to new variables $(\vec{I}, \vec{\Phi})$ with the generating function of the new momenta and the old coordinates

$$F(\vec{I}, \vec{\phi}, s) = \vec{\phi} \cdot \vec{I} + G(\vec{I}, \vec{\phi}, s) \quad . \tag{A.2}$$

The new coordinates are defined by $\vec{\Phi} = \vec{\phi} + G_{\vec{I}}, \vec{J} = \vec{I} + G_{\vec{\phi}}$ and the new Hamiltonian is

$$H_4 = H_o(\vec{I} + G_{\vec{\phi}}, s) + V(\vec{I} + G_{\vec{\phi}}, \vec{\phi}, s) + G_s \quad , \tag{A.3}$$

where a subscript indicate partial differentiation with respect to the subscript. Since for a small perturbation V, the function G is also small, we can expand H_4

$$H_4 = H_o(\vec{I}, s) + \vec{\Omega} \cdot G_{\vec{\phi}} + G_{\vec{\phi}} \cdot \vec{\Omega}_{\vec{I}} \cdot G_{\vec{\phi}}/2 + V(\vec{I}, \vec{\phi}, s) + V_{\vec{I}} \cdot G_{\vec{\phi}} + G_s \quad , \tag{A.4}$$

where the angular "frequencies" denoted by $d\vec{\psi}/ds \equiv \vec{\Omega}(\vec{I}, s) = H_{o,\vec{I}}$ may be functions of both the amplitude $\vec{I}$ and the independent variable s.

By choosing a function $G(\vec{I}, \vec{\phi}, s)$ to satisfy the equation

$$\vec{\Omega}(\vec{I}, s) \cdot G_{\vec{\phi}} + V(\vec{I}, \vec{\phi}, s) + G_s = 0 \tag{A.5}$$

we can eliminate from the Hamiltonian all the terms of the order of V. The new momenta $\vec{I}$ then will be integrals of motion with the accuracy V^2.

Equation (A.5) for G is a first order differential equation, the solution of which can be found using the fact that all functions are periodic in angles. Substituting into the Fourier expansions

$$V(\vec{I}, \vec{\phi}, s) = \sum_{\vec{m}} v_{\vec{m}}(\vec{I}, s) \exp(i\vec{m} \cdot \vec{\phi}) \quad , \quad G(\vec{I}, \vec{\phi}, s) = \sum_{\vec{m}} g_{\vec{m}}(\vec{I}, s) \exp(i\vec{m} \cdot \vec{\phi}) \quad , \tag{A.6}$$

where the vector of integer coefficients $\vec{m} = (m_x, m_y)$ is introduced, we find a system of equations which are equivalent to Eq. (A.5):

$$(i\vec{m} \cdot \vec{\Omega}(s) + \frac{d}{ds})g_{\vec{m}} = -v_{\vec{m}} \quad . \tag{A.7}$$

Hence, the function G with the initial condition $G(\vec{I}, \vec{\phi}, 0) = 0$ is [6,13]

$$G = -\sum_{\vec{m}} \int_0^s ds' v_{\vec{m}}(\vec{I}, s') \exp\{i\vec{m} \cdot [\vec{\phi} + \vec{\psi}(s') - \vec{\psi}(s)]\} \quad , \tag{A.8}$$

where $\vec{\psi}(s') = \int_o^{s'} ds'' \vec{\Omega}(s''), \vec{\psi}(s) = \int_o^s ds' \vec{\Omega}(s')$. That is, in fact, the expression (22) which was given in the text.