CERN-TH.4976/88

LOOP CORRECTIONS TO THE $E_8 \times E_8$ HETEROTIC STRING EFFECTIVE LAGRANGIAN

M. Abe and H. Kubota

Department of Physics, Tokyo Institute of Technology
Oh-okayama, Meguro, Tokyo 152, Japan

and

N. Sakai*)

CERN - Geneva

A B S T R A C T

One-loop corrections to the effective Lagrangian for the $E_8 \times E_8$ heterotic string theory are worked out. The resulting terms are of order $\alpha'^3 g^2$ and quartic in gauge field strength and curvature tensor.

---

*) Address from April to October 1988: Blackett Lab., Imperial College, Prince Consort Road, London SW7 2AZ, UK.
Permanent address: Department of Physics, Tokyo Institute of Technology, Oh-okayama, Meguro, Tokyo 152, Japan.

CERN-TH.4976/88

February 1988

1. - INTRODUCTION

The heterotic string with $E_8 \times E_8$ gauge symmetry is the most promising candidate to unify all the fundamental interactions [1]. Low energy effective Lagrangians have been very useful to discover the anomaly cancellation [2] and to explore possible compactified ground states of the ten-dimensional space-time [3-10]. The effective Lagrangians can most directly be obtained by matching scattering amplitudes of the string theory with those in the corresponding field theory involving only massless particles [11-13, 6-8]. The result can be given in a double series in powers of $\alpha' k^2$ and g, where α', k and g are the slope parameter, a typical momentum, and the string loop expansion parameter. Using the general co-ordinate and other invariances, the effective Lagrangian can be expressed in terms of Riemann curvature tensors and gauge field strengths. At the string tree level, quartic terms have been obtained from the heterotic string and the type-II superstring [4-8], and the most thorough study has been done by Gross and Sloan [14]. One-loop corrections to the quartic curvature terms have been obtained for the type-II superstring [15] and the heterotic string [16,17]. Terms involving gauge fields have also been worked out for $\mathrm{Spin}(32)/Z_2$ gauge group [17].

The purpose of our paper is to obtain the string one-loop (torus topology) corrections to the quartic terms of gauge field strengths and curvature tensors in the $E_8 \times E_8$ heterotic string theory. In constructing the effective Lagrangian, we shall take into account the non-polynomial couplings of the dilaton by invoking the ten-dimensional dilatational symmetry and non-linear σ model considerations [16,1,14,18-20]. We find that the one-loop correction to the term quartic in the gauge field strength is consistent with the form conjectured from the anomaly cancellation [17]. We shall primarily use the bosonic formulation of current algebra for the gauge degree of freedom. It is complementary to the fermionic formulation used in Ref. [17] and seems to be more powerful especially for the $E_8 \times E_8$ gauge group. Only the $O(16) \times O(16)$ part of current algebra is realized linearly in the fermionic formulation and the remaining part is quite non-linear. In the bosonic formulation, the entire gauge group is realized by vertex operator constructions.

In Section 2 we summarize one-loop amplitudes in the heterotic string theory for four massless bosons (graviton, antisymmetric tensor, dilaton, and gauge bosons). The bosonic formulation is employed for gauge degrees of freedom and the lattice momentum summation is explicitly performed. In Section 3 we derive the string one-loop corrections to quartic terms in the low energy effective Lagrangian. Appendix A sketches the method to compute the gauge group factor. Appendix B lists useful formulas in performing the ν and τ integrations.

2. - ONE-LOOP AMPLITUDES AND LATTICE MOMENTUM SUM

The four massless boson amplitudes in the heterotic string theory have been computed to one-loop order in the light-cone gauge by means of the operator formalism [15,21,22]. We follow mostly the convention of Ref. [15]. The vertex operators for the emission of charged gauge bosons (A) and "gravitons" (G) are given by*)

$$V(\tau; k, \zeta \text{ or } \rho) = \int_0^\pi \frac{d\sigma}{\pi} \tilde{V}(\tau, \sigma; k, \zeta \text{ or } \rho)$$
$$\tilde{V}_A(\tau,\sigma; k, K, \zeta) = \zeta_i B^i e^{ik_i X^i} :e^{2iK^I X^I}: C(K)$$
$$\tilde{V}_G(\tau,\sigma; k, \rho) = \rho_{ij} B^i \tilde{P}^j e^{ik_i X^i} . \tag{2.1}$$

Here we use the convention $\alpha' = \frac{1}{2}$.

We shall refer to the graviton, antisymmetric tensor, and dilaton as "graviton" collectively. A convenient way to deal with these vertex operators is to use the following vertex operator [21]

$$\tilde{V}(\tau,\sigma; k, K, \zeta, \bar{\zeta}) = e^{\zeta B} e^{\bar{\zeta}\tilde{P}} e^{ikX} :e^{2iKX}: C(K) \tag{2.2}$$

with the understanding that gauge bosons are given by the linear term in ζ with $\bar{\zeta} = 0$, whereas "gravitons" are given by terms linear in ζ and $\bar{\zeta}$ with $K = 0$.

By factorization analysis of tree amplitudes and the sewing procedure, we can determine the absolute normalization of loop amplitudes [15,16]. The one-loop four-point amplitude with the vertex (2.2) is then given by [21]:

$$T_{1\text{-loop}}(1, \cdots, 4) = \frac{1}{4\pi^{10}} \left(\frac{\pi g}{2}\right)^4 \bar{\epsilon} K \int \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2} \int \prod_{r=1}^{3} \frac{d^2\nu_r}{\mathrm{Im}\,\tau}$$
$$\cdot \prod_{1\le r<A\le 4} (\chi_{rA})^{k_r k_A/2} \left(\Delta_{12}^{-1} \mathcal{L} \prod_{1\le r<A\le 4} \psi_{rA}^{K_r K_A}\right)^*$$
$$\cdot \exp\left(\frac{1}{2}\sum_{r=1}^{4} \bar{\zeta}_r Q_r + \sum_{1\le r<A\le 4} \bar{\zeta}_r \bar{\zeta}_A T(\nu_{Ar})\right)^* , \tag{2.3}$$

where g is the string loop expansion parameter, Δ_{12} is the weight twelve cusp form, and $\bar{\epsilon}$ and K are a simplified notation for the product of cocycle factors and the superstring kinematical factor

---

*) The gauge boson vertex operator in Eq. (2.7) of Ref. [15] was incorrectly multiplied by a factor of i.

$$\Delta_{12} = e^{2\pi i\tau} \prod_{n=1}^{\infty} (1 - e^{2\pi i n\tau})^{24} , \tag{2.4}$$

$$K = \zeta_1^{i_1} \cdots \zeta_4^{i_4} K^{i_1 \cdots i_4} , \quad K^{i_1\cdots i_4} = t^{i_1 j_1 \cdots i_4 j_4} k_1^{j_1} \cdots k_4^{j_4} ,$$

$$t^{i_1 j_1 \cdots i_4 j_4} = (-\epsilon^{i_1 j_1 \cdots i_4 j_4} - t_1^{i_1 j_1 \cdots i_4 j_4} + t_2^{i_1 j_1 \cdots i_4 j_4})/2 , \tag{2.5}$$

$$t_1^{i_1 j_1 \cdots i_4 j_4} = (\delta^{i_1 j_2} \delta^{i_2 j_1} - \delta^{i_1 i_2} \delta^{j_1 j_2})(\delta^{i_3 j_4} \delta^{i_4 j_3} - \delta^{i_3 i_4} \delta^{j_3 j_4})$$

$$+ (1\ 3)(2\ 4) + (1\ 4)(2\ 3) ,$$

$$t_2^{i_1 j_1 \cdots i_4 j_4} = \delta^{i_1 j_2} \delta^{i_2 j_3} \delta^{i_3 j_4} \delta^{i_4 j_1} + \delta^{i_1 j_3} \delta^{i_3 j_2} \delta^{i_2 j_4} \delta^{i_4 j_1}$$

$$+ \delta^{i_1 j_3} \delta^{i_3 j_4} \delta^{i_4 j_2} \delta^{i_2 j_1} + \text{antisymmetrization in } (i_1 j_1) \cdots (i_4 j_4) . \tag{2.6}$$

Functions doubly periodic in ν with periods 1 and τ are given*) in terms of the elliptic theta function θ [23]

$$\chi_{rA} = \chi(\nu_{Ar}) , \qquad \nu_{Ar} = \nu_A - \nu_r$$

$$\chi(\nu) = 2\pi \left| \theta_1(\nu) / \theta_1'(0) \right| \exp\left(-\pi (\mathrm{Im}\,\nu)^2 / \mathrm{Im}\,\tau\right) \tag{2.7}$$

$$Q_r = \sum_{A=1}^{4} k_A K(\nu_{rA})$$

$$K(\nu) = \frac{1}{i\pi} \frac{\partial}{\partial \nu} \ln \chi(\nu) = \frac{1}{2\pi i} \frac{\partial}{\partial \nu} \ln \theta_1(\nu) + \frac{\mathrm{Im}\,\nu}{\mathrm{Im}\,\tau}$$

$$T(\nu) = \frac{i}{2\pi} \frac{\partial}{\partial \nu} K(\nu) = \frac{1}{4\pi^2} \frac{\partial^2}{\partial \nu^2} \ln \theta_1(\nu) + \frac{1}{4\pi \mathrm{Im}\,\tau} \tag{2.8}$$

whereas ψ is pseudo-doubly-periodic in ν

$$\psi_{rA} = \psi(\nu_{Ar})$$

$$\psi(\nu) = -2\pi i \left(\theta_1(\nu) / \theta_1'(0) \right) \exp\left(\pi i \nu^2 / \tau\right) \tag{2.9}$$

The gauge group factor $\mathcal{L}$ is defined as a sum over points on the weight lattice Λ of the gauge group

$$\mathcal{L} = \sum_{L \in \Lambda} \exp\left(i\pi\tau \left(L + \sum_{r=1}^{4} \nu_r K_r / \tau \right)^2 \right) \tag{2.10}$$

Restoring the slope parameter α' to account for the correct dimension ($[T(1,\ldots,N)] = M^{-4N+10}$), we obtain the one-loop four-charged gauge boson amplitude

---

*) Our functions $K(\nu)$ and $T(\nu)$ correspond to $\hat{\eta}$ and $\hat{\Omega}$ of Ref. [21].

as

$$T_{4A} = \frac{(2\alpha')^5}{4\pi^{10}} \left(\frac{\pi g}{2}\right)^4 \mathcal{E}(K_1,K_2)\,\mathcal{E}(K_4,K_3)\,\delta_{\Sigma K_r,0}\,\zeta_1^{i_1}\cdots\zeta_4^{i_4}$$
$$\cdot K^{i_1\cdots i_4}\int\frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}\int\prod_{r=1}^{3}\frac{d^2\nu_r}{\mathrm{Im}\,\tau}\prod_{1\le r<A\le 4}(\chi_{rA})^{\alpha' k_r k_A}$$
$$\cdot\left[\Delta_{12}^{-1}\,\mathcal{L}(\tau,\nu_1,\nu_2,\nu_3)\prod_{1\le r<A\le 4}\psi_{rA}^{K_r K_A}\right]^* . \qquad (2.11)$$

The one-loop amplitude with three charged gauge bosons and a graviton is given by

$$T_{3A,G} = \frac{(2\alpha')^5}{4\pi^{10}} \left(\frac{\pi g}{2}\right)^4 \mathcal{E}(K_1,K_2)\,\delta_{\Sigma K_r,0}\,\zeta_1^{i_1}\cdots\zeta_3^{i_3}\,\rho_4^{i_4 j_4}$$
$$\cdot K^{i_1\cdots i_4}\int\frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}\int\prod_{r=1}^{3}\frac{d^2\nu_r}{\mathrm{Im}\,\tau}\prod_{1\le r<A\le 4}(\chi_{rA})^{\alpha' k_r k_A}$$
$$\cdot\left[\Delta_{12}^{-1}\,\mathcal{L}(\tau,\nu_{31},\nu_{21})\,\psi_{12}^{-1}\,\psi_{13}^{-1}\,\psi_{23}^{-1}\,Q_4^{j_4}\sqrt{\alpha'/2}\right]^* . \qquad (2.12)$$

The one-loop amplitude with two charged gauge bosons and two gravitons is*)

$$T_{2A,2G} = \frac{(2\alpha')^5}{4\pi^{10}} \left(\frac{\pi g}{2}\right)^4 \delta_{K_1,-K_2}\,\zeta_1^{i_1}\,\zeta_2^{i_2}\,\rho_3^{i_3 j_3}\,\rho_4^{i_4 j_4}$$
$$\cdot K^{i_1\cdots i_4}\int\frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}\int\prod_{r=1}^{3}\frac{d^2\nu_r}{\mathrm{Im}\,\tau}\prod_{1\le r<A\le 3}(\chi_{rA})^{\alpha' k_r k_A}$$
$$\cdot\left[\Delta_{12}^{-1}\,\mathcal{L}(\tau,\nu_{21})\,\psi_{12}^{-2}\left(\delta^{j_3 j_4}\,T(\nu_{43}) + Q_3^{j_3} Q_4^{j_4}\frac{\alpha'}{2}\right)\right]^* . \qquad (2.13)$$

The one-loop four graviton amplitude is given by

$$T_{4G} = \frac{(2\alpha')^5}{4\pi^{10}} \left(\frac{\pi g}{2}\right)^4 \rho_1^{i_1 j_1}\cdots\rho_4^{i_4 j_4}\,K^{i_1\cdots i_4}$$
$$\cdot\int\frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}\int\prod_{r=1}^{3}\frac{d^2\nu_r}{\mathrm{Im}\,\tau}\prod_{1\le r<A\le 3}(\chi_{rA})^{\alpha' k_r k_A}\left[\Delta_{12}^{-1}\,\mathcal{L}(\tau)\right]^*$$
$$\cdot\Big[\left(\delta^{j_1 j_2}\delta^{j_3 j_4}\,T(\nu_{21})\,T(\nu_{43}) + 2\text{ more terms}\right)$$
$$+\left(\frac{\alpha'}{2}Q_1^{j_1}Q_2^{j_2}\delta^{j_3 j_4}\,T(\nu_{43}) + 5\text{ other terms}\right)$$
$$+\left(\frac{\alpha'}{2}\right)^2 Q_1^{j_1}\cdots Q_4^{j_4}\Big]^* . \qquad (2.14)$$

---

*) We have corrected a sign error in Eq. (2.28) of Ref. [15] due to the factor of i in the vertex operator. The power of 2α' is also corrected compared to Eqs. (2.28) and (2.30) of Ref. [15].

Let us note that our convention for the cocycle is*)

$$\varepsilon(K,-K) = \varepsilon(K,K) = \varepsilon(K,0) = 1 \ . \tag{2.15}$$

We can perform the sum over the lattice momenta explicitly and express the gauge group factor $\mathcal{L}$ in Eq. (2.10) in terms of the elliptic theta functions. For $E_8 \times E_8$, $\mathcal{L}$ becomes a product of $\mathcal{L}_1$ and $\mathcal{L}_2$ corresponding to two E_8 factor groups. By using a set of orthonormal vectors e_i $(i=1,\ldots,8)$ in the eight-dimensional lattice space, we can parametrize the discrete momentum vectors L on the E_8 weight lattice as

$$\begin{aligned} L &= \sum_{i=1}^{8} m_i e_i \ , \qquad m_8 = n_8 - \sum_{i=1}^{7} m_i \\ m_i &= n_i + n_8/2 \ , \qquad i = 1, \cdots, 7 \end{aligned} \tag{2.16}$$

where $n_1,\ldots,n_8$ run over all integers. Let us take the case with two gauge bosons (K_1 and K_2) as an example. Since $K_1 = -K_2$ is on one of the E_8 lattice, we obtain

$$\mathcal{L}(\tau, \nu_{21}) = \mathcal{L}_1(\tau, \nu_{21}) \cdot \mathcal{L}_2(\tau) \tag{2.17}$$

The second E_8 gauge group factor simply gives the Eisenstein series of weight four E_4 [24]

$$\begin{aligned} \mathcal{L}_2(\tau) &= \sum_{n_k \in \mathbf{Z}} \exp\left(i\pi\tau \sum_{k=1}^{8} m_k^2 \right) \\ &= \frac{1}{2} \sum_{\beta=2}^{4} \left(\theta_\beta(0) \right)^8 = E_4(\tau) \ , \end{aligned} \tag{2.18}$$

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n} \ , \qquad q = e^{2\pi i \tau} \ . \tag{2.19}$$

The other E_8 gauge group factor with the external lattice momentum $K_1 = e_1 - e_2$ becomes

$$\mathcal{L}_1(\tau, \nu_{21}) = \sum_{n_k \in \mathbf{Z}} \exp\left[i\pi\tau \left\{ \sum_{k=1}^{8} m_k^2 - 2(m_1 - m_2)\frac{\nu_{21}}{\tau} + 2\nu_{21}^2/\tau^2 \right\} \right] . \tag{2.20}$$

As briefly described in Appendix A, we calculate cases with m_1, m_2 = odd, even, odd-$\frac{1}{2}$, and even-$\frac{1}{2}$ separately and sum them up to obtain

$$\mathcal{L}_1(\tau, \nu_{21}) = \frac{1}{2} \sum_{\alpha=2}^{4} \theta_\alpha^2(\nu_{21})\, \theta_\alpha^6(0) \exp\left(2\pi i \frac{\nu_{21}^2}{\tau} \right) . \tag{2.21}$$

When multiplied by ϕ_{12}^{-2} as in Eq. (2.13), it becomes

---

*) The first of our convention (2.15) has an opposite sign compared to Eq. (2.22) of the second of Ref. [1].

$$\mathcal{L}_1(\tau,\nu_{21})\psi_{12}^{-2} = \frac{1}{2}\sum_{\alpha=2}^{4} \theta_\alpha^8(0)\left(\frac{i}{2\pi}\frac{\theta_1'(0)\,\theta_\alpha(\nu_{21})}{\theta_1(\nu_{21})\,\theta_\alpha(0)}\right)^2$$

$$= \frac{1}{2}\sum_{\alpha=2}^{4} \theta_\alpha^8(0)\,\frac{1}{4\pi^2}\left(e_{\alpha-1} - \wp(\nu_{21})\right) \tag{2.22}$$

where the Weierstrass $\wp$-function [23] is given by our $T(\nu)$ in Eq. (2.8) as

$$\wp(\nu) = -4\pi^2\,T(\nu) - \pi^2\,\hat{E}_2(\tau)/3\,. \tag{2.23}$$

Here $\hat{E}_2(\tau)$ is of weight two under modular transformations, but is not holomorphic [24]

$$\hat{E}_2(\tau) = E_2(\tau) - \frac{3}{\pi\,\mathrm{Im}\,\tau}\,, \quad E_2(\tau) = 1 - 24\sum_{n=1}^{\infty}\frac{n q^n}{1-q^n}\,. \tag{2.24}$$

The term with $e_{\alpha-1}$ can be expressed by the Eisenstein series of weight six [23,24]

$$\sum_{\alpha=2}^{4}\theta_\alpha^8(0)\,e_{\alpha-1} = -2\pi^2\,E_6(\tau)/3\,,$$

$$E_6(\tau) = 1 - 504\sum_{n=1}^{\infty}\frac{n^5 q^n}{1-q^n}\,. \tag{2.25}$$

Hence we find an expression which is manifestly covariant under modular transformations

$$\mathcal{L}(\tau,\nu_{21})\psi_{12}^{-2} = \mathcal{L}_1(\tau,\nu_{21})\psi_{12}^{-2}\cdot\mathcal{L}_2(\tau)$$

$$= \left\{E_4\,T(\nu_{21}) + (\hat{E}_2E_4 - E_6)/12\right\}\cdot E_4\,. \tag{2.26}$$

An analogous procedure for the gauge group Spin(32)/Z_2 gives instead of Eq. (2.22)

$$\mathcal{L}(\tau,\nu_{21})\psi_{12}^{-2} = \frac{1}{2}\sum_{\alpha=2}^{4}\theta_\alpha^{16}(0)\left(\frac{i}{2\pi}\frac{\theta_1'(0)\,\theta_\alpha(\nu_{21})}{\theta_1(\nu_{21})\,\theta_\alpha(0)}\right)^2 \tag{2.27}$$

which agrees with the result of Ref. [17], where the fermionic formulation was used. Using an identity [17]

$$\sum_{\alpha=2}^{4}\theta_\alpha^{16}(0)\,e_{\alpha-1} = -2\pi^2\,E_4E_6/3\,,$$

we find that the final result for Spin(32)/Z_2 case is identical to Eq. (2.26) for $E_8\times E_8$. Hence we find that the $E_8\times E_8$ and the Spin(32)/Z_2 heterotic string theories give identical scattering amplitude for two gauge bosons and tow gravitons, in spite of the different gauge interactions. This is valid before taking any limit like $\alpha' \to 0$ and is perhaps more non-trivial that the well-known fact that the partition functions (vacuum amplitudes) are identical for the $E_8 \times E_8$ and the $Spin(32)/Z_2$ theories.

In the case of three gauge bosons, the gauge group factor can similarly be obtained and reads for $E_8 \times E_8$

$$\mathcal{L}(\tau, \nu_{31}, \nu_{21}) \psi_{12}^{-1} \psi_{13}^{-1} \psi_{23}^{-1} = \frac{1}{2} \sum_{\alpha=2}^{4} \theta_\alpha^8(0) \frac{i}{2\pi} \frac{\theta_1'(0)\, \theta_\alpha(\nu_{21})}{\theta_1(\nu_{21})\, \theta_\alpha(0)}$$

$$\cdot \frac{i}{2\pi} \frac{\theta_1'(0)\, \theta_\alpha(\nu_{31})}{\theta_1(\nu_{31})\, \theta_\alpha(0)} \cdot \frac{i}{2\pi} \frac{\theta_1'(0)\, \theta_\alpha(\nu_{32})}{\theta_1(\nu_{32})\, \theta_\alpha(0)} \times \frac{1}{2} \sum_{\beta=2}^{4} \theta_\beta^8(0). \tag{2.28}$$

For the gauge group $Spin(32)/Z_2$, we need to change $\theta_\alpha^8(0) \to \theta_\alpha^{16}(0)$ and to delete the last factor (sum over β).

For the four gauge bosons we define the lattice momenta invariants

$$S = (K_1 + K_2)^2, \quad T = (K_2 + K_3)^2, \quad U = (K_1 + K_3)^2$$
$$S + T + U = -8 = \sum_{r=1}^{4} K_r^2$$

and distinguish four cases of charge configurations:

$$\begin{aligned}
&\text{(i)} \quad (S, T, U) = (-4, -4, 0), \quad K_1 K_2 = 0, \quad K_3 = -K_1, \quad K_4 = -K_2, \\
&\text{(ii)} \quad (S, T, U) = (-4, -2, -2), \quad K_1 K_2 = 0, \quad K_2 K_3 = K_1 K_3 = -1, \\
&\text{(iii)} \quad (S, T, U) = (-6, -2, 0), \quad K_1 K_2 = -1, \quad K_3 = -K_1, \quad K_4 = -K_2, \\
&\text{(iv)} \quad (S, T, U) = (-8, 0, 0), \quad K_1 = K_2 = -K_3 = -K_4.
\end{aligned} \tag{2.29}$$

If all four lattice momenta K's belong to the same E_8, the gauge group factor becomes

$$\mathcal{L}(\tau, \nu_1, \nu_2, \nu_3) = \mathcal{L}_1(\tau, \nu_1, \nu_2, \nu_3) \cdot \mathcal{L}_2(\tau). \tag{2.30}$$

For the case (i), we find the first gauge group factor by a procedure similar to that in Appendix A (a convenient choice of K's are $K_1 = e_1 - e_2$ and $K_2 = e_3 - e_4$)

$$\mathcal{L}_1(\tau,\nu_1,\nu_2,\nu_3)\,\psi_{13}^{-2}\psi_{24}^{-2}$$
$$= \frac{1}{2}\sum_{\alpha=2}^{4}\theta_\alpha^8(0)\left(\frac{i}{2\pi}\frac{\theta_1'(0)\theta_\alpha(\nu_{31})}{\theta_1(\nu_{31})\theta_\alpha(0)}\right)^2\left(\frac{i}{2\pi}\frac{\theta_1'(0)\theta_\alpha(\nu_{42})}{\theta_1(\nu_{42})\theta_\alpha(0)}\right)^2$$
$$= \Big\{E_4^2 - E_6(2\hat{E}_2+12T(\nu_{31})+12T(\nu_{42}))$$
$$+ E_4(\hat{E}_2+12T(\nu_{31}))(\hat{E}_2+12T(\nu_{42}))\Big\}\Big/144 \qquad (2.31)$$

where we used an identity besides Eqs. (2.23) and (2.25) [23]

$$\sum_{\alpha=2}^{4}\theta_\alpha^8(0)\,e_{\alpha-1}^2 = 2\pi^4 E_4^2/9 \qquad (2.32)$$

For the case (ii), we find a convenient choice of K's: $K_1 = e_1-e_2$, $K_2 = -e_3+e_4$, $K_3 = -e_1+e_3$ and $K_4 = e_2-e_4$, and we obtain

$$\mathcal{L}_1(\tau,\nu_1,\nu_2,\nu_3)\,\psi_{13}^{-1}\psi_{14}^{-1}\psi_{23}^{-1}\psi_{24}^{-1}$$
$$= \frac{1}{2}\sum_{\alpha=2}^{4}\theta_\alpha^8(0)\cdot\frac{i}{2\pi}\frac{\theta_1'(0)\theta_\alpha(\nu_{31})}{\theta_1(\nu_{31})\theta_\alpha(0)}\cdot\frac{i}{2\pi}\frac{\theta_1'(0)\theta_\alpha(\nu_{41})}{\theta_1(\nu_{41})\theta_\alpha(0)}$$
$$\cdot\frac{i}{2\pi}\frac{\theta_1'(0)\theta_\alpha(\nu_{32})}{\theta_1(\nu_{32})\theta_\alpha(0)}\cdot\frac{i}{2\pi}\frac{\theta_1'(0)\theta_\alpha(\nu_{42})}{\theta_1(\nu_{42})\theta_\alpha(0)}\,. \qquad (2.33)$$

For the case (iii), a convenient choice is $K_1 = -K_3 = e_1-e_2$, $K_2 = -K_4 = -e_2+e_3$ and we obtain

$$\mathcal{L}_1(\tau,\nu_1,\nu_2,\nu_3)\,\psi_{12}\psi_{34}\psi_{13}^{-2}\psi_{24}^{-2}\psi_{14}^{-1}\psi_{23}^{-1}$$
$$= \frac{1}{2}\left(\frac{i}{2\pi}\right)^4\sum_{\alpha=2}^{4}\frac{\theta_\alpha(\nu_{31})\,\theta_\alpha(\nu_{42})\theta_\alpha(\nu_{42}+\nu_{31})\,\theta_1(\nu_{21})\theta_1(\nu_{43})}{\theta_1^2(\nu_{31})\,\theta_1^2(\nu_{42})\,\theta_1(\nu_{41})\,\theta_1(\nu_{32})\,\theta_\alpha^3(0)}$$
$$\times(\theta_1'(0))^2\,\theta_\alpha^8(0) \qquad (2.34)$$

For the case (iv), we can choose $K_1 = K_2 = -K_3 = -K_4 = e_1-e_2$ and find

$$\mathcal{L}_1(\tau,\nu_1,\nu_2,\nu_3)\,\psi_{12}^2\psi_{34}^2\psi_{13}^{-2}\psi_{14}^{-2}\psi_{23}^{-2}\psi_{24}^{-2}$$
$$= \frac{1}{2}\left(\frac{i}{2\pi}\right)^4\sum_{\alpha=2}^{4}\left(\frac{\theta_\alpha(\nu_{42}+\nu_{31})\,\theta_1(\nu_{21})\,\theta_1(\nu_{43})}{\theta_1(\nu_{31})\,\theta_1(\nu_{41})\,\theta_1(\nu_{32})\theta_1(\nu_{42})\,\theta_\alpha(0)}\right)^2 \qquad (2.35)$$
$$\times(\theta_1'(0))^4\,\theta_\alpha^8(0)$$

If the lattice momenta K's come from two different E8 factor groups, the case (i) in Eq. (2.29) is the only possibility. Taking $K_1 = -K_3 \in E_8$ and $K_2 = -K_4 \in E_8{}'$, we find

$$\mathcal{L}(\tau, \nu_1, \nu_2, \nu_3)\, \psi_{13}^{-2}\, \psi_{24}^{-2}$$
$$= \frac{1}{2}\sum_{\alpha=2}^{4} \theta_\alpha^8(0) \left(\frac{i}{2\pi} \frac{\theta_1'(0)\,\theta_\alpha(\nu_{31})}{\theta_1(\nu_{31})\,\theta_\alpha(0)} \right)^2 \cdot \frac{1}{2}\sum_{\beta=2}^{4} \theta_\beta^8(0) \left(\frac{i}{2\pi} \frac{\theta_1'(0)\,\theta_\beta(\nu_{42})}{\theta_1(\nu_{42})\,\theta_\beta(0)} \right)^2$$
$$= \left\{ E_6 - E_4(\hat{E}_2 + 12T(\nu_{31})) \right\} \left\{ E_6 - E_4(\hat{E}_2 + 12T(\nu_{42})) \right\} / 144 \,. \qquad (2.36)$$

If the gauge group is $\mathrm{Spin}(32)/Z_2$, we simply need to change $\theta_\alpha^8(0) \to \theta_\alpha^{16}(0)$ in Eqs. (2.31) and (2.33)-(2.35), and to delete the second factor $\mathcal{L}_2(\tau)$ in Eq. (2.30). For instance, one finds for the case (i), instead of Eq. (2.31),

$$\left[\mathcal{L}(\tau, \nu_1, \nu_2, \nu_3)\, \psi_{13}^{-2}\, \psi_{24}^{-2} \right]^{\mathrm{Spin}(32)/Z_2}$$
$$= \left[\mathcal{L}(\tau, \nu_1, \nu_2, \nu_3)\, \psi_{13}^{-2}\, \psi_{24}^{-2} \right]^{E_8 \times E_8} - 2^3 \Delta_{12} \qquad (2.37)$$

where Δ_{12} is defined in Eq. (2.4) and the following identity [17] is used

$$\sum_{\alpha=2}^{4} \theta_\alpha^{16}(0)\, e_{\alpha-1}^2 = 2\pi^4 \left(E_4^3 - 2^7 \cdot 3^2 \Delta_{12} \right) / 9 \qquad (2.38)$$

We have also confirmed the above results by the fermionic formulation of the current algebra [1,17], except Eqs. (2.34) and (2.35) which seem to be difficult in the fermionic formulation.

3. - ONE-LOOP CORRECTIONS TO THE EFFECTIVE LAGRANGIAN

One-loop amplitudes with three or less external massless particles vanish in the heterotic string theory [1,21]. Hence the one-loop amplitude with four external massless particles has no massless particle poles and no subtraction is needed to obtain the effective Lagrangian. We simply need to expand Eqs. (2.11)-(2.14) in powers of α' and to integrate over ν and τ.

We have reported [16,17] the result of quartic curvature terms evaluated from Eq. (2.14) up to the α'^3 corrections*)

---

*) Our result appears to be different from Eq. (3.17) of Ref. [17] by a factor of 16, presumably due to the fact that their convention of Riemann tensor in their Eq. (3.15) is factor two bigger than our Riemann tensor in Eq. (3.3). Our convention gives the standard Einstein action at the leading order.

$$L_{4G} = (2\alpha')^3\, 2^{-18}\, 3^{-2}\, \pi^{-5}\, (g/\kappa)^2\, e^{\kappa D/\sqrt{2}}$$
$$\cdot \left\{ Y + 12\, t^{\mu_1 \cdots \mu_8}\, tr(\bar{R}_{\mu_1\mu_2} \bar{R}_{\mu_3\mu_4})\, tr(\bar{R}_{\mu_5\mu_6} \bar{R}_{\mu_7\mu_8}) \right\} \tag{3.1}$$

where the indices of t-tensors defined in Eq. (2.6) are understood here to run $\mu = 0,\ldots,9$ (the ε-tensor does not contribute) and Y is defined as

$$Y = t^{\mu_1 \cdots \mu_8}\, t^{\nu_1 \cdots \nu_8}\, \bar{R}_{\mu_1\mu_2\nu_1\nu_2} \cdots \bar{R}_{\mu_7\mu_8\nu_7\nu_8}\,. \tag{3.2}$$

Here we used the fact that "gravitons" appear as a generalized curvature $\bar{R}$ with a torsion [8,14,16]

$$\bar{R}_{\mu\nu}{}^{\lambda\rho} = R_{\mu\nu}{}^{\lambda\rho} + \kappa e^{-\kappa D/\sqrt{2}}\, \nabla_{[\mu} H_{\nu]}{}^{\lambda\rho} - \frac{\kappa}{\sqrt{8}}\, \eta_{[\mu}{}^{[\lambda}\, \nabla_{\nu]} \nabla^{\rho]} D\,,$$
$$R_{\mu\nu}{}^{\lambda}{}_{\rho} = \partial_\mu \Gamma^\lambda{}_{\nu\rho} + \Gamma^\lambda{}_{\mu\sigma} \Gamma^\sigma{}_{\nu\rho} - (\mu \leftrightarrow \nu)\,,$$
$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2}\, g^{\lambda\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})\,, \qquad g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu},$$
$$tr(\bar{R}_{\mu\nu} \bar{R}_{\lambda\rho}) = \bar{R}_{\mu\nu}{}^{\alpha}{}_{\beta}\, \bar{R}_{\lambda\rho}{}^{\beta}{}_{\alpha}$$
$$H_{\mu\nu\lambda} = \nabla_{[\mu} B_{\nu\lambda]} = \nabla_\mu B_{\nu\lambda} + \nabla_\nu B_{\lambda\mu} + \nabla_\lambda B_{\mu\nu} \tag{3.3}$$

where the graviton field is denoted as $h_{\mu\nu}$ and the gravitational coupling constant κ is given by the (dimensionless) string loop expansion parameter g as [1] $\kappa = g(2\alpha')^2/2$.

The non-polynomial dilaton coupling in Eq. (3.1) follows from the ten-dimensional conformal symmetry in the string theory [14,18-20]. In the non-linear sigma model, the string action contains a background "dilaton field" ϕ whose constant mode couples to the Euler character χ of the two-dimensional world sheet [19]

$$S_{NL\sigma M} = \frac{1}{4\pi\alpha'} \int d^2z \left(\sqrt{\gamma}\, \gamma^{ab}\, G_{\mu\nu}(X)\, \partial_a X^\mu \partial_b X^\nu - \frac{\alpha'}{2} \sqrt{\gamma}\, R^{(2)} \phi(X) \right)$$
$$\chi = \frac{1}{4\pi} \int d^2z\, \sqrt{\gamma}\, R^{(2)} \tag{3.4}$$

where γ^{ab} and $R^{(2)}$ are two-dimensional metric and curvature, and $G_{\mu\nu}$ is a ten-dimensional metric. Since string h-loop amplitudes have world sheets with $\chi = 2-2h$, they are proportional to $e^{(1-h)\phi}$. They are also proportional to g^{2h} where g is the dimensionless string loop expansion parameter. Thus we find that string h-loop contributions to the effective action are generically of the form

$$\Delta S_{h\text{-loop}} = \int d^{10}x \sqrt{-G}\ \tilde{O}\ e^{(1-h)\phi}\ g^{2h} \qquad (3.5)$$

where $\tilde{O}$ is an operator without the ϕ constant mode. The non-linear sigma model dilaton ϕ and the metric $G_{\mu\nu}$ is related to our dilaton D and metric $g_{\mu\nu}$ (in the S-matrix approach) through a ten-dimensional conformal rescaling [19]

$$G_{\mu\nu} = g_{\mu\nu} \exp(\kappa D/\sqrt{2}), \quad \phi = -2\sqrt{2}\,\kappa D. \qquad (3.6)$$

If the operator $\tilde{O}$ has the weight w under the conformal rescaling

$$\tilde{O} = O \exp(w\kappa D/\sqrt{2}) \qquad (3.7)$$

we obtain the general rule of the coupling of the ϕ constant mode for the weight w contribution at the h-loop order

$$\Delta S_{h\text{-loop}} = \int d^{10}x \sqrt{-g}\ O\ e^{(1+w)\kappa D/\sqrt{2}}\ (g^2 e^{2\sqrt{2}\kappa D})^h. \qquad (3.8)$$

From the one-loop amplitude (2.13) with two gauge bosons and two "gravitons", we can extract the string loop correction up to order α'^3 for the effective action

$$L^{E_8\times E_8}_{2A,2G} = \frac{-(2\alpha')^3 g^2_{YM}}{2^{15}\cdot 3\pi^5}\left(\frac{g}{\kappa}\right)^2 e^{\kappa D/\sqrt{2}}\, t^{\mu_1\cdots\mu_8}\cdot\frac{1}{30}$$
$$\cdot\left\{\mathrm{Tr}_1(F_{\mu_1\mu_2}F_{\mu_3\mu_4}) + \mathrm{Tr}_2(F_{\mu_1\mu_2}F_{\mu_3\mu_4})\right\}\cdot \mathrm{tr}(\bar{R}_{\mu_5\mu_6}\bar{R}_{\mu_7\mu_8}) \qquad (3.9)$$

where Tr_1 (Tr_2) denotes the trace of the adjoint representation of the first (the second) E_8 factor group and the gauge coupling g_{YM} is given by $g_{YM} = g(2\alpha')^{3/2}$. Our matrix T^a for the E_8 generator is antiHermitian and normalized as

$$\mathrm{Tr}(T^a T^b) = -60\,\delta^{ab}. \qquad (3.10)$$

In the case of $\mathrm{Spin}(32)/Z_2$, we usually express the gauge field in the fundamental representation whose trace is denoted as tr

$$\mathrm{tr}(T^a T^b) = -2\,\delta^{ab}. \qquad (3.11)$$

Our result for the $\mathrm{Spin}(32)/Z_2$ agrees*) with that of Ref. [17]

---

*) Apparent discrepancy of a factor of four is presumably due to the difference of the normalization of the Riemann tensor.

$$L_{2A,2G}^{Spin(32)/Z_2} = \frac{-(2\alpha')^3}{2^{15}\cdot 3\pi^5} g_{YM}^2 \left(\frac{g}{\kappa}\right)^2 e^{\kappa D/\sqrt{2}}\, t^{\mu_1\cdots\mu_8}$$
$$\cdot\, tr(F_{\mu_1\mu_2}F_{\mu_3\mu_4})\, tr(\bar{R}_{\mu_5\mu_6}\bar{R}_{\mu_7\mu_8})\,. \tag{3.12}$$

The method of ν and τ integrations to obtain the effective Lagrangian $L_{2A,2G}$ is entirely analogous to that of the four-graviton case. Moreover, the more extensive and useful exposition of the relevant techniques can be found in Refs. [25] and [17]. In Appendix B we describe our regularization method briefly and summarize some of the necessary formulas.

From the one-loop amplitude (2.12) with three gauge bosons and a "graviton", we find that the possible leading order term is of order α'^3k^5 and proportional to

$$\int \prod_{r=1}^{3} \frac{d^2\nu_r}{\mathrm{Im}\,\tau}\, Q_4^{j_4} \cdot \frac{\theta_\alpha(\nu_{12})\,\theta_\alpha(\nu_{13})\,\theta_\alpha(\nu_{23})}{\theta_1(\nu_{12})\,\theta_1(\nu_{13})\,\theta_1(\nu_{23})} = 0 \tag{3.13}$$

upon integration over ν_r. Therefore we find that the term cubic in $F_{\mu\nu}$ and linear in $R_{\mu\nu\lambda\rho}$ does not exist up to the string one-loop order.

We now turn to the case of four gauge bosons. In evaluating Eq. (2.11) to the leading order α'^3k^4, we note that a matrix X in the adjoint representation of E_8 satisfies [26,2]

$$\mathrm{Tr}\, X^4 = (\mathrm{Tr}\, X^2)^2/100\,. \tag{3.14}$$

Hence the effective Lagrangian quartic in the gauge field strength at the one-loop order for $E_8\times E_8$ can have only two independent terms whose coefficients are denoted as a and b

$$L_{4A}^{E_8\times E_8} = a\left\{(\mathrm{Tr}_1 F^2)^2 + (\mathrm{Tr}_2 F^2)^2\right\} + b\,\mathrm{Tr}_1 F^2\, \mathrm{Tr}_2 F^2,$$
$$(\mathrm{Tr}_1 F^2)^2 \equiv t^{\mu_1\cdots\mu_8}\, \mathrm{Tr}_1(F_{\mu_1\mu_2}F_{\mu_3\mu_4})\, \mathrm{Tr}_1(F_{\mu_5\mu_6}F_{\mu_7\mu_8}),$$
$$\mathrm{Tr}_1 F^2\, \mathrm{Tr}_2 F^2 \equiv t^{\mu_1\cdots\mu_8}\, \mathrm{Tr}_1(F_{\mu_1\mu_2}F_{\mu_3\mu_4})\, \mathrm{Tr}_2(F_{\mu_5\mu_6}F_{\mu_7\mu_8})\,. \tag{3.15}$$

The two coefficients a and b can be determined by evaluating the following two configurations of lattice momenta. For the case (i) in Eq. (2.29) of the previous section ($K_1K_2 = 0$, $K_3 = -K_1$, $K_4 = -K_2$), we have two possibilities: (a) all K_r belong to the same E_8 as in Eq. (2.31), and (b) K_1 and K_2 belong to different E_8's as in Eq. (2.36). The cases (a) and (b) determine the coefficients a and b in Eq. (3.15), respectively. Using the formulas in Appendix B, we find

$$L_{4A}^{E_8\times E_8} = \frac{(2\alpha')^3 g_{YM}^2}{2^{14}\cdot 3\pi^5}\left(\frac{g}{\kappa}\right)^2 e^{\kappa D/\sqrt{2}}\, t^{\mu_1\cdots\mu_8}\left(\frac{1}{30}\right)^2$$
$$\cdot\left\{\sum_{i=1}^{2} \mathrm{Tr}_i(F_{\mu_1\mu_2}F_{\mu_3\mu_4})\,\mathrm{Tr}_i(F_{\mu_5\mu_6}F_{\mu_7\mu_8})\right.$$
$$\left. - \mathrm{Tr}_1(F_{\mu_1\mu_2}F_{\mu_3\mu_4})\,\mathrm{Tr}_2(F_{\mu_5\mu_6}F_{\mu_7\mu_8})\right\} . \qquad (3.16)$$

This form of L_{4A} is different from that for Spin(32)/Z_2 and is consistent with the conjecture made in Ref. [17] based on the consideration of the Wess-Zumino term for the anomaly cancellation [27,25].

As a consistency check, we can also evaluate the amplitude for the case (ii) of the lattice momentum configuration: $K_1K_2 = 0$, $K_2K_3 = K_1K_3 = -1$. At least three dimensions are required to embed this lattice momentum configuration as shown in the Figure, whereas the identity (3.14) for E_8 implies that possible configurations in the $E_8\times E_8$ gauge theory are only those that can be embedded in two dimensions (as a subspace of the eight-dimensional weight lattice space)

$$(\mathrm{Tr}_1 F^2)^2 = 60^2\left(\sum_I (F^I)^2 + \sum_{K\in\Lambda} F^K F^{-K}\right)^2 \qquad (3.17)$$

where the first sum is over elements I of the Cartan subalgebra and the second is over the weight lattice Λ of E_8. Hence the configuration such as (ii) should not occur in the $E_8\times E_8$ gauge group and we find in fact vanishing contributions to order α'^3k^4 from the configuration (ii) as follows. Since the integrand has at most simple poles*) in ν, the result of the ν integration must be a modular form of weight twelve because of the modular invariance. Since there are only two independent modular forms of weight twelve [24], the result can unambiguously be determined by finding the first two coefficients of the expansion in powers of $q = e^{2\pi i\tau}$. In this way we find from Eqs. (2.30) and (2.33)

$$\int \prod_{r=1}^{3} \frac{d^2\nu_r}{\mathrm{Im}\,\tau}\, \mathcal{L}(\tau,\nu_1,\nu_2,\nu_3)\,\psi_{13}^{-1}\psi_{14}^{-1}\psi_{23}^{-1}\psi_{24}^{-1} = 0 \qquad (3.18)$$

Another way to see this result is to use identities given in Ref. [17] [Eqs. (E.18) and (E.28)]

---

*) If double poles are present, the result need not be purely holomorphic and the subsequent argument needs modifications.

$$\int \prod_{r=1}^{3} \frac{d^2\nu_r}{\mathrm{Im}\,\tau} \mathcal{L}(\tau, \nu_1, \nu_2, \nu_3)\, \psi_{13}^{-1} \psi_{14}^{-1} \psi_{23}^{-1} \psi_{24}^{-1}$$

$$= E_4 \cdot \left(\theta_2^8(0)\theta_3^4(0)\theta_4^4(0) - \theta_3^8(0)\theta_2^4(0)\theta_4^4(0) - \theta_4^8(0)\theta_2^4(0)\theta_3^4(0) \right) / 96$$

$$= 0 \tag{3.19}$$

because of the Riemann identity $\theta_2{}^4(0)-\theta_3{}^4(0) + \theta_4{}^4(0) = 0$.

Other lattice momentum configurations (iii) and (iv) are more difficult to deal with. Our result (3.16) implies that these cases should give non-vanishing results with definite coefficients. In principle one may be able to verify this by using the theorem that any doubly periodic function with poles can be expanded in terms of the Weierstrass ζ function and its derivatives. We have not completed the analysis of cases (iii) and (iv).

For the gauge group $Spin(32)/Z_2$, the effective Lagrangian quartic in gauge field strength has been worked out in Ref. [17]. Their result can be reproduced by using our Eq. (2.37) for the case (i) and another equation for the case (ii) [obtained from Eq. (2.33) by changing $\theta_\alpha^8(0) \to \theta_\alpha^{16}(0)$], since two possible terms $(\mathrm{tr}F^2)^2$ and $\mathrm{tr}F^4$ correspond to the cases (i) and (ii) respectively. It is interesting to note that the term of the form $(\mathrm{tr}F^2)^2$ turns out to be absent for the $Spin(32)/Z_2$ gauge group due to the additional contribution $-2^3\Delta_{12}$ in Eq. (2.37).

It is very interesting to examine the effect of the string loop correction to compactified solutions of the heterotic string. The kinematical structure of the loop correction is different from that at tree level, and it depends on the gauge group. However, we need to analyze more carefully in order to find, for instance, if the loop-corrected solution can be reduced to a non-local field redefinition of the Calabi-Yau solution [10] or not. We hope to examine this problem further.

ACKNOWLEDGEMENTS

One of us (N.S.) would like to thank Mike Freeman, Leah Mizrachi and Gabriele Veneziano for reading the manuscript and useful discussions. He also thanks W. Lerche, A. Schellekens and N. Warner for informing him of Ref. [25] which contains ν and τ integration methods developed by them. This work is partially supported by Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture of Japan (No. 61540200).

APPENDIX A - GAUGE GROUP FACTOR

Here we sketch our method to evaluate the lattice momentum summation for the gauge group factor $\mathcal{L}$ in Eq. (2.10), by taking the two-gauge boson case as an example.

First we need to specify the external lattice momenta in terms of the orthonormal vectors e_i $(i=1,\ldots,8)$. In the case of two gauge bosons, we can choose $K_1 = -K_2 = e_1 - e_2$. Taking into account our parametrization of the lattice momenta in Eq. (2.16), we find Eq. (2.20)

$$\mathcal{L}_1(\tau, \nu_{21}) = \sum_{n_k \in \mathbb{Z}} \exp\left[i\pi\tau \left\{ \sum_{k=1}^{8} m_k^2 - 2(m_1 - m_2)\frac{\nu_{21}}{\tau} + 2\nu_{21}^2/\tau^2 \right\}\right] \quad (A.1)$$

Since the lattice momenta are parametrized by eight integers or half-integers m_i $(i=1,\ldots,8)$, we need to distinguish sixteen cases depending on m_1 (m_2) being odd, even, odd-$\frac{1}{2}$, or even-$\frac{1}{2}$.

If m_1 and m_2 are both integers, Eq. (2.16) implies that n_8 must be even and all m_i should be also integers. If m_1 and m_2 are both odd, there must be an even number of odd m_k $(k=3,\ldots,8)$ in order for n_8 to be an even integer: $n_8 = \Sigma_{i=1}^{8} m_i$. Counting the number of possibilities, we obtain a contribution to $\mathcal{L}_1$, in Eq. (A.1) from the case of both m_1, m_2 being odd

$$\begin{aligned}
&[\mathcal{L}_1(\tau, \nu_{21})]_{odd,\, odd} \\
&= \Big[\sum_{m_1 = odd} e^{i\pi\tau(m_1^2 - 2m_1\frac{\nu_{21}}{\tau} + \frac{\nu_{21}^2}{\tau^2})} \Big] \Big[\sum_{m_2 = odd} e^{i\pi\tau(m_2^2 + 2m_2\frac{\nu_{21}}{\tau} + \frac{\nu_{21}^2}{\tau^2})} \Big] \\
&\times \Big(\big[\sum_{even} e^{i\pi m^2} \big]^6 + 15 \big[\sum_{even} e^{i\pi m^2} \big]^4 \big[\sum_{odd} e^{i\pi m^2} \big]^2 \\
&\qquad + 15 \big[\sum_{even} e^{i\pi m^2} \big]^2 \big[\sum_{odd} e^{i\pi m^2} \big]^4 + \big[\sum_{odd} e^{i\pi m^2} \big]^6 \Big) .
\end{aligned} \quad (A.2)$$

We can now recognize the elliptic theta functions [23]

$$\sum_{m = odd} e^{i\pi(\tau m^2 \pm 2m\nu)} = (\theta_3(\nu) - \theta_4(\nu))/2 , \quad (A.3)$$

$$\sum_{m = even} e^{i\pi(\tau m^2 \pm 2m\nu)} = (\theta_3(\nu) + \theta_4(\nu))/2 . \quad (A.4)$$

Hence we obtain an expression in terms of the theta functions

$$\begin{aligned}
[\mathcal{L}_1(\tau, \nu_{21})]_{odd,\, odd} &= \Big[\frac{1}{2}(\theta_3(\nu_{21}) - \theta_4(\nu_{21})) e^{\pi i \nu_{21}^2/\tau} \Big]^2 \\
&\times \frac{1}{2} \left\{ \theta_3^6(0) + \theta_4^6(0) \right\} .
\end{aligned} \quad (A.5)$$

Evaluation of other fifteen cases is completely analogous. By summing over all sixteen cases, we find that cross terms such as $\theta_3(\nu_{21})\times\theta_4(\nu_{21})$ cancel out and obtain Eq. (2.21), which exhibits the pattern of the spin structure summation encountered in the fermionic formulation.

The gauge group factor of other cases such as three or four gauge boson amplitudes can be obtained by an analogous method.

APPENDIX B - FORMULAS FOR ν AND τ INTEGRATIONS

Here we summarize the most relevant formulas for ν and τ integrations. Many useful formulas are contained in Ref. [17], and a more extensive discussion may be found in Ref. [25]. Some of the formulas are also found in Refs. [15] and [16].

Many terms drop out when integrating over ν because of the following identities [15,16]

$$\int d^2\nu\, K(\nu-\nu') = 0, \qquad \int d^2\nu\, T(\nu-\nu') = 0 \tag{B.1}$$

We need a careful evaluation of the integral around the pole at $\nu = \nu'$, especially in the case of double pole in T. We regularize the integral by cutting off an infinitesimal square of sides 2ε around the pole and by letting $\varepsilon \to 0$ at the end. Since both K and T can be written as derivatives in ν, we obtain only contributions from the boundary of the integration region (including sides of the infinitesimal square around the pole) and find that they cancel among themselves.

We can reduce the ν integration eventually to either one of the following types, if it does not vanish due to Eq. (B.1)

$$\int \prod_{r=1}^{3} \frac{d^2\nu_r}{\mathrm{Im}\,\tau} K(\nu_{21}) K(\nu_{24}) K(\nu_{31}) K(\nu_{34}) = \frac{E_4(\tau)}{720} \tag{B.2}$$

$$\int \prod_{r=1}^{3} \frac{d^2\nu_r}{\mathrm{Im}\,\tau} K(\nu_{21}) K(\nu_{21}) K(\nu_{43}) K(\nu_{43}) = \left(\frac{\hat{E}_2(\tau)}{12}\right)^2 \tag{B.3}$$

Equation (B.2) can be obtained by twice using the following formula valid for $\nu_1 \neq \nu_2$

$$\int \frac{d^2\nu}{\mathrm{Im}\,\tau} K(\nu-\nu_1) K(\nu-\nu_2) = \frac{1}{12} - 2\sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} \cos 2\pi n(\nu_1-\nu_2)$$
$$+ \frac{\mathrm{Im}(\nu_1-\nu_2)}{\mathrm{Im}\,\tau} K(\nu_1-\nu_2) + \frac{1}{2}\left(\frac{\mathrm{Im}(\nu_1-\nu_2)}{\mathrm{Im}\,\tau}\right)^2 . \tag{B.4}$$

To prove Eq. (B.4), we use the formula [16]

$$K(\nu) = \frac{1}{2i}\left(\cot \pi\nu + 4\sum_{n=1}^{\infty} \frac{q^n}{1-q^n} \sin 2\pi n\nu\right) + \frac{\mathrm{Im}\,\nu}{\mathrm{Im}\,\tau} \tag{B.5}$$

and expand the integrand in Fourier series such as

$$\frac{1}{2i}\cot\pi(\nu-\nu_1)=\begin{cases}\frac{1}{2}+\sum_{n=1}^{\infty}e^{-2\pi i n(\nu-\nu_1)}\,, & \mathrm{Im}\,\nu<\mathrm{Im}\,\nu_1\\ -\frac{1}{2}-\sum_{n=1}^{\infty}e^{2\pi i n(\nu-\nu_1)}\,, & \mathrm{Im}\,\nu>\mathrm{Im}\,\nu_1\end{cases}\tag{B.6}$$

and integrate first in Reν in order to exploit the periodicity under $\nu \to \nu+1$. We regularize the ν integral around poles at ν_1 and ν_2 by cutting off infinitesimal squares of sides 2ε around the poles and by letting $\varepsilon \to 0$ at the end. We find the simple poles at $\nu = \nu_1$ and $\nu = \nu_2$ harmless, and obtain Eq. (B.4) for $\nu_1 \neq \nu_2$. If $\nu_1 = \nu_2$, however, the double pole at $\nu = \nu_1$ (ν_2) needs to be treated with care. In particular, we obtain non-vanishing contributions from an infinitesimal strip I:

$$\mathrm{Im}\,\nu_1-\varepsilon<\mathrm{Im}\,\nu<\mathrm{Im}\,\nu_1+\varepsilon\,,$$

$$-\frac{1}{2}<\mathrm{Re}\,\nu<\mathrm{Re}\,\nu_1-\varepsilon\,,\quad \mathrm{Re}\,\nu_1+\varepsilon<\mathrm{Re}\,\nu<\frac{1}{2}\,.\tag{B.7}$$

We find a non-holomorphic piece from the strip I

$$\int_I\frac{d^2\nu}{\mathrm{Im}\tau}\left(\frac{\cot\pi(\nu-\nu_1)}{2i}\right)^2=\frac{-2}{\pi^2\mathrm{Im}\tau}\sum_{m=1}^{\infty}\frac{e^{-m\pi\varepsilon}}{m}\sin 2\pi m\varepsilon\,\sinh m\pi\varepsilon$$

$$\xrightarrow[\varepsilon\to 0]{}\;-\frac{1}{4\pi\,\mathrm{Im}\tau}\,.\tag{B.8}$$

Thus we obtain the additional contribution besides those coming from the $\nu_2 \to \nu_1$ limit of Eq. (B.4)

$$\int\frac{d^2\nu}{\mathrm{Im}\tau}\left(K(\nu-\nu_1)\right)^2=\frac{1}{12}-2\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}-\frac{1}{4\pi\,\mathrm{Im}\tau}\,.\tag{B.9}$$

The result of the ν integration transforms with the weight twelve under the modular transformation. Hence we only need to know the following τ integrals

$$I(p,q,r,s)=\int\frac{d^2\tau}{(\mathrm{Im}\tau)^2}\hat{E}_2^{\,p}E_4^{\,q}E_6^{\,r}\Delta_{12}^{\,s-1}\tag{B.10}$$

with $2p + 4q + 6r + 12s = 12$. They are given for $p \neq 0$

$$I(2,2,0,0)=48\pi\,,\quad I(1,1,1,0)=-48\pi\,.\tag{B.11}$$

For $p = 0$, two of the following three cases are independent

$$I(0,0,0,1)=\pi/3\,,\quad I(0,0,2,0)=-336\pi\,,$$

$$I(0,3,0,0)=240\pi\,.\tag{B.12}$$

One should note that our convention is $d^2\tau$ = dReτdImτ and is a factor of two smaller compared to Refs. [17] and [25].

REFERENCES

[1] D. Gross, J. Harvey, E. Martinec and R. Rohm - Nucl.Phys. B256 (1985) 253; B267 (1986) 75.

[2] M. Green and J. Schwarz - Phys.Lett. B149 (1984) 117.

[3] P. Candelas, G. Horowicz, A. Strominger and E. Witten - Nucl.Phys. B258 (1985) 46.

[4] M. Grisaru, A. Van de Ven and D. Zanon - Phys.Lett. B173 (1986) 423; Nucl.Phys. B277 (1986) 388;
M. Grisaru and D. Zanon - Phys.Lett. B177 (1986) 347.

[5] M. Freeman and C. Pope - Phys.Lett. B174 (1986) 48;
P. Howe, G. Papadopoulos and K. Stelle - Phys.Lett. B174 (1986) 405.

[6] D. Gross and E. Witten - Nucl.Phys. B277 (1986) 1.

[7] D. Chang and H. Nishino - Phys.Lett. B179 (1986) 257;
H. Nishino and S. Gates - Phys.Lett. B189 (1987) 45;
Y. Cai and C. Núñez - Nucl.Phys. B287 (1987) 279.

[8] Y. Kikuchi, C. Marzban and Y. Ng - Phys.Lett. B176 (1986) 57;
Y. Kikuchi and C. Marzban - Phys.Rev. D35 (1987) 1400;
M. Daniel, D. Hochberg and N. Mavromatos - Phys.Lett. B187 (1987) 79;
Z. Bern and T. Shimada - Phys.Lett. B197 (1987) 119.

[9] P. Candelas, M. Freeman, C. Pope, M. Sohnius and K. Stelle - Phys.Lett. B177 (1986) 341.

[10] D. Nemeschansky and A. Sen - Phys.Lett. B178 (1986) 365;
A. Sen - ibid 370.

[11] T. Yoneya - Nuovo Cimento Letters 8 (1973) 951.

[12] J. Scherk and J. Schwarz - Nucl.Phys. B81 (1974) 118.

[13] R. Nepomechie - Phys.Rev. D32 (1985) 3201.

[14] D. Gross and J. Sloan - Nucl.Phys. B291 (1986) 41.

[15] N. Sakai and Y. Tanii - Nucl.Phys. B287 (1987) 457.

[16] M. Abe, H. Kubota and N. Sakai - Phys.Lett. B200 (1988) 461.

[17] J. Ellis, P. Jetzer and L. Mizrachi - "One-loop string corrections to the effective field theory" CERN Preprint TH. 4829 (1987); Phys.Lett. B196 (1987) 492.

[18] D. Gross - in Unified String Theories, eds. M. Green and D. Gross (World Scientific, 1986), p. 357.

[19] C. Callan, D. Friedan, E. Martinec and M. Perry - Nucl.Phys. B262 (1985) 593.

[20] E. Fradkin and A. Tseytlin - Phys.Lett. B160 (1985) 69.

[21] M. Green, J. Schwarz and E. Witten - Superstring Theory I and II (Cambridge Univ. Press, 1987).

[22] P. Jetzer, J. Lacki and L. Mizrachi - "One-loop amplitudes in the heterotic string" UGVA-DPT 1986-09-517, to appear in Int.J.Mod.Phys.A.

[23] A. Erdelyi et al. - Higher Transcendental Functions, Vol. 2 (McGraw Hill, NY, 1953).

[24] B. Schoeneberg - Elliptic Modular Functions (Springer Verlag, NY, 1974); J.P. Serre - A Course in Arithmetic (Springer Verlag, NY, 1973).

[25] W. Lerche, B. Nilsson, A. Schellekens and N. Warner - "Anomaly cancelling terms from the elliptic genus", CERN Preprint TH. 4765 (1987).

[26] S. Okubo - J.Math.Phys. 20 (1979) 586.

[27] J. Ellis and L. Mizrachi - "On a possible Wess-Zumino term in the heterotic string", CERN Preprint TH. 4616 (1986).

FIGURE CAPTION

The lattice momentum configuration for the case (ii): $K_1 K_2 = 0$, $K_1 K_3 = K_2 K_3 = -1$. The plane containing K_1 and K_2 intersects with the plane containing K_3 and K_4 at a right angle. The intersection is denoted by a dashed line.