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QUANTUM CALCULATION OF BEAMSTRAHLUNG: THE SPINLESS CASE

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ABSTRACT

Bremsstrahlung energy losses by electrons and positrons in very high energy linear colliders are calculated in terms of Feynman graphs. A follow-up to a previous paper "Quantum Approach to Bremsstrahlung", which focuses upon radiation before bunch-crossing, this paper presents a calculation of the fractional energy loss of an electron as it radiates inside the positron bunch which it traverses. The calculation applies to the special conditions set by machines in the TeV range.

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1. - INTRODUCTION

Very high energy electron-positron linear colliders working in the TeV energy range, such as those presently envisaged[1)], have to use bunches of very small transverse dimensions and very high densities. This is necessary in order to achieve luminosities which have to compensate for the s^{-1} behaviour of potentially interesting annihilation cross-sections, where s is the centre-of-mass energy squared. One thus has to face a rather intense bremsstrahlung as an electron (positron) crosses a positron (electron) bunch. Since the radiated photons can take a large fraction X of the electron (positron) incident energy, a quantum treatment is appropriate.

In a previous paper[2)], we discussed the general features of this radiation process, referred to as beamstrahlung[3)]. It was argued that the relative values of the three usual characteristic lengths which come into play, namely the coherent radiation length L_c, the bunch length L_b and the virtual electron length L_e, change in a radical way when energy and design luminosity increase and, in particular, when one moves from the present SLC regime at 100 GeV to a potentially interesting future machine in the few TeV range[1)]. We have shown that the large value of L_e/L_c justifies a separate calculation of radiation before bunch-crossing, which was presented[2)]. This is special to such extreme energy colliders. In this paper, we present the calculation of radiation during bunch-crossing. We indeed work under conditions which are such that $L_c/L_b << 1$[2)], so that radiation can be considered as originating independently from different sections of the bunch in a way which we shall analyze. Here we are considering a highly simplified problem, namely that of an electron radiating as it is bent by the strong field met in the bunch. The full treatment of the problem involves a many-body approach which would have to be treated numerically. In order to proceed analytically, we also consider an idealized bunch of cylindrical shape and uniform density and neglect its granularity. Our purpose is to point out features which are special to machines in the TeV range.

Important parameters are Ne^2, the number of particles in a bunch times the square of the electric charge, which is typically of order 10^6, and the bunch radius, which is typically of order 10^{-9}m. We are therefore dealing with typical bending transverse momenta $2Ne^2/R$ of the order of 500 MeV, which are very large compared to the electron mass, while at such extreme energy the disruption parameter remains small. We thus deal with new kinematical conditions. While our approach shares some common features with previous ones[3)], and retains many features of the classical treatment of synchroton radiation[4)], it leads to the introduction of a coherent radiation length, independent of the electron mass, ℓ_c. It is defined as:

$$\ell_c = \left(L_c^2 L_e\right)^{\frac{1}{3}} \tag{1}$$

where L_c is the classical coherent radiation length, corresponding to the electron being bent by an angle γ^{-1}, and L_e is the virtual electron length.

In our previous paper we stressed that with the machine parameters proper to the TeV regime, $L_e >> L_c$. In this case, our coherent radiation length ℓ_c is therefore larger than the classical one L_c, and with the machine parameters used, it is so by a factor of order 20[2].

Once ℓ_c is introduced as the relevant coherent radiation length for such a new machine regime, the fractional energy loss is simply given by

$$\delta = K \frac{\alpha}{\pi} \frac{L_e}{\ell_c} \tag{2}$$

where K is a purely numerical factor, sensitive to the approximations made and, in particular, to our neglecting here complications due to spin. It is of order 1.

The calculation procedure is the one presented in our previous paper[2]. It is a distorted wave Born approximation approach, where we essentially calculate the rate associated with the Feynman graph of Fig. 1. The radiation amplitude is written as:

$$\mathcal{M} = \varepsilon^{\mu} M_{\mu} \tag{3}$$

$$M_{\mu}\left(\vec{k}_i, \vec{k}_f, \vec{k}_\gamma\right) = -ie \int d^3\vec{x}\, e^{-i\vec{k}_\gamma \cdot \vec{x}}\, \bar{\psi}_f(x)\, \gamma_\mu\, \psi_i(x) \tag{4}$$

ε is the polarization vector of the photon with momentum $\vec{k}_\gamma$. The initial and final momenta of the electron are denoted by $\vec{k}_i$ and $\vec{k}_f$ respectively. The wave functions ψ_i and ψ_f describe the electron in the presence of the bunch in the initial and final states respectively. These wave functions can be obtained by studying the Dirac (Klein-Gordon) equation at high energy in the bending field provided by the bunch[5),6)]. This was the starting point of our previous paper[2].

We have simplified the problem, considering merely the radiation of one highly energetic particle in an external field. The bunch is further assumed to be cylindrical with uniform density. It has a length L_b and a radius R, and with N particles in the bunch (in practice $N \sim 10^8$) the charge density is

$$\rho = \frac{Ne}{\pi L_\ell R^2} \tag{5}$$

Figure 2 presents the idealized bunch, with constant density (5), together with the classical path of an incoming electron with a particular impact parameter $r < R$. In practice, $L_b >> R$[1,2]. The z axis is taken along the bunch axis with the origin at the centre of the bunch, the inside region being defined by $|z| < L_b/2$. We define the xz plane by the directions of the incident and final electron momenta, $\vec{k}_i$ and $\vec{k}_f$. The photon momentum $\vec{k}_\gamma$ will in general have a component $k_{\gamma y}$ normal to that plane, but it is relatively small in practice. The classical electron trajectory is bent in a plane. A photon, when radiated with fractional energy X, emerges to a good approximation tangentially to the electron path while the electron, with energy $k_i(1-X)$, continues on its bending trajectory. While k_{fx} and $k_{\gamma x}$ result primarily from the bending experienced over the bunch length, $k_{\gamma y}$ implies a recoil of the electron which, as seen later, is compensated by bending over a coherent radiation length only, $\ell_c << L_b$.

Such bunch properties are not very realistic. A Gaussian distribution should offer a better approximation. However, our purpose is to treat a simplified, yet meaningful, problem analytically so that one can more clearly assess the importance of the different parameters involved. Complications due to spin will also be discussed separately[7]. The last term in the integral in Eq. (4) is the particle current at point x in the presence of the bunch, for which we shall here consider instead the Klein-Gordon version, namely

$$J_\mu(x) = \psi_f^*(\partial_\mu \psi_i) + (\partial_\mu \psi_f^*)\psi_i \tag{6}$$

While we therefore do not stress at this stage the actual value found for K, we emphasize the very simple form found for δ [Eq. (2)] in terms of two characteristic machine parameters. It applies to this new regime of energy and luminosity.

This paper is organized as follows. In Section 2, we discuss the coherence conditions using the high-energy approximation of the electron wave function in the presence of the bunch. We thus obtain the key properties of the space integral in Eq. (4). This naturally leads to the introduction of Airy functions in close analogy with the classical treatment of synchroton radiation[4]. However, and as previously mentioned, we come upon the coherent radiation length ℓ_c [Eq. (1)]. In Section 3, we introduce the radiation matrix element and perform the space integrals in order to obtain the two radiation amplitudes corresponding to polarization in the bending plane and normal to it. We use here the spinless current (6). All calculations are carried out analytically and explicitly so that a numerical approach, using more realistic bunch properties, could be implemented[8]. In Section 4, we compute the radiation rate and the fractional energy loss. This is also done analytically. Our main results have been summarized in Eq. (2).

The present approach should be complemented by a separate treatment of the Dirac case[7] and a separate discussion of radiation at the edges of the bunch, for which some of the approximations made in the following do not apply. Since $\ell_c \ll L_b$ in the case of interest, it is legitimate to restrict oneself to radiation "deep" inside the bunch, a length ℓ_c away from the boundary, where some bending has already occurred. This is what we do in this paper.

2. - COHERENCE CONDITIONS

In the spinless case, the asymptotic plane wave function of the electron has only to be modified by the introduction of phases in order to obtain ϕ_i and ϕ_f. In the high-energy regime and for small disruption parameters, the full phases are given by[5),6]

$$\phi_i = k_i z - \left\{ S_i(x,y,z) + \frac{1}{2k_i} \int_{-\infty}^{z} dz' \left(\left(\frac{\partial S_i(x,y,z')}{\partial x} \right)^2 + \left(\frac{\partial S_i(x,y,z')}{\partial y} \right)^2 \right) \right\}$$

$$\phi_f = \vec{k}_f . \vec{x} + \left\{ S_f(x,y,z) - \frac{1}{2k_f} \int_{z}^{\infty} dz' \left(-2k_{fx}(z-z') \frac{\partial eV}{\partial x} - \left(\frac{\partial S_f(x,y,z')}{\partial x} \right)^2 - \left(\frac{\partial S_f(x,y,z')}{\partial y} \right)^2 \right) \right\} \tag{7}$$

for ϕ_i and ϕ_f respectively.

We have introduced

$$S_i(x,y,z) = \int_{-\infty}^{z} dz' \, eV(x,y,z')$$

$$S_f(x,y,z) = \int_{z}^{\infty} dz' \, eV(x,y,z') \tag{8}$$

where V is the potential due to the bunch.

In our previous paper[2] focusing on radiation before bunch-crossing, we could neglect the $1/k$ terms in (7); they have to be kept when strong bending occurs. We work throughout in the rest system of the bunch. The potential created by the positrons in the bunch is calculated taking into account the fact that $R \ll L_b$. At a point of cylindrical co-ordinates z, r inside the bunch, it is found to be

$$V(r,z) = \frac{Ne}{L_b}\left(\ln\left(\frac{1}{R^2}\left(L_b^2 - 4z^2\right)\right) - \frac{r^2}{R^2} \right) \tag{9}$$

The first term is practically constant when compared to the second and leads to an overall phase of no consequence. We can thus work with the approximation

$$eV = \begin{cases} 0 & z < -\frac{L_b}{2} \\ \frac{N\alpha}{L_b}\frac{r^2}{R^2} & |z| < \frac{L_b}{2} \\ 0 & z > \frac{L_b}{2} \end{cases} \tag{10}$$

We write $N\alpha = n$ with a value which is in practice typically of the order of 10^6 [1,2].

From Eq. (10), we obtain ϕ_i and ϕ_f as defined by Eqs. (7) and (8). Inside the bunch, namely for $|z| < L_b/2$, they are:

$$\begin{aligned} \phi_i &= k_i z - \frac{n}{L_b}\frac{r^2}{R^2}\left(z + \frac{L_b}{2}\right)\left(1 + \frac{2}{3k_i}\frac{n}{L_b}\frac{1}{R^2}\left(z + \frac{L_b}{2}\right)^2\right) \\ \phi_f &= \vec{k}_f \cdot \vec{x} + \frac{n}{L_b}\frac{r^2}{R^2}\left(\frac{L_b}{2} - z\right) \\ &\quad + \frac{1}{2k_f}\left(\frac{2k_{fx} n}{L_b}\frac{x}{R^2}\left(\frac{L_b}{2} - z\right)^2 + \frac{4}{3}\frac{n^2}{L_b^2}\frac{r^2}{R^4}\left(\frac{L_b}{2} - z\right)^3\right) \end{aligned} \tag{11}$$

We work with the condition $L_b \gg \ell_c$ and can therefore focus separately on radiation produced inside the bunch. In practice, the radiation originates from a limited zone, defined according to the final momenta observed $\vec{k}_f$ and $\vec{k}_\gamma$ and the coherence conditions. We can, as is done in the following, consider the radiation "deep" inside the bunch, avoiding the edge zones and yet extend integrals to infinity in the calculation.

The phase of the radiation amplitude (3) is, up to a constant term,

$$\phi = \phi_i - \phi_f - \vec{k}_\gamma \cdot \vec{x} \tag{12}$$

Combining (11) and (12) we obtain, for $|z| < L_b/2$,

$$\phi = -n\frac{r^2}{R^2} - \frac{1}{2k_f}\left(\frac{2k_{fx}n}{L_b}\frac{x}{R^2}\left(\frac{L_b}{2}-z\right)^2 + \frac{4}{3}\frac{n^2}{L_b^2}\frac{r^2}{R^4}\left(\frac{L_b}{2}-z\right)^3\right)$$
$$- \frac{2}{3k_i}\frac{n^2}{L_b^2}\frac{r^2}{R^4}\left(z+\frac{L_b}{2}\right)^3 + k_i z - \vec{k}_f\cdot\vec{x} - \vec{k}_\gamma\cdot\vec{x} \tag{13}$$

The radiation process will be damped out by the rapid variation of the phase but for a radiation zone where the phase is stationary.

We first impose that $\partial\phi/\partial x = 0$ and that $\partial\phi/\partial y = 0$. This defines transverse radiation co-ordinates which are, to a good approximation, the transverse location of the radiation with a final electron and photon of momenta $\vec{k}_f$ and $\vec{k}_\gamma$ respectively. They are

$$x_0 = -\frac{R^2}{2n}\left(k_{fx}+k_{\gamma x}\right) - \frac{1}{2k_f}\left(\frac{k_{fx}}{L_b}\left(\frac{L_b}{2}-z\right)^2 - \frac{2}{3}\frac{k_{fx}+k_{\gamma x}}{L_b^2}\left(\frac{L_b}{2}-z\right)^3\right) + \frac{1}{3k_i}\frac{1}{L_b^2}\left(k_{fx}+k_{\gamma x}\right)\left(z+\frac{L_b}{2}\right)^3$$

$$y_0 = -\frac{R^2}{2n}k_{\gamma y} + \frac{1}{3}\frac{k_{\gamma y}}{L_b^2}\left(\frac{1}{k_f}\left(\frac{L_b}{2}-z\right)^3 + \frac{1}{k_i}\left(\frac{L_b}{2}+z\right)^3\right) \tag{14}$$

Since the initial and final electron momenta define the x,z plane, the radiation zone is slightly off that plane to the extent that $k_{\gamma y}$, while small, is in general different from zero.

The global transverse momentum is

$$\vec{\Delta}_T = -\vec{k}_{f_T} - \vec{k}_{\gamma_T} \tag{15}$$

The global longitudinal momentum transfer

$$\Delta_L = k_i - k_{fz} - k_{\gamma z} \tag{16}$$

can be written neglecting the electron mass in front of relatively large bending momenta, something specific to the regime considered:

$$\Delta_L = \frac{1}{2k_i}\left(\frac{k_{fx}^2}{1-X} + \frac{k_{\gamma_T}^2}{X}\right) \tag{17}$$

This is valid away from the front of the bunch, a zone which we neglect here insofar as $\ell_c \ll L_b$. Approximations made later lead us at this stage to keep clear of the edges of the bunch.

We now substitute (14) into (13) and obtain the phase at the stationary point in x and y. We write it using scaled variables, scaling in particular all transverse momenta according to the classical bending momentum $2n/R$[2]:

$$\tilde{\Delta}_T = \frac{\Delta_T R}{2m} \quad \ldots . \tag{18}$$

We also write, since it is small,

$$\varepsilon = \frac{n}{R^2}\frac{L_e}{k_i} \tag{19}$$

and

$$\tilde{z} = \frac{2z}{L_e} \tag{20}$$

We neglect terms at order ε^2 (19) and write

$$\Phi = \Delta_L z + n\Big(\tilde{\Delta}_T^2 - \frac{\varepsilon}{2}\frac{1}{1-X}\tilde{k}_{fx}\tilde{\Delta}_{Tx}(1-\tilde{z})^2 - \tilde{\Delta}_T^2\Big(\frac{(1-\tilde{z})^3}{1-X} + (1+\tilde{z})^3\Big)\tag{21}$$

the derivative of which with respect to z is

$$\frac{\partial\phi}{\partial\zeta} = \frac{n\Sigma}{L_e}\,\frac{1}{2X(1-X)}$$
$$\left\{\left(2\tilde{k}_{\gamma x} + \tilde{\Delta}_{Tx}\,X(1+\tilde{\zeta})\right)^2 + \tilde{k}^2_{\gamma y}\left(2 - X(1+\tilde{\zeta})\right)^2\right\} \tag{22}$$

where all quantities inside the bracket are a priori of order 1. In general, there is no real point of stationary phase!

However, $\tilde{k}_{\gamma y}$ is small as compared to $\tilde{k}_{\gamma x}$ and $\tilde{k}_{fx}$, except near the front of the bunch. We have already mentioned that fact and it will be shown explicitly later. It therefore makes sense to consider first the special case of $k_{\gamma y} = 0$. There is then a real point of stationary phase. It occurs for

$$\tilde{\zeta}_0 = -1 - \frac{2}{X}\,\frac{\tilde{k}_{\gamma x}}{\tilde{\Delta}_{Tx}} \tag{23}$$

and we can rewrite (22) as

$$\frac{\partial\phi}{\partial\zeta} = \frac{n\Sigma}{L_e}\,\frac{2X}{1-X}\,\tilde{\Delta}^2_T\left(\zeta-\zeta_0\right)^2 \tag{24}$$

The phase of the amplitude (3) is then, up to a constant term:

$$\phi = \frac{n\Sigma}{L^3_e}\,\frac{2X}{3(1-X)}\,\tilde{\Delta}^2_T\left(\zeta-\zeta_0\right)^3 \tag{25}$$

We can use this phase variation to define a coherent radiation length as

$$\ell_c = \frac{L_e}{(n\Sigma)^{\frac{1}{3}}} = \left(\frac{R^2 L^2_e k_i}{n^2}\right)^{\frac{1}{3}} \tag{26}$$

This is the length given by (1). It is different from the classical coherent radiation length L_c. Using the explicit form of $\bar{L}_c$ and $\bar{L}_e$, the bar referring to centre-of-mass variables, we have indeed[2)]

$$\bar{L}_c = \bar{L}_\ell \frac{Rm}{n} \qquad \bar{L}_e = \frac{2\bar{E}_i}{m^2} \tag{27}$$

hence

$$\bar{L}_c^2 \bar{L}_e = 2\frac{R^2 \bar{L}_\ell^2}{n^2} \bar{E}_i$$

which is (26), when also expressed in terms of centre-of-mass variables. Coming back to bunch frame variables, we can write

$$\ell_c^3 = L_c^2 L_e \tag{1}$$

Given E_i and the size of the bunch in its rest frame, ℓ_c is independent of the electron mass, although both L_c and L_e depend on it.

Radiation coherence thus implies a larger coherence length than the one considered classically for the radiation of a deflected charge. This is not surprising, since we are dealing with bending transverse momenta which are large as compared to the electron mass ($2n/R \sim 10^3 m$), and have thus to define a coherent radiation length independent of the mass. Since coherence occurs over this longer length ℓ_c, this is the cut-off length which should be used for radiation before and after bunch-crossing, as mentioned in Ref. 2). The condition $\ell_c \ll L_b$ still applies to typical multi-TeV machine parameters, since $L_b \sim 10^3 L_c$, and therefore $\ell_c \sim 10^{-2} L_b$.

While the radiation predominantly occurs near the bending plane, the radiation matrix element involves $k_{\gamma y}$, and we must therefore keep the $\tilde{k}^2_{\gamma y}$ term in (22).

We consider the point where the second derivative of ϕ vanishes. This occurs for

$$\tilde{z}_1 = -1 - \frac{2}{X} \frac{\tilde{k}_{\gamma x}\tilde{\Delta}_{Tx} - \tilde{k}^2_{\gamma y}}{\tilde{\Delta}^2_T} \tag{28}$$

with

$$\frac{\partial^2 \phi}{\partial z^2} = \frac{4n\varepsilon}{L_\ell^3} \frac{X}{1-X} \tilde{\Delta}_T^2 (z - z_1) \tag{29}$$

Up to a constant term, the phase (21) can now be written as

$$\Phi = \frac{2n\Sigma}{L_\ell^3}\,\frac{1}{X(1-X)}\left(\frac{X^2\tilde{\Delta}_T^2}{3}\left(z-z_1\right)^3 + L_\ell^2\,\frac{\tilde{k}_{fx}^2\,\tilde{k}_{\gamma y}^2}{\tilde{\Delta}_T^2}\left(z-z_1\right)\right) \tag{30}$$

The radiation amplitude (3), (4) involves an integral over space with a phase which is a function of z (30) and of r (13). Indeed, in the latter case, integration picks the neighbourhood of the stationary point (14), and in the former case a length ℓ_c around z_1. We thus remain inside the bunch, provided that we stay away from the front and back by ℓ_c.

3. - THE RADIATION AMPLITUDES

The radiation matrix element is defined as $\varepsilon^\mu j_\mu$, where j_μ is given by (6). In order to follow closely the classical approach[4)], we consider separately two polarization components in the xz bending plane and perpendicular to it respectively. We write

$$\varepsilon_\parallel = \left(1\ ,\ 0\ ,\ -\frac{k_{\gamma x}}{X k_i}\right)$$
$$\varepsilon_\perp = \left(0\ ,\ 1\ ,\ -\frac{k_{\gamma y}}{X k_i}\right) \tag{31}$$

The photon three-momentum is by definition

$$\vec{k} = \left(k_{\gamma x},\ k_{\gamma y},\ X k_i\right)$$

and the condition $\vec{\varepsilon}\cdot\vec{k} = 0$ is satisfied. Since we are primarily interested in hard photons (X not small), normalization conditions are satisfied, neglecting terms of order k_i^{-2}.

Using (7) and (14), while keeping only leading terms in k_i, we obtain the three-vector part of the current (6), namely

$$\vec{j} = \left(k_{fx}(1+\tilde{z}) + k_{\gamma x}\tilde{z}\ ,\ k_{\gamma y}\tilde{z}\ ,\ (2-X)k_i\right) \tag{32}$$

From (31) and (32) we get the matrix elements for polarization in the bending plane and normal to it respectively

$$\mathcal{M}_{\parallel} = k_{fx}\left(1+\tilde{z}\right) + k_{\gamma x}\left(\tilde{z} - \frac{2-X}{X}\right)$$

$$\mathcal{M}_{\perp} = k_{\gamma y}\left(\tilde{z} - \frac{2-X}{X}\right)$$

(33)

We now use (28) and keep only the leading components in both terms. We use the facts that $k_{\gamma y}$ is small compared with k_{fx} and $k_{\gamma x}$ and that the relevant range of $\tilde{z}-\tilde{z}_1$ is of order ℓ_c/L_b and therefore also small. This gives

$$\mathcal{M}_{\parallel} \approx -\Delta_T\left(\tilde{z}-\tilde{z}_1\right) = -\frac{2\Delta_T}{L_b}\left(z-z_1\right) \tag{34}$$

$$\mathcal{M}_{\perp} \approx 2k_{\gamma y}\frac{k_{fx}}{X\Delta_T}$$

(35)

In order to obtain the radiation amplitude (3), we therefore have two space integrals to calculate:

$$\mathcal{J}_{\parallel} = \int_{-\infty}^{\infty} dz \int_{r=0}^{\infty} dx\, dy\, \left(z-z_1\right) e^{i\phi(x,y,z)}$$

(36)

and

$$\mathcal{J}_{\perp} = \int_{-\infty}^{\infty} dz \int_{r=0}^{\infty} dx\, dy\, e^{i\phi(x,y,z)}$$

(37)

where ϕ is given by (30) and (12). One recognizes an Airy integral in (37) and its derivative in (36). This respective association is well known from the classical case[4]. The coherence condition indeed keeps us close to the classical treatment. Calculating from Eq. (11) the electron momentum $\vec{\nabla}\phi$ separately for ϕ_i and ϕ_f at the stationary point (x_0, y_0, z_1), one indeed finds that, except for the fact that $k_{\gamma y} \neq 0$, everything is as if the photon were emitted tangentially at that point, defined according to the values of $\vec{k}_f$ and $\vec{k}_\gamma$. The electron actually takes a recoil of order $k_{\gamma y}$. As shown later, studying the argument of the Airy function, this is the classical bending momentum which it takes over a distance ℓ_c.

We now proceed with the space integrals in (36) and (37). The integrals over x and y are easily done using the leading term in (13) while integrating over the whole plane, since one actually picks up only the contribution close to the stationary point (14). Up to a global phase, independent of z, this gives a factor $\pi R^2/n$. Using Eq. (30), we write (37) as

$$J_\perp = \frac{\pi R^2}{n} \int_{-\infty}^{\infty} dz \; \exp i \left\{ C_3 (z - z_1)^3 + C_1 (z - z_1) \right\}$$

with
$$C_3 = \frac{X}{6(1-X)} \; \frac{\Delta_T^2}{L_\ell^2 k_i}$$
$$C_1 = \frac{1}{2X(1-X)} \; \frac{k_{fx}^2 \, k_{\gamma y}^2}{\Delta_T^2 \, k_i} \tag{38}$$

Only the even term survives in the integration. $J_\perp$ can then be rewritten in terms of an Airy function[9)]

$$A_i(u) = \int_0^{\infty} dt \, \cos\left(t^3 + 3ut\right) \tag{39}$$

We obtain

$$J_\perp = \frac{2\pi R^2 L_\ell}{n} \left(\frac{6(1-X)}{X} \; \frac{k_i}{\Delta_T^2 L_\ell} \right)^{\frac{1}{3}} A_i(u) \tag{40}$$

where
$$u = \frac{k_{fx}^2 \, k_{\gamma y}^2}{\Delta_T^4} \left(\frac{L_\ell \, \Delta_T^2}{6X^2(1-X) \, k_i} \right)^{\frac{2}{3}} \tag{41}$$

We can now see the limits on $k_{\gamma y}$, which are imposed by the exponential asymptotic dependence of the Airy function. We can take as an estimate of the typical value of the component of the photon momentum normal to the bending plane

$$k_{\gamma y} \sim \frac{2n}{R}\,\frac{1}{L_\ell}\left(\frac{R^2 L_\ell^2 k_i}{n^2}\right)^{\frac{1}{3}} \tag{42}$$

which, using (26), is:

$$k_{\gamma y} \sim \frac{2n}{R}\,\frac{\ell_c}{L_\ell} \tag{43}$$

It corresponds, as previously said, to the bending momentum collected by the electron over the coherent radiation length ℓ_c only and is small as compared to the transverse momentum of the outgoing electron and of the photon in the bending plane, provided that we stay away from the front and back of the bunch. With the machine parameter used, $k_{\gamma y}$ is typically two orders of magnitude smaller than k_{fx}. The radiation is therefore strongly localized in the neighbourhood of the bending plane. Combining (33) and (37), the radiation amplitude with polarization normal to the bending plane can now be written, up to a global phase factor, as:

$$M_\perp = 4e\,k_{\gamma y}\,\frac{k_{fx}}{x\,\Delta_T}\,\frac{\pi R^2 L_\ell}{n}\,\omega^{\frac{1}{3}}\,A_i(u) \tag{44}$$

where u is given by (41), and

$$\omega = \frac{6(1-x)}{x}\,\frac{k_i}{\Delta_T^2 L_\ell} \tag{45}$$

Also for $J_\parallel$ [Eq. (36)], only the even term survives in the integration and we can write $J_\parallel$ in terms of a derivative of an Airy function:

$$A_i'(u) = -3\int_0^\infty dt\;t\,\sin\left(t^3+3ut\right) \tag{46}$$

This gives

$$J_\parallel = 2\frac{\pi R^2 L_\ell}{3n}\,\omega^{\frac{2}{3}}\,A_i'(u) \tag{47}$$

which, combined with (34), gives the amplitudes for radiation with polarization in the bending plane. Up to a global phase, it reads:

$$M_{\parallel} = e\,\Delta_T\, 4\, \frac{\pi R^2 L_b}{3h}\, \omega^{\frac{2}{3}}\, A_i'(u) \tag{48}$$

4. - RADIATION RATE

The full radiation rate is given by[2)]

$$I = \frac{1}{8k_i} \int \left(|M_\perp|^2 + |M_\parallel|^2 \right) 2\pi\, \delta(k_i - k_f - k_\gamma)\, \frac{d^3\vec{k}_f\, d^3\vec{k}_\gamma}{(2\pi)^6\, k_f\, k_\gamma}$$

$$= \frac{1}{8k_i^2 (2\pi)^4} \int \left(|M_\perp|^2 + |M_\parallel|^2 \right) k_{fx}\, dk_{fx}\, dk_{\gamma x}\, dk_{\gamma y}\, \frac{dX}{X(1-X)} \tag{49}$$

We first perform the integration over $k_{\gamma y}$, which is strongly limited in range by the Airy function. We treat separately the two polarization components. For the normal component, we have from (44)

$$\int_{-\infty}^{\infty} |M_\perp|^2\, dk_{\gamma y} =$$

$$32 e^2 \left(\frac{k_{fx}}{X\Delta_T}\, \frac{\pi R^2 L_b}{h} \right)^2 \omega^{\frac{2}{3}} \int_0^{\infty} k_{\gamma y}^2\, dk_{\gamma y}\, A_i^2(u) \tag{50}$$

and a contribution to I (49):

$$I_\perp = \frac{6\alpha}{\pi^3 k_i}\left(\frac{\pi R^2 L_b}{h}\right)^2 \int_0^\infty u^2 du\, A_i^2(u^2)$$

$$\int \omega^{\frac{2}{3}} \frac{\Delta_T^2}{L_\ell}\, dk_{fx}\, dk_{\gamma x}\, \frac{dX}{X} \tag{51}$$

where u is here a dimensionless dummy variable.

The integral over the Airy function squared can be performed, using the relation[9)]

$$A_i(u^2) = \left(\frac{u^2}{3}\right)^{\frac{1}{2}} K_{\frac{1}{3}}\left(2u^3\right) \tag{52}$$

$$\int_0^\infty u^2 du\, A_i^2(u^2) = \frac{1}{18}\left(\frac{1}{2}\right)^{\frac{2}{3}} \int_0^\infty u^{\frac{2}{3}} K_{\frac{1}{3}}^2(u)\, du \tag{53}$$

$$= \frac{\pi^{\frac{3}{2}}}{216} \frac{\Gamma(5/6)}{\Gamma(5/3)} \tag{54}$$

which is numerically close to $10^{-2}\pi$. For the component in the bending plane, we have from (48)

$$\int_{-\infty}^{\infty} (M_{\parallel})^2 dk_{\gamma y} =$$

$$32\, e^2 \Delta_T^2 \left(\frac{\pi R^2 L_b}{h}\right)^2 \omega^{\frac{4}{3}} \frac{1}{9}\int_0^\infty dk_{\gamma y}\, A_i'^2(u) \tag{55}$$

with u and w given by (41) and (45) respectively, and a contribution to I (49)

$$I_{\parallel} = \frac{6\alpha}{\pi^3 k_i} \left(\frac{\pi R^2 L_\ell}{n}\right)^2 \frac{1}{9} \int_0^\infty du\, A_i'^2(u^2)$$

$$\int \omega^{\frac{2}{3}} \frac{\Delta_T^2}{L_\ell} \, dk_{fx} \, dk_{\gamma x} \, \frac{dX}{X} \tag{56}$$

We thus obtain the same dependence on all variables in $I_{\perp}$ and $I_{\parallel}$, as expected.

The integral over the square of the derivative of the Airy function can be performed using the relation[9]: [u in (56) is again a dimensionless dummy variable]

$$A_i'^2(u^2) = 3u^4 K_{\frac{2}{3}}^2(2u^3)$$

$$\frac{1}{9}\int_0^\infty du\, A_i'^2(u^2) = \frac{1}{18}\left(\frac{1}{2}\right)^{\frac{2}{3}} \int_0^\infty u^{\frac{2}{3}} K_{\frac{2}{3}}^2(u)\, du \tag{57}$$

The integral is known[9] and (57) is equal to

$$\frac{1}{72} \pi^{\frac{3}{2}} \frac{\Gamma(5/6)}{\Gamma(5/3)}$$

which is numerically close to $3\,\pi \cdot 10^{-2}$. The total radiation rate is obtained from (51) and (56) which both involve the same integral over k_{fx}, $k_{\gamma x}$ and X. Neglecting $k_{\gamma y}$ as compared with $k_{\gamma x}$, we integrate over $k_{\gamma x}$ which varies from 0 to $X\Delta_T$ ($|\tilde{z}| < 1$) and over Δ_T which varies from 0 to $2n/R$. One finds

$$I = 1.94 \frac{\alpha}{\pi} \left\{ \left(\frac{n}{R}\right)^{\frac{2}{3}} \left(\frac{L_\ell}{k_i}\right)^{\frac{1}{3}} \right\} \pi R^2 \int_0^1 \left(\frac{1-X}{X}\right)^{\frac{2}{3}} dX \tag{58}$$

In order to obtain the fractional energy loss we have to normalize the rate to the bunch cross-section πR^2 and weight the integrand in the last integral by X. We recognize in the bracket the ratio L_b/ℓ_c (26).

We therefore obtain

$$\delta = 1.94 \ \frac{\alpha}{\pi} \left(\frac{L_b}{\ell_c}\right) \int_0^1 (1-X)^{\frac{2}{3}} X^{\frac{1}{3}} dX \tag{59}$$

The integral over X is a β function which is easily calculated. It is numerically close to 0.4, hence

$$\delta = 0.78 \ \frac{\alpha}{\pi} \left(\frac{L_b}{\ell_c}\right) \tag{60}$$

where, with the machine parameter considered, $L_b/\ell_c \approx 60$. The fractional energy loss is then of the order of 10%.

The expression arrived at is therefore particularly simple. The bunch length is scaled according to the coherent radiation length ℓ_c, which is defined by (1).

The picture that emerges for the reaction

$$e + \text{bunch} \rightarrow e + \text{bunch} + \gamma$$

in a multi-TeV linear e^+e^- collider, for which $L_e >> L_b >> \ell_c >> L_c$, considered in the approximation of a uniform cylindrical bunch, is as follows when analyzed according to the transverse momentum of the photon. From this point of view we have two peaks at both small and maximum values, corresponding to radiation before and after bunch-crossing and a flat distribution in between, corresponding to radiation produced during bunch-crossing. When these contributions are integrated, the two peaks yield a radiated energy proportional to the beam energy with a coefficient of order $(\alpha/\pi)\ \ell n\ (L_e/\ell_c)$ and the flat region a radiated energy proportional to the beam energy but with a coefficient $(\alpha/\pi)\ L_b/\ell_c$. This picture is expected to hold not only for uniform cylindrical bunches but also for more realistic charge distributions, such as Gaussian bunches.

With the machine parameters considered, these two contributions are of similar orders of magnitude. Our second-order approximation (7) would not be sufficiently accurate for the calculation of terms of order α/π.

Our approach bears clear similarities to that of Blankenbecler and Drell[3]. There are, however, two main differences. Firstly, we go directly to the regime of very small C, as they define it[3]. This is also referred to as the extreme quantum case, which is appropriate for a multi-TeV machine. In so doing, we emphasize the importance of ℓ_c (1) as the proper scaling length for beamstrahlung in that regime. Secondly, we emphasize here and in Ref. 2) the fact that L_e becomes larger that L_b in such machines, whereas they kept L_b as the longest length. Consequently they have only a term linear in L_b.

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FIGURE CAPTIONS

Fig. 1 : Feynman graph for the emission of a photon by an electron in the presence of a bunch.

Fig. 2 : Radiation of a photon by an electron while its trajectory is bent by the presence of the bunch which extends from $-L_b/2$ to $L_b/2$. Fixing the final momenta $\vec{k}_f$ and $\vec{k}_\gamma$ implies radiation from a zone fixed by the coherence conditions around x_0, y_0 and z_0. The transverse component of the photon $k_{\gamma x}$ is primarily due to the bending of the electron trajectory before emission. In practice $L_b \gg R$. For "typical" super parameters $L_b \sim 3$m, $R \sim 10^{-9}$m. The disruption effect shown in the figure is overemphasized. In practice $D \sim 0.1$.

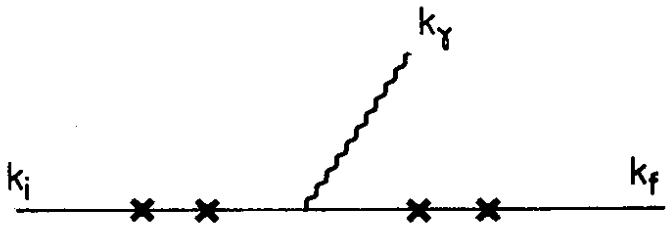

Fig. 1

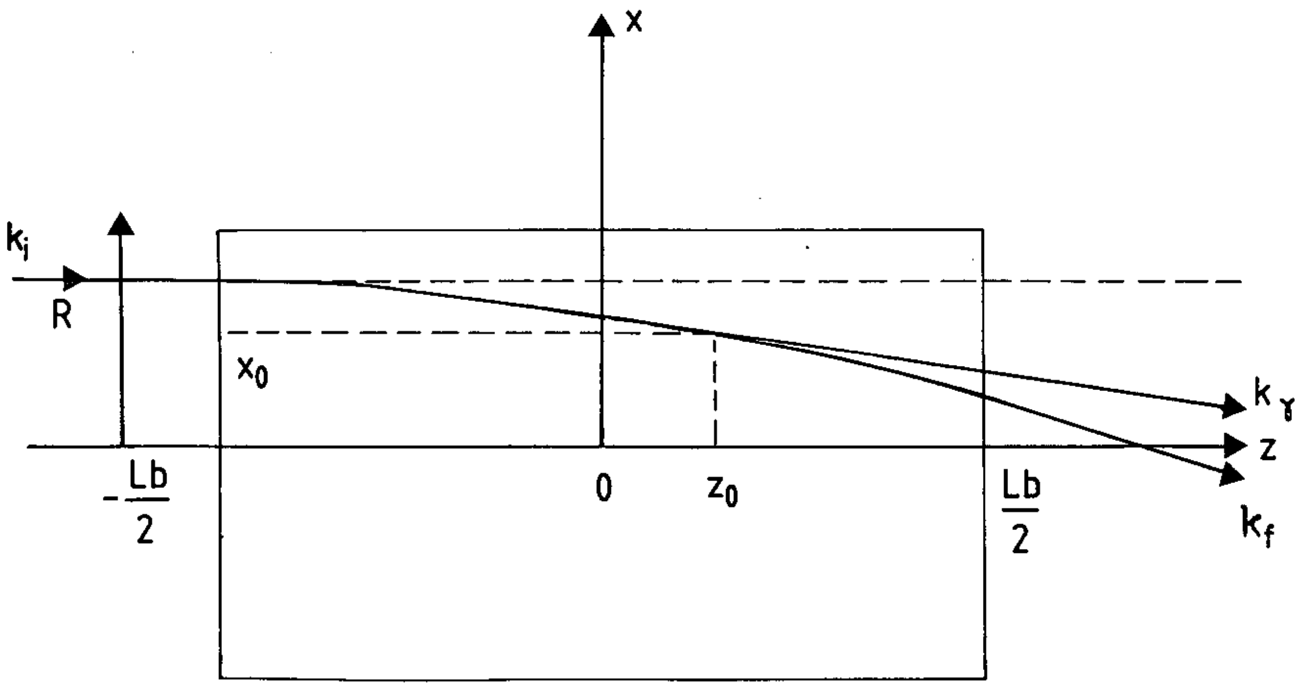

Fig. 2