

F-theory on all toric hypersurface fibrations and its Higgs branches

Denis Klevers,^{a,b} Damián Kaloni Mayorga Peña,^c Paul-Konstantin Oehlmann,^c Hernan Piragua^a and Jonas Reuter^c

^a*Department of Physics and Astronomy, University of Pennsylvania, Philadelphia, PA 19104-6396, U.S.A.*

^b*Theory Group, Physics Department, CERN, CH-1211, Geneva 23, Switzerland*

^c*Bethe Center for Theoretical Physics, Physikalisches Institut der Universität Bonn, Nussallee 12, 53115 Bonn, Germany*

E-mail: klevers@sas.upenn.edu, damian@th.physik.uni-bonn.de, oehlmann@th.physik.uni-bonn.de, hpiragua@sas.upenn.edu, jreuter@th.physik.uni-bonn.de

ABSTRACT: We consider F-theory compactifications on genus-one fibered Calabi-Yau manifolds with their fibers realized as hypersurfaces in the toric varieties associated to the 16 reflexive 2D polyhedra. We present a base-independent analysis of the codimension one, two and three singularities of these fibrations. We use these geometric results to determine the gauge groups, matter representations, 6D matter multiplicities and 4D Yukawa couplings of the corresponding effective theories. All these theories have a non-trivial gauge group and matter content. We explore the network of Higgsings relating these theories. Such Higgsings geometrically correspond to extremal transitions induced by blow-ups in the 2D toric varieties. We recover the 6D effective theories of all 16 toric hypersurface fibrations by repeatedly Higgsing the theories that exhibit Mordell-Weil torsion. We find that the three Calabi-Yau manifolds without section, whose fibers are given by the toric hypersurfaces in \mathbb{P}^2 , $\mathbb{P}^1 \times \mathbb{P}^1$ and the recently studied $\mathbb{P}^2(1, 1, 2)$, yield F-theory realizations of SUGRA theories with discrete gauge groups \mathbb{Z}_3 , \mathbb{Z}_2 and \mathbb{Z}_4 . This opens up a whole new arena for model building with discrete global symmetries in F-theory. In these three manifolds, we also find codimension two I_2 -fibers supporting matter charged only under these discrete gauge groups. Their 6D matter multiplicities are computed employing ideal techniques and the associated Jacobian fibrations. We also show that the Jacobian of the biquadric fibration has one rational section, yielding one U(1)-gauge field in F-theory. Furthermore, the elliptically fibered Calabi-Yau manifold based on dP_1 has a U(1)-gauge field

induced by a non-toric rational section. In this model, we find the first F-theory realization of matter with $U(1)$ -charge $q = 3$.

KEYWORDS: F-Theory, Differential and Algebraic Geometry, Discrete and Finite Symmetries, Supergravity Models

ARXIV EPRINT: [1408.4808](https://arxiv.org/abs/1408.4808)

Contents

| | | |
|----------|--|-----------|
| 1 | Introduction & summary of results | 1 |
| 1.1 | Summary of results | 3 |
| 1.2 | Structure of the paper | 6 |
| 2 | Geometry & physics of F-theory backgrounds | 7 |
| 2.1 | Genus-one, Jacobian and elliptic fibrations with Mordell-Weil groups | 7 |
| 2.2 | Divisors on genus-one fibrations and their intersections | 9 |
| 2.3 | The spectrum of F-theory on genus-one fibrations | 10 |
| 2.4 | Explicit examples: Calabi-Yau hypersurfaces in 2D toric varieties | 14 |
| 3 | Analysis of F-theory on toric hypersurface fibrations | 16 |
| 3.1 | Three basic ingredients: the cubic, biquadric and quartic | 17 |
| 3.1.1 | Constructing toric hypersurface fibrations | 17 |
| 3.1.2 | Fibration by cubic curves: X_{F_1} and its specializations | 18 |
| 3.1.3 | Fibration by the biquadric: X_{F_2} | 20 |
| 3.1.4 | Fibration by the quartic: X_{F_4} | 22 |
| 3.2 | Fibration with discrete gauge symmetry | 23 |
| 3.2.1 | Polyhedron F_1 : $G_{F_1} = \mathbb{Z}_3$ | 23 |
| 3.2.2 | Polyhedron F_2 : $G_{F_2} = \text{U}(1) \times \mathbb{Z}_2$ | 26 |
| 3.2.3 | Polyhedron F_4 : $G_{F_4} = \text{SU}(2) \times \mathbb{Z}_4$ | 32 |
| 3.3 | Fibration with gauge groups of rank 1, 2 and no discrete gauge symmetry | 36 |
| 3.3.1 | Polyhedron F_3 : $G_{F_3} = \text{U}(1)$ | 36 |
| 3.3.2 | Polyhedron F_5 : $G_{F_5} = \text{U}(1)^2$ | 42 |
| 3.3.3 | Polyhedron F_6 : $G_{F_6} = \text{SU}(2) \times \text{U}(1)$ | 44 |
| 3.4 | Fibrations with gauge groups of rank 3: selfdual polyhedra | 50 |
| 3.4.1 | Polyhedron F_7 : $G_{F_7} = \text{U}(1)^3$ | 50 |
| 3.4.2 | Polyhedron F_8 : $G_{F_8} = \text{SU}(2)^2 \times \text{U}(1)$ | 52 |
| 3.4.3 | Polyhedron F_9 : $G_{F_9} = \text{SU}(2) \times \text{U}(1)^2$ | 58 |
| 3.4.4 | Polyhedron F_{10} & the generalized Tate form: $G_{F_{10}} = \text{SU}(3) \times \text{SU}(2)$ | 61 |
| 3.5 | Fibrations with gauge groups of rank 4, 5 and no MW-torsion | 66 |
| 3.5.1 | Polyhedron F_{11} : $G_{F_{11}} = \text{SU}(3) \times \text{SU}(2) \times \text{U}(1)$ | 66 |
| 3.5.2 | Polyhedron F_{12} : $G_{F_{12}} = \text{SU}(2)^2 \times \text{U}(1)^2$ | 69 |
| 3.5.3 | Polyhedron F_{14} : $G_{F_{14}} = \text{SU}(3) \times \text{SU}(2)^2 \times \text{U}(1)$ | 72 |
| 3.6 | Fibrations with gauge groups of rank 5 and 6 and MW-torsion | 78 |
| 3.6.1 | Polyhedron F_{13} : $G_{F_{13}} = (\text{SU}(4) \times \text{SU}(2)^2)/\mathbb{Z}_2$ | 78 |
| 3.6.2 | Polyhedron F_{15} : $G_{F_{15}} = \text{SU}(2)^4/\mathbb{Z}_2 \times \text{U}(1)$ | 81 |
| 3.6.3 | Polyhedron F_{16} : $G_{F_{16}} = \text{SU}(3)^3/\mathbb{Z}_3$ | 87 |

| | | |
|----------|--|------------|
| 4 | The toric Higgs branch of F-theory | 89 |
| 4.1 | Toric Higgsing: an example | 91 |
| 4.1.1 | Matching the charged spectrum | 94 |
| 4.1.2 | Matching of the neutral spectrum: Higgsing & Euler numbers | 96 |
| 4.1.3 | Allowed regions for base \mathbb{P}^2 | 97 |
| 4.2 | Higgsings to theories with discrete gauge symmetries | 99 |
| 4.3 | The complete Higgsing chain | 102 |
| 5 | Conclusions | 103 |
| A | Anomaly cancellation conditions in 6D | 106 |
| B | Additional data on toric hypersurface fibrations | 108 |
| C | Euler numbers of the Calabi-Yau threefolds X_{F_i} | 115 |
| D | The full Higgs chain of toric hypersurface fibrations | 116 |
| E | Group theoretical decomposition of representations | 118 |

1 Introduction & summary of results

F-theory [1–3] is a non-perturbative formulation of Type IIB string theory with backreacted 7-branes, that is manifestly invariant under the $SL(2, \mathbb{Z})$ -duality symmetry of the theory. String backgrounds constructed via F-theory are not only located in the heart of the web of string dualities, but also allow for the construction of phenomenologically appealing local GUT-models [4–7], which has recently rekindled a lot of interest into the subject. The basic idea of F-theory is to replace the axio-dilaton $\tau = C_0 + ie^{-\phi}$, that is only defined up to $SL(2, \mathbb{Z})$ -transformations, by a quantity, that only depends on the $SL(2, \mathbb{Z})$ -equivalence class of τ . The canonical geometrical object with this property is a two-torus $T^2(\tau)$, whose complex structure is identified with τ . Thus, replacing τ by this auxiliary $T^2(\tau)$ provides an $SL(2, \mathbb{Z})$ -invariant formulation of Type IIB. Non-trivial backgrounds of τ , which are sourced by 7-branes, on manifolds B are mapped under this replacement to torus-fibrations over B . In particular, for a supersymmetric and tadpole-canceling setup of 7-branes on a complex Kähler manifold B the total space of this $T^2(\tau)$ -fibration is a Calabi-Yau manifold $\pi : X \rightarrow B$.

Most of the torus-fibered Calabi-Yau manifolds X that have been studied are algebraic, that is they are realized as complete intersections in some ambient space.¹ In these constructions, the torus fiber over B is realized as an algebraic curve \mathcal{C} of genus one. In addition, many examples considered in the literature are elliptically fibered, meaning that

¹For recent advances on Calabi-Yau manifolds constructed as determinantal and Pfaffian varieties, see [8, 9].

X has a section $B \rightarrow X$, which was traditionally assumed to be holomorphic. These elliptically fibered Calabi-Yau manifolds have fruitful applications, e.g. for the construction of semi-realistic GUTs in global F-theory compactifications starting with [10, 11] or the classification and study of 6D SCFTs [12–14].

Despite these successes, addressing open conceptual questions e.g. regarding the finiteness of the F-theory landscape² or which consistent 6D and 4D supergravity (SUGRA) theories can be realized in F-theory,³ as well as the understanding of the geometric origins of discrete symmetries or analogous field theoretic mechanisms, crucial for the phenomenology of F-theory models, requires to broaden the class of Calabi-Yau manifolds X used for F-theory compactifications. In fact, using the well-developed map between the geometry of F-theory and SUGRA theories, see [20, 21] for the complete map in 6D and [22] for results about certain topological terms in 4D,⁴ one finds that the Calabi-Yau manifolds realizing many known consistent SUGRA theories, in particular those with $U(1)$ symmetries [24] or discrete gauge groups, are still unknown [25].⁵ For the search of an F-theory realization of these theories it is crucial to construct new classes of Calabi-Yau manifolds X admitting new geometric features and to deduce the general SUGRA theories that arise in F-theory compactifications on these X .⁶

There has been a lot of recent progress in systematically extending the set of Calabi-Yau manifolds X that can be used for F-theory compactifications. The different approaches can be roughly sorted into two groups. The first group of approaches focuses on the classification and construction of all bases B that are admissible for F-theory [28–30]. The second group, to which this work belongs, focuses on generalizing the type of fiber \mathcal{C} and the ways in which it can be fibered in a Calabi-Yau manifold X . There are three major extensions in this direction:

1. Elliptic fibers with an increasing number of rational points and their corresponding elliptically fibered Calabi-Yau manifolds X with a *Mordell-Weil group* (MW-group) of rational sections of increasing rank have been systematically constructed and studied [31–46].⁷ The free part of the MW-group leads to $U(1)$ -gauge fields in F-theory⁸ [2] and the torsion part yields non-simply connected gauge groups [53].
2. Elliptic fibrations X with a *non-holomorphic zero section* have been considered re-

²See [15] and the recent [16] for a finiteness proof in related Type I compactifications.

³F-theory compactifications to 8D are well-studied and classified, see e.g. the recent toric analysis of [17] and the classification of elliptic fibrations on K3-surfaces in [18]. Last subtleties in the understanding of the gauge group of a generic K3 have been understood in [19].

⁴SUGRA theories from string theory can also be constrained using tools from heterotic/F-theory duality [23].

⁵Of course it is a logical possibility that some of the SUGRA theories without a known F-theory realization are not consistent effective theories due to a violation of consistency constraints that are unknown at this point.

⁶Compactifications of F-theory on Spin 7 manifolds, considered recently [26, 27], have not yet produced SUGRA theories that cannot be obtained by a Calabi-Yau compactification.

⁷Certain aspects of models with a higher rank MW-group are studied already in [47–49], see also [50] for an analysis of models with D_5 -fiber.

⁸See [35, 51, 52] for a discussion of (geometrically) massive $U(1)$'s.

cently, starting with [37, 39, 43]. This permits the introduction of discrete degrees of freedom in the construction of the fibration of the elliptic curve over the base B yielding a finite number of strata in the moduli space of X .

3. Algebraic curves \mathcal{C} of one without any (rational) point have been used to construct *genus-one fibrations* [54–56]. These are Calabi-Yau manifolds X which do not have a section, but only multi-sections. These models can be analyzed employing their associated Jacobian fibration $J(X)$, which does exhibit a zero section, and its Weierstrass form.

1.1 Summary of results

The Calabi-Yau manifolds X we consider in this work invoke all of the three aforementioned extensions. We study all F-theory compactifications on Calabi-Yau manifolds X_{F_i} with genus-one fiber \mathcal{C}_{F_i} given as a hypersurface in the toric varieties associated to the 16 2D reflexive polyhedra, denoted by F_i , $i = 1, \dots, 16$.⁹ We refer to these Calabi-Yau manifolds as *toric hypersurface fibrations*. We determine the generic and intrinsic features of these X_{F_i} that are relevant to F-theory: the generic gauge group, the corresponding matter spectrum and the 4D Yukawa couplings corresponding to the codimension one, two and three singularities of X_{F_i} . These geometric results completely determine the 6D and non-chiral 4D SUGRA theories obtained by compactifying F-theory on Calabi-Yau threefolds and fourfolds without G_4 -flux. We prove completeness of our analysis of codimension one and two singularities by checking cancellation of all 6D anomalies. All these results are base-independent in the sense that they follow directly from the geometry of the fiber \mathcal{C}_{F_i} . The only dependence on the base B enters through the choice of two divisors on B that label the possible Calabi-Yau fibrations of \mathcal{C}_{F_i} [39].

We highlight the following interesting geometrical findings of our analysis of F-theory on the Calabi-Yau manifolds X_{F_i} :

- Every X_{F_i} has an associated minimal gauge group G_{F_i} that is completely determined by the polyhedron F_i . In other words this gauge group is present without tuning the complex structure of X_{F_i} by means of Tate’s algorithm [59–61] (see [62, 63] for recent refinements) or upon addition of tops. The gauge groups G_{F_i} and $G_{F_i^*}$ associated to F_i and its dual polyhedron F_i^* obey the rank relation

$$\text{rk}(G_{F_i}) + \text{rk}(G_{F_i^*}) = 6. \tag{1.1}$$

- We consider three Calabi-Yau manifolds X_{F_i} , $i = 1, 2, 4$, without section. Their fibers are the general cubic in \mathbb{P}^2 , the general biquadric in $\mathbb{P}^1 \times \mathbb{P}^1$ and the general quartic in $\mathbb{P}^2(1, 1, 2)$, respectively, where the latter is also studied in [54–56]. The fibrations X_{F_i} , $i = 1, 2, 4$, only have a genus-one fibration with a three-, a two- and a two-section, respectively. As a direct consequence of this absence of sections, F-theory has discrete gauge group factors given by \mathbb{Z}_3 , \mathbb{Z}_2 and \mathbb{Z}_4 , respectively. Using prime

⁹These genus-one curves have also been used in [57, 58] as mirror curves for the computation of refined stable pair invariants in the refined topological string.

ideal and elimination theory techniques from computational algebraic geometry, we show that these Calabi-Yau manifolds, most notably the fibration of the cubic, X_{F_1} , have I_2 -fibers at codimension two that support singlet matter with charge 1 under the respective discrete gauge groups. We explain how the charge of all matter fields under these discrete groups are computed from the intersections of the multi-sections with the relevant codimension two fibers.

- We show that both X_{F_2} and X_{F_3} give rise to one $U(1)$ -factor, namely $G_{F_2} = U(1) \times \mathbb{Z}_2$ and $G_{F_3} = U(1)$. To this end, we show that unlike X_{F_2} , the Weierstrass form of the Jacobian $J(X_{F_2})$ has one rational section, whereas already X_{F_3} has two sections: a toric and a non-toric one. In both cases, we determine the coordinates of all sections explicitly.
- For the first time, we find F-theory compactifications with matter of $U(1)$ -charge three. This matter is supported at a codimension two locus of X_{F_3} with an I_2 -fiber where both the zero section and the non-toric rational section are ill-behaved and each “wrap” one irreducible fiber component.

We note that the 16 toric hypersurface fibrations X_{F_i} were considered in [41], where a thorough classification of their toric MW-groups was carried out. Further specializations of the X_{F_i} corresponding to toric tops [64, 65] (see [66] for a systematic approach based on Tate’s algorithm for elliptic fibers in $\text{Bl}_1\mathbb{P}^2(1, 1, 2)$) permitted the engineering of toric F-theory models with certain gauge groups, in particular with an $SU(5)$ GUT-group. Some 4D examples of chiral $SU(5)$ GUTs were constructed in this manner [42, 43]. Since we determine here the intrinsic gauge groups and the non-toric MW-groups, as well as the full matter spectrum and the Yukawa couplings of X_{F_i} , our approach is complementary to these previous works.

We have to remark that none of the fibrations X_{F_i} yield an $SU(5)$ gauge factor in their low-energy effective theories. Hence, strictly speaking, the intrinsic gauge symmetries associated to the toric hypersurface fibrations do not suffice to engineer $SU(5)$ F-theory GUTs. There are, however, some arguments that challenge the simplest GUT picture in F-theory, and therefore, draw our attention towards alternative schemes which may be promising for particle physics models. In this spirit, we would like to briefly highlight some of the effective theories we obtain, which can potentially be used to construct promising particle physics models in F-theory. We find models with discrete symmetries and up to three $U(1)$ factors. These additional symmetries can be used in order to forbid dangerous operators which would render the theory incompatible with observations, e.g. by mediating fast proton decay. In addition, we observe theories with interesting gauge groups and spectra. In fact, $X_{F_{11}}$ precisely leads to an effective theory with the Standard Model gauge group and the usual representations,¹⁰ and we further identify the trinification group for $X_{F_{16}}$ as well as the Pati-Salam group for $X_{F_{13}}$. As we demonstrate explicitly, the matter spectra we obtain are very close to those one usually has in both of these theories.

¹⁰See [67] for a different realization of a standard model like theory based on tops of X_{F_5} .

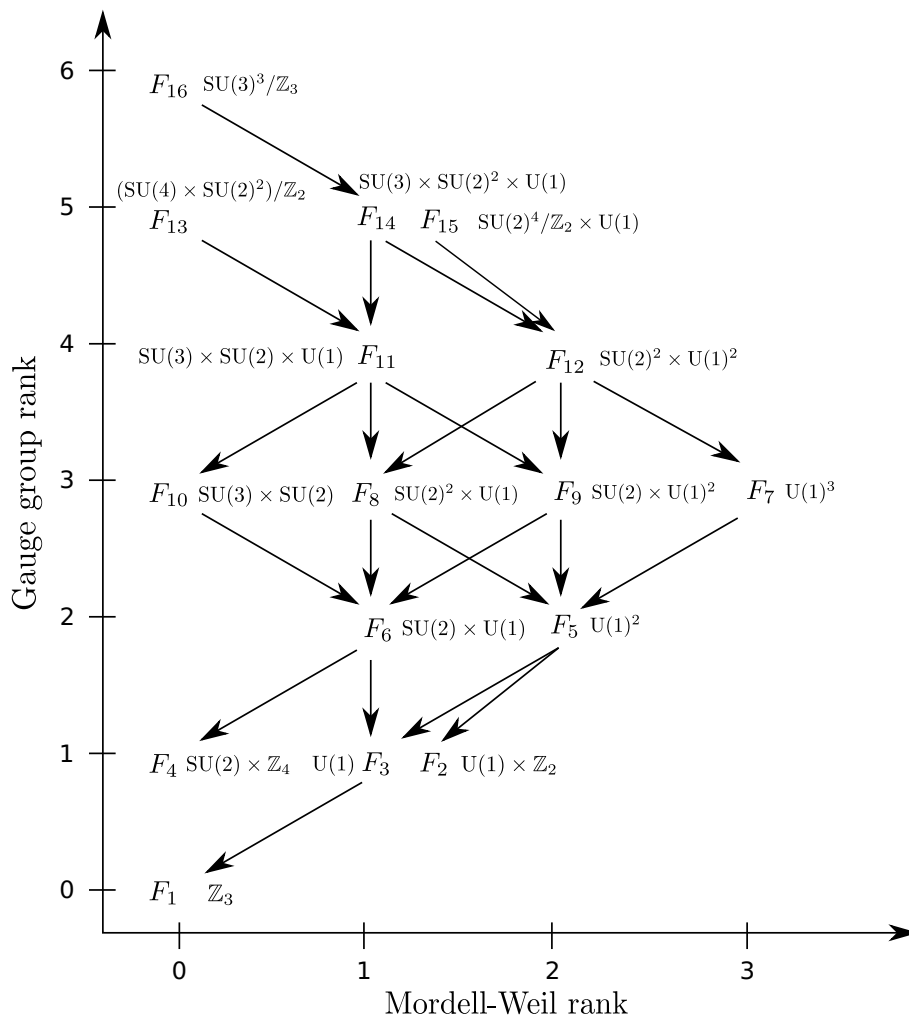


Figure 1. The network of Higgsings between all F-theory compactifications on toric hypersurface fibrations X_{F_i} . The axes show the rank of the MW-group and the total rank of the gauge group of X_{F_i} . Each Calabi-Yau X_{F_i} is abbreviated by F_i and its corresponding gauge group is shown. The arrows indicate the existence of a Higgsing between two Calabi-Yau manifolds.

In this paper we also work out the entire network of Higgsings relating the effective theories of F-theory on the toric hypersurface fibrations X_{F_i} . It is well-known that the toric varieties corresponding to the 16 2D reflexive polyhedra F_i are related by blow-downs. Consequently, the Calabi-Yau manifolds X_{F_i} are related by the extremal transitions induced by these birational maps and a subsequent toric complex structure deformation. These transitions can be understood as Higgsings in the effective SUGRA theories arising from F-theory on the X_{F_i} : given two polyhedra F_i and $F_{i'}$ related by a blow-down as $F_i \rightarrow F_{i'}$, we explicitly determine the Higgsing that relates the effective theory of F-theory on X_{F_i} to that on $X_{F_{i'}}$. The resulting diagram of all those Higgsings is given in figure 1. Since this chain of Higgsings is only a sub-branch of the full Higgs branch of the effective SUGRA theories of F-theory on X_{F_i} , we refer to it as the *toric Higgs branch*. We check that both the charged and the neutral spectrum of the SUGRA theories in 6D match. This

involves the computation of the number of neutral hyper multiplets, that in turn can be obtained from the Euler numbers of all X_{F_i} , which we also compute explicitly.

We point out some interesting observations about the Higgsing diagram in figure 1:

- All effective theories can be obtained by appropriately Higgsing the theories with maximal gauge groups and matter spectra obtained from F-theory on X_{F_i} , $i = 13, 15, 16$. We note that these are precisely the theories with non-trivial Mordell-Weil torsion.
- The network of Higgsings is symmetric around the horizontal line where the total rank of the gauge group is 3. Reflection along this line exchanges the polyhedron with its dual. This symmetry of figure 1 reflects the rank condition (1.1). We emphasize that this symmetry maps theories with discrete gauge groups to theories with non-simply connected group, suggesting that multi-sections are somehow “dual” to MW-torsion.
- The three theories with discrete gauge groups arise by Higgsing theories with $U(1)$'s. It is also remarkable that all discrete symmetries found are surviving remnants of $U(1)$ symmetries. It seems that discrete symmetries in F-theory are automatically in agreement with the early observation [68–70] that in a consistent theory of gravity, discrete global symmetries must be always embeddable into a local continuous symmetry.
- The toric Higgsings cannot change the rank of the F-theory gauge group by more than 1. This explains why there are no arrows with slope below 45 degrees.

1.2 Structure of the paper

This paper is organized as follows. Section 2 contains a summary of the geometry of Calabi-Yau manifolds constructed as genus-one fibrations and the physics of F-theory compactified on them. We also present a basic account on toric geometry. This section serves mostly as review preparing for the main analysis of the paper. It can be skipped by readers familiar with these matters.

Section 3 contains the central results of this work. To set the stage we review the construction and the different types of toric hypersurface fibrations X_{F_i} over an arbitrary base in section 3.1. The remainder of section 3 is organized as a catalog (ordered mostly according to the order of the 2D reflexive polyhedra F_i) with a separate subsection for the analysis of each of the 16 toric hypersurface fibrations X_{F_i} . The codimension one, two and three singularities of the X_{F_i} are analyzed, and the number of their complex structure moduli is computed in the threefold case. The F-theory gauge group, matter spectrum and Yukawa couplings are extracted from these results. The gauge groups of all X_i are summarized at the beginning of section 3 and the matter spectra and Yukawa couplings of every X_{F_i} are summarized in tables in the corresponding subsection, which can be directly consulted by readers less interested in the details of the calculations.

Section 4 is devoted to the study of the physics of the toric Higgs branch of F-theory compactified on the X_{F_i} . One particular Higgsing is discussed in detail in order to illustrate the relevant techniques. Here we also present the particularly important Higgsings leading to the effective theories with discrete gauge groups.

Our conclusions can be found in section 5. This work contains additional appendices on 6D anomalies (appendix A), additional geometrical data of the X_{F_i} (appendix B) and the explicit Euler numbers of all Calabi-Yau threefolds X_{F_i} (appendix C) as well as further details on the entire Higgsing chain in appendices D and E.

2 Geometry & physics of F-theory backgrounds

In this section, we summarize the key geometrical properties of Calabi-Yau manifolds X that are genus-one fibrations over a base B which are relevant for the study of F-theory compactifications, see sections 2.1 and 2.2. The structure of the 6D effective SUGRA theories obtained by compactifying F-theory on these manifolds is discussed in section 2.3. Since we study in this work Calabi-Yau manifolds X with their genus-one fibers realized as toric hypersurfaces, we introduce the necessary elements of toric geometry in section 2.4.

Readers familiar with the tools and definitions presented here can safely skip this section and continue directly with section 3.

2.1 Genus-one, Jacobian and elliptic fibrations with Mordell-Weil groups

We consider a smooth Calabi-Yau manifold $\pi : X \rightarrow B$ with general fiber given by an algebraic curve \mathcal{C} of genus-one. \mathcal{C} is a non-singular curve defined over a field K that is not necessarily algebraically closed. In particular, we can think of the fibration X as an algebraic curve \mathcal{C} defined over the field K of meromorphic functions on the base B , which is clearly in general not algebraically closed. Thus, there are two qualitatively different situations to consider.

Curves with points

First, if the curve \mathcal{C} has points with coordinates in K , then it is called an elliptic curve, which we denote by \mathcal{E} , and X is called an elliptic fibration. The points on \mathcal{E} form an Abelian group under addition: one point can be chosen as the zero point, denoted by P_0 , and the additional points P_m , $m = 1, \dots, r$, (more precisely the differences $P_m - P_0$) are the generators of the Mordell-Weil group of rational points of \mathcal{E} . The Mordell-Weil theorem states that this group is finitely generated [71, 72]. Thus, it splits into a free part isomorphic to \mathbb{Z}^r and a torsion subgroup, where the latter has been fully classified for $K = \mathbb{Q}$ by Mazur [73, 74], see [71] for a review.¹¹ Every point on \mathcal{E} gives rise to a section of the fibration X , i.e. rational maps from the base B into X . The section associated to P_0 is the zero section, denoted by $\hat{s}_0 : B \rightarrow X$, and the r rational points P_m induce the rational sections $\hat{s}_m : B \rightarrow X$. The set $\{\hat{s}_m\}$ can be seen to form a group, the MW-group of rational sections of X , by defining the addition of rational sections by addition of their corresponding points on \mathcal{E} . The free part of the MW-group gives rise to Abelian gauge symmetry in F-theory [2] and its torsion part yields non-simply connected gauge groups [53], see also [46] for a recent discussion of MW-torsion.

¹¹For the field of meromorphic functions on B , there are more torsion subgroups possible than for $K = \mathbb{Q}$ [53].

Every elliptic fibration X can be written in Weierstrass form (WSF) [75], i.e. as a hypersurface in the weighted projective bundle $\mathbb{P}^{(1,2,3)}(\mathcal{O}_B \oplus \mathcal{L}^2 \oplus \mathcal{L}^3)$ over B of the form

$$y^2 = x^3 + fxz^4 + gz^6. \tag{2.1}$$

Here, \mathcal{O}_B is the trivial bundle on B and the line-bundle \mathcal{L} is fixed by the Calabi-Yau condition of X as $\mathcal{L} = K_B^{-1}$, with K_B denoting the canonical bundle of the base B . Then, the coefficients f and g have to be sections of K_B^{-4} and K_B^{-6} , respectively. The map from the canonical presentation of X inherited from the canonical presentation of the elliptic curve \mathcal{E} to the Weierstrass form (2.1) is birational. The zero section \hat{s}_0 of X maps to the holomorphic zero section $[z : x : y] = [0 : \lambda^2 : \lambda^3]$ in (2.1) and the rational sections \hat{s}_m map to rational sections in (2.1) with certain coordinates $[z_{P_m} : x_{P_m} : y_{P_m}]$, that are rational expressions in K (we can clear denominators to obtain holomorphic coordinates).

Curves without points

Second, if the genus-one curve \mathcal{C} does not have a point, the fibration $\pi : X \rightarrow B$ is without section. Such a fibration is referred to as a genus-one fibration [54]. Given a genus-one curve \mathcal{C} , one can construct an associated elliptic curve $\mathcal{E} = J(\mathcal{C})$, that is the Jacobian of the curve \mathcal{C} , i.e. the group of degree zero line bundles on \mathcal{C} . The zero point on $J(\mathcal{C})$ is the trivial line bundle. Thus, there exists an elliptic fibration $\pi : J(X) \rightarrow B$ associated to X with general fiber given by $J(\mathcal{C})$. This implies that $J(X)$ can be represented as a Weierstrass model (2.1). Furthermore, it is a key property for F-theory that the τ -function and the discriminant of X and $J(X)$ are identical [54].

In this work, we consider concrete genus-one curves \mathcal{C} with K -rational divisors of degree $n > 1$, respectively.¹² The corresponding fibration X does not have a section, but an n -section, that we denote by $\hat{s}^{(n)} : B \rightarrow X$. Locally, at a point p on B , the function field K reduces to \mathbb{C} and the n -section $\hat{s}^{(n)}$ maps p to n points in the fiber \mathcal{C} . Globally, however, upon moving around branch loci in B the individual points are exchanged by a monodromy action, so that only the collection of all n points together induce a well-defined divisor in X .

As we will see explicitly for concrete genus-one fibrations X , the map from X to the Weierstrass form (2.1) of $J(X)$ can be obtained by an algebraic field extension L of K . This field extension is only necessary as an intermediate step, i.e. the final WSF (2.1) of $J(X)$ is again defined over K . In sections 3.2.1, 3.2.2 and 3.2.3 we explicitly work out the maps from X to the WSF of their Jacobian fibrations $J(X)$, namely for the fibration of the cubic in \mathbb{P}^2 , that has a three-section, and the two fibrations of the biquadric curve in $\mathbb{P}^1 \times \mathbb{P}^1$ and the quartic curve in $\mathbb{P}^2(1, 1, 2)$, that both have a two-section.¹³ We note that genus-one fibrations X by the quartic in $\mathbb{P}^2(1, 1, 2)$ have been considered recently in an F-theory context in [54–56].

¹²We expect that there always exists a degree n divisor on a given algebraic genus-one curve \mathcal{C} .

¹³These examples have been considered in the mathematics literature in [76].

2.2 Divisors on genus-one fibrations and their intersections

In F-theory we are particularly interested in Calabi-Yau manifolds X that arise as a crepant resolution of singular genus-one or elliptic fibrations. These resolved manifolds exhibit three different classes of divisors, that we discuss in the following.

The first set of divisors on X is formed by the vertical divisors, i.e. divisors that arise as pullbacks of divisors on B under the projection map $\pi : X \rightarrow B$. Hence, there are $h^{(1,1)}(B)$ such divisors on X . We denote the preimage under π of a vertical divisor D on X by D^b so that $D = \pi^*(D^b)$. Thus, D is a fibration of the curve \mathcal{C} (or its degenerations) over the base D^b .

The second class of divisors are the exceptional divisors of X . In more detail, if the discriminant $\Delta = -16(4f^3 + 27g^2)$ of the WSF (2.1) of X or of its Jacobian $J(X)$ vanishes to order higher than 1 at one of its irreducible components

$$\mathcal{S}_{G_I}^b := \{\Delta_I = 0\}, \quad I = 1, \dots, N, \quad (2.2)$$

in B , then the total space of the WSF is singular. These codimension one singularities are classified in [59, 60]. In the resolution X the fiber over each $\mathcal{S}_{G_I}^b$ splits into a number of rational curves whose intersection pattern often agrees with the Dynkin diagram of a Lie group G_I .¹⁴ The shrinkable irreducible components of the fiber at $\mathcal{S}_{G_I}^b$ represent the simple roots of G_I and are denoted by $c_{-\alpha_i}^{G_I}$ for $i = 1, \dots, \text{rk}(G_I)$. Thus, X has a set of exceptional divisors $D_i^{G_I}$ given as the fibration of $c_{-\alpha_i}^{G_I}$ over $\mathcal{S}_{G_I}^b$ for every I , to which we refer to as Cartan divisors of G_I . The $D_i^{G_I}$ intersect the curves $c_{-\alpha_i}^{G_I}$ as

$$D_i^{G_I} \cdot c_{-\alpha_j}^{G_I} = -C_{ij}^{G_I} \delta_{IJ}, \quad (2.3)$$

where $C_{ij}^{G_I}$ denotes the Cartan matrix of G_I . The F-theory gauge group is then given by the product of all G_I , as discussed in section 2.3.

Finally, the third class of divisors is induced by the independent sections and n -sections of the fibration of X . We denote the divisor classes of the zero section \hat{s}_0 and the generators of the MW-group of rational sections \hat{s}_m by S_0 and S_m , respectively. The class of a multi-section $\hat{s}^{(n)}$ is denoted by $S^{(n)}$. Then, the intersections of these divisors with the fiber $f \cong \mathcal{C}$, \mathcal{E} read

$$S_0 \cdot f = S_m \cdot f = \frac{1}{n} S^{(n)} \cdot f = 1. \quad (2.4)$$

The divisor classes that support Abelian gauge fields in F-theory [34, 39, 77], see also [22, 31, 78], are obtained from the Shioda map σ of the rational sections \hat{s}_m . To a given generator of the MW-group \hat{s}_m the Shioda map assigns the divisor

$$\sigma(\hat{s}_m) := S_m - S_0 + [K_B] - \pi(S_m \cdot S_0) + \sum_{I=1}^N (S_m \cdot c_{-\alpha_i}^{G_I}) (C_{G_I}^{-1})^{ij} D_j^{G_I}. \quad (2.5)$$

Here $\pi(\cdot)$ denotes the projection of a codimension two variety in X to a divisor in the base B and $[K_B]$ is the canonical bundle of B . The last term encodes contributions from

¹⁴The fibers that do not have an associated group G_I are the unconventional fibers in table 2 of [61].

non-Abelian gauge groups G_I in F-theory with $(C_{G_I}^{-1})^{ij}$ denoting the inverse of the Cartan matrix $C_{ij}^{G_I}$.

The Shioda map (2.5) enables us to define the Néron-Tate height pairing of two rational sections \hat{s}_m, \hat{s}_n as

$$\begin{aligned} \pi(\sigma(\hat{s}_m) \cdot \sigma(\hat{s}_n)) &= \pi(S_m \cdot S_n) + [K_B] - \pi(S_m \cdot S_0) - \pi(S_n \cdot S_0) \\ &\quad + \sum_I (C_{G_I}^{-1})^{ij} (S_m \cdot c_{-\alpha_i}^{G_I})(S_n \cdot c_{-\alpha_j}^{G_I}) \mathcal{S}_{G_I}^b, \end{aligned} \quad (2.6)$$

where $C_{ij}^{G_I}$ is the coroot matrix of G_I . We note that for evaluating this pairing in a concrete situation, the universal intersection relations

$$\pi(S_P^2 + [K_B^{-1}] \cdot S_P) = \pi(S_m^2 + [K_B^{-1}] \cdot S_m) = 0 \quad (2.7)$$

prove useful (cf. [35, 39, 42, 77] for details), whereas $\pi(S_m \cdot S_n)$ and $\pi(S_m \cdot S_0)$ are model-dependent.

We note that in F-theory both the vertical divisor (2.2) and the matrix (2.6) of vertical divisors enter the coefficients of the Green-Schwarz terms [22, 34, 35] and are thus essential for anomaly cancellation, cf. appendix A for details.

2.3 The spectrum of F-theory on genus-one fibrations

After the geometric preludes of sections 2.1 and 2.2, we are prepared to extract the spectrum of F-theory on a genus-one fibration X . The following discussion applies most directly to F-theory compactifications to 6D with effective theory given by an $\mathcal{N} = (1, 0)$ SUGRA theory. However certain statements directly generalize to 4D F-theory vacua without G_4 -flux.

For a more detailed derivation of some of the following results, that oftentimes require M-/F-theory duality, we refer [78–80] and references therein.

Codimension one singularities

All vector fields and certain hyper multiplets in F-theory arise from the singularities of the WSF of X that are induced by codimension one singularities of its fibration. Over a given irreducible discriminant component $\mathcal{S}_{G_I}^b$ defined in (2.2), the fiber of X is reducible. We assume that there is a Lie group G_I associated to this codimension one fiber of X . Then, the shrinkable holomorphic curves $c_{-\alpha}^{G_I}$ in the fiber over $\mathcal{S}_{G_I}^b$ represent all the positive roots of G_I . By quantization of the moduli space of an M2-brane wrapping such a curve $c_{-\alpha}^{G_I}$ one finds BPS-states transforming in one charged vector multiplet and $2g_I$ charged half-hyper multiplets with charge-vector $-\alpha$ [81, 82]. Another vector multiplet and $2g_I$ half-hyper multiplets with charges $+\alpha$ are contributed by an M2-brane wrapping $c_{-\alpha}^{G_I}$ with the opposite orientation. Here g_I is the genus of the curve $\mathcal{S}_{G_I}^b$ in B , that is computed as

$$g_I = 1 + \frac{1}{2} \mathcal{S}_{G_I}^b \cdot (\mathcal{S}_{G_I}^b + [K_B]). \quad (2.8)$$

All these charged states become massless in the F-theory limit, when the volume of the class of the genus-one fiber of X is taken to zero. In this limit, these BPS-states fall into representations of the group G_I as follows.

First, we focus on the vector multiplets. All vector multiplets for every root α of G_I are completed into one massless vector multiplet transforming in the adjoint representation $\mathbf{adj}(G_I)$ of G_I . The additional vector multiplets are provided by the KK-reduction of the M-theory three-form C_3 along the harmonic $(1,1)$ -forms in X that are dual to the Cartan divisor $D_I^{G_I}$ of G_I . Thus, every irreducible component (2.2) of the discriminant with respective codimension one fiber classified by a Lie group G_I , $I = 1, \dots, N$, gives rise to a G_I gauge symmetry in F-theory [2, 3, 61]. Furthermore, if X has a MW-group of rank r , there are additional $(1,1)$ -forms on X , which are the duals of the divisors (2.5), that give rise to vector multiplets of Abelian gauge groups [3]. Thus, the total gauge group G_X of F-theory on X is

$$G_X = \mathrm{U}(1)^r \times \prod_{I=1}^N G_I. \tag{2.9}$$

This discussion and the results of section 2.2 imply further that the rank of G_X can be directly computed in terms of the Hodge numbers $h^{(1,1)}(X)$ and $h^{(1,1)}(B)$ of X and B , respectively, as

$$\mathrm{rk}(G_X) = h^{(1,1)}(X) - h^{(1,1)}(B) - 1. \tag{2.10}$$

These results (2.9) and (2.10) hold in compactifications to eight,¹⁵ six and four dimensions.

Second, we turn to the massless half-hyper multiplets over $\mathcal{S}_{G_I}^b$. In fact, also these fields are completed into the adjoint representation $\mathbf{adj}(G_I)$ of G_I . In order to see this, we first note that there are neutral hyper multiplets induced by the complex structure moduli of X . Their total number, denoted by H_{neut} , is computed by the Hodge number $h^{(2,1)}(X)$ (or equivalently the Euler number $\chi(X)$) of X as

$$H_{\mathrm{neut}} = h^{(2,1)}(X) + 1 = h^{(1,1)}(B) + 2 + \mathrm{rk}(G_X) - \frac{1}{2}\chi(X). \tag{2.11}$$

Then, for every group G_I , the $2g_I$ half-hyper multiplets with charges $-\alpha$ for all roots of G_I combine with $g_I \cdot \mathrm{rk}(G_I)$ neutral hyper multiplets from (2.11) into g_I hyper multiplets in the adjoint $\mathbf{adj}(G_I)$ of G_I . Thus, the number of hyper multiplets transforming in $\mathbf{adj}(G_I)$ is given by (2.8) for every group G_I .

Let us emphasize that this discussion implies that $h^{(2,1)}(X)$ contains information about the gauge groups G_I of X . Furthermore, also parts of the charged matter content from codimension two fibers are counted by $h^{(2,1)}(X)$, however of *another* theory related to the considered one by Higgsing. In section 4, for F-theory compactified on all toric hypersurface fibrations X , we identify the part of $h^{(2,1)}(X)$ that comes from matter fields in all theories related by Higgsing.

Codimension two singularities

The rest of the charged spectrum of F-theory on X is encoded in the codimension two singularities of the WSF of X .

Non-Abelian charged matter is located at loci in $\mathcal{S}_{G_I}^b$, where the vanishing order of the discriminant Δ of X enhances. These loci are typically complete intersections of $\mathcal{S}_{G_I}^b$ with

¹⁵In 8D vacua, no non-split fibers are possible, i.e. all gauge groups are of *ADE*-type [61].

another divisor in B , that can be read off from Δ . The fiber of X at these codimension two loci contains additional shrinkable rational curves c that are not present in codimension one. These curves correspond to the weights of a representation $\mathbf{R}_{\underline{q}}$, under the gauge group G_X in (2.9), where $\underline{q} = (q_1, \dots, q_r)$ denotes the vector of $U(1)$ -charges. The Dynkin labels $\lambda_i^{G_I}$ of \mathbf{R} are computed according to

$$\lambda_i^{G_I} = D_i^{G_I} \cdot c, \quad (2.12)$$

and the m^{th} $U(1)$ -charge q_m is computed using (2.5) as [34, 77]

$$q_m = \sigma(\hat{s}_m) \cdot c = (S_m \cdot c) - (S_0 \cdot c) + \sum_I (S_m \cdot c_{-\alpha_I}) (C_{(I)}^{-1})^{i_I j_I} (D_{j_I} \cdot c). \quad (2.13)$$

We emphasize that these charges are automatically quantized, but not necessarily integers due to the usually fractional contribution from the last term in (2.13).

We note that in the presence of $U(1)$'s, we automatically have additional matter that does not originate from intersections of codimension one discriminant components. In fact, the WSF of X automatically has codimension two singularities for every rational section \hat{s}_m with coordinates $[z_{P_m} : x_{P_m} : y_{P_m}]$ at the following locus in B :

$$y_{P_m} = fz_{P_m}^4 + x_{P_m}^2 = 0, \quad m = 1, \dots, r. \quad (2.14)$$

This can be seen by inserting $[z_{P_m} : x_{P_m} : y_{P_m}]$ into (2.1), which implies a relation between f and g which allows for a factorization of (2.1) that reveals the presence of conifold singularities in the WSF of X precisely at (2.14), see e.g. [39, 77] for details. In the crepant resolution X , there is a reducible I_2 -fiber with one isolated rational curve at the codimension two loci (2.14). The matter at the loci (2.14) are charged singlets $\mathbf{1}_{\underline{q}}$ with their $U(1)$ -charges computed according to (2.13). This is clear as generically (2.14) does not intersect any discriminant component, which are the loci where the Cartan divisors $D_i^{G_I}$ are supported, so that (2.12) is trivial.

In concrete applications, the complete intersection (2.14) describes a reducible variety in B supporting multiple singlets with different charges. Matter at a generic point of (2.14) has $U(1)$ -charge one, whereas matter at non-generic points, i.e. points along which other, oftentimes simpler constraints vanish, too, has different $U(1)$ -charges. From a technical point of view, we are interested in all *prime ideals*, denoted throughout the paper by $I_{(k)}$, of the loci (2.14) for every m . These are obtained by a primary decomposition of the complete intersection (2.14), cf. [42] for details. The codimension two variety in B associated to $I_{(k)}$ is denoted by $V(I_{(k)})$, which is the standard notation in algebraic geometry for an algebraic set, i.e. the set of points in B so that all constraints in $I_{(k)}$ vanish. Then, we explicitly analyze the I_2 -fibers of X over all these irreducible varieties $V(I_{(k)})$ in order to compute the respective $U(1)$ -charges via (2.13).

In general, the multiplicity of matter in the representation $\mathbf{R}_{\underline{q}}$ is given by the homology class of the corresponding codimension two locus in B . If the base B is two-dimensional, which is the case in compactifications to 6D, this is just a set of points and the multiplicity is the number of these points. In F-theory compactifications to 4D, the homology class of a codimension two locus is the class of the corresponding matter curve.

More specifically, the multiplicity of non-Abelian charged matter is computed easily as the homology class of the complete intersection with $\mathcal{S}_{G_I}^b$. However, the determination of the multiplicity of singlets $\mathbf{1}_q$ is more involved since they are located on the varieties $V(I_{(k)})$ associated to the usually very complex prime ideals $I_{(k)}$ of the complete intersection (2.14). The respective matter multiplicities are then again given as the homology class of the variety $V(I_{(k)})$. It can be computed by the following procedure, see [39, 42, 44] for more details: we first compute the homology class of the reducible complete intersection (2.14). Given the list of its associated prime ideals $I_{(k)}$, we then subtract the multiplicities (homology classes) of those matter loci $V(I_{(k')})$, $\{k'\} \subset \{k\}$, we already know. Here we have to take into account the order $n_{k'}$ of the matter locus $V(I_{(k')})$ inside the complete intersection (2.14). The order $n_{k'}$ is computed using the *resultant technique* [39]. In all the cases considered below in section 3, this strategy yields the homology classes of all singlets $\mathbf{1}_q$.

In summary, the 6D $\mathcal{N} = (1, 0)$ SUGRA theory obtained by compactifying F-theory on a Calabi-Yau threefold X has

- a total number of vector multiplets V reading

$$V = \mathbf{adj}(G_X) = \sum_I \dim(\mathbf{adj}(G_I)) + r, \tag{2.15}$$

where $\mathbf{adj}(G_X)$ and $\mathbf{adj}(G_I)$ denote the adjoint representations of G_I and G_X , respectively, and r denotes the MW-rank of X ,

- a total number of hyper multiplets H given by

$$\begin{aligned} H &= H_{\text{codim}=2} + H_{\text{codim}=1} + H_{\text{mod}} \\ &= H_{\text{codim}=2} + \sum_{I=1}^n g_I (\dim(\mathbf{adj}(G_I)) - \text{rk}(G_I)) + h^{(2,1)}(X) + 1, \end{aligned} \tag{2.16}$$

where we split into contributions $H_{\text{codim}=2}$, $H_{\text{codim}=1}$ and H_{mod} from codimension two fibers, from codimension one fibers over higher genus Riemann surfaces in B and from complex structure moduli of X (plus 1), respectively,

- and a number of tensor multiplets T counted by

$$T = h^{(1,1)}(B) - 1 = 9 - [K_B^{-1}]^2. \tag{2.17}$$

For the second equality in (2.17) we have employed the identity

$$[K_B^{-1}] \cdot [K_B^{-1}] = \int_B c_1(B)^2 = 10 - h^{(1,1)}(B). \tag{2.18}$$

Here we used in the last equality the Euler number $\chi(B) = (2 + h^{(1,1)}(B))$ of a simply-connected base B with $h^{(2,0)} = 0$ and the index formula for the arithmetic genus $\chi_0(B) = 1$,

$$1 = \chi_0(B) = \frac{1}{12} \int (c_2(B) + c_1(B)^2) = \frac{1}{12} \left(2 + h^{(1,1)}(B) + \int_B c_1(B)^2 \right), \tag{2.19}$$

where $c_i(B)$, $i = 1, 2$, denote the Chern classes of B .

Codimension three singularities

For completeness we note that codimension three singularities of the WSF of a Calabi-Yau fourfold X support Yukawa points in F-theory compactifications to 4D. The codimension three singularities are at the points in the threefold base B of further enhancement of the vanishing order of the discriminant Δ . All such enhancement points are given as intersections of three matter curves in B , including self-intersections. Technically, given three matter curves $V(I_{(1)})$, $V(I_{(2)})$ and $V(I_{(3)})$ we have to check that the variety $V(I_{(1)}) \cap V(I_{(2)}) \cap V(I_{(3)})$ contains a codimension three component in B . This is achieved by checking that the ideal $I_{(1)} \cup I_{(2)} \cup I_{(3)}$ is codimension three in the ring of appropriate polynomials on B , where we used the well-known equality $\bigcap_k V(I_{(k)}) = V(\bigcup_k I_{(k)})$ for a family of algebraic sets $V(I_{(k)})$ [83].

As we see in section 3, all gauge-invariant Yukawa couplings are realized for the case of toric hypersurface fibrations X_{F_i} .

2.4 Explicit examples: Calabi-Yau hypersurfaces in 2D toric varieties

All Calabi-Yau manifolds X considered in this work are constructed as fibrations of genus-one curves \mathcal{C} that have a natural presentation as hypersurfaces in 2D toric varieties. These fibrations are automatically smooth, if the toric ambient spaces of the fiber \mathcal{C} are fully resolved. In this section we present a very brief account on the construction of Calabi-Yau hypersurfaces in 2D toric varieties that are the basis for the rest of this work. For a more complete account, we refer to standard text books [84, 85].

A toric almost Fano surface is associated to each of the 16 two-dimensional reflexive polyhedra F_i , $i = 1, \dots, 16$, in a lattice $N = \mathbb{Z}^2$.¹⁶ These 16 reflexive polyhedra are given in a convenient presentation in figure 2 [17]. As indicated there, the polyhedra F_i and F_{17-i} for $i = 1, \dots, 6$ are dual to each other, $F_i^* = F_{17-i}$, and the F_i for $i = 7, \dots, 10$ are self-dual, $F_i = F_i^*$, where the dual polyhedron F_i^* is defined in the dual lattice $M = \mathbb{Z}^2$ of N as

$$F_i^* = \{q \in M \otimes \mathbb{R} \mid \langle y, q \rangle \geq -1, \forall y \in F_i\}, \tag{2.20}$$

where $\langle \cdot, \cdot \rangle$ is the pairing between N and M .

For a given polyhedron F_i , we denote the associated toric variety by \mathbb{P}_{F_i} . Toric varieties are generalizations of weighted projective spaces [86]: to each integral point v_k , $k = 1, \dots, m + 2$, except the origin of F_i , we associate a coordinate x_k in \mathbb{C} . Next, we introduce the lattice of relations between the v_k with generators $\ell^{(a)}$ defined by

$$\sum_{k=1}^{m+2} \ell_k^{(a)} v_k = 0, \quad a = 1, \dots, m. \tag{2.21}$$

Then, a smooth toric variety \mathbb{P}_{F_i} is defined as the $(\mathbb{C}^*)^m$ -quotient

$$\mathbb{P}_{F_i} = \frac{\mathbb{C}^{m+2} \setminus \text{SR}}{(\mathbb{C}^*)^m} = \left\{ x_k \sim \prod_{a=1}^m \lambda_a^{\ell_k^{(a)}} x_k \mid \underline{x} \notin \text{SR}, \lambda_a \in \mathbb{C}^* \right\}, \tag{2.22}$$

where the points $\underline{x} := (x_1, \dots, x_{m+2})$ are not allowed to lie in the Stanley-Reisner ideal SR.

¹⁶We refrain from the common notation Δ for a polyhedron in order to avoid confusion with the discriminant.

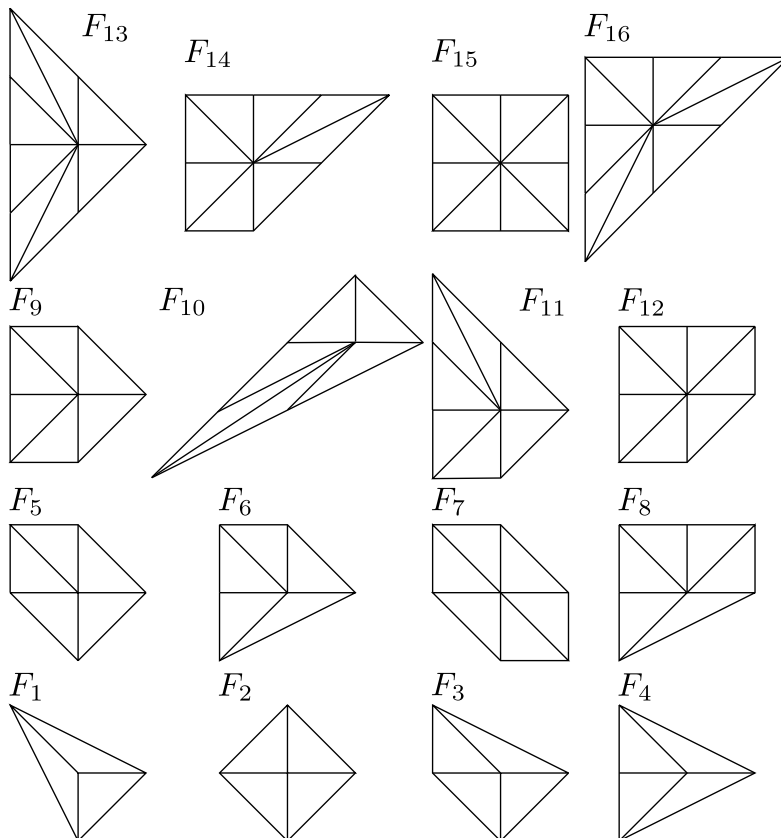


Figure 2. The 16 two dimensional reflexive polyhedra [17]. The polyhedron F_i and F_{17-i} are dual for $i = 1 \dots 6$ and self-dual for $i = 7 \dots 10$.

The construction (2.22) provides a dictionary between the combinatorics of the polyhedron F_i and the geometry of \mathbb{P}_{F_i} . For example, the toric divisor group on \mathbb{P}_{F_i} is generated by the divisors $D_k = \{x_k = 0\}$ and the intersections of the D_k are encoded in the SR ideal. A full basis of the divisor group on \mathbb{P}_{F_i} can be obtained using the linear equivalences between the D_k . Due to the relevance for the smoothness of a toric hypersurface fibration X , we stress here that points that are not vertices in F_i correspond to exceptional divisors resolving orbifold singularities in \mathbb{P}_{F_i} .

The polyhedra F_1, F_3, F_5, F_7 describe the generic del Pezzo surfaces \mathbb{P}^2 and dP_i , $i = 1, 2, 3$, respectively, F_2 yields $\mathbb{P}^1 \times \mathbb{P}^1$, F_4 describes $\mathbb{P}^2(1, 1, 2)$ and F_{10} yields $\mathbb{P}^2(1, 2, 3)$. In fact all other toric varieties \mathbb{P}_{F_i} can be viewed as higher del Pezzo surfaces at a special point in their respective complex structure moduli spaces.

Every toric variety \mathbb{P}_{F_i} has an associated Calabi-Yau hypersurface, i.e. a genus-one curve \mathcal{C}_{F_i} . It is defined as the generic section of its anti-canonical bundle $K_{\mathbb{P}_{F_i}}^{-1}$. The Calabi-Yau hypersurface in \mathbb{P}_{F_i} is obtained by the Batyrev construction as the following polynomial [87]

$$p_{F_i} = \sum_{q \in F_i^* \cap M} a_q \prod_k x_k^{\langle v_k, q \rangle + 1}, \quad (2.23)$$

where q denotes all integral points in F_i^* and the a_q are coefficients in the field K .

We note that points v_i interior to edges in F_i are usually excluded in the product (2.23) because the corresponding divisors do not intersect the hypersurface \mathcal{C}_{F_i} . However, when considering Calabi-Yau fibrations X_{F_i} of \mathcal{C}_{F_i} , as in section 3, these divisors intersect X_{F_i} and resolve singularities of X_{F_i} induced by singularities of its fibration, i.e. these divisors are related to Cartan divisors D_i^{GI} discussed above in section 2.2.

3 Analysis of F-theory on toric hypersurface fibrations

In this section we analyze the geometry of the Calabi-Yau manifolds X_{F_i} , that are constructed as fibration of the genus-one curves \mathcal{C}_{F_i} over a generic base B . For each manifold we calculate the effective theory resulting from compactifying F-theory on it. We calculate the gauge group, the charged and neutral matter spectrum and the Yukawa couplings.

We start with a quick summary of some interesting results of this study. There are three polyhedra leading to manifolds X_{F_i} without a section: F_1 , F_2 and F_4 , see sections 3.2.1, 3.2.2 and 3.2.3, respectively. They yield the discrete gauge groups in F-theory. For three polyhedra we find associated gauge groups with Mordell-Weil torsion, giving rise to non-simply connected gauge groups: F_{13} , F_{15} and F_{16} , see sections 3.6.1, 3.6.2 and 3.6.3, respectively. The analysis of the hypersurface X_{F_3} and the corresponding effective theory of F-theory, whose spectrum contains a charged singlet with U(1)-charge three, can be found in section 3.3.1.

We obtain the following list of gauge groups G_{F_i} of F-theory on the X_{F_i} :

| | | | | | |
|-----------|-----------------------------|--------------|----------------------------------|--------------|---|
| G_{F_1} | \mathbb{Z}_3 | G_{F_7} | $U(1)^3$ | | |
| G_{F_2} | $U(1) \times \mathbb{Z}_2$ | G_{F_8} | $SU(2)^2 \times U(1)$ | $G_{F_{13}}$ | $(SU(4) \times SU(2)^2) / \mathbb{Z}_2$ |
| G_{F_3} | $U(1)$ | G_{F_9} | $SU(2) \times U(1)^2$ | $G_{F_{14}}$ | $SU(3) \times SU(2)^2 \times U(1)$ |
| G_{F_4} | $SU(2) \times \mathbb{Z}_4$ | $G_{F_{10}}$ | $SU(3) \times SU(2)$ | $G_{F_{15}}$ | $SU(2)^4 / \mathbb{Z}_2 \times U(1)$ |
| G_{F_5} | $U(1)^2$ | $G_{F_{11}}$ | $SU(3) \times SU(2) \times U(1)$ | $G_{F_{16}}$ | $SU(3)^3 / \mathbb{Z}_3$ |
| G_{F_6} | $SU(2) \times U(1)$ | $G_{F_{12}}$ | $SU(2)^2 \times U(1)^2$ | | |

From this and as a simple consequence of (2.10), we see that there is the following rule of thumb for computing the rank of a gauge group G_{F_i} : given a polyhedron F_i with $3 + n$ integral points without the origin, we have a gauge group with total rank n .

Let us outline the structure of this section. In the first subsection 3.1 we briefly discuss the three different representations of genus-one curves \mathcal{C}_{F_i} realized as toric hypersurfaces: the cubic, the biquadric and the quartic. There, we define the line bundles of the base B in which the coefficients in these constraints have to take values in order to obtain a genus-one fibered Calabi-Yau manifold. The functions f and g of the Weierstrass form (2.1) for the cubic, the biquadric and the quartic can be found in appendix B. By appropriate specializations of the coefficients, one can obtain f , g and $\Delta = 4f^3 + 27g^2$ for all toric hypersurface fibration X_{F_i} .

In sections 3.2 to 3.6 we proceed to describe in detail each Calabi-Yau manifold X_{F_i} . In each case we first discuss the genus-one curve \mathcal{C}_{F_i} realized as a toric hypersurface in \mathbb{P}_{F_i} .

We then construct the corresponding toric hypersurface fibration X_{F_i} and analyze its codimension one and two singularities from which we extract the non-Abelian gauge group and matter spectrum. If X_{F_i} has a non-trivial MW-group, we determine all its generators, their Shioda maps and the height pairing. For genus-one fibrations, we determine their discrete gauge groups. For completeness, we also determine the Yukawa couplings from codimension three singularities. In each case we show as a consistency check that the necessary 6D anomalies (pure Abelian, gravitational-Abelian, pure non-Abelian, non-Abelian gravitational, non-Abelian-Abelian and purely gravitational) are canceled implying consistency of the considered effective theories.

We organize the Calabi-Yau manifolds X_{F_i} into five categories: those with discrete gauge symmetries (section 3.2), those with a gauge group of rank one and two but without discrete gauge groups (section 3.3), those with a gauge group of rank three, whose fiber polyhedra happen to be also self-dual (section 3.4), those with gauge groups of rank four and five without MW-torsion (section 3.5) and those X_{F_i} with MW-torsion (section 3.6). This arrangement is almost in perfect agreement with the labeling of the polyhedra F_i in figure 2 which facilitates the navigation through this section. We name the subsection containing the analysis of the specific manifold X_{F_i} by its corresponding fiber polyhedron F_i .

3.1 Three basic ingredients: the cubic, biquadric and quartic

3.1.1 Constructing toric hypersurface fibrations

In this section we explain the general construction of the Calabi-Yau manifolds X_{F_i} with toric hypersurface fiber \mathcal{C}_{F_i} and base B . The following discussion applies to Calabi-Yau n -folds X_{F_i} with a general $(n - 1)$ -dimensional base B . The cases of most relevance for F-theory and for this work are $n = 3, 4$. We refer to [39, 42] for more details on the following discussion.

The starting point of the construction of the genus-one fibered Calabi-Yau manifold X_{F_i} is the hypersurface equation (2.23) of the curve \mathcal{C}_{F_i} . In order to obtain the equation of X_{F_i} , the coefficients a_q and the variables x_i of (2.23) have to be promoted to sections of appropriate line bundles of the base B . We determine these line bundles, by first constructing a fibration of the 2D toric variety \mathbb{P}_{F_i} , which is the ambient space of \mathcal{C}_{F_i} , over the same base B ,

$$\begin{array}{ccc} \mathbb{P}_{F_i} & \longrightarrow & \mathbb{P}_{F_i}^B(D, \tilde{D}) . \\ & & \downarrow \\ & & B \end{array} \tag{3.1}$$

Here $\mathbb{P}_{F_i}^B(D, \tilde{D})$ denotes the total space of this fibration. The structure of its fibration is parametrized by two divisors in B , denoted by D and \tilde{D} . This can be seen by noting that all $m + 2$ coordinates x_k on the fiber \mathbb{P}_{F_i} are in general non-trivial sections of line bundles on B . Then, we can use the $(\mathbb{C}^*)^m$ -action of the toric variety \mathbb{P}_{F_i} to set m variables to transform in the trivial bundle of B . The divisors dual to the two remaining line bundles are precisely D, \tilde{D} .

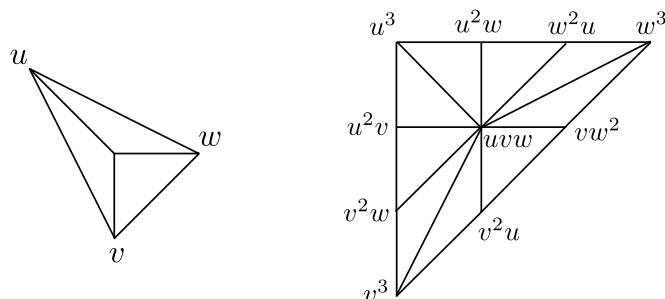


Figure 3. Polyhedron F_1 with choice of projective coordinates and its dual with corresponding monomials.

Next we impose equation (2.23) in $\mathbb{P}_{F_i}(D, \tilde{D})$. Consistency fixes the line bundles in which the coefficients a_q have to take values in terms of the two divisors D and \tilde{D} . Then, we require (2.23) to be a section of the anti-canonical bundle $K_{\mathbb{P}_{F_i}^B}^{-1}$, which is the Calabi-Yau condition. In addition, equation (2.23) imposed in $\mathbb{P}_{F_i}^B(D, \tilde{D})$ clearly describes a genus-one fibration over B , since for every generic point on B , the hypersurface (2.23) describes exactly the curve \mathcal{C}_{F_i} in \mathbb{P}_{F_i} . The total Calabi-Yau space resulting from the fibration of the toric hypersurface \mathcal{C}_{F_i} is denoted by X_{F_i} in the following. It enjoys the fibration structure

$$\begin{array}{ccc} \mathcal{C}_{F_i} & \longrightarrow & X_{F_i} \cdot \\ & & \downarrow \\ & & B \end{array} \quad (3.2)$$

In principle, this procedure has to be carried out for all Calabi-Yau manifolds X_{F_i} associated to the 16 2D toric polyhedra F_i . However, we observe that all the hypersurface constraints of the X_{F_i} , except for X_{F_2} and X_{F_4} , can be obtained from the hypersurface constraint for X_{F_1} , after setting appropriate coefficients to zero. This is possible because if F_1 is a sub-polyhedron of F_i , then the corresponding toric variety \mathbb{P}_{F_i} is the blow-up of $\mathbb{P}_{F_1} = \mathbb{P}^2$ at a given number of points, with the additional rays in F_i corresponding to the blow-up divisors. However, adding rays to the polyhedron F_1 removes rays from its dual polyhedron $F_1^* = F_{16}$. By means of (2.23), this removes coefficients from hypersurface equation for X_{F_1} , i.e. the hypersurface for X_{F_i} is a certain specialization of the hypersurface of X_{F_1} with some $a_q \equiv 0$. We will be more explicit about this in the following subsection (section 3.1.2).

Thus, we only have to explicitly carry out the construction of the toric hypersurfaces separately for the two Calabi-Yau manifolds X_{F_2} and X_{F_4} . The details of this are given in sections 3.1.3 and 3.1.4.

3.1.2 Fibration by cubic curves: X_{F_1} and its specializations

We proceed to construct the Calabi-Yau manifold $\mathcal{C}_{F_1} \rightarrow X_{F_1} \rightarrow B$ with fiber given by the curve \mathcal{C}_{F_1} in the toric variety \mathbb{P}_{F_1} . In addition, we argue how the Calabi-Yau manifolds X_{F_i} , whose fiber polyhedron F_i contains F_1 , can be obtained from X_{F_1} .

The polyhedron F_1 and its dual are shown in figure 3. The toric variety \mathbb{P}_{F_1} , constructed using (2.22), is the well-known projective space \mathbb{P}^2 . We introduce the projective coordinates $[u : v : w]$ on \mathbb{P}^2 . In terms of these coordinates, we can read off the SR-ideal from figure 3 as

$$SR = \{uvw\}. \tag{3.3}$$

The divisor group of \mathbb{P}^2 is generated by the hyperplane class H . The Calabi-Yau onefold in \mathbb{P}^2 is the degree three \mathcal{C}_{F_1} in $3H$. Its defining equation, constructed using (2.23) and figure 3, is the most general cubic

$$p_{F_1} = s_1u^3 + s_2u^2v + s_3uv^2 + s_4v^3 + s_5u^2w + s_6uvw + s_7v^2w + s_8uw^2 + s_9vw^2 + s_{10}w^3, \tag{3.4}$$

where the coefficients s_i take values in the field K .

Next, we follow the discussion of section 3.1.1 to construct the toric hypersurface fibration X_{F_1} . We first construct the ambient space (3.1), which in the case at hand is a \mathbb{P}^2 -fibration over the base B ,

$$\begin{array}{ccc} \mathbb{P}^2 & \longrightarrow & \mathbb{P}^2(\mathcal{S}_7, \mathcal{S}_9) \\ & & \downarrow \\ & & B \end{array} \tag{3.5}$$

The two divisors parametrizing this fibration are \mathcal{S}_7 and \mathcal{S}_9 , cf. [39, 42]. Upon imposing the constraint (3.4) and requiring the Calabi-Yau condition for X_{F_1} , we see that these two divisors are precisely the classes of the coefficients s_7 and s_9 , respectively. Indeed, as mentioned above, we can use the \mathbb{C}^* -action on \mathbb{P}^2 to turn e.g. w into a section of the trivial line bundle of the base. Then, we choose the variables u and v as sections of the bundles

$$u \in \mathcal{O}_B(\mathcal{S}_9 + [K_B]), \quad v \in \mathcal{O}_B(\mathcal{S}_9 - \mathcal{S}_7). \tag{3.6}$$

This allows us to compute the anti-canonical bundle of the \mathbb{P}^2 -fibration (3.5) using adjunction as

$$K_{\mathbb{P}^2(\mathcal{S}_7, \mathcal{S}_9)}^{-1} = \mathcal{O}(3H + 2\mathcal{S}_9 - \mathcal{S}_7). \tag{3.7}$$

Finally, we impose the Calabi-Yau condition on the constraint (3.4) for X_{F_1} which fixes the divisor classes of the coefficients s_i . We summarize the divisor classes of the homogeneous coordinates $[u : v : w]$ and the coefficients s_i in the following tables:

| Section | Divisor Class |
|----------|---|
| s_1 | $3[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_9$ |
| s_2 | $2[K_B^{-1}] - \mathcal{S}_9$ |
| s_3 | $[K_B^{-1}] + \mathcal{S}_7 - \mathcal{S}_9$ |
| s_4 | $2\mathcal{S}_7 - \mathcal{S}_9$ |
| s_5 | $2[K_B^{-1}] - \mathcal{S}_7$ |
| s_6 | $[K_B^{-1}]$ |
| s_7 | \mathcal{S}_7 |
| s_8 | $[K_B^{-1}] + \mathcal{S}_9 - \mathcal{S}_7$ |
| s_9 | \mathcal{S}_9 |
| s_{10} | $2\mathcal{S}_9 - \mathcal{S}_7$ |

| Section | Divisor Class |
|---------|-------------------------------------|
| u | $H + \mathcal{S}_9 + [K_B]$ |
| v | $H + \mathcal{S}_9 - \mathcal{S}_7$ |
| w | H |

(3.8)

X_{F_i} as specialized cubics

As mentioned in the previous subsection, the equations of the Calabi-Yau manifolds X_{F_i} with $i \neq 2, 4$ can be expressed as specialized versions of the cubic hypersurface equation (3.4) of X_{F_1} .

In order to find the hypersurface equation for an X_{F_i} we begin by calculating the anti-canonical class of the fibration $\mathbb{P}_{F_i}^B(D, \tilde{D})$ defined in (3.1). To this end, we first note that toric ambient spaces \mathbb{P}_{F_i} are obtained from \mathbb{P}_{F_1} by a certain number of blow-ups at points P_j . Assuming the number of blow-ups is k , we have

$$\mathbb{P}_{F_i} = \text{Bl}_{P_1, \dots, P_k} \mathbb{P}_{F_1}. \tag{3.9}$$

Each blow-up adds a \mathbb{P}^1 with an associated new variable e_j and divisor class E_j . From the combinatorial point of view, this means that there is an additional \mathbb{C}^* -action on \mathbb{P}_{F_i} .

Next we note that the fibration $\mathbb{P}_{F_i}^B(D, \tilde{D})$ can be parametrized by the same base divisors \mathcal{S}_7 and \mathcal{S}_9 as the fibration (3.5), i.e. we identify $D = \mathcal{S}_7$ and $\tilde{D} = \mathcal{S}_9$. Indeed, this is possible since we can use (\mathbb{C}^*) -actions, including the new (\mathbb{C}^*) -actions from the k blow-ups, to make the variables w and e_j transform in the trivial bundle on B while maintaining the assignments (3.6) for u and v . Employing these results we calculate the anti-canonical bundle of $\mathbb{P}_{F_i}^B(\mathcal{S}_7, \mathcal{S}_9)$, using the adjunction formula, yielding

$$K_{\mathbb{P}_{F_i}^B(\mathcal{S}_7, \mathcal{S}_9)}^{-1} = \mathcal{O}(3\tilde{H} - E_1 - E_2 - \dots - E_k + 2\mathcal{S}_9 - \mathcal{S}_7). \tag{3.10}$$

Here, \tilde{H} denotes the pull-back $\tilde{H} = \tilde{\pi}^*(H)$ of the hyperplane class H on \mathbb{P}^2 under the blow-down map $\tilde{\pi} : \mathbb{P}_{F_i} \rightarrow \mathbb{P}^2$. By abuse of notation, we will denote it throughout this work simply by H . It is to be observed that if the coefficient s_i is present in the constraint of X_{F_i} , i.e. if it is not removed by the k blow-ups, its corresponding class $[s_i]$ remains unaltered from the one given in table (3.8).

This relation of the hypersurface constraints of the X_{F_i} for $i \neq 2, 4$ and all the bundles entering it to the hypersurface equation (3.4) and bundles (3.8) of X_{F_1} will facilitate our following presentation. In particular, in the respective subsections on X_{F_i} for $i \neq 2, 4$ only the classes for the variables u, v, w and e_j have to be given explicitly.

3.1.3 Fibration by the biquadric: X_{F_2}

We construct the Calabi-Yau manifold $\mathcal{C}_{F_2} \rightarrow X_{F_2} \rightarrow B$ as the fibration of the curve \mathcal{C}_{F_2} in the toric variety \mathbb{P}_{F_2} over B . As mentioned before, its hypersurface equation cannot be described as a cubic. Thus, X_{F_2} has to be analyzed separately.

The polyhedron F_2 and its dual are presented in figure 4. The toric variety associated to it is $\mathbb{P}_{F_2} = \mathbb{P}^1 \times \mathbb{P}^1$ and we have introduced the projective coordinates $[x : t]$ and $[y : s]$ on the two \mathbb{P}^1 's, respectively. The Stanley-Reisner ideal of \mathbb{P}_{F_2} is given by

$$SR = \{xt, ys\}. \tag{3.11}$$

There are two divisor classes on \mathbb{P}_{F_2} , that we denote by H_1 and H_2 with respective representatives $\{x = 0\}$ and $\{y = 0\}$. The Calabi-Yau onefold in \mathbb{P}_{F_2} is the curve \mathcal{C}_{F_2} in the class $2H_1 + 2H_2$. It is a biquadric of the form

$$p_{F_2} = (b_1y^2 + b_2sy + b_3s^2)x^2 + (b_5y^2 + b_6sy + b_7s^2)xt + (b_8y^2 + b_9sy + b_{10}s^2)t^2, \tag{3.12}$$

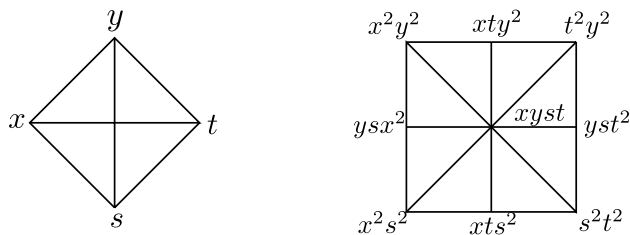


Figure 4. Polyhedron F_2 with choice of projective coordinates and its dual with corresponding monomials.

as can be shown using (2.23) and figure 4. Here the b_i denote coefficients in the field K .

In order to find X_{F_2} we proceed to construct $\mathbb{P}_{F_2}^B(D, \tilde{D})$, the fibration of \mathbb{P}_{F_2} introduced in (3.1). It is possible to consistently parametrize this fibration in terms of the same divisor classes $D = \mathcal{S}_7$ and $\tilde{D} = \mathcal{S}_9$ as in (3.5). In the hypersurface constraint (3.12), they correspond to the classes of the coefficients b_7 and b_9 respectively.¹⁷ This will facilitate the matching of the effective theories via Higgsings, as discussed in section 4. Next, we use the $(\mathbb{C}^*)^2$ -actions on \mathbb{P}_{F_2} to achieve that the variables x and y transform in the trivial line bundle on B . The other two variables s and t take values in the following line bundles on B :

$$t \in \mathcal{O}_B([K_B^{-1}] - \mathcal{S}_9), \quad s \in \mathcal{O}_B([K_B^{-1}] - \mathcal{S}_7). \tag{3.13}$$

With this assignment of line bundles to the coordinates on \mathbb{P}_{F_2} , the anti-canonical class of $\mathbb{P}_{F_2}^B(\mathcal{S}_7, \mathcal{S}_9)$ is readily calculated as

$$K_{\mathbb{P}_{F_2}^B}^{-1} = \mathcal{O}(2H_1 + 2H_2 + 3[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_9). \tag{3.14}$$

Finally, we require that the hypersurface (3.12) is Calabi-Yau, which fixes the divisor classes of the coefficients b_i in terms of \mathcal{S}_7 , \mathcal{S}_9 and $[K_B^{-1}]$. In summary, we obtain that the coordinates on \mathbb{P}_{F_2} and the coefficients b_i have the following divisor classes:

| Section | Divisor Class |
|----------|---|
| b_1 | $3[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_9$ |
| b_2 | $2[K_B^{-1}] - \mathcal{S}_9$ |
| b_3 | $[K_B^{-1}] + \mathcal{S}_7 - \mathcal{S}_9$ |
| b_5 | $2[K_B^{-1}] - \mathcal{S}_7$ |
| b_6 | $[K_B^{-1}]$ |
| b_7 | \mathcal{S}_7 |
| b_8 | $[K_B^{-1}] + \mathcal{S}_9 - \mathcal{S}_7$ |
| b_9 | \mathcal{S}_9 |
| b_{10} | $\mathcal{S}_9 + \mathcal{S}_7 - [K_B^{-1}]$ |

| Section | Divisor Class |
|---------|------------------------------------|
| x | H_1 |
| t | $H_1 + [K_B^{-1}] - \mathcal{S}_9$ |
| y | H_2 |
| s | $H_2 + [K_B^{-1}] - \mathcal{S}_7$ |

(3.15)

We emphasize that the classes of the coefficients b_i , except for b_{10} , agree with the classes of s_i of the cubic X_{F_1} , cf. (3.8), as expected.

¹⁷The consistency of this assignment can be seen by noting that \mathbb{P}_{F_2} is related to \mathbb{P}^2 by the blow-up at $x = y = 0$ setting $b_{10} = 0$ and the subsequent blow-downs $x = y = 1$. Then, (3.12) precisely yields (3.4).

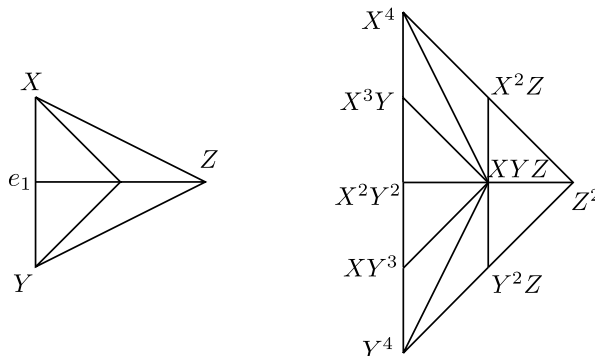


Figure 5. Polyhedron F_4 with choice of projective coordinates and its dual with corresponding monomials where the blow-up variable e_1 is suppressed.

3.1.4 Fibration by the quartic: X_{F_4}

We proceed to construct the Calabi-Yau manifold $\mathcal{C}_{F_4} \rightarrow X_{F_4} \rightarrow B$ with general fiber given by the curve \mathcal{C}_{F_4} in the toric variety \mathbb{P}_{F_4} . As we have mentioned above, its hypersurface equation is not a special case of a cubic, which requires a separate analysis of X_{F_4} .

The polyhedron F_4 and its dual polyhedron are shown in figure 5. Its associated toric variety is $\mathbb{P}_{F_4} = \mathbb{P}^2(1, 1, 2)$ and we introduce the homogeneous coordinates $[X : Y : Z : e_1]$. The Stanley-Reisner ideal of this toric variety can be read off from figure 5 as

$$SR = \{XY, Ze_1\}. \quad (3.16)$$

There are two divisor classes on \mathbb{P}_{F_4} , that are denoted by H and E_1 with representatives $\{X = 0\}$ and $\{e_1 = 0\}$ respectively. We note that \mathbb{P}_{F_4} automatically contains an exceptional divisor E_1 corresponding to the point interior to the edge of F_4 . The equation for the Calabi-Yau onefold \mathcal{C}_{F_4} in \mathbb{P}_{F_4} , which is a degree two curve, is the quartic equation in the class $4H - 2E_1$. It reads explicitly

$$p_{F_4} = d_1 e_1^2 X^4 + d_2 e_1^2 X^3 Y + d_3 e_1^2 X^2 Y^2 + d_4 e_1^2 X Y^3 + d_5 e_1^2 Y^4 + d_6 e_1 X^2 Z + d_7 e_1 X Y Z + d_8 e_1 Y^2 Z + d_9 Z^2, \quad (3.17)$$

as we infer from (2.23) and figure 5. The coefficients d_i take values in the field K .

In order to find X_{F_4} we construct the fibration $\mathbb{P}_{F_4}^B(\mathcal{S}_7, \mathcal{S}_9)$, the fibration of \mathbb{P}_{F_4} over B introduced in (3.1). Again, it is possible to parametrize the fibration by the identical divisors \mathcal{S}_7 and \mathcal{S}_9 as in the fibration (3.5) relevant for the X_{F_1} .¹⁸ As before, we can use the two \mathbb{C}^* -actions on P_{F_4} to make two of its homogeneous coordinates transform in the trivial bundles. However, it turns out that a convenient assignment of sections, avoiding fractions, is given by

$$X \in \mathcal{O}_B(\mathcal{S}_9 - [K_B^{-1}]), \quad Y \in \mathcal{O}_B(\mathcal{S}_9 - \mathcal{S}_7), \quad Z \in \mathcal{O}_B(\mathcal{S}_9 - [K_B^{-1}]). \quad (3.18)$$

¹⁸We note that this assignment is a consequence of the birational map between \mathbb{P}_{F_4} and \mathbb{P}_{F_1} induced by the blow-up at $X = Z = 0$, setting $d_5 = 0$, and the blow-downs $X = 1$ and $e_1 = 1$.

We use this to compute the anti-canonical bundle on $\mathbb{P}_{F_4}^B(\mathcal{S}_7, \mathcal{S}_9)$ as

$$K_{\mathbb{P}_{F_4}^B}^{-1} = \mathcal{O}(4H - 2E_1 + 3\mathcal{S}_9 - \mathcal{S}_7 - [K_B^{-1}]) . \tag{3.19}$$

Imposing the Calabi-Yau condition on the constraint (3.17), we fix the classes of all coefficients d_i . The assignments of divisor classes to the coordinates on \mathbb{P}_{F_4} and the d_i in summary read

| Section | Divisor Class |
|---------|--|
| d_1 | $3[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_9$ |
| d_2 | $2[K_B^{-1}] - \mathcal{S}_9$ |
| d_3 | $[K_B^{-1}] + \mathcal{S}_7 - \mathcal{S}_9$ |
| d_4 | $2\mathcal{S}_7 - \mathcal{S}_9$ |
| d_5 | $-[K_B^{-1}] + 3\mathcal{S}_7 - \mathcal{S}_9$ |
| d_6 | $2[K_B^{-1}] - \mathcal{S}_7$ |
| d_7 | $[K_B^{-1}]$ |
| d_8 | \mathcal{S}_7 |
| d_9 | $[K_B^{-1}] - \mathcal{S}_7 + \mathcal{S}_9$ |

| Section | Divisor Class |
|---------|---|
| X | $H - E_1 + \mathcal{S}_9 - [K_B^{-1}]$ |
| Y | $H - E_1 - \mathcal{S}_7 + \mathcal{S}_9$ |
| Z | $2H - E_1 + \mathcal{S}_9 - [K_B^{-1}]$ |
| e_1 | E_1 |

(3.20)

We emphasize that the classes of the coefficients d_i slightly differ from (3.8) in the cubic X_{F_1} due to the slightly different assignments (3.18) of classes to the coordinates.

3.2 Fibration with discrete gauge symmetry

In this section we analyze the toric hypersurface fibrations based on the fiber polyhedra F_1 , F_2 and F_4 . Since their fibrations do not have a section, but only multi-sections, they are genus-one fibrations. We analyze the codimension one, two and three singularities of these models, employing also their respective associated Jacobian fibrations. We show that the effective theories of F-theory on these Calabi-Yau manifolds exhibit discrete gauge groups and include matter that is charged only under the respective discrete group.

3.2.1 Polyhedron F_1 : $G_{F_1} = \mathbb{Z}_3$

We consider the genus-one fibration X_{F_1} over an arbitrary base B with genus-one fiber \mathcal{C}_{F_1} realized as the Calabi-Yau hypersurface in $\mathbb{P}_{F_1} = \mathbb{P}^2$. The toric data of $\mathbb{P}_{F_1} = \mathbb{P}^2$ and the construction of the Calabi-Yau manifolds X_{F_1} have been discussed in section 3.1.2. The hypersurface equation for X_{F_1} is given by (3.4) with the relevant divisor classes of the coordinates $[u : v : w]$ and the coefficients s_i summarized in (3.8).

The fibration $\pi : X_{F_1} \rightarrow B$ does not have a section, but only a three-section. Thus, X_{F_1} is only a genus-one fibration, cf. the general discussion in section 2.1. In order to obtain the WSF of X_{F_1} , given the absence of sections of its fibration, we have to calculate the associated Jacobian fibration $J(X_{F_1})$. The algorithm for computing $J(X_{F_1})$ is well known in the mathematics literature, see for example [76], from where we calculate f and g , given explicitly in (B.1) and (B.2), and subsequently the discriminant Δ . The discriminant does not factorize, which shows the absence of codimension one singularities

of X_{F_1} and therefore, the absence of non-Abelian gauge groups in the corresponding F-theory compactification.

The fibration X_{F_1} has a three-section that is given by

$$\hat{s}^{(3)} = X_{F_1} \cap \{u = 0\} : s_4 v^3 + s_7 v^2 w + s_9 v w^2 + s_{10} w^3 = 0, \quad (3.21)$$

as follows from the Calabi-Yau constraint (3.4). We denote its divisor class, that agrees with $H + \mathcal{S}_9 + [K_B]$, by $S^{(3)}$. Under the degree nine map from X_1 to its Jacobian this three-section is mapped to the canonical zero section $z = 0$ in the WSF of $J(X_{F_1})$. However, in X_{F_1} , the three-section $\hat{s}^{(3)}$ locally maps a point on the base B to three points on the fiber \mathcal{C}_{F_1} . Globally, there exists a monodromy group that interchanges these three points, upon moving on the base B . This fact, together with the existence of I_2 -fibers in X_{F_1} at codimension two on which the monodromy group acts non-trivially, as we present next, and the results from Higgsing the $U(1)$ gauge group in the effective theory associated to X_{F_3} , see section 4.2, leads us to postulate the following *discrete* gauge group of X_{F_1} :

$$G_{F_1} = \mathbb{Z}_3. \quad (3.22)$$

In order to compute the charges of matter under this discrete group, we have to associate a divisor class to the three-section. As certain models with multi-section are related to models with multiple rational sections by conifold transitions, see [55, 56], a natural proposal for such a divisor class is an expression similar to the Shioda map (2.5). We recall the three defining properties of a Shioda map summarized on page 21 in [34]. Imposing these conditions on the divisor class associated to (3.21), we obtain the following divisor class,

$$\sigma_{\mathbb{Z}_3}(\hat{s}^{(3)}) = S^{(3)} + [K_B] + \frac{4}{3}\mathcal{S}_9 - \frac{2}{3}\mathcal{S}_7. \quad (3.23)$$

We propose that matter charges under the discrete group \mathbb{Z}_3 should be computed using this class. In fact, we demonstrate next, that the class (3.23) allows us to compute \mathbb{Z}_3 -charges of matter-representations on X_{F_1} , that are consistent with 6D anomaly cancellation and the Higgsing from the model X_{F_3} , discussed in section 4.2.

Charged and uncharged matter in X_{F_1}

We proceed with determining the codimension two singularities of the WSF of $J(X_{F_1})$. This analysis is most easily carried out directly in the smooth fibration X_{F_1} . The same techniques presented here will also be used in a slightly modified form for the analysis of the fibration X_{F_2} , X_{F_3} and X_{F_4} . We note that the same technique has been used recently in [54] and [56].

Finding the loci of I_2 -fibers using elimination ideals

We are looking for loci of B that support I_2 -fibers in X_{F_1} . At these loci, the genus-one fiber \mathcal{C}_{F_1} of X_{F_1} has to degenerate into two \mathbb{P}^1 's, i.e. the hypersurface equation (3.4) has to factor into two smooth polynomials. For a smooth cubic the only factorization with this property is the one into a conic and a line, i.e. a factorization of (3.4) of the form

$$p_{F_1} \stackrel{!}{=} s_1(u + \alpha_1 v + \alpha_2 w)(u^2 + \beta_1 v^2 + \beta_2 w^2 + \beta_3 uv + \beta_4 vw + \beta_5 uw), \quad (3.24)$$

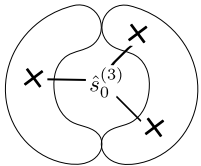
| Representation | Multiplicity | Fiber | Locus |
|----------------|---|---|--------------|
| $\mathbf{1}_1$ | $3(6[K_B^{-1}]^2 - \mathcal{S}_7^2 + \mathcal{S}_7\mathcal{S}_9 - \mathcal{S}_9^2 + [K_B^{-1}](\mathcal{S}_7 + \mathcal{S}_9))$ |  | $V(I_{(1)})$ |

Table 1. Charged matter representation under \mathbb{Z}_3 and corresponding codimension two fiber of X_{F_1} .

where α_j and β_k are seven unknown polynomials on B . We note that we can assume $s_1 \neq 0$, because otherwise we would obtain a locus of codimension three or higher. Making a comparison of coefficients on both sides of (3.24), we obtain a set of constraints that defines an ideal in the ring $K[s_i, \alpha_j, \beta_k]$, where s_i are the coefficients in (3.4). We denote this ideal by $I_{(s_i, \alpha, \beta)}$. We emphasize that there are two more constraints, namely nine, in $I_{(s_i, \alpha, \beta)}$ than unknowns α_j, β_k , i.e. the system is over-determined. Thus, there only exists a solution for α_j, β_k satisfying (3.24), if two additional constraints on the s_i are obeyed. This implies that the ideal $I_{(s_i, \alpha, \beta)}$ describes a codimension two locus of the s_i .

In order to obtain the constraints that the s_i have to obey for the factorization (3.24) to exist, we compute the elimination ideal $I_{(s_i)} = I_{(s_i, \alpha, \beta)} \cap K[s_i]$,¹⁹ where $K[s_i]$ is the polynomial ring only in the variables s_i . We compute $I_{(s_i)}$, in the following abbreviated as $I_{(1)} \equiv I_{(s_i)}$, explicitly using Singular [88] and obtain an ideal with 50 generators. Furthermore, we calculate its codimension in the ring $K[s_i]$ to be two. Thus, its vanishing locus $V(I_{(1)})$ describes a codimension two variety in B . In summary, we have shown that the factorization (3.24) corresponding to an I_2 -fiber in X_{F_1} happens at the codimension two locus $V(I_{(1)})$ in B .

We note that (3.24) is the only type of factorization that can occur. Thus, we do not expect any further codimension two fibers and corresponding matter representation in X_{F_1} . The spectrum of X_{F_1} is summarized in table 1. Next we argue how to compute the charge of the matter located at $V(I_{(1)})$ under the discrete gauge group $G_{F_1} = \mathbb{Z}_3$. Due to the absence of a zero section on X_{F_1} , there is no preferred curve in the I_2 -fiber in table 1. As can be observed from (3.24), the two \mathbb{P}^1 's in this I_2 -fiber have intersection numbers one and two with the three-section $\hat{s}_0^{(3)}$. By naively applying (2.13) using the divisor class (3.23), we compute the charges $q = 1$ and $q = 2$ for the two rational curves, respectively. Thus, it seems that there is no meaningful way to assign a discrete charge to the matter located at $V(I_{(1)})$. However, this seeming contradiction is resolved by noting that a 6D hyper multiplet of charge $q = 1$ is the same as one with charge $q = -1$. In addition, employing the discrete \mathbb{Z}_3 symmetry, we have $-1 = 2 \pmod 3$, showing that a 6D hyper multiplet of charge $q = 1$ under a \mathbb{Z}_3 symmetry is physically equivalent to one with charge $q = 2$. Thus, the matter at $V(I_{(1)})$ has charge $q = 1$ which is the same as $q = 2$ under the discrete gauge group $G_{F_1} = \mathbb{Z}_3$.

We proceed to calculate the multiplicity of $V(I_{(1)})$. Unfortunately given the size and number of polynomials in the ideal $I_{(1)}$, we are unable to obtain its multiplicity geometri-

¹⁹Here we deviate from the notation in mathematics literature, where the subscripts of the elimination ideal indicate the eliminated variables.

| Yukawa | Locus |
|--|----------------------------------|
| $\mathbf{1}_1 \cdot \mathbf{1}_1 \cdot \mathbf{1}_1$ | $V(I_1) \cap V(I_1) \cap V(I_1)$ |

Table 2. Codimension three locus and corresponding Yukawa coupling for X_{F_1} .

cally with the available computing power. Instead, we invoke the results for its multiplicity that is obtained in section 4.2 using the Higgs transition $X_{F_3} \rightarrow X_{F_1}$. It is shown in table 1 for completeness.

We complete the discussion of the matter spectrum of X_{F_1} by calculating the number of neutral hyper multiplets. We use (2.11) and the explicit expression for the Euler number $\chi(X_{F_1})$ of X_{F_1} in (C.1) to obtain

$$H_{\text{neut}} = 12 + 11[K_B^{-1}]^2 - 3[K_B^{-1}]\mathcal{S}_7 + 3\mathcal{S}_7^2 - 3[K_B^{-1}]\mathcal{S}_9 - 3\mathcal{S}_7\mathcal{S}_9 + 3\mathcal{S}_9^2. \quad (3.25)$$

Employing this together with the number of vector multiplets $V = 0$ and the charged spectrum in table 1 we check cancellation of the 6D gravitational anomaly in (A.1).

Yukawa couplings in X_{F_1}

We conclude this section by noting that there is only one gauge-invariant Yukawa coupling possible, listed in table 2. Again, we cannot check for its presence explicitly due to the complexity of the ideal $I_{(1)}$.²⁰

3.2.2 Polyhedron F_2 : $G_{F_2} = \mathbf{U}(1) \times \mathbb{Z}_2$

Here, we analyze the genus-one fibration X_{F_2} constructed as a fibration of the Calabi-Yau onefold \mathcal{C}_{F_2} in $\mathbb{P}_{F_2} = \mathbb{P}^1 \times \mathbb{P}^1$. The toric data of $\mathbb{P}^1 \times \mathbb{P}^1$ and the construction of the toric hypersurface fibration X_{F_2} have been presented in section 3.1.3. The hypersurface constraint for X_{F_2} is given in (3.12) and the relevant divisor classes are summarized in (3.15).

First, we note that the fibration $\pi : X_{F_2} \rightarrow B$ does not have a section, i.e. it is a genus-one fibration. We obtain its WSF by computing its associated Jacobian fibration $J(X_{F_2})$, employing again the straightforward algorithms from the mathematics literature [76]. The results for the functions f and g can be found in (B.3) and (B.4), from which the discriminant can be readily computed. The discriminant does not factorize, which again shows the absence of codimension one singularities. Thus, there is no non-Abelian gauge symmetry for this F-theory compactification.

The fibration of X_{F_2} has two independent two-sections, that are given by

$$\begin{aligned} \hat{s}_0^{(2)} &= X_{F_2} \cap \{x = 0\} : b_8y^2 + b_9sy + b_{10}s^2 = 0, \\ \hat{s}_1^{(2)} &= X_{F_2} \cap \{y = 0\} : b_3x^2 + b_7xt + b_{10}t^2 = 0, \end{aligned} \quad (3.26)$$

where we used the hypersurface constraint (3.12) and the SR-ideal (3.11). We denote the two corresponding divisor classes, that agree with H_1 and H_2 , by $S_0^{(2)}$ and $S_1^{(2)}$, respectively.

²⁰The presence of this coupling can be deduced considering the Higgsing from X_{F_3} to X_{F_1} (see section 4.2). Decomposing the states in X_{F_3} in terms of states in X_{F_1} , we observe that after Higgsing, the Yukawa coupling $\mathbf{1}_1 \cdot \mathbf{1}_1 \bar{\mathbf{1}}_2$ in X_{F_3} (see table 8) gives rise to $\mathbf{1}_1 \cdot \mathbf{1}_1 \cdot \mathbf{1}_1$ in X_{F_3} .

Analogous to the previous section 3.2.1, we expect a discrete \mathbb{Z}_2 gauge group associated to the two-section $S_0^{(2)}$, cf. the similar discussion in [55]. We will provide independent evidence for this by the analysis in section 4.2 of the Higgsing of the effective theory of F-theory on X_{F_5} , that has a $U(1)^2$ gauge group, to the one arising from X_{F_2} .

The role of the other two-section $\hat{s}_1^{(2)}$, however, is less clear in the biquadric representation. Its meaning for F-theory is unraveled by transforming the biquadric (3.12) defining X_{F_2} into a cubic hypersurface and then by computing its Weierstrass form, which is precisely the WSF of the Jacobian fibration of X_{F_2} , as we show. This detour via the cubic yields a direct map to the Jacobian fibration $J(X_{F_2})$, which allows us to follow the two-section $\hat{s}_1^{(2)}$ in (3.26).

Map to the cubic in \mathbb{P}_{F_5} & the MW-group of $J(X_{F_2})$

The curve \mathcal{C}_{F_2} given as the biquadric (3.12) in \mathbb{P}_{F_2} can be treated as the cubic in \mathbb{P}_{F_5} after an appropriate change of variables. Indeed, by applying the transformation $x \rightarrow x + \alpha t$ or $y \rightarrow y + \beta s$, we can set the coefficient of the monomial $s^2 t^2$ (3.12) to zero for an appropriate α or β . We note that both α and β have to involve square roots of the coefficients b_i in (3.12), i.e. the two variable transformations are only defined in a field extension. As we will see, this field extension will only be an intermediate step, since all square roots will drop out in the final result of our computation. After the change of variables, we obtain a polynomial of the following form:

$$\tilde{p} = (\tilde{s}_1 y^2 + \tilde{s}_2 s y + \tilde{s}_3 s^2) x^2 + (\tilde{s}_5 y^2 + \tilde{s}_6 s y + \tilde{s}_7 s^2) x t + (\tilde{s}_8 y^2 + \tilde{s}_9 s y) t^2, \tag{3.27}$$

where the redefined coefficients \tilde{s}_i depend on the variables b_i and are explicitly given in (B.5). We note that \tilde{p} is precisely of the form of the cubic (3.72) in \mathbb{P}_{F_5} after identifying

$$t \rightarrow w, \quad s \rightarrow v, \quad x \rightarrow e_2, \quad y \rightarrow e_1, \quad u = 1. \tag{3.28}$$

Since the curve $\tilde{p} = 0$ is an elliptic curve, we can compute its WSF, in particular the functions f and g . Inserting the explicit expressions (B.5) for the sections \tilde{s}_i in (3.27) in terms of the b_i in (3.12) into the expressions for f and g , we precisely recover (B.3) and (B.4) obtained from the WSF of the Jacobian fibration $J(X_{F_2})$. Most notably, all square roots in the coefficients \tilde{s}_i have dropped out, as claimed.

Next, we note that the two-section $\hat{s}_1^{(2)}$ in (3.26) formally maps to the section

$$s = 1, \quad t = -\frac{\tilde{s}_3}{\tilde{s}_7} \quad y = 0, \quad x = 1, \tag{3.29}$$

in (3.27). Under the identification of coordinates (3.28), this is precisely the section \hat{s}_1 of X_{F_5} given in (3.73). Inserting the explicit expressions (B.5) for the \tilde{s}_i into the WS-coordinates (B.9) of \hat{s}_1 , we obtain

$$\begin{aligned} z_1 &= 1, \\ x_1 &= \frac{1}{12} (8b_1 b_{10} + b_6^2 - 4b_5 b_7 + 8b_3 b_8 - 4b_2 b_9), \\ y_1 &= \frac{1}{2} (b_{10} b_2 b_5 - b_1 b_{10} b_6 + b_3 b_6 b_8 - b_2 b_7 b_8 - b_3 b_5 b_9 + b_1 b_7 b_9). \end{aligned} \tag{3.30}$$

We emphasize that all square roots in the coefficients b_i in (B.5) have dropped out and we obtain completely rational WS-coordinates for the two-section $\hat{s}_1^{(2)}$. We double-check that (3.30) solves the WSF of the Jacobian $J(X_{F_2})$.

In summary, we have shown for the first time that the associated Jacobian fibration $J(X_{F_2})$ exhibits a *rank one* MW-group of rational sections generated by the section in (3.30), which is precisely the image of the two-section $\hat{s}_1^{(2)}$ in X_{F_2} under the map $X_{F_2} \rightarrow J(X_{F_2})$. This means that there is an associated Abelian gauge field in the F-theory compactified on X_{F_2} . We note that application of the same logic to the two-section $\hat{s}_0^{(2)}$, which formally maps to the section \hat{s}_2 defined in (3.73) in X_{F_3} , does not lead to a rational section of the Jacobian $J(X_{F_2})$ since its WS-coordinates (B.10) after inserting (B.5) still contain square roots. Hence, $\hat{s}_0^{(2)}$ does not yield an additional U(1)-factor, but corresponds to a discrete group \mathbb{Z}_2 , as claimed.

Having proven the presence of a MW-group on $J(X_{F_2})$, we compute the Shioda map of its generator. We note that the usual expression (2.5) has to be modified since $\hat{s}_1^{(2)}$ is a two-section. It can be shown that the following expression obeys all conditions listed in [34] that have to be obeyed by a Shioda map:

$$\sigma(\hat{s}_1^{(2)}) = S_1^{(2)} - S_0^{(2)} + \frac{1}{2}([K_B] - \mathcal{S}_7 + \mathcal{S}_9) \tag{3.31}$$

Then we obtain the corresponding height pairing, using (2.6), as

$$b_{11} = -\pi(\sigma(\hat{s}_1^{(2)}) \cdot \sigma(\hat{s}_1^{(2)})) = 2[K_B^{-1}]. \tag{3.32}$$

Here, we used the following intersections

$$\begin{aligned} \pi((S_0^{(2)})^2) &= -2([K_B^{-1}] - \mathcal{S}_9), \\ \pi((S_1^{(2)})^2) &= -2([K_B^{-1}] - \mathcal{S}_7), \\ \pi(S_0^{(2)} \cdot S_1^{(2)}) &= \mathcal{S}_7 + \mathcal{S}_9 - [K_B^{-1}]. \end{aligned} \tag{3.33}$$

The first two equalities are just a translation of the SR-ideal (3.11) into intersection relations of divisor classes on X_{F_2} , employing (3.15) and (2.4) for $n = 2$. The third relation follows by noting that according to (3.26) the two two-sections $\hat{s}_0^{(2)}$ and $\hat{s}_1^{(2)}$ intersect precisely at $b_{10} = 0$, whose class is $[b_{10}] = \mathcal{S}_7 + \mathcal{S}_9 - [K_B^{-1}]$, cf. (3.15).

We conclude by summarizing the full gauge group of the theory:

$$G_{F_2} = \text{U}(1) \times \mathbb{Z}_2. \tag{3.34}$$

We highlight again that the U(1) corresponds to a rational section in the Jacobian fibration $J(X_{F_2})$, that is the image of the two-section $\hat{s}_1^{(2)}$ under the degree four map $X_{F_2} \rightarrow J(X_{F_2})$.

As mentioned before, the discrete gauge \mathbb{Z}_2 symmetry is induced by the two-section $\hat{s}_0^{(2)}$. For the computation of charges of matter w.r.t. the \mathbb{Z}_2 , we have to associate a divisor class to it. Imposing conditions on this divisor class similar to the one that have lead to the Shioda map (2.5) [34], we obtain

$$\sigma_{\mathbb{Z}_2}(\hat{s}_0^{(2)}) = S_0^{(2)} + [K_B^{-1}] - \mathcal{S}_9. \tag{3.35}$$

We use this divisor class to successfully compute the \mathbb{Z}_2 -charges of matter in the following.

Charged and uncharged matter in X_{F_2}

Now that we know the gauge group of the theory, we proceed to derive first the matter representation and then the corresponding 6D matter multiplicities. As in section 3.2.1, we use the elimination ideal technique to show directly the presence of three matter representations in X_{F_2} , namely $\mathbf{1}_{(1,+)}$, $\mathbf{1}_{(1,-)}$ and $\mathbf{1}_{(0,-)}$, where \pm denote the two possible \mathbb{Z}_2 -eigenvalues. Then, we compute their multiplicities, where we also invoke the equivalent presentation of X_{F_2} as a quartic.

In order to find the three I_2 -fibers at codimension two in X_{F_2} , we first note that there are three different possible ways to factorize the biquadric (3.12), that correspond to the three inequivalent ways to split its degree $(2, 2)$ w.r.t. the classes H_1 and H_2 in $\mathbb{P}^1 \times \mathbb{P}^1$, namely as $(2, 2) = (1, 1) + (1, 1)$, $(2, 2) = (1, 0) + (1, 2)$ and $(2, 2) = (0, 1) + (2, 1)$ respectively.

The first type of factorization of (3.12) corresponding to $(2, 2) = (1, 1) + (1, 1)$ is given by

$$p_{F_2} \stackrel{\perp}{=} b_1 [(y + \alpha_1 s)x + (\alpha_2 y + \alpha_3 s)t] [(y + \beta_1 s)x + (\beta_2 y + \beta_3 s)t]. \quad (3.36)$$

Clearly, both factors are bilinear in $[x : t]$ and $[y : s]$, respectively, as required. As before, we can factor out b_1 because it must not vanish at a codimension two locus. We note that there are six unknown polynomials α_j and β_k and eight non-trivial constraints, as can be seen by a comparison of coefficients on both sides. Thus, the ideal of constraints is over-determined and imposes a codimension two condition on the coefficients b_i for a solution to (3.36) to exist. The elimination ideal, that we call $I_{(1)}$, obtained by eliminating the unknowns α_j and β_k from the ideal of constraints is generated by 50 polynomials. It is checked to be codimension two in the ring, as expected, proving the existence of the factorization (3.36) at codimension two. We denote the zero set of $I_{(1)}$ by $V(I_{(1)})$, which is the geometric codimension two locus in B .

Next, we note that each curve of the I_2 -fiber described by (3.36) has intersection one with both two-sections $\hat{s}_0^{(2)}$ and $\hat{s}_1^{(2)}$. The $U(1)$ -charge computed using (2.13) and (3.31) is zero. We also note that the representation has charge $(-)$ under the discrete symmetry because the two intersection points of $\hat{s}_0^{(2)}$ with the fiber are interchanged under a monodromy action. Formally, the charge under \mathbb{Z}_2 is computed using (2.13) together with the divisor class (3.35), showing that both curves in the I_2 -fiber have \mathbb{Z}_2 -charge $(-)$. Thus, the representation at the locus $V(I_{(1)})$ is $\mathbf{1}_{(0,-)}$ as shown in table 3.

The second type of factorization of (3.12) into two polynomials of degrees $(1, 0)$ and $(1, 2)$, respectively, takes the following explicit form

$$p_{F_2} \stackrel{\perp}{=} b_1 [y + \alpha_1 s] [(y + \beta_1 s)x^2 + (\beta_2 y + \beta_3 s)xt + (\beta_4 y + \beta_5 s)t^2], \quad (3.37)$$

where α_1 and the β_k are six unknown polynomials. We compute again the elimination ideal, denote $I_{(2)}$, that is generated by eight polynomials in the b_i and check that it is codimension two in the ring. The corresponding codimension two locus in B supporting this type of I_2 -fiber is denoted by $V(I_{(2)})$. The intersection pattern of the two-sections with the I_2 -fiber is shown in the second entry of table 3. The $U(1)$ - and \mathbb{Z}_2 -charges readily follow as discussed before and we find the representation at this locus to be $\mathbf{1}_{(1,-)}$.

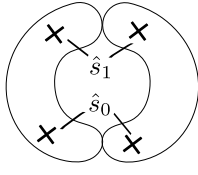
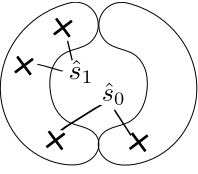
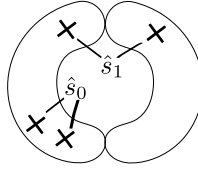
| Representation | Multiplicity | Fiber | Locus |
|----------------------|--|---|--------------|
| $\mathbf{1}_{(0,-)}$ | $6[K_B^{-1}]^2 + 4[K_B^{-1}](\mathcal{S}_7 + \mathcal{S}_9) - 2\mathcal{S}_7^2 - 2\mathcal{S}_9^2$ |  | $V(I_{(1)})$ |
| $\mathbf{1}_{(1,-)}$ | $6[K_B^{-1}]^2 + 4[K_B^{-1}](\mathcal{S}_9 - \mathcal{S}_7) + 2\mathcal{S}_7^2 - 2\mathcal{S}_9^2$ |  | $V(I_{(2)})$ |
| $\mathbf{1}_{(1,+)}$ | $6[K_B^{-1}]^2 + 4[K_B^{-1}](\mathcal{S}_7 - \mathcal{S}_9) - 2\mathcal{S}_7^2 + 2\mathcal{S}_9^2$ |  | $V(I_{(3)})$ |

Table 3. Charged matter representations under $U(1) \times \mathbb{Z}_2$ and corresponding codimension two fibers of X_{F_2} .

Finally, the last type of factorization corresponds to a split of (3.12) into two polynomials of degrees (0, 1) and (2, 1). It can be written down explicitly and takes a similar form as (3.37). The codimension two elimination ideal corresponding to this factorization, denoted by $I_{(3)}$, is generated by eight polynomials and its vanishing set is denoted by $V(I_{(3)})$. The intersection pattern of the two-sections with this type of I_2 -fiber is shown in the last entry of table 3. Using the charge formula (2.13) and the Shioda map (3.31), as well as (3.35) we show that the representation at $V(I_{(3)})$ is $\mathbf{1}_{(1,+)}$.

As a confirmation of the completeness of our analysis of codimension two singularities of X_{F_2} supporting $U(1)$ -charged matter, we recall that the codimension two locus supporting all I_2 -singularities associated to a $U(1)$ is given by (2.14). In the case at hand we have to evaluate this constraint for the rational sections of $J(X_2)$ with coordinates $[x_1 : y_1 : z_1]$ given in (3.30). We calculate all associated prime ideals of the obtained complete intersection using Singular [88] and indeed find precisely the two prime ideals $I_{(2)}$ and $I_{(3)}$ corresponding to the two representations $\mathbf{1}_{(1,-)}$ and $\mathbf{1}_{(1,+)}$ found previously using the elimination ideal technique.

As a next step, we calculate the homology classes in B for the three codimension two loci supporting the I_2 -fibers, which determine, according to section 2.3, the multiplicities of 6D charged hyper multiplets in the corresponding representations. We begin with the variety $V(I_{(3)})$, whose multiplicity we denote by $x_{\mathbf{1}_{(1,+)}}$, supporting the representation $\mathbf{1}_{(1,+)}$. Its homology class is computed by taking two constraints of the ideal $I_{(3)}$ and computing the homology class of the complete intersection described by them. Then, we subtract (with their corresponding orders) those components that are inside this complete

intersection but do not satisfy the other generators of the ideal $I_{(3)}$. We obtain:

$$\begin{aligned} x_{\mathbf{1}_{(1,+)}} &= [b_2^2 b_{10}^2] \cdot [b_{10} b_2 b_5] - 2([b_2 b_{10}] \cdot [b_2 b_7] - [b_2] \cdot [b_3]), \\ &= 6[K_B^{-1}]^2 + 4[K_B^{-1}](\mathcal{S}_7 - \mathcal{S}_9) - 2\mathcal{S}_7^2 + 2\mathcal{S}_9^2. \end{aligned} \quad (3.38)$$

The multiplicity of $V(I_{(2)})$, denoted by $x_{\mathbf{1}_{(1,-)}}$, can be calculated in a similar way. It is given in the third row of table 3. As a consistency check, we calculate the sum of both multiplicities and it agrees with $[y_1] \cdot [fz_1^4]$ as it should, because the $I_{(2)}$ and $I_{(3)}$ are the two associated prime ideals of (2.14) for the section (3.30).

For the computation of the multiplicity of the variety $V(I_{(1)})$, denoted by $x_{\mathbf{1}_{(0,-)}}$, we cannot carry out the previously mentioned algorithm, due to the size and complexity of the ideal $I_{(1)}$. Instead, $x_{\mathbf{1}_{(0,-)}}$ is obtained by first calculating the multiplicity of all hyper multiplets charged under the discrete symmetry, namely the $\mathbf{1}_{(0,-)}$ and $\mathbf{1}_{(1,-)}$, and then subtracting the number of hyper multiplets in the representation $\mathbf{1}_{(1,-)}$, that we already know.

We begin by noting that the total number of charge $(-)$ hyper multiplets under the \mathbb{Z}_2 symmetry was calculated geometrically in [55, 56] for a genus-one fibration given by the quartic curve in $\mathbb{P}^2(1, 1, 2)$. Indeed, we can directly use their results since we can transform the biquadric (3.12) to a quartic presentation. This quartic is to be obtained by taking the discriminant of the biquadric (3.12) with respect to y . To this end, we rewrite the biquadric in the suggestive form

$$p = A(x, t)y^2 + B(x, t)ys + C(x, t)s^2 \quad (3.39)$$

and then take the discriminant of this quadric in y (we also set $s = 1$). We construct a genus-one curve as the double cover over this discriminant, which is then a quartic in $[x : t]$ and a new variable w of weight two of the form

$$w^2 = B(x, t)^2 - 4A(x, t)C(x, t) \equiv e_0x^4 + e_1x^3t + e_2x^2t^2 + e_3xt^3 + e_4t^4. \quad (3.40)$$

Here we used the conventions of [55] in the last equality. The coefficients e_i can be expressed in terms of the b_i in (3.12) by a comparison of coefficients.

In this form the reason for choosing the quadric w.r.t. to y in order to construct the quartic (3.40) is evident, because the two-section $\hat{s}_0^{(2)} = X_{F_2} \cap \{x = 0\}$ is mapped to the two-section $x = 0$, $w^2 = e_4t^4$ in (3.40). Using the results in [55, 56], we calculate the multiplicities of all charge $(-)$ hyper multiplets, both $\mathbf{1}_{(0,-)}$, $\mathbf{1}_{(1,-)}$, using the one-to-one correspondence between the loci of I_2 -fibers in (3.40) with the following complete intersection, cf. equation (2.22) in [55],

$$\{e_1^4 - 8e_0e_1^2e_2 + 16e_0^2e_2^2 - 64e_0^3e_4 = 0\} \cap \{e_3 = 0\}. \quad (3.41)$$

Its homology class is readily given as $[4e_1] \cdot [e_3]$, which has to agree as mentioned before with the sum $x_{\mathbf{1}_{(1,-)}} + x_{\mathbf{1}_{(0,-)}}$. Thus, the multiplicity $x_{\mathbf{1}_{(0,-)}}$ follows by subtracting the multiplicity $x_{\mathbf{1}_{(1,-)}}$ calculated previously from $[4e_1] \cdot [e_3]$. The result is given in the second row of table 3.

| Yukawa | Locus |
|---|--|
| $\mathbf{1}_{(1,-)} \cdot \overline{\mathbf{1}}_{(1,+)} \cdot \mathbf{1}_{(0,-)}$ | $V(I_{(1)}) \cap V(I_{(2)}) \cap V(I_{(3)})$ |

Table 4. Codimension three locus and corresponding Yukawa coupling for X_{F_2} .

To complete the matter spectrum we calculate the number of neutral hyper multiplets. Using (2.11) and the explicit formula for the Euler number of X_{F_2} in (C.1) of appendix C, we obtain

$$H_{\text{neut}} = 13 + 11[K_B^{-1}]^2 - 4[K_B^{-1}]\mathcal{S}_7 + 2\mathcal{S}_7^2 - 4[K_B^{-1}]\mathcal{S}_9 + 2\mathcal{S}_9^2. \quad (3.42)$$

Using this together with the charged matter spectrum in table 3, the number of vector multiplets $V = 1$ and the height pairing (3.32) we confirm that all anomalies, including the purely gravitational one, are canceled.

Yukawa couplings in X_{F_2}

We conclude this section by stating the geometrically realized Yukawa couplings. We find the single Yukawa coupling in table 4, by checking explicitly that the corresponding varieties intersect at codimension three, i.e. that the ideal $I_{(1)} \cup I_{(2)} \cup I_{(3)}$ is codimension three in the ring generated by the coefficients b_i .

3.2.3 Polyhedron F_4 : $G_{F_4} = \text{SU}(2) \times \mathbb{Z}_4$

In this section we study the genus one-fibration X_{F_4} that is constructed as the toric hypersurface fibration of \mathcal{C}_{F_4} in $\mathbb{P}_{F_4} = \mathbb{P}^2(1, 1, 2)$. The toric data of $\mathbb{P}^2(1, 1, 2)$ as well as the construction of the toric hypersurface fibration X_{F_4} have been discussed in section 3.1.4. The hypersurface constraint of X_{F_4} is shown in (3.17) and the relevant divisor classes can be found in (3.20). This model has recently received a lot of attention [54–56]. Here we provide additional insights in the nature of the \mathbb{Z}_4 discrete gauge group of F-theory on X_{F_4} as well as in the computation of the charges of matter under this discrete group. We also check 6D anomaly cancellation, which requires knowledge of all multiplicities of 6D charged and uncharged hyper multiplets.

We begin by noting that the fibration $\pi : X_{F_4} \rightarrow B$ does not have a section, but only two- and four-sections, i.e. X_{F_4} is a genus-one fibration, see section 2.1. The three multi-sections induced by the ambient space $\mathbb{P}^2(1, 1, 2)$ of the fiber are

$$\begin{aligned} \hat{s}_1^{(2)} = X_{F_4} \cap \{X = 0\} : \quad & d_5 e_1^2 + d_8 e_1 Z + d_9 Z^2 = 0, \\ \hat{s}_2^{(2)} = X_{F_4} \cap \{Y = 0\} : \quad & d_1 e_1^2 + d_6 e_1 Z + d_9 Z^2 = 0, \\ \hat{s}_3^{(4)} = X_{F_4} \cap \{Z = 0\} : \quad & d_1 X^4 + d_2 X^3 Y + d_3 X^2 Y^2 + d_4 X Y^3 + d_5 Y^4 = 0, \end{aligned} \quad (3.43)$$

where we used the SR-ideal (3.16). We denote the one independent divisor classes of $\hat{s}_1^{(2)}$ by $S_1^{(2)}$. It agrees with $H - E_1 + \mathcal{S}_9 - [K_B^{-1}]$ according to (3.20), where E_1 is the class of the exceptional divisor on $\mathbb{P}^2(1, 1, 2)$.

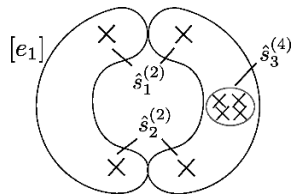


Figure 6. Codimension one fibers of X_{F_4} . The crosses denote the intersections with the sections.

Since this fibration does not have a section one has to utilize its associated Jacobian fibration $J(X_{F_4})$ in order to find its WSF. We readily compute the functions f and g in (2.1) using the algorithm in [76]. From the discriminant of this WSF, we find one I_2 -singularity over the divisors $\mathcal{S}_{\text{SU}(2)}^b = \{d_9 = 0\} \cap B$ in B . Along this divisor, the constraint (3.17) factorizes as

$$\text{SU}(2) : \quad p_{F_4}|_{d_9=0} = e_1 \cdot q_3, \tag{3.44}$$

where q_3 is the polynomial that remains after factoring out e_1 . The corresponding I_2 -fiber is depicted in figure 6. Due to the absence of a zero section, there is no preferred rational curve in this I_2 -fiber. Thus, a possible choice for the Cartan-divisor D_1 of the $\text{SU}(2)$ is given by

$$D_1 = [e_1]. \tag{3.45}$$

In summary, the gauge group of X_{F_4} is given by

$$G_{F_4} = \text{SU}(2) \times \mathbb{Z}_4. \tag{3.46}$$

The discrete symmetry stems from the multi-sections. In order to calculate the charges under the discrete symmetry we have to orthogonalize the $\text{SU}(2)$ such that it is not charged under the discrete symmetry. This is done in a similar way as for $\text{U}(1)$'s via the Shioda map (2.5). For the two two-sections \hat{s}_m with $m = 1, 2$, we propose

$$\sigma_{\mathbb{Z}_4}(\hat{s}_m^{(2)}) = S_m^{(2)} + \frac{1}{2}D_1 + \frac{3}{4}[K_B^{-1}] - \frac{3}{4}\mathcal{S}_7 - \frac{1}{4}\mathcal{S}_9, \tag{3.47}$$

as the appropriate divisor class to compute charges under the discrete gauge group \mathbb{Z}_4 . Here we used that both two-sections intersect each node in figure 6 precisely once. For the four-section $\hat{s}_3^{(4)}$ we note that the node corresponding to the simple root of $\text{SU}(2)$ in figure 6 is not intersected. Thus, the appropriate class for computing \mathbb{Z}_4 charges based on $\hat{s}_3^{(4)}$ is

$$\sigma_{\mathbb{Z}_4}(\hat{s}_3^{(4)}) = S_3^{(4)} + \frac{1}{2}[K_B^{-1}] - \frac{3}{2}\mathcal{S}_7 + \frac{1}{2}\mathcal{S}_9. \tag{3.48}$$

It is straightforward to check that the divisors (3.47) and (3.48) obey all properties of a Shioda map [34].

Next, we analyze the codimension two singularities of the WSF of $J(X_{F_4})$ to determine the charged matter spectrum. We find two codimension two singularities leading to the matter representations and the corresponding codimension two fibers in X_{F_4} that are given

| Representation | Multiplicity | Fiber | Locus |
|----------------|---|----------|--------------------------------|
| $\mathbf{2}_1$ | $(6[K_B^{-1}] + 2\mathcal{S}_7 - 2\mathcal{S}_9) \times ([K_B^{-1}] - \mathcal{S}_7 + \mathcal{S}_9)$ | | $V(I_{(1)})$, given by (3.49) |
| $\mathbf{1}_2$ | $6[K_B^{-1}]^2 + 13[K_B^{-1}]\mathcal{S}_7 - 3\mathcal{S}_7^2 - 5[K_B^{-1}]\mathcal{S}_9 - 2\mathcal{S}_7\mathcal{S}_9 + \mathcal{S}_9^2$ | | $V(I_{(2)})$, given by (3.50) |
| $\mathbf{3}$ | $1 + ([K_B^{-1}] - \mathcal{S}_7 + \mathcal{S}_9) \frac{\mathcal{S}_9 - \mathcal{S}_7}{2}$ | Figure 6 | $d_9 = 0$ |

Table 5. Charged matter representations under $SU(2) \times \mathbb{Z}_4$ and corresponding codimension two fibers of X_{F_4} . The adjoint matter is included for completeness.

in the first and second entry of table 5, respectively.²¹ We have also added the adjoint matter at the divisor $\mathcal{S}_{SU(2)}^b = \mathcal{S}_9$ for completeness. We have checked the representation content at the two codimension two loci explicitly by computation of the Dynkin labels using (2.12) with D_1 given in (3.45) and using the charge formula (2.13) for (3.47) or (3.48). We note that the charges calculated from the two two-sections are half the integral charges computed from the four-section. However, both charges are physically equivalent since in the case of the two-section we have to calculate modulo two, whereas for the four-section, we calculate modulo four. In other words, we obtain charges in two different conventions. Here we choose the charge convention where all discrete charges are integer, which agrees with the charges computed using $\sigma_{\mathbb{Z}_4}(\hat{s}_3^{(4)})$.

The codimension two locus supporting the representation $\mathbf{2}_1$ is given as the following complete intersection, that can be read off directly from the discriminant of $J(X_{F_4})$:

$$\begin{aligned}
 I_{(1)} := & \{d_9, d_5^2 d_6^4 - d_4 d_5 d_6^3 d_7 + d_3 d_5 d_6^2 d_7^2 - d_2 d_5 d_6 d_7^3 + d_1 d_5 d_7^4 + d_4^2 d_6^3 d_8 - 2d_3 d_5 d_6^3 d_8 \\
 & - d_3 d_4 d_6^2 d_7 d_8 + 3d_2 d_5 d_6^2 d_7 d_8 + d_2 d_4 d_6 d_7^2 d_8 - 4d_1 d_5 d_6 d_7^2 d_8 - d_1 d_4 d_7^3 d_8 + d_3^2 d_6^2 d_8^2 \\
 & - 2d_2 d_4 d_6^2 d_8^2 + 2d_1 d_5 d_6^2 d_8^2 - d_2 d_3 d_6 d_7 d_8^2 + 3d_1 d_4 d_6 d_7 d_8^2 + d_1 d_3 d_7^2 d_8^2 + d_2^2 d_6 d_8^3 \\
 & - 2d_1 d_3 d_6 d_8^3 - d_1 d_2 d_7 d_8^3 + d_1^2 d_8^4\}.
 \end{aligned} \tag{3.49}$$

This ideal is easily checked to be prime. The discrete charge of the $SU(2)$ doublet can be computed from the intersection of $\sigma_{\mathbb{Z}_4}(\hat{s}_3^{(4)})$ with the irreducible fiber components, which are depicted in table 5. There one observes that the four-section intersects one node three times

²¹We note that on the charged matter spectrum in table 5, the \mathbb{Z}_2 subgroup of \mathbb{Z}_4 is realized as the center of $SU(2)$. We thank Iñaki García-Etxebarria for discussions on this subtlety.

and another one just once. Recall again that 6D hyper multiplets with discrete charges 1 and 3 are physically equivalent since a hyper multiplet contains two half-hypers with charges 1 and -1 and charge -1 is identified with charge 3 under the discrete \mathbb{Z}_4 -symmetry.

The complete intersection supporting the representation $\mathbf{1}_2$ is obtained by directly searching for the loci of degeneration of the fiber of X_{F_4} to I_2 . One makes a general factorization ansatz of (3.17) and determines the ideal of constraints imposed by this factorization [55, 56]. Then one eliminates the unknown variables introduced in this ansatz by computation of the elimination ideal, see sections 3.2.1 and 3.2.2 for an explanation of this technique. The obtained elimination ideal, denoted by $I_{(2)}$, is prime and codimension two in the ring of coefficients d_i . It reads

$$\begin{aligned}
 I_{(2)} := & \{ (d_8^2 - 4d_5d_9)^3 d_1 - (d_5d_6d_7^2d_8^3 - d_5d_6^2d_8^4 - d_4d_6d_7d_8^4 + d_3d_6d_8^5 - d_5^2d_7^4d_9 \\
 & - d_3^2d_8^4d_9 + 2d_4d_5d_7^3d_8d_9 + 8d_5^2d_6^2d_8^2d_9 + 4d_4d_5d_6d_7d_8^2d_9 - d_4^2d_7^2d_8^2d_9 - 2d_3d_5d_7^2d_8^2d_9 \\
 & + d_4^2d_6d_8^3d_9 - d_4^4d_9^3 - 8d_3d_5d_6d_8^3d_9 + 2d_3d_4d_7d_8^3d_9 - 16d_5^3d_6^2d_9^2 - 2d_4^2d_5d_7^2d_9^2 \\
 & + 8d_3d_5^2d_7^2d_9^2 - 4d_4^2d_5d_6d_8d_9^2 + 8d_3d_4^2d_5d_9^3 + 16d_3d_5^2d_6d_8d_9^2 + 2d_4^3d_7d_8d_9^2 \\
 & - 8d_3d_4d_5d_7d_8d_9^2 - 2d_3d_4^2d_8^2d_9^2 + 8d_3^2d_5d_8^2d_9^2 - 16d_3^2d_5^2d_9^3 - 4d_5^2d_6d_7^2d_8d_9), \\
 & (d_8^2 - 4d_5d_9)^2 d_2 - (d_5d_7^3d_8 - 2d_5d_6d_7d_8^2 - d_4d_7^2d_8^2 + d_4d_6d_8^3 + d_3d_7d_8^3 - 2d_4^3d_9^2 \\
 & + 8d_5^2d_6d_7d_9 - 2d_4d_5d_7^2d_9 - 4d_4d_5d_6d_8d_9 + 3d_4^2d_7d_8d_9 - 4d_3d_5d_7d_8d_9 \\
 & - 2d_3d_4d_8^2d_9 + 8d_3d_4d_5d_9^2) \} : I_{\text{rest}}^7, \tag{3.50}
 \end{aligned}$$

where we have indicated the quotient by the ideal $I_{\text{rest}} := \{2d_4d_9 - d_7d_8, 4d_5d_9 - d_8^2\}$ by “:”.

The multiplicity of 6D hyper multiplets in the representation $\mathbf{2}_1$ is computed as the product of the classes of the two constraints in the complete intersection $I_{(1)}$. The multiplicity of matter in the $\mathbf{1}_2$ representation is more involved since the locus described by $I_{(2)}$, $V(I_{(2)})$, is one of three irreducible components of the complete intersection in the first ideal in (3.50). Using the resultant technique [39] we decompose this complete intersection into $V(I_{(2)}) + V(\{s_8, s_9\}) + 8 \cdot V(\{2d_4d_9 - d_7d_8, 4d_5d_9 - d_8^2, d_4d_8 - 2d_5d_7\})$. This allows us to obtain the multiplicity shown in table 5 that we double check following the arguments in [55] explained around (3.41).

We complete the matter spectrum of X_{F_4} by the number of neutral hyper multiplets, which is computed from (2.11) using the Euler number of X_{F_4} in (C.1). It reads

$$H_{\text{neut}} = 13 + 11[K_B^{-1}]^2 - 4[K_B^{-1}]\mathcal{S}_7 + 6\mathcal{S}_7^2 - 4[K_B^{-1}]\mathcal{S}_9 - 4\mathcal{S}_7\mathcal{S}_9 + 2\mathcal{S}_9^2. \tag{3.51}$$

Finally, we use $S_{\text{SU}(2)}^b = \{d_9 = 0\}$, the charged spectrum in table 5 and (3.51) together with the number of vector multiplets $V = 1$ to check that all 6D anomalies (A.1) are canceled.

We conclude this section with the computation of the Yukawa couplings. We find the single Yukawa coupling given in table 6. In order to check that it is realized at codimension three in B we compute the associated prime ideals of the ideal $I_{(1)} \cup I_{(2)}$. Indeed, we find that it is codimension three in the ring, as required for the existence of the Yukawa coupling.

| Yukawa | Locus |
|---|--|
| $\mathbf{2}_1 \cdot \mathbf{2}_1 \cdot \overline{\mathbf{1}}_2$ | $V(I_{(1)}) \cap V(I_{(1)}) \cap V(I_{(2)})$ |

Table 6. Codimension three locus and corresponding Yukawa coupling for X_{F_4} .

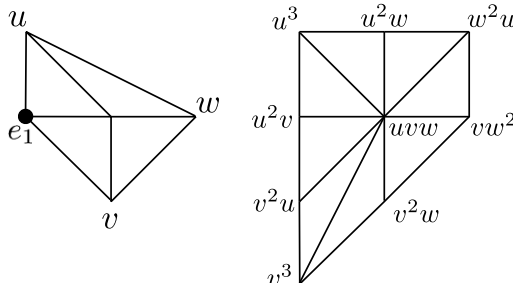


Figure 7. Polyhedron F_3 with a choice of projective coordinates and its dual F_{14} with the corresponding monomials. We have set $e_1 = 1$ for brevity of our notation. The zero section is indicated by the dot.

3.3 Fibration with gauge groups of rank 1, 2 and no discrete gauge symmetry

In this section we analyze all toric hypersurface fibrations X_{F_i} with gauge groups of rank one and two, but without discrete gauge symmetries. They are constructed using the fiber polyhedra F_3 , F_5 and F_6 . Apart from X_{F_3} , that possesses a non-toric section, all other X_{F_i} considered here can be analyzed using techniques already developed e.g. in [39, 42, 44].

3.3.1 Polyhedron F_3 : $G_{F_3} = \mathbf{U}(1)$

We construct a Calabi-Yau manifold, denoted X_{F_3} , as a fibration of the toric hypersurface in $\mathbb{P}_{F_3} = dP_1$ over a base B . The polyhedron of F_3 along with a choice of projective coordinates as well as its dual polyhedron are depicted in figure 7. The coordinate e_1 vanishes on the exceptional divisor E_1 of dP_1 and $[u : v : w]$ are the pullback under the blow-down map $dP_1 \rightarrow \mathbb{P}^2$ of the \mathbb{P}^2 -coordinates. The SR-ideal of dP_1 reads

$$SR_{F_3} = \{uv, we_1\}. \quad (3.52)$$

Using (2.23) we construct the Calabi-Yau manifold X_{F_3} as the hypersurface

$$p_{F_3} = s_1 u^3 e_1^2 + s_2 u^2 v e_1^2 + s_3 u v^2 e_1^2 + s_4 v^3 e_1^2 + s_5 u^2 w e_1 + s_6 u v w e_1 + s_7 v^2 w e_1 + s_8 u w^2 + s_9 v w^2, \quad (3.53)$$

in the ambient space (3.1), that in the case at hand is a dP_1 -fibration over B . The coordinates $[u : v : w : e_1]$ and the coefficients s_i take values in the line bundles in (3.8).

The Calabi-Yau manifold X_{F_3} is an elliptic fibration. This is clear because for a generic point on B there is one marked point P_0 on its fiber, which is the intersection of $e_1 = 0$ with (3.53). This point gives rise to a section of X_{F_3} , which we choose as the zero section. Its generic coordinates read

$$\hat{s}_0 = X_{F_3} \cap \{e_1 = 0\} : [s_9 : -s_8 : 1 : 0]. \quad (3.54)$$

There always exists a second section of X_{F_3} , which generates a rank one MW-group.²² We emphasize that this second section is not toric, i.e. not given as the intersection of a toric divisor in the fiber \mathbb{P}_{F_3} with the hypersurface (3.53), in contrast to the zero section (3.54).

This can be seen as follows. Without loss of generality, set $e_1 = 1$ in (3.53) and consider it as an elliptic curve \mathcal{E} over a field K . Then construct the tangent t_P to the point P_0 which now is at $[u : v : w] = [0 : 0 : 1]$. It is determined by requiring that along t_P both p_{F_3} and its first derivative vanish at P_0 , i.e. that P_0 is a point of intersection two of \mathcal{E} and t_P . It is described by

$$t_P = s_8 u + s_9 v. \tag{3.55}$$

Since (3.53) is a curve of degree three, every line has to intersect it at three points. Thus, $t_P = 0$ intersects \mathcal{E} at a third point, denoted by P_1 , which is automatically rational. It gives rise to a *rational section* of X_{F_3} , with generic coordinates

$$\hat{s}_1 = X_{F_3} \cap \{t_P = 0\} : [-s_9 : s_8 : s_1 s_9^3 - s_4 s_8^3 + s_3 s_9 s_8^2 - s_2 s_9^2 s_8 : s_7 s_8^2 - s_6 s_9 s_8 + s_5 s_9^2]. \tag{3.56}$$

Thus, the elliptic fibration X_{F_3} indeed has a rank one MW-group with a non-toric generator, as claimed. The Shioda map (2.5) of the section \hat{s}_1 reads

$$\sigma(\hat{s}_1) = S_1 - S_0 + 3[K_B] + \mathcal{S}_7 - 2\mathcal{S}_9, \tag{3.57}$$

where S_1, S_0 are the divisor classes of the rational sections \hat{s}_1 and \hat{s}_0 .

This result allows us to compute the height pairing of the section \hat{s}_1 . We obtain

$$b_{11} = -2(3[K_B] + \mathcal{S}_7 - 2\mathcal{S}_9), \tag{3.58}$$

where we employed (2.6) along with the self-intersection (2.7) for the section \hat{s}_1 as well as

$$\pi(S_1 \cdot S_0) = [z_1] = 2[K_B^{-1}] + 2\mathcal{S}_9 - \mathcal{S}_7. \tag{3.59}$$

This follows by noting that $\pi(S_0 \cdot S_1)$ is the locus in B where the coordinates (3.54) and (3.56) of the two sections agree, which happens at $z_1 := s_7 s_8^2 - s_6 s_8 s_9 + s_5 s_9^2 = 0$, that is precisely the z -coordinate of \hat{s}_1 in the WSF, cf. (3.60). The divisor class of z_1 is read off from (3.8).

Weierstrass form and gauge group

We can apply Nagell's algorithm to the cubic (3.53) with respect to the point P_0 to obtain a birational map to its WSF. We plug the coordinates of the rational section (3.56) into this map to obtain its coordinates in WSF,

$$z_1 = s_7 s_8^2 - s_6 s_8 s_9 + s_5 s_9^2, \quad x_1 = s_4^2 s_8^6 + \dots = p_8(\underline{s}), \quad y_1 = -s_4^3 s_8^9 + \dots = p_{12}(\underline{s}). \tag{3.60}$$

Here $p_8(\underline{s})$ and $p_{12}(\underline{s})$ are two homogeneous polynomials in the coefficients s_i of degree eight and twelve, respectively. We have written out only one monomial in x_{Q_1} and y_{Q_1} ,

²²We note that this is not in contradiction with the results of [41]. There the toric Mordell-Weil group is computed, which is indeed trivial.

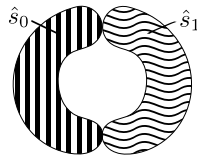
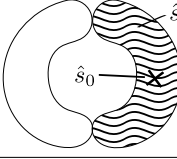
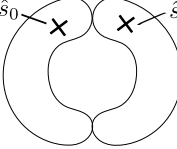
| Representation | Multiplicity | Fiber | Locus |
|----------------|--|---|--|
| $\mathbf{1}_3$ | $\mathcal{S}_9([K_B^{-1}] + \mathcal{S}_9 - \mathcal{S}_7)$ |  | $V(I_{(3)}) := \{s_8 = s_9 = 0\}$ |
| $\mathbf{1}_2$ | $6[K_B^{-1}]^2 + [K_B^{-1}](4\mathcal{S}_9 - 5\mathcal{S}_7) + \mathcal{S}_7^2 + 2\mathcal{S}_7\mathcal{S}_9 - 2\mathcal{S}_9^2$ |  | $V(I_{(2)}) := \{s_4 s_8^3 - s_3 s_8^2 s_9 + s_2 s_8 s_9^2 - s_1 s_9^3 = s_7 s_8^2 + s_5 s_9^2 - s_6 s_8 s_9 = 0$ with $(s_8, s_9) \neq (0, 0)\}$ |
| $\mathbf{1}_1$ | $12[K_B^{-1}]^2 + [K_B^{-1}](8\mathcal{S}_7 - \mathcal{S}_9) - 4\mathcal{S}_7^2 + \mathcal{S}_7\mathcal{S}_9 - \mathcal{S}_9^2$ |  | $V(I_{(1)}) := \{(3.62)\} \setminus (V(I_{(2)}) \cup V(I_{(3)}))$ |

Table 7. Charged matter representation under U(1) and codimension two fibers of X_{F_3} .

respectively, in order to be able to determine their divisor classes. We refer the reader to (B.8) in appendix B for the explicit and lengthy expressions for $p_8(\underline{s})$ and $p_{12}(\underline{s})$.

Furthermore, we determine the functions f , g and the discriminant Δ of the WSF for X_{F_3} . They are given by specializing (B.1) as $s_{10} = 0$. We observe that there is no factorization of Δ indicating the absence of codimension one singularities and a non-Abelian gauge group. Thus, the full gauge group on X_{F_3} is given by the single U(1) associated to its rank one MW-group,

$$G_{F_3} = \text{U}(1). \quad (3.61)$$

We emphasize again that the generator (3.56) of the MW-group of X_{F_3} is not toric.

Charged and uncharged matter

Since the Calabi-Yau manifold X_{F_3} has a non-trivial MW-group, it automatically has I_2 -fibers at codimension two in B , that support U(1)-charged matter.

We first summarize the charged matter spectrum of X_{F_3} before we discuss its derivation in detail. The full charged matter spectrum is shown in table 7, which includes the U(1)-charges and the multiplicities of 6D charged hyper multiplets, as well as a schematic presentation of the corresponding reducible fibers and the full expressions for the codimension two loci.

The starting point for the derivation of the matter spectrum of X_{F_3} is, as discussed in section 2.3, the complete intersection (2.14) in B with the WS-coordinates (3.56) of the section \hat{s}_1 inserted:

$$y_1 = fz_1^4 + 3x_1^2 = 0. \quad (3.62)$$

We show that (3.62) is a reducible variety with three irreducible components supporting matter with charges one, two and three. The corresponding codimension two loci are

denoted $V(I_{(1)})$, $V(I_{(2)})$ and $V(I_{(3)})$, respectively, with $I_{(1)}$, $I_{(2)}$ and $I_{(3)}$ denoting the corresponding prime ideals, cf. table 7. In order to strictly prove that these three varieties are all irreducible components of the complete intersection (3.62), we have to compute all its associated prime ideals. Unfortunately, this is unfeasible with the available computing power and computer algebra programs, due to the high degree of the two polynomials in (3.62). However, we explain that X_{F_3} has three possible types of I_2 -fibers corresponding to the three possible factorizations of (3.53) and that these factorization happen precisely at the codimension two loci $V(I_{(1)})$, $V(I_{(2)})$ and $V(I_{(3)})$. Thus, we claim that the corresponding ideals $I_{(1)}$, $I_{(2)}$ and $I_{(3)}$ are the only associated prime ideals of (3.62). We will further substantiate this claim by checking 6D anomaly cancellation at the end of this section as well as by reproducing the spectrum of X_{F_3} by Higgsing the effective theories of X_{F_5} and X_{F_6} , see section 4.

We begin by analyzing the fiber at the first two codimension two loci in table 7. These are precisely the loci where the coordinates (3.56) of the section \hat{s}_1 are ill-defined, since they are forbidden by the SR-ideal (3.52). This indicates, that the section \hat{s}_1 does not mark a point on the elliptic fiber of X_{F_3} , but does wrap an entire \mathbb{P}^1 . Since the rational section is non-toric, determining the wrapped \mathbb{P}^1 is slightly more involved than usual, as we demonstrate next.

First, we consider the locus $V(I_{(3)}) = \{s_8 = s_9 = 0\}$, which we readily check to obey (3.62). At this locus the constraint (3.53) factorizes as

$$p_{F_3}|_{s_8=s_9=0} = e_1(s_1u^3e_1 + s_2u^2ve_1 + s_3uv^2e_1 + s_4v^3e_1 + s_5u^2w + s_6uvw + s_7v^2w). \quad (3.63)$$

Clearly, $V(I_{(3)})$ is the only codimension two locus where this factorization can occur. We immediately observe that the zero section \hat{s}_0 defined by (3.54) has wrapped the entire rational curve $e_1 = 0$ in (3.63). The rational section \hat{s}_1 can be identified at this locus by recalling the definition of the point (3.56) as the second intersection point of the tangent to P_0 with \mathcal{E} . However, at $s_8 = s_9 = 0$ the curve (3.63) is singular (after setting $e_1 = 1$) precisely at P_0 . Thus, every line through P_0 is automatically tangential at P_0 . This simply means that P_1 has become the entire singular fiber at $s_8 = s_9 = 0$, since given any point on (3.63) (for $e_1 = 1$) we can construct a tangent at P_0 that passes through that point. Thus, at $s_8 = s_9 = 0$ the section \hat{s}_1 wraps the rational curve described by the parenthesis in (3.63). The resulting fiber at $V(I_{(3)})$ is shown in the second column of table 7. We readily compute using the charge formula (2.13) that the U(1) charge of the matter is indeed $q = 3$ and its multiplicity is given by $[s_8] \cdot [s_9]$, which after using (3.8), yields the result shown in table 7. We emphasize that this is the first occurrence of matter with charge $q > 2$ in models with Abelian gauge symmetry in F-theory.

Second, we consider the locus $V(I_{(2)})$. The complete intersection in $V(I_{(2)})$ shown in table 7 has two irreducible components, one of which given by $V(I_{(3)})$, that we forbid by requiring $(s_8, s_9) \neq (0, 0)$, and a second one described by a prime ideal $I_{(2)}$ with ten generators.²³ The variety $V(I_{(2)})$ supports matter of charge two. We can check this locally by solving the complete intersection inside $V(I_{(2)})$ e.g. for s_3 and s_6 and by plugging this solution

²³As all prime ideals in this work, it is computed by the primary decomposition function in Singular [88].

into (3.53). Indeed, the fiber splits into a line and a non-singular quadric $q_2(e_1u, e_1v, w)$,

$$p_{F_3} \rightarrow (s_8u + s_9v)q_2(e_1u, e_1v, w). \quad (3.64)$$

Furthermore, we prove that $V(I_{(2)})$ is the only locus that can yield an I_2 -fiber of this type by computing the elimination ideal of the ideal of constraints necessary for the factorization (3.64). We see that the zero section (3.54) is well-defined at $V(I_{(2)})$ and passes through the line. However, the rational section (3.56) is ill-defined. This is clear because the line in (3.64) is precisely the tangent t_P at P_0 defined in (3.55) and since the section \hat{s}_1 is defined as the intersection of t_P with \mathcal{E} . Thus, the section \hat{s}_1 at $V(I_{(2)})$ wraps the entire rational curve given by the line in (3.64). Again we use (2.13) to show that the $U(1)$ -charge is $q = 2$, as claimed in table 7. The multiplicity of a 6D hyper multiplet in the representation $\mathbf{1}_2$ is given by the homology class of $V(I_{(2)})$. It is computed by first computing the homology class of the complete intersection in $V(I_{(2)})$ in table 7 using (3.8) and by subtracting the class of the unwanted component $V(I_{(3)})$ with the appropriate order. We determine it to be six using the resultant technique of [39], which precisely yields the multiplicity in the third row of table 7.

Finally, we turn to the codimension two locus $V(I_{(1)})$ supporting matter of charge one. In order for the charge formula (2.13) to produce charge one for an I_2 -fiber, both \hat{s}_0 and \hat{s}_1 have to be regular and pass through different rational curves in the I_2 -fiber. This can only happen for a factorization of (3.53) of the form (we can set $e_1 = 1$)

$$p_{F_3} \rightarrow (d_1u + d_2v + d_3w)q_2(u, v, w), \quad (3.65)$$

with $q_2(u, v, w)$ denoting a quadric without the monomial w^2 . We note that all coefficients d_i , $i = 1, 2, 3$, have to be non-vanishing since $d_1 = 0$, $d_2 = 0$ or $d_3 = 0$ lead to a factorization in (3.65) that cannot happen at codimension two. We see that \hat{s}_0 intersects the quadric $q_2 = 0$ and \hat{s}_1 intersects the line, as required for matter with charge one. Furthermore, we compute the elimination ideal, denoted by $I_{(1)}$, of the ideal of constraints necessary for the factorization (3.65). It is prime and of codimension two in the ring, that means that the factorization (3.65), indeed, occurs in codimension two in B . In addition, we check that the complete intersection (2.14) is inside the ideal $I_{(1)}$ and that $I_{(1)}$ is in turn not contained in $I_{(3)}$ or $I_{(2)}$, as required. Thus, we identify $I_{(1)}$ as the third and last associated prime ideal of (3.62).

Under the well-motivated assumption that $I_{(3)}$, $I_{(2)}$ and $I_{(1)}$ are the only associated prime ideals of (3.62), we determine the multiplicity of the $\mathbf{1}_1$ -matter as follows. First, we determine the orders of the loci $V(I_{(3)})$ and $V(I_{(2)})$ in the complete intersection (2.13). Using the resultant technique of [39] and random integers for some of the s_i we find the orders 81 and 16 for these loci, respectively. Then, we subtract their multiplicities with these orders from the class of the complete intersection (3.62) and obtain, using (3.8), the multiplicity in the last row of table 7.

The matter spectrum of X_{F_3} is completed by the number of neutral hyper multiplets H_{neut} . Employing (2.11) and the Euler number $\chi(X_{F_3})$ of X_{F_3} given in (C.1), we obtain

$$H_{\text{neutral}} = 13 + 11[K_B^{-1}]^2 - 3[K_B^{-1}]\mathcal{S}_7 + 3\mathcal{S}_7^2 - 4[K_B^{-1}]\mathcal{S}_9 - 2\mathcal{S}_7\mathcal{S}_9 + 2\mathcal{S}_9^2. \quad (3.66)$$

| Yukawa | Locus |
|---|--|
| $\mathbf{1}_1 \cdot \mathbf{1}_1 \cdot \overline{\mathbf{1}}_2$ | $V(I_{(1)}) \cap V(I_{(2)})$ |
| $\mathbf{1}_1 \cdot \mathbf{1}_2 \cdot \overline{\mathbf{1}}_3$ | $V(I_{(1)}) \cap V(I_{(2)}) \cap V(I_{(3)})$ |

Table 8. Codimension three loci and corresponding Yukawa couplings for X_{F_3} .

Finally, we check anomaly-freedom of the full 6D SUGRA theory. To this end we use (3.58), the charged spectrum in table 7 and (3.66), together with $V = 1$, to show that all relevant anomalies of the 6D SUGRA theory in (A.1) are canceled.

There is another quantum consistency condition the spectrum in table 7 has to pass. In order to have an effective theory that makes sense also in a quantum gravity model, it has been argued in [89] that all charges allowed by Dirac quantization have to be present in the spectrum. Indeed, it is clear from the multiplicity formulas in table 7 (e.g. by evaluation for a concrete base B , see section 4.1.3) that if matter with a maximal charge q is present in the spectrum, also matter with all lower charges $q' < q$ is automatically there, as required.

For completeness, we include a discussion of the Yukawa couplings. Forming the union of the ideals and computing their codimension to be three in the polynomial ring $K[s_i]$, we find the two Yukawa couplings given in table 8.

An alternative perspective: X_{F_3} from X_{F_5} by an extremal transition

There is a second perspective on X_{F_3} that provides an alternative explanation for the presence of the rational point (3.56) and that will be useful for the understanding of the Higgs transition in section 4. The following can be skipped on a first reading, as it is not important for the main thread of this work.

We begin by noting that (3.53) becomes singular if we tune the complex structure so that $s_4 \equiv 0$. The induced I_2 -singularities occur at codimension two and can be resolved by the blow-up in the fiber at $u = w = 0$. The Calabi-Yau manifold after this extremal transition is precisely X_{F_5} , that we discuss below in section 3.3.2. It has been shown that X_{F_5} has a rank two Mordell-Weil group [38, 39].

In the singular fibration with all exceptional divisors blown down, the three rational points on the fiber \mathcal{C}_{F_5} are the three intersection points with the line $u = 0$. One point agrees with the origin (3.54) of X_{F_3} . We denote the other two points by Q_1, Q_2 . This implies that the point $Q_1 + Q_2$ is precisely given by (3.56), in the limit $s_4 \equiv 0$. Indeed, the group law on a cubic curve is defined so that the point $Q_1 + Q_2$ is found by first constructing the third intersection point of the line through Q_1 and Q_2 and then by forming the line through that point and the origin P_0 . This line again has a third intersection point with the curve, which is defined to be $Q_1 + Q_2$. In our situation, the line through Q_1 and Q_2 is $u = 0$. Thus, the third intersection point of $u = 0$ with \mathcal{E} is the origin P_0 . Consequently, the point $Q_1 + Q_2$ is the second intersection point of the *tangent* through P_0 with the elliptic curve. In fact, it can be checked by performing this addition on the fiber of X_{F_5} explicitly that the coordinates of the point $Q_1 + Q_2$ on the fiber of X_{F_3} agree with the coordinates (3.56)

after setting $s_4 \equiv 0$. Furthermore, we compute the Weierstrass coordinates of $Q_1 + Q_2$ that also agree with (3.60) after setting $s_4 \equiv 0$.²⁴

This is not surprising since we recall that the P_1 in X_{F_3} has been constructed as the second intersection of the tangent (3.55) to P_0 . Thus, we see that the section \hat{s}_1 can be understood as the sum of the sections $\hat{s}_1 + \hat{s}_2$ on X_{F_5} , which *survives* the extremal transition $X_{F_5} \leftrightarrow X_{F_3}$, i.e. the complex structure deformation associated to switching on s_4 . In contrast, the individual sections \hat{s}_1 and \hat{s}_2 on X_{F_5} do not map to rational sections on X_{F_3} . As consequence, cf. section 4, the U(1)-charges of matter in X_{F_3} are given by the sum of the U(1)-charges $q_1 + q_2$ on X_{F_5} .

We can make these statements even more explicit by mapping X_{F_3} to X_{F_5} . The shift

$$w \mapsto w - \frac{s_7 + \sqrt{s_7^2 - 4s_4s_9}}{2s_9} e_1 v \tag{3.67}$$

precisely cancels the monomial proportional to s_4 in (3.53). Clearly, this requires an extension of the field of meromorphic functions on B by the square root $\sqrt{s_7^2 - 4s_4s_9}$. Thus, this map is certainly not birational. After this shift, we precisely obtain the hypersurface of X_{F_5} , cf. (3.72) for $e_2 = 1$. Due to the shift (3.67), the coefficients s_i in (3.72) have to be replaced by

$$\begin{aligned} s_2 \mapsto s_2 - s_5 \frac{s_7 + \sqrt{s_7^2 - 4s_4s_9}}{2s_9}, \quad s_3 \mapsto s_3 - \frac{s_4s_8}{s_9} + \frac{(s_7s_8 - s_6s_9)(s_7 + \sqrt{s_7^2 - 4s_4s_9})}{s_9^2}, \\ s_6 \mapsto s_6 - s_8 \frac{s_7 + \sqrt{s_7^2 - 4s_4s_9}}{s_9}, \quad s_7 \mapsto -\sqrt{s_7^2 - 4s_4s_9}, \end{aligned} \tag{3.68}$$

with s_1, s_5, s_8 and s_9 unchanged. If we insert this variable transformation into the expressions for \hat{s}_1 or \hat{s}_2 in (3.73), we introduce square roots, i.e. these sections do not map to rational sections on X_{F_3} . However, if we insert (3.68) into the coordinates for $\hat{s}_1 + \hat{s}_2$ on X_{F_5} , we precisely reproduce (3.56), i.e. all square roots cancel.

Furthermore, we can re-derive the Weierstrass coordinates (3.60) of \hat{s}_1 on X_{F_3} by first computing the Weierstrass coordinates of $Q_1 + Q_2$ and then inserting (3.68). In addition, f and g of the WSF of X_{F_3} can be obtained from the WSF for X_{F_5} by insertion of (3.68).

3.3.2 Polyhedron F_5 : $G_{F_5} = \text{U}(1)^2$

The toric hypersurface fibration X_{F_5} is constructed as the fibration of the elliptic curve in $\mathbb{P}_{F_5} = dP_2$. As it is completely analyzed in [39, 43], we only state the results here for completeness.

The toric diagram of F_5 along with a choice of homogeneous coordinates as well as its dual polyhedron are depicted in figure 8. In the monomials corresponding to the integral points of F_{12} by (2.23) we have set $e_i = 1, \forall i$. The toric variety \mathbb{P}_{F_5} is the blow-up of \mathbb{P}^2 , cf. section 3.1.2, at two points, i.e. dP_2 . The blow-up map reads

$$u \rightarrow e_1 e_2 u, \quad v \rightarrow e_2 v, \quad w \rightarrow e_1 w. \tag{3.69}$$

²⁴The coordinates of $Q_1 + Q_2$ in WSF are obtained by inserting its coordinates into the birational map from X_{F_5} to its WSF. We note that the result agrees with the WS-coordinates of $Q_1 + (-Q_2)$, *not* $Q_1 + Q_2$, where ‘+’ denotes here the addition in the WSF of X_{F_5} .

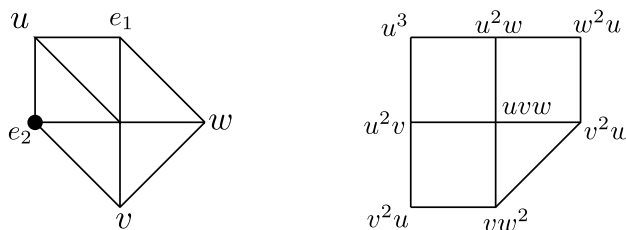


Figure 8. The toric diagram of polyhedron F_5 (dP_2) and its dual. The zero section is indicated by the dot.

The homogeneous coordinates on dP_2 after this blow-up are $[u : v : w : e_1 : e_2]$ and are sections in line bundles associated to the divisors classes

| Section | Divisor class | |
|---------|---|--------|
| u | $H - E_1 - E_2 + \mathcal{S}_9 + [K_B]$ | (3.70) |
| v | $H - E_2 + \mathcal{S}_9 - \mathcal{S}_7$ | |
| w | $H - E_1$ | |
| e_1 | E_1 | |
| e_2 | E_2 | |

The Stanley-Reisner ideal of \mathbb{P}_{F_5} is given by

$$SR = \{we_2, wu, ve_1, e_2e_1, vu\}. \tag{3.71}$$

By use of (2.23) the hypersurface equation for X_{F_5} in the dP_2 -fibration (3.1) is given by

$$p_{F_5} = s_1e_2^2e_1^2u^3 + s_2e_2^2e_1u^2v + s_3e_2^2uv^2 + s_5e_2e_1^2u^2w + s_6e_2e_1uvw + s_7e_2v^2w + s_8e_1^2uw^2 + s_9e_1vw^2, \tag{3.72}$$

where the sections s_i take values in the line bundles shown in (3.8). We see that (3.72) can also be obtained from (3.4) by the specialization $s_4 = s_{10} = 0$ and the map (3.69).

There are three rational sections of the fibration of X_{F_5} with the coordinates

$$\begin{aligned} \hat{s}_0 &= X_{F_5} \cap \{e_2 = 0\} : [s_9 : -s_8 : 1 : 1 : 0], \\ \hat{s}_1 &= X_{F_5} \cap \{e_1 = 0\} : [s_7 : 1 : -s_3 : 0 : 1], \\ \hat{s}_2 &= X_{F_5} \cap \{u = 0\} : [0 : 1 : 1 : s_7 : -s_9], \end{aligned} \tag{3.73}$$

where we choose \hat{s}_0 as the zero section.

The Weierstrass form (2.1) of (3.72) can be computed using Nagell’s algorithm. The WS-coordinates of the sections \hat{s}_1 and \hat{s}_2 are given by (B.9) and (B.10), respectively. The functions f and g are given by (B.1) and (B.2), respectively, after setting $s_4 = s_{10} = 0$. After using this to calculate the discriminant we do not find any codimension one singularities. Then the total gauge group of X_{F_5} is

$$G_{F_5} = U(1)^2. \tag{3.74}$$

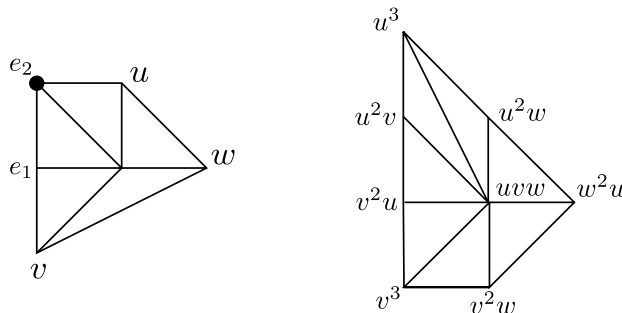


Figure 9. Polyhedron F_6 and its dual F_{11} . The zero section is indicated by the dot.

Thus, the corresponding Shioda maps (2.5) for \hat{s}_1 and \hat{s}_2 read

$$\begin{aligned}\sigma(\hat{s}_1) &= (S_1 - S_0 - [K_B^{-1}]), \\ \sigma(\hat{s}_2) &= (S_2 - S_0 - [K_B^{-1}] - [s_9]),\end{aligned}\tag{3.75}$$

which allows us to compute the corresponding height pairing (2.6) as

$$b_{mn} = \begin{pmatrix} 2[K_B^{-1}] & [K_B^{-1}] + \mathcal{S}_9 - \mathcal{S}_7 \\ [K_B^{-1}] + \mathcal{S}_9 - \mathcal{S}_7 & 2[K_B^{-1}] + 2\mathcal{S}_9 \end{pmatrix}_{mn}.\tag{3.76}$$

To determine the 6D spectrum of charged hyper multiplets we analyze the codimension two singularities of the WSF of X_{F_5} . There are six singularities leading to the matter representations and the corresponding codimension two fibers in X_{F_5} given in the first and third column of table 9. The detailed derivation of these results can be found in [38, 39, 43, 44].

We complete the matter spectrum of X_{F_5} by the number of neutral hyper multiplets, which is computed from (2.11) using the Euler number (C.1). It reads

$$H_{\text{neut}} = 14 + 11[K_B^{-1}]^2 - 4[K_B^{-1}]\mathcal{S}_7 + 2\mathcal{S}_7^2 - 4[K_B^{-1}]\mathcal{S}_9 - \mathcal{S}_7\mathcal{S}_9 + 2\mathcal{S}_9^2.\tag{3.77}$$

The number T of tensor multiplets is given by (2.17) and the number of vector multiplets is $V = 2$. Using the above results it can be checked that all 6D anomalies are canceled. Finally we summarize the codimension three singularities of the WSF of X_{F_5} . This leads to the Yukawa points given in table 10, see [42].

3.3.3 Polyhedron F_6 : $G_{F_6} = \text{SU}(2) \times \text{U}(1)$

We consider an elliptically fibered Calabi-Yau manifold X_{F_6} with an arbitrary base B and general elliptic fiber given by the elliptic curve \mathcal{E} in \mathbb{P}_{F_6} . The toric data of \mathbb{P}_{F_6} is summarized in figure 9, where the polyhedron F_6 along with a choice of homogeneous coordinates as well as its dual polyhedron F_{11} are shown. For brevity, we have set $e_i = 1, \forall i$, in the monomials that are associated to the integral points of F_{11} by (2.23). We note that \mathbb{P}_{F_6} is the blow-up of \mathbb{P}^2 , cf. section 3.1.2, at two points, defined by

$$u \rightarrow e_1 e_2^2 u, \quad v \rightarrow e_1 e_2 v.\tag{3.78}$$

| Representation | Multiplicity | Fiber | Locus |
|------------------------|--|-------|---|
| $\mathbf{1}_{(1,-1)}$ | $\mathcal{S}_7([K_B^{-1}] + \mathcal{S}_7 - \mathcal{S}_9)$ | | $V(I_{(1)}) := \{s_3 = s_7 = 0\}$ |
| $\mathbf{1}_{(1,0)}$ | $6[K_B^{-1}]^2 + [K_B^{-1}](4\mathcal{S}_7 - 5\mathcal{S}_9) - 2\mathcal{S}_7^2 + \mathcal{S}_7\mathcal{S}_9 + \mathcal{S}_9^2$ | | $V(I_{(2)}) := \{s_2s_7^2 + s_3^2s_9 - s_3s_6s_7 = 0, s_5s_3s_7 - s_3^2s_8 - s_7^2s_1 = 0\} \setminus V(I_{(1)})$ |
| $\mathbf{1}_{(-1,-2)}$ | $\mathcal{S}_9([K_B^{-1}] - \mathcal{S}_7 + \mathcal{S}_9)$ | | $V(I_{(3)}) := \{s_8 = s_9 = 0\}$ |
| $\mathbf{1}_{(-1,-1)}$ | $6[K_B^{-1}]^2 + [K_B^{-1}](-5\mathcal{S}_7 + 4\mathcal{S}_9) + \mathcal{S}_7^2 + \mathcal{S}_7\mathcal{S}_9 - 2\mathcal{S}_9^2$ | | $V(I_{(4)}) := \{s_2s_8s_9 - s_3s_8^2 - s_9^2s_1 = 0, s_5s_9^2 - s_6s_8s_9 + s_8^2s_7 = 0\} \setminus V(I_{(3)})$ |
| $\mathbf{1}_{(0,2)}$ | $\mathcal{S}_7\mathcal{S}_9$ | | $V(I_{(5)}) := \{s_9 = s_7 = 0\}$ |
| $\mathbf{1}_{(0,1)}$ | $6[K_B^{-1}]^2 + [K_B^{-1}](4\mathcal{S}_7 + 4\mathcal{S}_9) - 2\mathcal{S}_7^2 - 2\mathcal{S}_9^2$ | | $V(I_{(6)}) := \{s_1s_9^4s_7^2 + (s_3s_9^2 + s_7 \times (-s_6s_9 + s_8s_7))(s_3s_8s_9^2 + s_7 \times (-s_6s_8s_9 + s_8^2s_7 + s_9^2s_5)) = 0, s_2s_9^3s_7^2 + s_3^2s_9^4 - s_3s_6s_9^3s_7 - s_7^3(-s_6s_8s_9 + s_8^2s_7 + s_9^2s_5) = 0\} \setminus (V(I_{(1)}) \cup V(I_{(3)}) \cup V(I_{(4)}) \cup V(I_{(5)}))$ |

Table 9. Charged matter representations under $U(1)^2$ and corresponding codimension two fibers of X_{F_5} .

The homogeneous coordinates on the fiber after this blow-up are $[u : v : w : e_1 : e_2]$ and take values in the line bundles associated to the following divisor classes:

| Section | Divisor class |
|---------|---|
| u | $H - E_1 - E_2 + \mathcal{S}_9 + [K_B]$ |
| v | $H - E_1 - \mathcal{S}_7 + \mathcal{S}_9$ |
| w | H |
| e_1 | $E_1 - E_2$ |
| e_2 | E_2 |

(3.79)

| Yukawa | Locus |
|---|---|
| $\mathbf{1}_{(-1,-2)} \cdot \mathbf{1}_{(0,2)} \cdot \mathbf{1}_{(1,0)}$ | $s_7 = s_8 = s_9 = 0$ |
| $\mathbf{1}_{(0,2)} \cdot \mathbf{1}_{(1,-1)} \cdot \mathbf{1}_{(-1,-1)}$ | $s_3 = s_7 = s_9 = 0$ |
| $\mathbf{1}_{(-1,-1)} \cdot \mathbf{1}_{(1,0)} \cdot \mathbf{1}_{(0,1)}$ | $V(I_{(2)}) \cup V(I_{(4)}) \cup V(I_{(6)})$ |
| $\mathbf{1}_{(1,-1)} \cdot \overline{\mathbf{1}_{(1,0)}} \cdot \mathbf{1}_{(0,1)}$ | $V(I_{(1)}) \cup V(I_{(2)}) \cup V(I_{(6)})$ |
| $\overline{\mathbf{1}_{(-1,-1)}} \cdot \mathbf{1}_{(-1,-2)} \cdot \mathbf{1}_{(0,1)}$ | $V(I_{(3)}) \cup V(I_{(4)}) \cup V(I_{(6)})$ |
| $\mathbf{1}_{(0,1)} \cdot \mathbf{1}_{(0,1)} \cdot \overline{\mathbf{1}_{(0,2)}}$ | $s_7 = s_9 = 4s_1s_3s_8 - s_3s_5^2 + s_2s_5s_6 - s_1s_6^2 - s_2^2s_8 = 0$ |

Table 10. Codimension three loci and corresponding Yukawa couplings for polyhedron F_5 . For the complicated loci we refer to the literature [42, 43]

Here H denotes the pullback of the hyperplane class on \mathbb{P}^2 and the E_i are the exceptional divisors of the blow-up (3.78). The Stanley-Reisner ideal of \mathbb{P}_{F_6} then reads

$$SR = \{uv, ue_1, we_1, we_2, ve_2\}. \tag{3.80}$$

Employing (2.23) the hypersurface equation for X_{F_6} in the \mathbb{P}_{F_6} -fibration (3.1) is

$$p_{F_6} = s_1e_1^2e_2^4u^3 + s_2e_1^2e_2^3u^2v + s_3e_1^2e_2^2uv^2 + s_4e_1^2e_2v^3 + s_5e_1e_2^2u^2w + s_6e_1e_2uvw + s_7e_1v^2w + s_8uw^2, \tag{3.81}$$

where the sections s_i take values in the line bundles shown in (3.8). We note that (3.81) is readily obtained from (3.4) by the specialization $s_9 = s_{10} = 0$ and the map (3.78).

There are two rational sections of the fibration of X_{F_6} . Their coordinates are

$$\begin{aligned} \hat{s}_0 &= X_{F_6} \cap \{e_2 = 0\} : \quad [-s_7 : 1 : s_8 : 1 : 0], \\ \hat{s}_1 &= X_{F_6} \cap \{u = 0\} : \quad [0 : 1 : s_4 : 1 : -s_7], \end{aligned} \tag{3.82}$$

where we choose \hat{s}_0 as the zero section. The corresponding points on \mathcal{E} are denoted P_0 and P_1 , respectively.

We compute the Weierstrass form (2.1) of (3.81) using Nagell’s algorithm. The WS-coordinates of the section \hat{s}_1 are given by (B.8) after setting $s_9 = 0$. Furthermore, the functions f and g take the form of (B.1) and (B.2), respectively, after setting $s_9 = s_{10} = 0$. From this the discriminant Δ is readily computed. This allows us to find all codimension one singularities of the WSF of X_{F_6} . We find one I_2 -singularity over the divisor $\mathcal{S}_{\text{SU}(2)}^b = \{s_8 = 0\} \cap B$ in B . Along this divisor the constraint (3.81) factorizes as

$$\text{SU}(2) : \quad p_{F_6}|_{s_8=0} = e_1 \cdot q_3, \tag{3.83}$$

where q_3 is the polynomial that remains after factoring out e_1 . This is clearly an I_2 -fiber, cf. figure 10, giving rise to an $\text{SU}(2)$ gauge group. In summary, the gauge group of X_{F_6} is

$$G_{F_6} = \text{SU}(2) \times \text{U}(1). \tag{3.84}$$

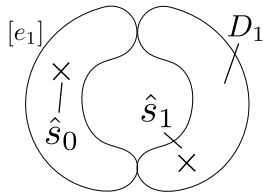


Figure 10. Codimension one fiber of X_{F_6} at $s_8 = 0$ in B . The crosses denote the intersections with the two sections.

The rational curve of the I_2 -fiber in figure 10 that is intersected by the zero section \hat{s}_0 is the affine node and the other rational curve, $c_{-\alpha_1}$, corresponds to the simple root of $SU(2)$. Thus, the class of the $SU(2)$ Cartan divisor in X_{F_6} , which is the fibration of $c_{-\alpha_1}$ over $S_{SU(2)}^b$, reads

$$D_1 = [s_8] - [e_1]. \quad (3.85)$$

This can be seen by noting that $[s_8]$ is the class of the complete I_2 -fiber fibered over the base divisor $S_{SU(2)}^b$, whereas $[e_1]$ is the class of the affine node fibered over $S_{SU(2)}^b$.

With these results, we compute the Shioda map (2.5) of the section \hat{s}_1 as

$$\sigma(\hat{s}_1) = S_1 - S_0 + [K_B] - \mathcal{S}_7 + \frac{1}{2}D_1. \quad (3.86)$$

Here S_0, S_1 denote the divisor classes of the sections \hat{s}_0, \hat{s}_1 , respectively, and we use

$$S_1 \cdot c_{-\alpha_1} = 1, \quad (3.87)$$

which can be deduced from figure 10. Using (3.86), we compute the height pairing (2.6),

$$b_{11} = \frac{3}{2}[K_B^{-1}] + \frac{5}{2}\mathcal{S}_7 - \frac{1}{2}\mathcal{S}_9, \quad (3.88)$$

where we use (2.7) as well as

$$\pi(S_1 \cdot S_0) = \mathcal{S}_7, \quad (3.89)$$

that follows since the coordinates (3.82) of the two sections agree at $s_7 = 0$.

Next, we analyze the codimension two singularities of the WSF of X_{F_6} to determine the charged matter spectrum. Here, the corresponding representations under the gauge group are determined following the general procedure outlined in section 2.1 for the computation of Dynkin labels and $U(1)$ -charges. We find five codimension two singularities. Four of these lead to the matter representations and the corresponding codimension two fibers in X_{F_6} given in the first and third column of table 11, respectively. At the remaining locus $s_8 = s_6^2 - 4s_5s_7 = 0$, the fiber is of Type *III*, i.e. it is a degenerate version of the I_2 -fiber in figure 10 with the two \mathbb{P}^1 's intersecting in one point. Thus it does not support any additional matter.

The spectrum of charged singlets is determined starting from the complete intersection (2.14) for the section \hat{s}_1 , see (3.90) for its explicit expression. By computing its

| Representation | Multiplicity | Fiber | Locus |
|---------------------|---|-----------|--|
| $\mathbf{2}_{-3/2}$ | $\mathcal{S}_7([K_B^{-1}] - \mathcal{S}_7 + \mathcal{S}_9)$ | | $V(I_{(1)}) := \{s_8 = s_7 = 0\}$ |
| $\mathbf{2}_{1/2}$ | $([K_B^{-1}] - \mathcal{S}_7 + \mathcal{S}_9) \times (6[K_B^{-1}] - 2\mathcal{S}_9 + \mathcal{S}_7)$ | | $V(I_{(2)}) := \{s_8 = s_4^2 s_5^3 - s_3 s_4 s_5^2 s_6 + s_2 s_4 s_5 s_6^2 - s_1 s_4 s_6^3 + s_3^2 s_5^2 s_7 - 2s_2 s_4 s_5^2 s_7 - s_2 s_3 s_5 s_6 s_7 + 3s_1 s_4 s_5 s_6 s_7 + s_1 s_3 s_6^2 s_7 + s_2^2 s_5 s_7^2 - 2s_1 s_3 s_5 s_7^2 - s_1 s_2 s_6 s_7^2 + s_1^2 s_7^3 = 0\}$ |
| $\mathbf{1}_2$ | $\mathcal{S}_7(-\mathcal{S}_9 + 2\mathcal{S}_7)$ | | $V(I_{(3)}) := \{s_4 = s_7 = 0\}$ |
| $\mathbf{1}_1$ | $6[K_B^{-1}]^2 + 13[K_B^{-1}]\mathcal{S}_7 - 3\mathcal{S}_7^2 - 5[K_B^{-1}]\mathcal{S}_9 - 2\mathcal{S}_7\mathcal{S}_9 + \mathcal{S}_9^2$ | | $V(I_{(4)}) := \{y_Q = fz_Q^4 + 3x_Q^2 = 0\} \setminus (V(I_{(1)}) \cup V(I_{(3)}))$ with x_Q, y_Q, f given in (3.90) |
| $\mathbf{3}_0$ | $1 + ([K_B^{-1}] - \mathcal{S}_7 + \mathcal{S}_9) \frac{(-\mathcal{S}_7 + \mathcal{S}_9)}{2}$ | Figure 10 | $s_8 = 0$ |

Table 11. Charged matter representations under $SU(2) \times U(1)$ and corresponding codimension two fibers of X_{F_6} . The adjoint matter is included for completeness.

primary decomposition, we find two associated prime ideals, denoted $I_{(3)}$ and $I_{(4)}$, corresponding to two different matter representations $\mathbf{1}_2$ and $\mathbf{1}_1$. We observe that the ideal $I_{(3)}$ describes precisely the locus, where the section 3.82 is ill-defined and has to acquire a fiber component.

There is one subtlety since the constraint (3.81) does not factorize further at the locus $V(I_{(2)})$. In order to see the I_3 -fiber of the representation $\mathbf{2}_{1/2}$ we have to compute the associated prime ideals of (3.81) at the locus $V(I_{(2)})$. Indeed, we find three prime ideals with the right intersections and, they thus, correspond to the three irreducible components of an I_3 -fiber. Two fiber components are described by prime ideals generated by more than three constraints.

The multiplicities of the charged hyper multiplets are presented in table 11. These have been computed following section 2.3. For the case of the representations $\mathbf{2}_{-3/2}$, $\mathbf{2}_{1/2}$ and $\mathbf{1}_2$, the multiplicities can be directly computed from (3.8) and table 11, as the corresponding varieties $V(I_{(1)})$, $V(I_{(2)})$ and $V(I_{(3)})$ are irreducible complete intersections. In contrast, $V(I_{(4)})$ supporting the $\mathbf{1}_1$ -matter is not a complete intersection. However, note that the varieties $V(I_{(4)})$, $V(I_{(1)})$ and $V(I_{(3)})$ are the three irreducible components of the complete

| Yukawa | Locus |
|--|--|
| $\mathbf{2}_{-3/2} \cdot \mathbf{2}_{1/2} \cdot \mathbf{1}_1$ | $s_7 = s_8 = 0$ $s_4 s_5^3 - s_3 s_5^2 s_6 + s_2 s_5 s_6^2 - s_1 s_6^3 = 0$ |
| $\mathbf{2}_{-3/2} \cdot \overline{\mathbf{2}}_{1/2} \cdot \mathbf{1}_2$ | $s_4 = s_7 = s_8 = 0$ |
| $\overline{\mathbf{1}}_1 \cdot \overline{\mathbf{1}}_1 \cdot \mathbf{1}_2$ | $s_4 = s_7 = 0$ $s_3 s_5^2 - s_2 s_5 s_6 + s_1 s_6^2 + s_2^2 s_8 - 4 s_1 s_3 s_8 = 0$ |
| $\mathbf{2}_{1/2} \cdot \mathbf{2}_{1/2} \cdot \overline{\mathbf{1}}_1$ | $V(I_{(2)}) \cap V(I_{(2)}) \cap V(I_{(4)})$ |

Table 12. Codimension three loci and corresponding Yukawa couplings for X_{F_6} .

intersection (2.14) for the section \hat{s}_1 . Using its WS-coordinates, given by (B.8) for $s_9 = 0$, it reads

$$\begin{aligned}
 y_1 &= s_4 s_6^2 s_7^2 - s_4 s_5 s_7^3 - s_3 s_6 s_7^3 + s_2 s_7^4 - 3 s_4^2 s_6 s_7 s_8 + 2 s_3 s_4 s_7^2 s_8 + 2 s_4^3 s_8^2 = 0, \\
 f z_1^4 + 3 x_1^2 &= -12 s_4^3 s_6 s_7 s_8^2 + 6 s_4^4 s_8^3 + s_4^2 s_7^2 s_8 (7 s_6^2 - 4 s_5 s_7 + 8 s_3 s_8) + s_7^5 (-s_2 s_6 + 2 s_1 s_7) \\
 &\quad + s_7^4 (s_3 (s_6^2 - 2 s_5 s_7) + 2 s_3^2 s_8) + s_4 s_7^3 (-s_6^3 + 3 s_5 s_6 s_7 - 8 s_3 s_6 s_8 + 2 s_2 s_7 s_8) = 0.
 \end{aligned}
 \tag{3.90}$$

Thus, the homology class of this complete intersection minus the classes of $V(I_{(1)})$ and $V(I_{(3)})$ (with their respective orders inside (3.90)) yields the multiplicity of the $\mathbf{1}_1$ -matter.

The spectrum of charged matter is completed by the matter in the adjoint representation $\mathbf{3}_0$ given in the last row of table 11. We recall that it does not originate from codimension two fibers of X_{F_6} , but is present if the divisor $S_{\text{SU}(2)}^b$ is a higher genus curve in B , cf. section 2.3. The multiplicity of charged hyper multiplets in the adjoint is calculated using (2.8).

We complete the matter spectrum of X_{F_6} by the number of neutral hyper multiplets, which is computed from (2.11) using the Euler number (C.1) of X_{F_6} . It reads

$$H_{\text{neut}} = 14 + 11[K_B^{-1}]^2 - 4[K_B^{-1}]\mathcal{S}_7 + 4\mathcal{S}_7^2 - 4[K_B^{-1}]\mathcal{S}_9 - 3\mathcal{S}_7\mathcal{S}_9 + 2\mathcal{S}_9^2.
 \tag{3.91}$$

The number T of tensor multiplets is given by (2.17) and we have $V = 4$. Finally, we use $S_{\text{SU}(2)}^b = \{s_8 = 0\}$, (3.88), the charged spectrum in table 11 and (3.91) to check, following appendix A, that all 6D anomalies are canceled.

We conclude this section by analyzing codimension three singularities of the WSF of X_{F_6} . This determines the Yukawa points in a compactification to 4D. All geometrically allowed Yukawa couplings of the charged matter spectrum of X_{F_6} are given in table 12. In order to check the last Yukawa coupling in table 12, we compute the minimal associated primes of $I_{(2)} \cup I_{(4)}$. Indeed, it has a codimension three associated prime, which confirms the presence of the Yukawa coupling. We emphasize that all Yukawa couplings allowed by gauge symmetry are indeed realized.

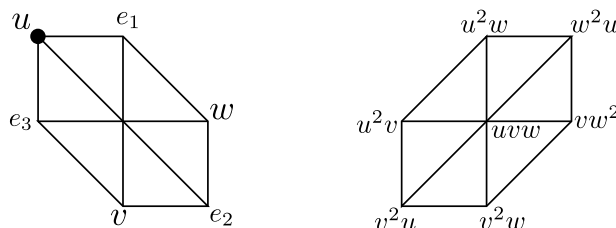


Figure 11. The toric diagram of polyhedron F_7 and its dual. The zero section is indicated by the dot.

3.4 Fibrations with gauge groups of rank 3: selfdual polyhedra

In the following section we analyze all toric hypersurface fibrations constructed from the four self-dual polyhedra F_7 , F_8 , F_9 and F_{10} . The rank of the gauge group of all these models is three and the rank of the MW-group assumes all values from zero to three. We encounter one novelty in the analysis of codimension two fibers in X_{F_8} and $X_{F_{10}}$. There we find matter representations from non-split fibers at codimension two. The Calabi-Yau manifold $X_{F_{10}}$ is also a generalization of the Tate form, allowing for non-trivial coefficients of the monomials x^3 and y^2 . The vanishing loci of these coefficients support $SU(2)$ and $SU(3)$ gauge groups, respectively.

3.4.1 Polyhedron F_7 : $G_{F_7} = U(1)^3$

We consider the elliptically fibered Calabi-Yau manifold X_{F_7} with base B and general elliptic fiber given by the elliptic curve \mathcal{E} in \mathbb{P}_{F_7} . The toric diagram of $\mathbb{P}_{F_7} = dP_3$ is depicted in figure 11, where the polyhedron F_7 along with a choice of homogeneous coordinates as well as its dual polyhedron F_7 are shown. For brevity, we have set $e_i = 1, \forall i$, in the monomials associated to the integral points in the dual polyhedron by (2.23). The toric variety \mathbb{P}_{F_7} is the del Pezzo surface dP_3 , that is the blow-up of \mathbb{P}^2 , cf. section 3.1.2, at three points with blow-down map defined by

$$u \rightarrow e_1 e_3 u, \quad w \rightarrow e_1 e_2 w, \quad v \rightarrow e_2 e_3 v. \tag{3.92}$$

The homogeneous coordinates on the fiber after this blow-up are $[u : v : w : e_1 : e_2 : e_3]$ and take values in the line bundles associated to the following divisor classes:

| Section | Divisor class | |
|---------|---|--------|
| u | $H - E_1 - E_3 + \mathcal{S}_9 + [K_B]$ | |
| v | $H - E_2 - E_3 + \mathcal{S}_9 - \mathcal{S}_7$ | |
| w | $H - E_1 - E_2$ | (3.93) |
| e_1 | E_1 | |
| e_2 | E_2 | |
| e_3 | E_3 | |

The Stanley-Reisner ideal of \mathbb{P}_{F_7} is given by

$$SR = \{uw, ue_2, uv, e_1 e_2, e_1 v, e_1 e_3, vw, we_3, e_2 e_3\}. \tag{3.94}$$

Using (2.23) the hypersurface equation for X_{F_7} in the ambient space of the \mathbb{P}_7 -fibration (3.1) is obtained as

$$\begin{aligned}
 p_{F_7} = & s_2 e_1 e_3^2 u^2 v + s_3 e_2 e_3^2 u v^2 + s_5 e_1^2 e_3 u^2 w + s_6 e_1 e_2 e_3 u v w \\
 & + s_7 e_2^2 e_3 v^2 w + s_8 e_1^2 e_2 u w^2 + s_9 e_1 e_2^2 w^2 v.
 \end{aligned} \tag{3.95}$$

Here the sections s_i take values in the line bundles shown in (3.8). We observe that (3.95) can be obtained from (3.4) by the specialization $s_1 = s_4 = s_{10} = 0$ and the map (3.92).

In total there are six rational sections of the elliptic fibration of X_{F_7} with four of them being linearly independent [41]. Their coordinates are

$$\begin{aligned}
 \hat{s}_0 &= X_{F_7} \cap \{u = 0\} : [0 : 1 : 1 : s_7 : 1 : -s_9], \\
 \hat{s}_1 &= X_{F_7} \cap \{e_1 = 0\} : [s_7 : 1 : -s_3 : 0 : 1 : 1], \\
 \hat{s}_2 &= X_{F_7} \cap \{e_2 = 0\} : [1 : s_5 : -s_2 : 1 : 0 : 1], \\
 \hat{s}_3 &= X_{F_7} \cap \{e_3 = 0\} : [s_9 : -s_8 : 1 : 1 : 1 : 0], \\
 & X_{F_7} \cap \{v = 0\} : [1 : 0 : 1 : 1 : s_5 : -s_8], \\
 & X_{F_7} \cap \{w = 0\} : [1 : 1 : 0 : s_3 : -s_2 : 1],
 \end{aligned} \tag{3.96}$$

where we choose \hat{s}_0 as the zero section and the sections \hat{s}_m , $m = 1, 2, 3$, as the generators of the MW-group of X_{F_7} .

Using Nagell's algorithm we compute the Weierstrass form (2.1) of (3.95). The WS-coordinates of the section \hat{s}_1 , \hat{s}_2 and \hat{s}_3 are given by (B.11), (B.12) and (B.13), respectively. Furthermore, we can obtain the functions f and g from (B.1) and (B.2), respectively, by setting $s_1 = s_4 = s_{10} = 0$. Since we do not find any codimension one singularity the total gauge group of X_{F_7} is

$$G_{F_7} = \mathrm{U}(1)^3. \tag{3.97}$$

Thus we compute the Shioda map (2.5) of all rational sections \hat{s}_m , $m = 1, 2, 3$, as

$$\sigma(\hat{s}_m) = S_m - S_0 + [K_B] - \pi(S_m \cdot S_0), \tag{3.98}$$

where we use the following intersection relations:

$$\begin{aligned}
 \pi(S_1 \cdot S_0) &= \mathcal{S}_7, & \pi(S_2 \cdot S_0) &= 0, & \pi(S_3 \cdot S_0) &= \mathcal{S}_9, \\
 \pi(S_1 \cdot S_2) &= 0, & \pi(S_1 \cdot S_3) &= 0, & \pi(S_2 \cdot S_3) &= 0.
 \end{aligned} \tag{3.99}$$

Using (3.98) and these intersection relations, together with (2.7), we compute the height pairing (2.6) as

$$b_{mn} = -\pi(\sigma(\hat{s}_m)\sigma(\hat{s}_n)) = \begin{pmatrix} 2[K_B^{-1}] + 2\mathcal{S}_7 & [K_B^{-1}] + \mathcal{S}_7 & [K_B^{-1}] + \mathcal{S}_7 + \mathcal{S}_9 \\ [K_B^{-1}] + \mathcal{S}_7 & 2[K_B^{-1}] & [K_B^{-1}] + \mathcal{S}_9 \\ [K_B^{-1}] + \mathcal{S}_7 + \mathcal{S}_9 & [K_B^{-1}] + \mathcal{S}_9 & 2[K_B^{-1}] + 2\mathcal{S}_9 \end{pmatrix}_{mn}. \tag{3.100}$$

Next, we analyze the codimension two singularities of the WSF of X_{F_7} to determine the charged matter spectrum. We find ten codimension two singularities, which lead to the

matter representations and the corresponding codimension two fibers in X_{F_7} given in the first and third column of table 13, respectively. Given all these codimension two fibers and the positions of the rational sections, we readily compute the $U(1)$ -charges using (2.13). Here, the starting point of the analysis, that has led to the complete charged matter spectrum in table 13, are the three complete intersections (2.14) evaluated for the three rational sections \hat{s}_1, \hat{s}_2 and \hat{s}_3 . Then, we determine all their minimal associated prime ideals using Singular [88], which precisely produces all the ten ideals $I_{(k)}$, shown in table 13. We note that the varieties $V(I_{(k)})$, $k = 1, \dots, 6$, are precisely the loci where the six rational sections given in (3.96) are ill-defined and have to acquire a fiber component.

The multiplicities of the charged hyper multiplets are presented in table 13. These have been computed following section 2.3 as the homology classes of the respective codimension two varieties $V(I_{(k)})$, $k = 1, \dots, 10$. In the cases where $V(I_{(k)})$ is not a complete intersection, the multiplicities are calculated using the resultant technique similarly as described in section 3.3.3.

To complete the matter spectrum of X_{F_7} we compute the number of neutral hyper multiplets from (2.11) using the Euler number $\chi(X_{F_6})$ given in (C.1). It reads

$$H_{\text{neut}} = 15 + 7[K_B^{-1}]^2 - 2[K_B^{-1}]\mathcal{S}_7 + 2\mathcal{S}_7^2 - 2[K_B^{-1}]\mathcal{S}_9 - 2\mathcal{S}_7\mathcal{S}_9 + 2\mathcal{S}_9^2. \quad (3.101)$$

We note that there are T tensor multiplets and $V = 3$ vector multiplets. These results together with the charged spectrum in table 13 and (3.101) as well as the height pairing (3.100) allows us, following appendix A, to check cancelation of all 6D anomalies.

To conclude this section we analyze the codimension three singularities of the WSF of X_{F_7} . This determines the Yukawa points in a compactification to 4D. The geometrically allowed Yukawa couplings of X_{F_7} are given in table 14. Here we confirm the presence of each Yukawa coupling by checking that the intersection of the relevant varieties is codimension three in B .

3.4.2 Polyhedron F_8 : $G_{F_8} = \text{SU}(2)^2 \times \text{U}(1)$

In this section, we consider the elliptically fibered Calabi-Yau manifold X_{F_8} over an arbitrary base B and with general elliptic fiber given by the elliptic curve \mathcal{E} in \mathbb{P}_{F_8} . In figure 12 the toric diagram of F_8 and of its dual polyhedron are depicted. For brevity, we have set $e_i = 1, \forall i$, in the monomials that are associated by (2.23) to the integral points of the dual polyhedron. The toric variety \mathbb{P}_{F_8} is \mathbb{P}^2 , cf. section 3.1.2, blown-up at three non-generic points. The blow-down map reads

$$u \rightarrow e_1 e_2^2 e_3 u, \quad v \rightarrow e_1 v, \quad w \rightarrow e_3 w. \quad (3.102)$$

| Representation | Multiplicity | Fiber | Locus |
|---------------------------|--|-------|---|
| $\mathbf{1}_{(1,1,0)}$ | $(2[K_B^{-1}] - \mathcal{S}_9)$ $\times ([K_B^{-1}] + \mathcal{S}_7 - \mathcal{S}_9)$ | | $V(I_{(1)}) := \{s_2 = s_3 = 0\}$ |
| $\mathbf{1}_{(0,-1,0)}$ | $(2[K_B^{-1}] - \mathcal{S}_9)$ $\times (2[K_B^{-1}] - \mathcal{S}_7)$ | | $V(I_{(2)}) := \{s_2 = s_5 = 0\}$ |
| $\mathbf{1}_{(2,1,1)}$ | $\mathcal{S}_7([K_B^{-1}] + \mathcal{S}_7 - \mathcal{S}_9)$ | | $V(I_{(3)}) := \{s_3 = s_7 = 0\}$ |
| $\mathbf{1}_{(0,1,1)}$ | $(2[K_B^{-1}] - \mathcal{S}_7)$ $\times ([K_B^{-1}] - \mathcal{S}_7 + \mathcal{S}_9)$ | | $V(I_{(4)}) := \{s_5 = s_8 = 0\}$ |
| $\mathbf{1}_{(-2,-1,-2)}$ | $\mathcal{S}_7 \mathcal{S}_9$ | | $V(I_{(5)}) := \{s_7 = s_9 = 0\}$ |
| $\mathbf{1}_{(1,1,2)}$ | $\mathcal{S}_9([K_B^{-1}] - \mathcal{S}_7 + \mathcal{S}_9)$ | | $V(I_{(6)}) := \{s_8 = s_9 = 0\}$ |
| $\mathbf{1}_{(1,0,0)}$ | $2[K_B^{-1}](4[K_B^{-1}] - 2\mathcal{S}_7 + \mathcal{S}_9)$ $- 2(2[K_B^{-1}] - \mathcal{S}_7)$ $\times ([K_B^{-1}] - \mathcal{S}_7 + \mathcal{S}_9)$ | | $V(I_{(7)}) := \{s_3 s_8 - s_5 s_7 = 0$ $s_2 s_8^2 - s_5 s_6 s_8 + s_3^2 s_9 = 0\}$ |
| $\mathbf{1}_{(0,0,1)}$ | $2[K_B^{-1}](2[K_B^{-1}] - \mathcal{S}_7 + 2\mathcal{S}_9)$ $- 2\mathcal{S}_9([K_B^{-1}] - \mathcal{S}_7 + \mathcal{S}_9)$ | | $V(I_{(8)}) := \{s_3 s_8 - s_2 s_9 = 0$ $s_5 s_9^2 - s_6 s_8 s_9 + s_7 s_8^2 = 0\}$ |
| $\mathbf{1}_{(1,0,1)}$ | $2[K_B^{-1}]([K_B^{-1}] + \mathcal{S}_7$ $+ \mathcal{S}_9) - 2\mathcal{S}_7 \mathcal{S}_9$ | | $V(I_{(9)}) := \{s_5 s_7 - s_2 s_9 = 0$ $s_3 s_9^2 - s_6 s_7 s_9 + s_7^2 s_8 = 0\}$ |
| $\mathbf{1}_{(1,1,1)}$ | $4[K_B^{-1}]^2 + 2[K_B^{-1}](\mathcal{S}_7 + \mathcal{S}_9)$ $+ 2(-\mathcal{S}_7^2 + \mathcal{S}_7 \mathcal{S}_9 - \mathcal{S}_9^2)$ | | $V(I_{(10)}) := \{s_2^2 s_5 s_7 + s_3^2 s_5^2$ $- s_2 s_3 s_5 s_6 + s_2^2 s_3 s_8 = 0$ $s_2 s_5^2 s_9 + s_3 s_5^2 s_8$ $- s_2 s_5 s_6 s_8 + s_2^2 s_8^2 = 0\}$ |

Table 13. Charged matter representations under $U(1)^3$ and corresponding codimension two fibers of X_{F_7} .

| Yukawa | Locus |
|--|---|
| $\mathbf{1}_{(1,1,0)} \cdot \mathbf{1}_{(0,-1,0)} \cdot \overline{\mathbf{1}_{(1,0,0)}}$ | $s_2 = s_3 = s_5 = 0$ |
| $\mathbf{1}_{(1,1,0)} \cdot \overline{\mathbf{1}_{(2,1,1)}} \cdot \mathbf{1}_{(1,0,1)}$ | $s_2 = s_3 = s_7 = 0$ |
| $\mathbf{1}_{(1,1,0)} \cdot \mathbf{1}_{(0,0,1)} \cdot \overline{\mathbf{1}_{(1,1,1)}}$ | $s_2 = s_3 = s_5 s_9^2 - s_6 s_8 s_9 + s_7 s_8^2 = 0$ |
| $\mathbf{1}_{(0,-1,0)} \cdot \mathbf{1}_{(0,1,1)} \cdot \overline{\mathbf{1}_{(0,0,1)}}$ | $s_2 = s_5 = s_8 = 0$ |
| $\mathbf{1}_{(0,-1,0)} \cdot \overline{\mathbf{1}_{(1,0,1)}} \cdot \mathbf{1}_{(1,1,1)}$ | $s_2 = s_5 = s_3 s_9^2 - s_6 s_7 s_9 + s_7^2 s_8 = 0$ |
| $\mathbf{1}_{(2,1,1)} \cdot \mathbf{1}_{(-2,-1,-2)} \cdot \mathbf{1}_{(0,0,1)}$ | $s_3 = s_7 = s_9 = 0$ |
| $\overline{\mathbf{1}_{(2,1,1)}} \cdot \mathbf{1}_{(1,0,0)} \cdot \mathbf{1}_{(1,1,1)}$ | $s_3 = s_7 = s_2 s_8^2 - s_5 s_6 s_8 + s_5^2 s_9 = 0$ |
| $\mathbf{1}_{(0,1,1)} \cdot \overline{\mathbf{1}_{(1,1,2)}} \cdot \mathbf{1}_{(1,0,1)}$ | $s_5 = s_8 = s_9 = 0$ |
| $\mathbf{1}_{(-2,-1,-2)} \cdot \mathbf{1}_{(1,1,2)} \cdot \mathbf{1}_{(1,0,0)}$ | $s_7 = s_8 = s_9 = 0$ |
| $\mathbf{1}_{(-2,-1,-2)} \cdot \mathbf{1}_{(1,0,1)} \cdot \mathbf{1}_{(1,1,1)}$ | $s_7 = s_9 = s_3 s_5^2 - s_2 s_5 s_6 + s_2^2 s_8 = 0$ |
| $\overline{\mathbf{1}_{(1,1,2)}} \cdot \mathbf{1}_{(0,0,1)} \cdot \mathbf{1}_{(1,1,1)}$ | $s_8 = s_9 = s_2^2 s_7 - s_2 s_3 s_6 + s_3^2 s_5 = 0$ |
| $\mathbf{1}_{(1,0,0)} \cdot \mathbf{1}_{(0,0,1)} \cdot \overline{\mathbf{1}_{(1,0,1)}}$ | $s_2 s_9^3 + s_7^2 s_8^2 - s_6 s_7 s_8 s_9 = 0$ $s_3 s_9^2 - s_6 s_7 s_9 + s_7^2 s_8 = 0$ $s_5 s_9^2 - s_6 s_8 s_9 + s_7 s_8^2 = 0$ |

Table 14. Codimension three loci and corresponding Yukawa couplings for X_{F_7} .

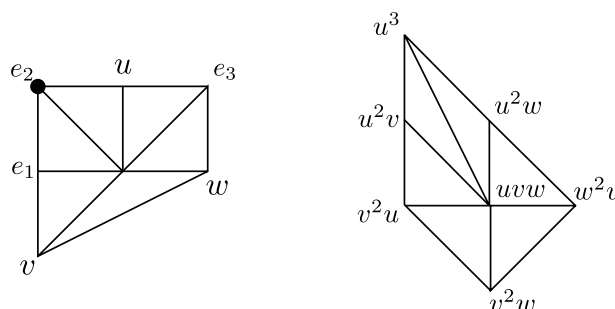


Figure 12. The toric diagram of polyhedron F_8 and its dual. The zero section is indicated by the dot.

The homogeneous coordinates on the fiber after this blow-up are $[u : v : w : e_1 : e_2 : e_3]$ and take values in the line bundles associated to the divisors given by:

| Section | Divisor class |
|---------|---|
| u | $H - E_1 - E_2 - E_3 + \mathcal{S}_9 + [K_B]$ |
| v | $H - E_1 - \mathcal{S}_7 + \mathcal{S}_9$ |
| w | $H - E_3$ |
| e_1 | $E_1 - E_2$ |
| e_2 | E_2 |
| e_3 | E_3 |

(3.103)

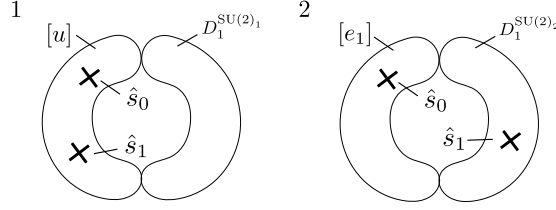


Figure 13. Codimension one fibers of X_{F_8} at $s_7 = 0$ and $s_8 = 0$ in B . The crosses denote the intersections with the two sections.

The Stanley-Reisner ideal of \mathbb{P}_{F_8} reads

$$SR = \{uw, uv, ue_1, e_3v, e_3e_1, e_3e_2, we_1, we_2, ve_2\}. \quad (3.104)$$

Using (2.23), the hypersurface equation for X_{F_8} in the ambient space (3.1) with $F_i = F_8$ is

$$p_{F_8} = s_1e_1^2e_2^4e_3^2u^3 + s_2e_1^2e_2^3e_3u^2v + s_3e_1^2e_2^2uv^2 + s_5e_1e_2^2e_3^2u^2w + s_6e_1e_2e_3uvw + s_7e_1v^2w + s_8e_3^2uw^2, \quad (3.105)$$

where the classes of the sections s_i are given in (3.8). We note that restriction of (3.4) as $s_4 = s_9 = s_{10} = 0$ and application of the map (3.102) also leads to (3.105).

There are two rational sections of the fibration of X_{F_8} . Their coordinates are

$$\begin{aligned} \hat{s}_0 &= X_{F_8} \cap \{e_2 = 0\} : [s_7 : 1 : 1 : -s_8 : 0 : 1], \\ \hat{s}_1 &= X_{F_8} \cap \{e_3 = 0\} : [s_7 : 1 : -s_3 : 1 : 1 : 0], \end{aligned} \quad (3.106)$$

where we choose \hat{s}_0 as the zero section.

We compute the Weierstrass form (2.1) of (3.105) using Nagell's algorithm. The WS-coordinates of the section \hat{s}_1 are given by (B.8) after setting $s_4 = s_9 = 0$. Additionally, the functions f and g take the form of (B.1) and (B.2), respectively, after setting $s_4 = s_9 = s_{10} = 0$. This allows us to find all codimension one singularities of the WSF of X_{F_8} . We find two I_2 -singularities over the divisors $\mathcal{S}_{\text{SU}(2)_1}^b = \{s_7 = 0\} \cap B$ and $\mathcal{S}_{\text{SU}(2)_2}^b = \{s_8 = 0\} \cap B$ in B . Along these divisors the constraint (3.105) factorizes as

$$\begin{aligned} \text{SU}(2)_1 : \quad p_{F_8}|_{s_7=0} &= u \cdot q_2, \\ \text{SU}(2)_2 : \quad p_{F_8}|_{s_8=0} &= e_1 \cdot q_3. \end{aligned} \quad (3.107)$$

Here q_2, q_3 are the polynomials of degree n in $[u : v : w]$ that remain after factoring out u and e_1 , respectively. These are clearly I_2 -fibers, cf. figure 13, giving rise to two $\text{SU}(2)$ gauge groups. In summary, the gauge group of X_{F_8} is

$$G_{F_8} = \text{SU}(2)^2 \times \text{U}(1). \quad (3.108)$$

Similar as in section 3.3.3, we obtain the classes of the $\text{SU}(2)$ Cartan divisors in X_{F_8} given by

$$D_1^{\text{SU}(2)_1} = [s_7] - [u], \quad D_1^{\text{SU}(2)_2} = [s_8] - [e_1]. \quad (3.109)$$

This allows us to compute the Shioda map (2.5) of the section \hat{s}_1 as

$$\sigma(\hat{s}_1) = S_1 - S_0 + [K_B] + \frac{1}{2}D_1^{\text{SU}(2)_2}, \quad (3.110)$$

where we use that the two sections in (3.106) do not intersect, implying $\pi(S_1 \cdot S_0) = 0$, and

$$S_1 \cdot C_{-\alpha_1}^{\text{SU}(2)_1} = 0, \quad S_1 \cdot C_{-\alpha_1}^{\text{SU}(2)_2} = 1, \quad (3.111)$$

which follows from figure 13. We compute the height pairing (2.6) of \hat{s}_1 using (2.7) as

$$b_{11} = \frac{3}{2}[K_B^{-1}] + \frac{1}{2}\mathcal{S}_7 - \frac{1}{2}\mathcal{S}_9. \quad (3.112)$$

In order to determine the charged matter spectrum we analyze the codimension two singularities of the WSF of X_{F_8} . For the singlets we compute the associated prime ideals of the complete intersection (2.14) associated to \hat{s}_1 . We find seven codimension two singularities.

Five of these lead to the matter representations and the corresponding codimension two fibers in X_{F_8} given in the first and third column of table 15, respectively.

The representations under G_{F_8} have been determined following the general procedure outlined in section 2.3. The remaining two loci, $s_7 = s_6^2 - 4s_3s_8 = 0$ and $s_8 = s_6^2 - 4s_5s_7 = 0$, support Type III singularities, which do not lead to additional matter.

We note the following subtlety. The fiber supporting the matter in the representation $(\mathbf{2}, \mathbf{1})_0$ is non-split in the sense of [61].²⁵ This means that the constraint (3.105) of the elliptic fiber at the codimension two locus $V(I_{(4)})$ does not fully factorize, as one expects, over the field K of meromorphic functions on B . It only factorizes in a field extension where certain square roots of the coefficients s_i are allowed. In fact, the fiber we obtain at the locus $V(I_{(4)})$ allowing only for factorizations in K is a line and a singular conic. However, a singular conic describes two lines, i.e. the conic has to be factorized into two linear constraints describing two lines. This factorization requires introducing square roots of some combinations of the s_i . Geometrically, this means that these lines are interchanged by a codimension three monodromy (that occur only on threefold bases B).²⁶ The two lines of the non-split fiber that are interchanged by this monodromy are the dashed \mathbb{P}^1 's in the fourth row of table 15.

The multiplicities of the charged hyper multiplets are presented in table 15 in the second row. These have been computed following section 2.3. Since the locus of the representation $(\mathbf{1}, \mathbf{1})_1$ is not a complete intersection we compute its multiplicity as described in section 3.3.3 by subtraction of the locus $V(I_{(3)})$ with its appropriate order.

The number of neutral hyper multiplets completes the matter spectrum of X_{F_8} . It is computed from (2.11) using the Euler number (C.1) of X_{F_8} and reads

$$H_{\text{neut}} = 15 + 11[K_B^{-1}]^2 - 5[K_B^{-1}]\mathcal{S}_7 + 3\mathcal{S}_7^2 - 4[K_B^{-1}]\mathcal{S}_9 - 2\mathcal{S}_7\mathcal{S}_9 + 2\mathcal{S}_9^2. \quad (3.113)$$

²⁵We note that non-split fibers at codimension two are not classified.

²⁶We thank Dave Morrison and Ron Donagi for explanations related to non-split codimension two fibers and singular conics. We also thank Sakura Schäfer-Nameki and Craig Lawrie for explaining to us the corresponding box graphs [90].

| Representation | Multiplicity | Fiber | Locus |
|-----------------------------------|---|-----------|--|
| $(\mathbf{2}, \mathbf{2})_{-1/2}$ | $\mathcal{S}_7([K_B^{-1}] - \mathcal{S}_7 + \mathcal{S}_9)$ | | $V(I_{(1)}) := \{s_7 = s_8 = 0\}$ |
| $(\mathbf{1}, \mathbf{2})_{1/2}$ | $2([K_B^{-1}] - \mathcal{S}_7 + \mathcal{S}_9)$ $(3[K_B^{-1}] - \mathcal{S}_9)$ | | $V(I_{(2)}) := \{s_8 = 0$ $s_3^2 s_5^2 - s_2 s_3 s_5 s_6 + s_1 s_3 s_6^2 + s_2^2 s_5 s_7$ $- 2s_1 s_3 s_5 s_7 - s_1 s_2 s_6 s_7 + s_1^2 s_7^2 = 0\}$ |
| $(\mathbf{2}, \mathbf{1})_1$ | $\mathcal{S}_7([K_B^{-1}] + \mathcal{S}_7 - \mathcal{S}_9)$ | | $V(I_{(3)}) := \{s_3 = s_7 = 0\}$ |
| $(\mathbf{2}, \mathbf{1})_0$ | $\mathcal{S}_7(5[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_9)$ | | $V(I_{(4)}) := \{s_7 = 0$ $s_3 s_5^2 - s_2 s_5 s_6 + s_1 s_6^2$ $+ s_2^2 s_8 - 4s_1 s_3 s_8 = 0\}$ |
| $(\mathbf{1}, \mathbf{1})_1$ | $6[K_B^{-1}]^2 + 3[K_B^{-1}]\mathcal{S}_7$ $-\mathcal{S}_7^2 - 5[K_B^{-1}]\mathcal{S}_9 + \mathcal{S}_9^2$ | | $V(I_{(5)}) := \{s_3 s_6 - s_2 s_7 = 0$ $s_7(-s_2 s_6 + 2s_1 s_7) + s_3(s_6^2$ $- 2s_5 s_7) + 2s_3^2 s_8 = 0\} \setminus V(I_{(3)})$ |
| $(\mathbf{3}, \mathbf{1})_0$ | $1 + \mathcal{S}_7 \frac{\mathcal{S}_7 - [K_B^{-1}]}{2}$ | Figure 13 | $s_7 = 0$ |
| $(\mathbf{1}, \mathbf{3})_0$ | $1 + \frac{\mathcal{S}_9 - \mathcal{S}_7}{2}$ $\times ([K_B^{-1}] - \mathcal{S}_7 + \mathcal{S}_9)$ | Figure 13 | $s_8 = 0$ |

Table 15. Charged matter representations under $SU(2)^2 \times U(1)$ and corresponding codimension two fibers of X_{F_8} . The adjoint matter is included for completeness.

The base-dependent number T of tensor multiplets is given by (2.17) and we have $V = 7$. Finally, we use this together with $\mathcal{S}_{SU(2)_1}^b = \mathcal{S}_7^b$, $\mathcal{S}_{SU(2)_2}^b = \mathcal{S}_8^b$, (3.112), the charged spectrum in table 15 and (3.113) to check cancelation of all 6D anomalies in (A.1).

To obtain the Yukawa points in a compactification to 4D we analyze codimension three singularities of the WSF of X_{F_8} . All Yukawa couplings of the charged matter spectrum of X_{F_8} are given in table 16. Clearly, all relevant loci here are codimension three.

| Yukawa | Locus |
|---|---|
| $(\mathbf{2}, \mathbf{2})_{-1/2} \cdot \overline{(\mathbf{1}, \mathbf{2})_{1/2}} \cdot (\mathbf{2}, \mathbf{1})_1$ | $s_8 = s_7 = s_3 = 0$ |
| $(\mathbf{2}, \mathbf{2})_{-1/2} \cdot (\mathbf{1}, \mathbf{2})_{1/2} \cdot \overline{(\mathbf{2}, \mathbf{1})_0}$ | $s_8 = s_7 = s_2 s_5^2 - s_1 s_5 s_6 + s_0 s_6^2 = 0$ |
| $(\mathbf{2}, \mathbf{1})_1 \cdot \overline{(\mathbf{2}, \mathbf{1})_0} \cdot (\mathbf{1}, \mathbf{1})_1$ | $s_7 = s_2 = -s_1 s_5 s_6 + s_0 s_6^2 + s_1^2 s_8 = 0$ |
| $(\mathbf{2}, \mathbf{2})_{-1/2} \cdot \overline{(\mathbf{2}, \mathbf{2})_{-1/2}} \cdot (\mathbf{1}, \mathbf{1})_1$ | $s_6 = s_7 = s_8 = 0$ |
| $(\mathbf{1}, \mathbf{2})_{1/2} \cdot \overline{(\mathbf{1}, \mathbf{2})_{1/2}} \cdot (\mathbf{1}, \mathbf{1})_1$ | $s_8 = 0 = s_1 s_6 - s_2 s_5 = s_1 s_7 - s_3 s_5 = s_2 s_7 - s_3 s_6$ |

Table 16. Codimension three loci and corresponding Yukawa couplings for X_{F_8} .

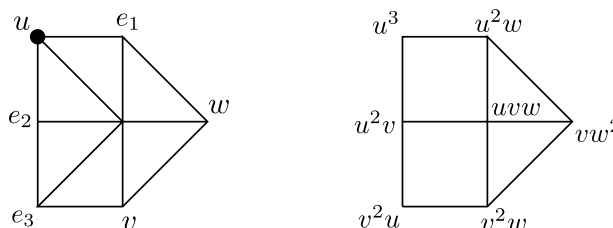


Figure 14. The toric diagram of polyhedron F_9 and its dual. The zero section is indicated by the dot.

3.4.3 Polyhedron F_9 : $G_{F_9} = \text{SU}(2) \times \text{U}(1)^2$

Here, we consider the elliptically fibered Calabi-Yau manifold X_{F_9} over a base B and with general elliptic fiber given by the toric hypersurface \mathcal{C}_{F_9} in \mathbb{P}_{F_9} . The toric data of \mathbb{P}_{F_9} is depicted in figure 14. Here the polyhedron F_9 along with a choice of homogeneous coordinates as well as its dual polyhedron is shown. For brevity, we have set $e_i = 1, \forall i$, in the monomials associated to the integral points of the dual by (2.23). The toric variety \mathbb{P}_{F_9} is obtained from \mathbb{P}^2 , cf. section 3.1.2, by the three non-generic blow-ups

$$u \rightarrow e_1 e_2 e_3 u, \quad w \rightarrow e_1 w, \quad v \rightarrow e_2 e_3^2 v. \tag{3.114}$$

After these blow-ups the homogeneous coordinates on the fiber are $[u : v : w : e_1 : e_2 : e_3]$, which take values in the line bundles associated to the divisors:

| Section | Divisor class |
|---------|---|
| u | $H - E_1 - E_2 + \mathcal{S}_9 + [K_B]$ |
| v | $H - E_2 - E_3 + \mathcal{S}_9 - \mathcal{S}_7$ |
| w | $H - E_1$ |
| e_1 | E_1 |
| e_2 | $E_2 - E_3$ |
| e_3 | E_3 |

(3.115)

The Stanley-Reisner ideal of \mathbb{P}_{F_9} then reads

$$SR = \{uw, uv, ue_3, e_1v, e_1e_3, e_1e_2, we_3, we_2, ve_2\}. \tag{3.116}$$

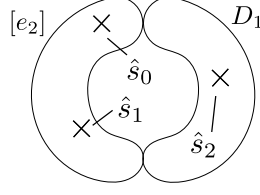


Figure 15. Codimension one fiber of X_{F_9} at $s_9 = 0$ in B . The crosses denote the intersections with the sections.

Using (2.23) the hypersurface equation for X_{F_9} in the ambient space (3.1) with $F_i = F_9$ is

$$p_{F_9} = s_1 e_1^2 e_2^2 e_3 u^3 + s_2 e_1 e_2^2 e_3^2 u^2 v + s_3 e_2^2 e_3^3 u v^2 + s_5 e_1^2 e_2 u^2 w + s_6 e_1 e_2 e_3 u v w + s_7 e_2 e_3^2 v^2 w + s_9 e_1 v w^2, \quad (3.117)$$

where the divisor classes of the sections s_i are given in (3.8). We see that (3.117) can also be obtained from (3.4) by the specialization $s_4 = s_8 = s_{10} = 0$ and the map (3.114).

There are four rational sections on X_{F_6} with one linear relation between them. Their coordinates are

$$\begin{aligned} \hat{s}_0 &= X_{F_9} \cap \{u = 0\} : [0 : 1 : 1 : s_7 : -s_9 : 1], \\ \hat{s}_1 &= X_{F_9} \cap \{e_3 = 0\} : [1 : s_5 : 1 : 1 : -s_9 : 0], \\ \hat{s}_2 &= X_{F_9} \cap \{e_1 = 0\} : [s_7 : 1 : -s_3 : 0 : 1 : 1], \\ X_{F_9} \cap \{v = 0\} &: [1 : 0 : s_1 : 1 : 1 : -s_5]. \end{aligned} \quad (3.118)$$

We choose \hat{s}_0 as the zero section and \hat{s}_m , $m = 1, 2$, as the generators of the MW-group of X_{F_9} .

Employing Nagell's algorithm we compute the Weierstrass form (2.1) of (3.117). The WS-coordinates of the sections \hat{s}_1 and \hat{s}_2 are given by (B.9) and (B.10), respectively, after setting the appropriate sections s_i to zero. To get the functions f and g we specialize (B.1) and (B.2) as $s_4 = s_8 = s_{10} = 0$. Using f and g we can compute the discriminant Δ to find all codimension one singularities of the WSF of X_{F_9} . We find one I_2 -singularity over the divisor $\mathcal{S}_{\text{SU}(2)}^b = \{s_9 = 0\} \cap B$ in B . Along this divisor the constraint (3.117) factorizes as

$$\text{SU}(2) : p_{F_9}|_{s_9=0} = e_2 \cdot q_3, \quad (3.119)$$

where q_3 is the polynomial that remains after factoring out e_2 . This is an I_2 -fiber, cf. figure 15, giving rise to an $\text{SU}(2)$ gauge group. In summary, the gauge group of X_{F_9} is

$$G_{F_9} = \text{SU}(2) \times \text{U}(1)^2. \quad (3.120)$$

Similar as in section 3.3.3, we compute the divisor class of the Cartan divisor as

$$D_1 = [s_9] - [e_2] = [s_9] - E_2 + E_3. \quad (3.121)$$

Employing this we obtain the Shioda map (2.5) of the sections as

$$\sigma(\hat{s}_m) = S_m - S_0 + [K_B] - \delta_{m,2} \mathcal{S}_7 + \frac{1}{2} \delta_{m,2} D_1. \quad (3.122)$$

Here S_0, S_m denote the divisor classes of the sections $\hat{s}_0, \hat{s}_m, m = 1, 2$, respectively, and we used the relation

$$S_m \cdot C_{-\alpha_1} = \delta_{m,2}, \quad \pi(S_0 \cdot S_m) = \delta_{m,2} \mathcal{S}_7, \quad \pi(S_1 \cdot S_2) = 0, \quad (3.123)$$

which can be deduced from figure 15 and the coordinates (3.118) of the sections, respectively. Using these relations and (2.7), we compute the height pairing (2.6) of the \hat{s}_m as

$$b_{mn} = \begin{pmatrix} 2[K_B^{-1}] & [K_B^{-1}] + \mathcal{S}_7 \\ [K_B^{-1}] + \mathcal{S}_7 & 2[K_B^{-1}] + 2\mathcal{S}_7 - \frac{1}{2}\mathcal{S}_9 \end{pmatrix}_{mn}. \quad (3.124)$$

Turning to the charged matter spectrum we analyze the codimension two singularities of the WSF of X_{F_9} . The matter in non-trivial representations of the non-Abelian part of G_{F_9} follows directly from the discriminant with $s_9 = 0$, whereas the charged singlets are most easily seen from the primary decomposition of the complete intersections (2.14) corresponding to the two sections \hat{s}_1, \hat{s}_2 , respectively. We find eight codimension two singularities. Seven of these lead to the matter representations and the corresponding codimension two fibers in X_{F_9} given in the first and third column of table 17, respectively. Here, the corresponding representation under the gauge group are determined following the general procedure explained in section 2.1. At the remaining locus $s_9 = s_6^2 - 4s_5s_7 = 0$, the fiber is of Type III, cf. the discussion in section 3.3.3. Thus it does not support any additional matter.

We note that at the locus $V(I_{(2)})$ the linearly dependent section in (3.118) is singular. At the locus $V(I_{(7)})$ corresponding to matter in the representation $\mathbf{2}_{(0,-1/2)}$ the elliptic curve (3.117) does not naively factor into three rational curves. To correctly derive its splitting one needs to compute the associated prime ideals of the elliptic fiber at this locus. We find three ideals corresponding to three rational curves, which indeed intersect as an I_3 -fiber.

The multiplicities of the charged hyper multiplets are presented in table 17. These have been computed following section 2.3. The multiplicities of $V(I_{(3)})$ and $V(I_{(4)})$ must be calculated by appropriately subtracting the multiplicities of the loci $V(I_{(1)})$, $V(I_{(5)})$ and $V(I_{(6)})$, respectively, as described in section 3.3.3 and indicated in table 17.

We complete the matter spectrum of X_{F_9} by the number of neutral hyper multiplets, which is computed from (2.11) using the Euler number $\chi(X_{F_9})$ given in (C.1). It reads

$$H_{\text{neut}} = 15 + 11[K_B^{-1}]^2 - 4[K_B^{-1}]\mathcal{S}_7 + 2\mathcal{S}_7^2 - 6[K_B^{-1}]\mathcal{S}_9 + 2\mathcal{S}_9^2. \quad (3.125)$$

The number of tensor multiplets T can be obtained by (2.17) and we have $V = 5$ vector multiplets. To check that the anomalies are canceled we use $S_{\text{SU}(2)}^b = \{s_9 = 0\}$, (3.124), the charged spectrum in table 17 and (3.125), following the discussion in appendix A.

We conclude this section by analyzing codimension three singularities of the WSF of X_{F_9} determining the Yukawa points in a compactification to 4D. In table 18 all geometrically allowed Yukawa couplings of the charged matter spectrum of X_{F_9} are given.

| Representation | Multiplicity | Fiber | Locus |
|--------------------------|--|-----------|---|
| $\mathbf{1}_{(1,2)}$ | $\mathcal{S}_7([K_B^{-1}] + \mathcal{S}_7 - \mathcal{S}_9)$ | | $V(I_{(1)}) := \{s_7 = s_3 = 0\}$ |
| $\mathbf{1}_{(1,0)}$ | $(2[K_B^{-1}] - \mathcal{S}_7)$ $(3[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_9)$ | | $V(I_{(2)}) := \{s_5 = s_1 = 0\}$ |
| $\mathbf{1}_{(0,1)}$ | $(3[K_B^{-1}] - \mathcal{S}_9)$ $\times (2[K_B^{-1}] + 2\mathcal{S}_7 - \mathcal{S}_9)$ $- 2\mathcal{S}_7([K_B^{-1}] + \mathcal{S}_7 - \mathcal{S}_9)$ | | $V(I_{(3)}) := \{s_2 s_7^2 + s_3^2 s_9 - s_3 s_6 s_7 = 0$ $s_5 s_3 - s_7 s_1 = 0\} \setminus V(I_{(1)})$ |
| $\mathbf{1}_{(1,1)}$ | $6[K_B^{-1}]^2 + [K_B^{-1}]$ $\times (4\mathcal{S}_7 - 2\mathcal{S}_9) - 2\mathcal{S}_7^2$ | | $V(I_{(4)}) := \{s_2 s_9 s_7^2 + s_3^2 s_9^2$ $- s_3 s_6 s_9 s_7 - s_7^3 s_5 = 0$ $s_1 s_9 s_7 + s_5 (s_3 s_9 - s_7 s_6) = 0\}$ $\setminus (V(I_{(1)}) \cup V(I_{(5)}) \cup V(I_{(6)}))$ |
| $\mathbf{2}_{(-1,-1/2)}$ | $\mathcal{S}_9(2[K_B^{-1}] - \mathcal{S}_7)$ | | $V(I_{(5)}) := \{s_9 = s_5 = 0\}$ |
| $\mathbf{2}_{(1,3/2)}$ | $\mathcal{S}_7 \mathcal{S}_9$ | | $V(I_{(6)}) := \{s_9 = s_7 = 0\}$ |
| $\mathbf{2}_{(0,-1/2)}$ | $2\mathcal{S}_9(3[K_B^{-1}] - \mathcal{S}_9)$ | | $V(I_{(7)}) := \{s_9 = 0$ $s_3^2 s_5^2 + s_3 (-s_6 s_2 s_5$ $+ s_6^2 s_1 - 2s_7 s_5 s_1) + s_7 (s_2^2 s_5$ $- s_6 s_2 s_1 + s_7 s_1^2) = 0\}$ |
| $\mathbf{3}_{(0,0)}$ | $1 + \mathcal{S}_9 \frac{(s_9 - [K_B^{-1}])}{2}$ | Figure 15 | $s_9 = 0$ |

Table 17. Charged matter representations under $SU(2) \times U(1)^2$ and corresponding codimension two fibers of X_{F_9} . The adjoint matter is included for completeness.

3.4.4 Polyhedron F_{10} & the generalized Tate form: $G_{F_{10}} = SU(3) \times SU(2)$

The elliptically fibered Calabi-Yau manifold $X_{F_{10}}$ is constructed as the fibration of the elliptic curve \mathcal{E} in $\mathbb{P}_{F_{10}} = \mathbb{P}^2(1, 2, 3)$ over an arbitrary base B . Thus, the generic fiber in $\mathbb{P}_{F_{10}}$ is just the elliptic curve \mathcal{E} in the Tate form of the WS equation (2.1), however, with non-trivial coefficients in front of the terms x^3 and y^2 , that are usually set to one.

The toric data of $\mathbb{P}_{F_{10}}$ is encoded in the polyhedron F_{10} , that is shown along with a choice of homogeneous coordinates and together with its dual polyhedron in figure 16. In the dual polyhedron, we have set $e_i = 1, \forall i$, in the monomials that are associated to its

| Yukawa | Locus |
|--|---|
| $\mathbf{1}_{(1,0)} \cdot \mathbf{1}_{(0,1)} \cdot \overline{\mathbf{1}_{(1,1)}}$ | $s_1 = s_5 = s_2 s_7^2 + s_3^2 s_9 - s_3 s_6 s_7 = 0$ |
| $\overline{\mathbf{1}_{(0,1)}} \cdot \mathbf{2}_{(-1,-1/2)} \cdot \mathbf{2}_{(1,3/2)}$ | $s_5 = s_7 = s_9 = 0$ |
| $\mathbf{1}_{(1,1)} \cdot \mathbf{2}_{(-1,-1/2)} \cdot \mathbf{2}_{(0,-1/2)}$ | $s_5 = s_9 = s_1^3 (s_3 s_6^2 + s_1 s_7^2 - s_2 s_6 s_7) = 0$ |
| $\mathbf{1}_{(1,0)} \cdot \mathbf{2}_{(-1,-1/2)} \cdot \overline{\mathbf{2}_{(0,-1/2)}}$ | $s_5 = s_9 = s_1 = 0$ |
| $\mathbf{1}_{(1,2)} \cdot \overline{\mathbf{2}_{(1,3/2)}} \cdot \mathbf{2}_{(0,-1/2)}$ | $s_7 = s_9 = s_3 = 0$ |
| $\overline{\mathbf{1}_{(1,1)}} \cdot \mathbf{2}_{(1,3/2)} \cdot \mathbf{2}_{(0,-1/2)}$ | $s_7 = s_9 = s_1^2 (s_3 s_5^2 + s_1 s_6^2 - s_2 s_5 s_6) = 0$ |
| $\mathbf{2}_{(0,-1/2)} \cdot \overline{\mathbf{2}_{(0,-1/2)}} \cdot \mathbf{1}_{(0,1)}$ | $s_9 = s_2 s_7 - s_3 s_6 = s_1 s_7 - s_3 s_5 = s_1 s_6 - s_2 s_5 = 0$ |

Table 18. Codimension three loci and corresponding Yukawa couplings for X_{F_9} .

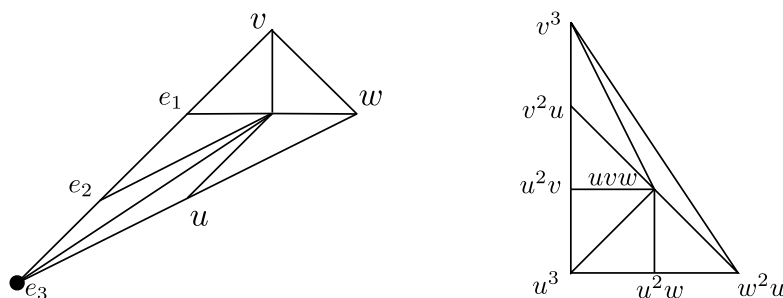


Figure 16. Polyhedron F_{10} and its dual. The zero section is indicated by the dot.

integral points by (2.23). We obtain $\mathbb{P}_{F_{10}} = \mathbb{P}^2(1, 2, 3)$ by blowing-up \mathbb{P}^2 , see section 3.1.2, in the following way:

$$u \rightarrow e_1 e_2^2 e_3^3 u, \quad v \rightarrow e_1 e_2 e_3 v. \quad (3.126)$$

After these blow-ups, the homogeneous coordinates on the fiber are $[u : v : w : e_1 : e_2 : e_3]$ and take values in the line bundles associated to the divisor classes given by:

| Section | Divisor class |
|---------|---|
| u | $H - E_1 - E_2 - E_3 + \mathcal{S}_9 + [K_B]$ |
| v | $H - E_1 + \mathcal{S}_9 - \mathcal{S}_7$ |
| w | H |
| e_1 | $E_1 - E_2$ |
| e_2 | $E_2 - E_3$ |
| e_3 | E_3 |

(3.127)

The Stanley-Reisner ideal of $\mathbb{P}_{F_{10}}$ reads

$$SR = \{ue_2, ue_1, uv, e_3e_1, e_3v, e_3w, e_2v, e_2w, e_1w\}. \quad (3.128)$$

Employing (2.23) we obtain the hypersurface equation for $X_{F_{10}}$ in the ambient space (3.53) with general fiber given by $\mathbb{P}^2(1, 2, 3)$ given by

$$p_{F_{10}} = s_1 e_1^2 e_2^4 e_3^6 u^3 + s_2 e_1^2 e_2^3 e_3^4 u^2 v + s_3 e_1^2 e_2^2 e_3^2 uv^2 + s_4 e_1^2 e_2 v^3 + s_5 e_1 e_2^2 e_3^3 u^2 w + s_6 e_1 e_2 e_3 uvw + s_8 uw^2. \quad (3.129)$$

Here the sections s_i take values in the line bundles associated to the divisor classes shown in (3.8). The hypersurface equation (3.129) can also be obtained by the specialization $s_7 = s_9 = s_{10} = 0$ and the map (3.126) applied to (3.4). There is one rational section of the fibration of $X_{F_{10}}$. Its coordinate is

$$\hat{s}_0 = X_{F_{10}} \cap \{e_3 = 0\} : [s_4 : 1 : 1 : 1 : -s_8 : 0]. \tag{3.130}$$

Since this is the only section we naturally choose it as the zero section.

Comparison with the Tate form

Before proceeding with the analysis of $X_{F_{10}}$, let us pause to compare with the standard elliptic fibration with fiber in $\mathbb{P}^2(1, 2, 3)$, that is the Tate form. We emphasize that (3.129) describing $X_{F_{10}}$ can be viewed as a two-fold generalization of the standard Tate form of an elliptic fibration studied e.g. in [2, 3, 91], which is produced in the special case $s_8 = 1$ and $s_4 = 1$.

First, we identify the usual projective coordinates $[z : x : y]$ on $\mathbb{P}^2(1, 2, 3)$ and the Tate coefficients a_i . They read

$$\begin{aligned} z &\equiv e_3, & x &\equiv v, & y &\equiv w, \\ a_1 &\equiv s_6, & a_2 &\equiv s_3, & a_3 &\equiv s_5, & a_4 &\equiv s_2, & a_6 &\equiv s_1. \end{aligned} \tag{3.131}$$

Using this, we see that (3.129) is indeed in Tate form. However, we note that there are two additional coefficients, namely s_8 and s_4 , that do not have an analog in the standard Tate form, because they correspond to the coefficients of y^2 and x^3 , that are typically set to one. As we see below, at the vanishing loci of these sections we find a $SU(3)$ - and a $SU(2)$ -singularity, respectively. Thus, allowing for non-trivial s_4, s_8 , is the first of the two aforementioned generalizations of $X_{F_{10}}$, compared to the standard Tate form.

In addition, consistently imposing $s_4 = s_8 = 1$ fixes the degrees of freedom in constructing the fibration of the elliptic curve $\mathcal{C}_{F_{10}}$ over the base B . Indeed, setting $s_4 = s_8 = 1$ requires their divisor classes to be trivial, $[s_8] = 0, [s_4] = 0$. This fixes \mathcal{S}_7 and \mathcal{S}_9 according to (3.8) as

$$\mathcal{S}_7 \stackrel{!}{=} [K_B], \quad \mathcal{S}_9 \stackrel{!}{=} 2\mathcal{S}_7 = 2[K_B]. \tag{3.132}$$

Thus, the fibration $X_{F_{10}}$ is completely fixed in terms of the canonical bundle K_B of the base B . As we see from (3.127), the coordinates u and v transform as a section of the line bundles K_B^3 and K_B , respectively. Using the \mathbb{C}^* -action on $\mathbb{P}_{F_{10}}$, this is equivalent, employing (3.131), to

$$x \in \mathcal{O}_B([K_B^{-2}]), \quad y \in \mathcal{O}_B([K_B^{-3}]). \tag{3.133}$$

Thus, we see that by relaxing $s_4 = s_8 = 1$, we also get more freedom, parametrized in the divisors \mathcal{S}_7 and \mathcal{S}_9 , in constructing the fibration of $\mathcal{C}_{F_{10}}$ over a given base B . This is the second generalization of $X_{F_{10}}$ in contrast to the standard Tate model.

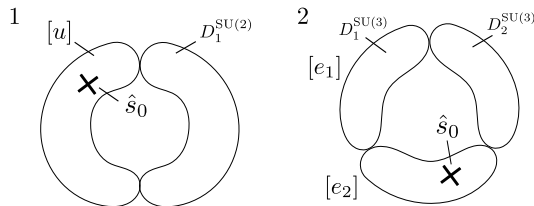


Figure 17. Codimension one fibers of $X_{F_{10}}$. The crosses denote the intersections with the zero section.

Higher codimension singularities & the spectrum of F-theory on $X_{F_{10}}$

We consider in the following the most general elliptic fibration $X_{F_{10}}$ with general, non-trivial coefficients s_4 and s_8 . In order to compute the Weierstrass form (2.1) of the general hypersurface equation (3.129) we apply Nagell’s algorithm. After setting $s_7 = s_9 = s_{10} = 0$ in (B.1) and (B.2) we obtain the functions f and g . From this we compute the discriminant Δ to find all codimension one singularities of the WSF of $X_{F_{10}}$. We find one I_2 -singularity over the divisor $\mathcal{S}_{\text{SU}(2)}^b = \{s_4 = 0\} \cap B$ in B and one I_3 -singularity over the divisor $\mathcal{S}_{\text{SU}(3)}^b = \{s_8 = 0\} \cap B$ in B . At the singularities the constraint (3.129) factorizes as

$$\begin{aligned} \text{SU}(2) : \quad p_{F_{10}}|_{s_4=0} &= u \cdot q_2, \\ \text{SU}(3) : \quad p_{F_{10}}|_{s_8=0} &= e_1 e_2 \cdot q_3, \end{aligned} \tag{3.134}$$

where q_2 and q_3 are the remaining polynomials after factoring out u and $e_1 e_2$, respectively. The reducible fibers at these loci are depicted in figure 17. Thus, the total gauge group of $X_{F_{10}}$ is

$$G_{F_{10}} = \text{SU}(3) \times \text{SU}(2). \tag{3.135}$$

The divisor classes of the corresponding Cartan divisors can be calculated in a similar fashion as in section 3.3.3. We obtain the classes

$$D_1^{\text{SU}(2)} = [s_4] - [u], \quad D_1^{\text{SU}(3)} = [e_1], \quad D_2^{\text{SU}(3)} = [s_8] - [e_1] - [e_2]. \tag{3.136}$$

Next, we turn to the charged matter spectrum, which is obtained by analyzing the codimension two singularities of the WSF of $X_{F_{10}}$. All loci of codimension two singularities directly follow from the behavior of the discriminant and the representation content under the gauge group $G_{F_{10}}$ is determined following the general procedure outlined in section 2.1. We find five codimension two singularities. Three of these lead to the matter representations and the corresponding codimension two fibers in $X_{F_{10}}$ in the first and third column of table 19, respectively. We note that the fiber corresponding to matter in the representation $(\mathbf{2}, \mathbf{1})$ is non-split, cf. the discussion in 3.4.2. The two nodes that are identified by codimension three monodromies are drawn with dashed lines in table 19. At the locus $s_4 = s_6^2 - 4s_3 s_8 = 0$ the fiber is of Type *III* and at the locus $s_6 = s_8 = 0$ it is of Type *IV*, i.e. the fiber is a degeneration of the I_3 -fiber at the locus $s_8 = 0$, where the three \mathbb{P}^1 ’s intersect in one point. Thus both loci do not support any additional matter. The matter in the adjoint representations has been added to table 19 for completeness.

| Representation | Multiplicity | Fiber | Locus |
|----------------|--|-----------|---|
| (2, 3) | $(2\mathcal{S}_7 - \mathcal{S}_9)$ $([K_B^{-1}] - \mathcal{S}_7 + \mathcal{S}_9)$ | | $V(I_{(1)}) :=$ $\{s_4 = s_8 = 0\}$ |
| (2, 1) | $(2\mathcal{S}_7 - \mathcal{S}_9)$ $(5[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_9)$ | | $V(I_{(2)}) := \{s_4 = 0 =$ $-s_3s_5^2 + s_2s_5s_6 - s_1s_6^2$ $-s_2^2s_8 + 4s_1s_3s_8\}$ |
| (1, 3) | $([K_B^{-1}] - \mathcal{S}_7 + \mathcal{S}_9)$ $(6[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_9)$ | | $V(I_{(3)}) := \{s_8 = 0$ $s_4s_5^3 - s_3s_5^2s_6$ $+s_2s_5s_6^2 - s_1s_6^3 = 0\}$ |
| (3, 1) | $1 + \frac{2\mathcal{S}_7 - \mathcal{S}_9 - [K_B^{-1}]}{2}$ $\times (2\mathcal{S}_7 - \mathcal{S}_9)$ | Figure 17 | $s_4 = 0$ |
| (1, 8) | $1 + \frac{\mathcal{S}_9 - \mathcal{S}_7}{2}$ $\times ([K_B^{-1}] - \mathcal{S}_7 + \mathcal{S}_9)$ | Figure 17 | $s_8 = 0$ |

Table 19. Charged matter representations under $SU(3) \times SU(2)$ and corresponding codimension two fibers of $X_{F_{10}}$. The adjoint matter is included for completeness.

In the second column of table 19 the multiplicities of the charged hyper multiplets are presented. They have been computed following section 2.3, directly from the classes of all varieties $V(I_k)$, $k = 1, 2, 3$. This is straightforward, employing (3.8), as these varieties are irreducible complete intersections.

Finally, the number of neutral hyper multiplets is computed from (2.11) using the Euler number (C.1) of $X_{F_{10}}$. It reads

$$H_{\text{neut}} = 15 + 11[K_B^{-1}]^2 - 6[K_B^{-1}]\mathcal{S}_7 + 6\mathcal{S}_7^2 - 3[K_B^{-1}]\mathcal{S}_9 - 6\mathcal{S}_7\mathcal{S}_9 + 3\mathcal{S}_9^2. \quad (3.137)$$

The number T of tensor multiplets is given by (2.17) and we have $V = 11$. We check that all 6D anomalies, cf. appendix A, are canceled using $\mathcal{S}_{SU(2)}^b = \{s_4 = 0\} \cap B$, $\mathcal{S}_{SU(3)}^b = \{s_8 = 0\} \cap B$, the charged spectrum in table 19 and (3.137).

We conclude this section by the analysis of codimension three singularities of the WSF of $X_{F_{10}}$ and the corresponding Yukawa points in compactifications to 4D. We find one possible Yukawa coupling of the charged matter spectrum of $X_{F_{10}}$, which is given in table 20.

| Yukawa | Locus |
|---|---|
| $(\mathbf{2}, \mathbf{3}) \cdot (\mathbf{2}, \mathbf{1}) \cdot \overline{(\mathbf{1}, \mathbf{3})}$ | $s_4 = s_8 = 0 = s_3 s_5^2 - s_2 s_5 s_6 + s_1 s_6^2$ |
| $(\mathbf{2}, \mathbf{3}) \cdot (\mathbf{2}, \mathbf{3}) \cdot (\mathbf{1}, \mathbf{3})$ | $s_4 = s_6 = s_8 = 0$ |
| $(\mathbf{1}, \mathbf{3}) \cdot (\mathbf{1}, \mathbf{3}) \cdot (\mathbf{1}, \mathbf{3})$ | $s_5 = s_6 = s_8 = 0$ |

Table 20. Codimension three loci and corresponding Yukawa couplings for $X_{F_{10}}$.

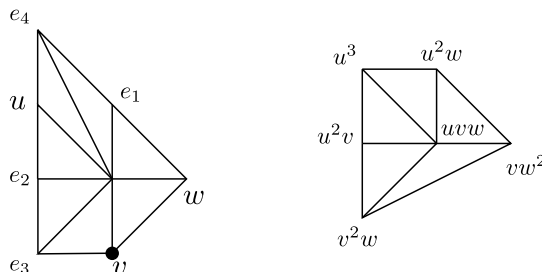


Figure 18. The toric diagram of polyhedron F_{11} and its dual. The zero section is indicated by the dot.

3.5 Fibrations with gauge groups of rank 4, 5 and no MW-torsion

In this section we analyze toric hypersurface fibrations based on the fiber polyhedra F_{11} , F_{12} and F_{14} . These are the fibrations that give rise to F-theory models with simply-connected gauge groups of maximal rank among all toric hypersurface fibrations, that is four and five. Most outstanding here is $X_{F_{11}}$ that exhibits the gauge group and the matter representations that coincide precisely with that of the *Standard Model*.

3.5.1 Polyhedron F_{11} : $G_{F_{11}} = \text{SU}(3) \times \text{SU}(2) \times \text{U}(1)$

We construct an elliptically fibered Calabi-Yau manifold $X_{F_{11}}$ with an arbitrary base B and general elliptic fiber given by the elliptic curve \mathcal{E} in $\mathbb{P}_{F_{11}}$. The toric data of $\mathbb{P}_{F_{11}}$ is encoded in figure 18, where the corresponding polyhedron F_{11} , a choice of homogeneous coordinates as well as its dual polyhedron F_6 are shown. In the monomials that are associated to the integral points of F_6 according to (2.23), we have set $e_i = 1, \forall i$, for brevity of our notation. Starting from \mathbb{P}^2 , we obtain the toric variety $\mathbb{P}_{F_{11}}$ as a blow-up at four non-generic points. The blow-down map reads

$$u \rightarrow e_1 e_2 e_3 e_4^2 u, \quad w \rightarrow e_1 e_4 w, \quad v \rightarrow e_2 e_3^2 v. \quad (3.138)$$

After these blow-ups, the homogeneous coordinates on the fiber, given by $[u : v : w : e_1 : e_2 : e_3 : e_4]$, take values in the line bundles associated to the following divisor classes:

| Section | Divisor class |
|---------|---|
| u | $H - E_1 - E_2 - E_4 + \mathcal{S}_9 + [K_B]$ |
| v | $H - E_2 - E_3 + \mathcal{S}_9 - \mathcal{S}_7$ |
| w | $H - E_1$ |
| e_1 | $E_1 - E_4$ |
| e_2 | $E_2 - E_3$ |
| e_3 | E_3 |
| e_4 | E_4 |

(3.139)

The Stanley-Reisner ideal of $\mathbb{P}_{F_{11}}$ can be read off from figure 18. It is given by

$$SR = \{ue_1, uw, uv, ue_3, e_4w, e_4v, e_4e_3, e_4e_2, e_1v, e_1e_3, e_1e_2, we_3, we_2, ve_2\}. \quad (3.140)$$

We obtain the hypersurface equation of $X_{F_{11}}$ in the ambient space given by the $\mathbb{P}_{F_{11}}$ -fibration (3.1) either by applying (2.23) or by specializing (3.4) as $s_4 = s_7 = s_8 = s_{10} = 0$ and applying the map (3.138). It reads

$$p_{F_{11}} = s_1e_1^2e_2^2e_3e_4^4u^3 + s_2e_1e_2^2e_3^2e_4^2u^2v + s_3e_2^2e_3^2uv^2 + s_5e_1^2e_2e_4^3u^2w + s_6e_1e_2e_3e_4uvw + s_9e_1vw^2, \quad (3.141)$$

where the sections s_i take values in the line bundles associated to the divisor classes in (3.8).

The elliptic fibration $X_{F_{11}}$ has three rational sections. Two of these are linear independent, that means the MW-group of $X_{F_{11}}$ has rank one. The coordinates of the sections read

$$\begin{aligned} \hat{s}_0 &= X_{F_{11}} \cap \{v = 0\} : [1 : 0 : s_1 : 1 : 1 : -s_5 : 1], \\ X_{F_{11}} \cap \{e_3 = 0\} &: [1 : s_5 : 1 : 1 : -s_9 : 0 : 1], \\ \hat{s}_1 &= X_{F_{11}} \cap \{e_4 = 0\} : [s_9 : 1 : 1 : -s_3 : 1 : 1 : 0], \end{aligned} \quad (3.142)$$

where we choose \hat{s}_0 as the zero section and \hat{s}_1 as the generator of the MW-group.

The WSF (2.1) of (3.141) is computed using Nagell's algorithm. The WS-coordinates of the section \hat{s}_1 are given by restricting (B.8) as $s_4 = s_7 = s_8 = 0$. The functions f and g of the WSF can be obtained by setting $s_4 = s_7 = s_8 = s_{10} = 0$ in (B.1) and (B.2), respectively. Using that we calculate the discriminant Δ . This allows us to find all codimension one singularities of the WSF of $X_{F_{11}}$. We find one I_2 -singularity over the divisor $\mathcal{S}_{\text{SU}(2)}^b = \{s_3 = 0\} \cap B$ and one I_3 -singularity over the divisor $\mathcal{S}_{\text{SU}(3)}^b = \{s_9 = 0\} \cap B$ in B . Along these divisors the constraint (3.141) factorizes as

$$\begin{aligned} \text{SU}(2) : \quad p_{F_{11}}|_{s_3=0} &= e_1 \cdot q_3, \\ \text{SU}(3) : \quad p_{F_{11}}|_{s_9=0} &= e_2u \cdot q_2, \end{aligned} \quad (3.143)$$

where q_2, q_3 are the homogeneous polynomials in $[u : v : w]$ of degree two and three that remain after factoring out e_1 and e_2u . The corresponding I_2 - and I_3 -fibers are depicted in figure 19. In summary, the total gauge group of $X_{F_{11}}$ is

$$G_{F_{11}} = \text{SU}(3) \times \text{SU}(2) \times \text{U}(1). \quad (3.144)$$

Following the path of section 3.3.3 we calculate the classes of the Cartan divisors as

$$D_1^{\text{SU}(2)} = [e_1], \quad D_1^{\text{SU}(3)} = [e_2] \quad D_2^{\text{SU}(3)} = [u]. \quad (3.145)$$

This enables the computation of the Shioda map (2.5) of the section \hat{s}_1 . It reads

$$\sigma(\hat{s}_1) = S_1 - S_0 + [K_B] + \frac{1}{2}D_1^{\text{SU}(2)} + \frac{1}{3}(D_1^{\text{SU}(3)} + 2D_2^{\text{SU}(3)}), \quad (3.146)$$

which follows since the section \hat{s}_1 does not intersect the zero section, see (3.142), implying

$$\pi(S_1 \cdot S_0) = 0, \quad (3.147)$$

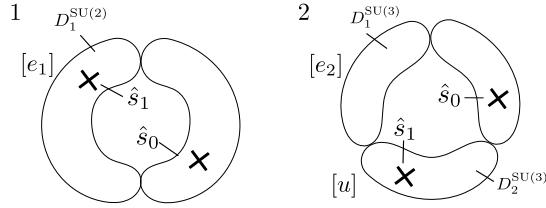


Figure 19. Codimension one fibers of $X_{F_{11}}$. The crosses denote the intersections with the two sections.

and from the intersections of \hat{s}_0 and \hat{s}_1 with the codimension one fibers in figure 19, yielding

$$S_1 \cdot C_{-\alpha_1}^{\text{SU}(2)} = 1, \quad S_1 \cdot C_{-\alpha_1}^{\text{SU}(3)} = 0, \quad S_1 \cdot C_{-\alpha_1}^{\text{SU}(3)} = 1. \quad (3.148)$$

The data of the MW-group is completed by the height pairing (2.6) of \hat{s}_1 . It is computed as

$$b_{11} = \frac{3}{2}[K_B^{-1}] - \frac{1}{2}\mathcal{S}_7 - \frac{1}{6}\mathcal{S}_9, \quad (3.149)$$

where we use the universal intersection relation (2.7) as well as (3.147).

Next, we turn to the codimension two singularities of the WSF of $X_{F_{11}}$ to calculate its charged matter spectrum. Here, all representations under the gauge group are determined using the methods outlined in section 2.1. The non-Abelian representations readily follow from the discriminant, whereas the charged singlets are determined from the primary decomposition of the complete intersection (2.14) for the section \hat{s}_1 . Using this, we find seven singularities in codimension two. Five of these lead to the matter representations and the corresponding codimension two fibers in $X_{F_{11}}$ given in the first and third column of table 21, respectively. The remaining singularities at $s_3 = s_6^2 - 4s_2s_9 = 0$ and $s_6 = s_9 = 0$ are of Type *III* and *IV*, respectively. Since they are just degenerations of the codimension one fibers in figure 19 without additional \mathbb{P}^1 's, they do not yield further matter representations. The adjoint representations in the last two rows of table 21 are shown for completeness.

The multiplicities of the charged hyper multiplets that are presented in table 21 are straightforwardly computed from the homology class of all complete intersections $V(I_{(k)})$, $k = 1, \dots, 5$.

We complete the matter spectrum of $X_{F_{11}}$ by the number of neutral hyper multiplets, which is computed from (2.11) using the Euler number $\chi(X_{F_{11}})$ given in (C.1). It reads

$$H_{\text{neut}} = 16 + 11[K_B^{-1}]^2 - 4[K_B^{-1}]\mathcal{S}_7 + 2\mathcal{S}_7^2 - 7[K_B^{-1}]\mathcal{S}_9 - \mathcal{S}_7\mathcal{S}_9 + 3\mathcal{S}_9^2. \quad (3.150)$$

There are T tensors computed by (2.17) and we have $V = 12$ vector multiplets. Using $S_{\text{SU}(2)}^b = \{s_3 = 0\}$, $S_{\text{SU}(3)}^b = \{s_9 = 0\}$, (3.149), the charged spectrum in table 21 and (3.150) we check cancelation of all 6D anomalies in (A.1), following the discussion of appendix A.

We conclude our analysis with the Yukawa couplings of the charged matter spectrum of $X_{F_{11}}$, corresponding to the codimension three singularities of its WSF. All Yukawa points of $X_{F_{11}}$ are presented in table 22.

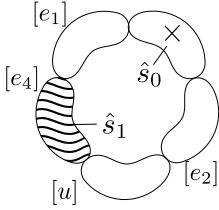
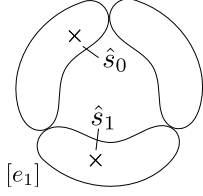
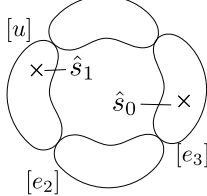
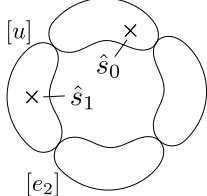
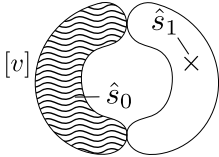
| Representation | Multiplicity | Splitting | Locus |
|-----------------------------------|--|---|---|
| $(\mathbf{3}, \mathbf{2})_{-1/6}$ | $\mathcal{S}_9([K_B^{-1}] + \mathcal{S}_7 - \mathcal{S}_9)$ |  | $V(I_{(1)}) := \{s_3 = s_9 = 0\}$ |
| $(\mathbf{1}, \mathbf{2})_{1/2}$ | $([K_B^{-1}] + \mathcal{S}_7 - \mathcal{S}_9)$ $(6[K_B^{-1}] - 2\mathcal{S}_7 - \mathcal{S}_9)$ |  | $V(I_{(2)}) := \{s_3 = 0$ $s_2 s_5^2 + s_1(s_1 s_9 - s_5 s_6) = 0\}$ |
| $(\mathbf{3}, \mathbf{1})_{-2/3}$ | $\mathcal{S}_9(2[K_B^{-1}] - \mathcal{S}_7)$ |  | $V(I_{(3)}) := \{s_5 = s_9 = 0\}$ |
| $(\mathbf{3}, \mathbf{1})_{1/3}$ | $\mathcal{S}_9(5[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_9)$ |  | $V(I_{(4)}) := \{s_9 = 0$ $s_3 s_5^2 + s_6(s_1 s_6 - s_2 s_5) = 0\}$ |
| $(\mathbf{1}, \mathbf{1})_{-1}$ | $(2[K_B^{-1}] - \mathcal{S}_7)$ $(3[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_9)$ |  | $V(I_{(5)}) := \{s_1 = s_5 = 0\}$ |
| $(\mathbf{8}, \mathbf{1})_0$ | $1 + \mathcal{S}_9 \frac{\mathcal{S}_9 - [K_B^{-1}]}{2}$ | Figure 19 | $s_9 = 0$ |
| $(\mathbf{1}, \mathbf{3})_0$ | $1 + \frac{\mathcal{S}_7 - \mathcal{S}_9}{2}$ $\times ([K_B^{-1}] + \mathcal{S}_7 - \mathcal{S}_9)$ | Figure 19 | $s_3 = 0$ |

Table 21. Charged matter representations under $SU(3) \times SU(2) \times U(1)$ and corresponding codimension two fibers of $X_{F_{11}}$. The adjoint matter is included for completeness.

3.5.2 Polyhedron F_{12} : $G_{F_{12}} = SU(2)^2 \times U(1)^2$

In this section, we analyze the elliptically fibered Calabi-Yau manifold $X_{F_{12}}$ with base B and general elliptic fiber given by the elliptic curve \mathcal{E} in $\mathbb{P}_{F_{12}}$. The toric data of $\mathbb{P}_{F_{12}}$ can be extracted from figure 20, where the fiber polyhedron F_{12} together with a choice of homogeneous coordinates as well as its dual polyhedron are shown. As before, we have set $e_i = 1, \forall i$, in the monomials associated to the integral points of F_5 by (2.23). The toric variety $\mathbb{P}_{F_{12}}$ is \mathbb{P}^2 blown-up at four non-generic points. In our conventions, the blow-down

| Yukawa | Locus |
|--|-------------------------------------|
| $(\mathbf{3}, \mathbf{2})_{-1/6} \cdot (\mathbf{3}, \mathbf{1})_{-2/3} \cdot (\mathbf{1}, \mathbf{2})_{1/2}$ | $s_3 = s_5 = s_9 = 0$ |
| $(\mathbf{3}, \mathbf{2})_{-1/6} \cdot (\mathbf{3}, \mathbf{1})_{1/3} \cdot (\mathbf{1}, \mathbf{2})_{1/2}$ | $s_3 = s_9 = 0 = s_1 s_6 - s_2 s_5$ |
| $(\mathbf{3}, \mathbf{1})_{-2/3} \cdot (\mathbf{3}, \mathbf{1})_{1/3} \cdot (\mathbf{1}, \mathbf{1})_{-1}$ | $s_1 = s_5 = s_9 = 0$ |
| $(\mathbf{3}, \mathbf{2})_{-1/6} \cdot (\mathbf{3}, \mathbf{2})_{-1/6} \cdot (\mathbf{3}, \mathbf{1})_{1/3}$ | $s_3 = s_9 = s_6 = 0$ |
| $(\mathbf{1}, \mathbf{2})_{1/2} \cdot (\mathbf{1}, \mathbf{2})_{1/2} \cdot (\mathbf{1}, \mathbf{1})_{-1}$ | $s_1 = s_5 = s_3 = 0$ |
| $(\mathbf{3}, \mathbf{1})_{1/3} \cdot (\mathbf{3}, \mathbf{1})_{1/3} \cdot (\mathbf{3}, \mathbf{1})_{-2/3}$ | $s_5 = s_6 = s_9$ |

Table 22. Codimension three loci and corresponding Yukawa couplings for $X_{F_{11}}$.

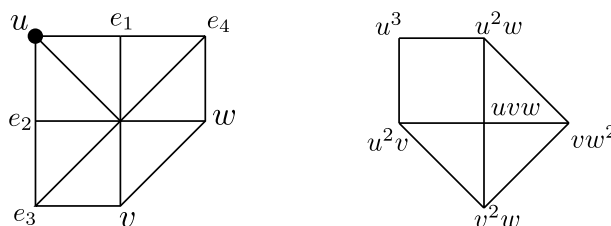


Figure 20. The toric diagram of polyhedron F_{12} and its dual. The zero section is indicated by the dot.

map takes the form

$$u \rightarrow e_1 e_2 e_3 e_4 u, \quad w \rightarrow e_1 e_4^2 w, \quad v \rightarrow e_2 e_3^2 v, \quad (3.151)$$

so that the homogeneous coordinates on the fiber after this blow-up are $[u : v : w : e_1 : e_2 : e_3 : e_4]$. In the total space of the $\mathbb{P}_{F_{12}}$ -fibration constructed as in (3.1), these coordinates have the divisor classes given by:

| Section | Divisor class |
|---------|---|
| u | $H - E_1 - E_2 + \mathcal{S}_9 + [K_B]$ |
| v | $H - E_2 - E_3 + \mathcal{S}_9 - \mathcal{S}_7$ |
| w | $H - E_1 - E_4$ |
| e_1 | $E_1 - E_4$ |
| e_2 | $E_2 - E_3$ |
| e_3 | E_3 |
| e_4 | E_4 |

(3.152)

The Stanley-Reisner ideal of $\mathbb{P}_{F_{12}}$ follows from figure 20 as

$$SR = \{ue_4, uw, uv, ue_3, e_1 w, e_1 v, e_1 e_3, e_1 e_2, e_4 v, e_4 e_3, e_4 e_2, we_3, we_2, ve_2\}. \quad (3.153)$$

The hypersurface equation for $X_{F_{12}}$ can be obtained employing (2.23). It reads

$$p_{F_{12}} = s_1 e_1^2 e_2^2 e_3 e_4 u^3 + s_2 e_1 e_2^2 e_3^2 u^2 v + s_5 e_1^2 e_2 e_4^2 u^2 w + s_6 e_1 e_2 e_3 e_4 u w w + s_7 e_2 e_3^2 v^2 w + s_9 e_1 e_4^2 v w^2, \quad (3.154)$$

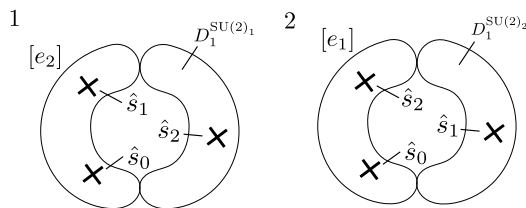


Figure 21. Codimension one fibers of $X_{F_{12}}$. The crosses denote the intersections with the three sections.

where the divisor classes of the sections s_i are fixed by the Calabi-Yau condition as shown in (3.8). We note that (3.154) can also be obtained from (3.4) by the specialization $s_3 = s_4 = s_8 = s_{10} = 0$ and the map (3.151).

There are five rational sections of the elliptic fibration of $X_{F_{12}}$. Their coordinates are

$$\begin{aligned}
 \hat{s}_0 &= X_{F_{12}} \cap \{u = 0\} : & [0 : 1 : 1 : s_7 : -s_9 : 1 : 1], \\
 \hat{s}_1 &= X_{F_{12}} \cap \{e_3 = 0\} : & [1 : s_5 : 1 : 1 : -s_9 : 0 : 1], \\
 \hat{s}_2 &= X_{F_{12}} \cap \{e_4 = 0\} : & [1 : 1 : s_2 : -s_7 : 1 : 1 : 0], \\
 X_{F_{12}} \cap \{v = 0\} &: & [1 : 0 : s_1 : 1 : 1 : -s_5 : 1], \\
 X_{F_{12}} \cap \{w = 0\} &: & [1 : s_1 : 0 : 1 : 1 : 1 : -s_2],
 \end{aligned} \tag{3.155}$$

where we choose \hat{s}_0 as the zero section. Clearly, only three of these sections are linearly independent. We choose \hat{s}_1 and \hat{s}_2 as the generators of the rank two MW-group of $X_{F_{12}}$.

We compute the Weierstrass form (2.1) of (3.154) using Nagell's algorithm. The WS-coordinates of the sections \hat{s}_1 and \hat{s}_2 are given by (B.9) and (B.10), respectively, in the limit $s_3 = s_4 = s_8 = s_{10} = 0$. Similarly, we obtain the functions f and g from (B.1) and (B.2) using this specialization. From this the discriminant Δ is readily computed. The factorization of Δ shows the presence of two I_2 -singularities in $X_{F_{12}}$ over the divisors $\mathcal{S}_{\text{SU}(2)_1}^b = \{s_7 = 0\} \cap B$ and $\mathcal{S}_{\text{SU}(2)_2}^b = \{s_9 = 0\} \cap B$ in B . At these loci, the constraint (3.154) factorizes as

$$\begin{aligned}
 \text{SU}(2)_1 : & \quad p_{F_{12}}|_{s_7=0} = e_1 q_3, \\
 \text{SU}(2)_2 : & \quad p_{F_{12}}|_{s_9=0} = e_2 q'_3,
 \end{aligned} \tag{3.156}$$

where q_3, q'_3 are the remaining polynomials after factoring out e_1 and e_2 . The corresponding I_2 -fibers are depicted in figure 21. In summary, the total gauge group of $X_{F_{12}}$ is

$$G_{F_{12}} = \text{SU}(2)^2 \times \text{U}(1)^2. \tag{3.157}$$

Analogous to section 3.3.3, we obtain the divisor classes of the Cartan divisors of $X_{F_{12}}$ as

$$D_1^{\text{SU}(2)_1} = [s_7] - [e_1], \quad D_1^{\text{SU}(2)_2} = [s_9] - [e_2]. \tag{3.158}$$

Using these results, we compute the Shioda map (2.5) of the sections \hat{s}_m , $m = 1, 2$, as

$$\sigma(\hat{s}_m) = S_m - S_0 + [K_B] + \frac{1}{2} \delta_{m,1} D_1^{\text{SU}(2)_1} + \frac{1}{2} \delta_{m,2} D_1^{\text{SU}(2)_2}. \tag{3.159}$$

Here S_0, S_m denote the divisor classes of the sections \hat{s}_0, \hat{s}_m , respectively, and we used

$$\pi(S_1 \cdot S_2) = \pi(S_1 \cdot S_0) = \pi(S_2 \cdot S_0) = 0, \tag{3.160}$$

which follows directly from (3.155) as well as

$$S_m \cdot C_{-\alpha_i}^{\text{SU}(2)_1} = \delta_{m,2}, \quad S_m \cdot C_{-\alpha_i}^{\text{SU}(2)_2} = \delta_{m,1}, \tag{3.161}$$

which can be deduced from figure 21. Employing these intersection relations along with (2.7), we obtain the height pairing (2.6) as

$$b_{mn} = \begin{pmatrix} 2[K_B^{-1}] - \frac{1}{2}\mathcal{S}_7 & [K_B^{-1}] \\ [K_B^{-1}] & 2[K_B^{-1}] - \frac{1}{2}\mathcal{S}_9 \end{pmatrix}_{mn}. \tag{3.162}$$

To obtain the charged matter spectrum we analyze the codimension two singularities of the WSF of $X_{F_{12}}$. The corresponding representation under the gauge group are determined following the general procedure outlined in section 2.1. As before all non-trivial representations of the non-Abelian part of $G_{F_{12}}$ are easily read off from the discriminant. The charged singlets are obtained by the primary decompositions of the two complete intersections (2.14) associated to the sections \hat{s}_1 and \hat{s}_2 . We find ten codimension two singularities, eight of which lead to the matter representations and the corresponding codimension two fibers in $X_{F_{12}}$ given in the first and third column of table 23, respectively. At the remaining loci, namely $s_7 = s_6^2 - 4s_2s_9 = 0$ and $s_9 = s_6^2 - 4s_5s_7$, we find Type III singularities and thus no additional matter, cf. section 3.3.3 for more details. We note that the matter locus $V(I_{(3)})$ agrees with the singular locus of the dependent rational section in (3.155). For completeness, matter in the adjoint representation of $G_{F_{12}}$ is also given in the last two rows of table 23.

The number of neutral hyper multiplets completes the matter spectrum of $X_{F_{12}}$. It is computed from (2.11) using the Euler number (C.1) of $X_{F_{12}}$. It reads

$$H_{\text{neut}} = 16 + 11[K_B^{-1}]^2 - 6[K_B^{-1}]\mathcal{S}_7 + 2\mathcal{S}_7^2 - 6[K_B^{-1}]\mathcal{S}_9 + \mathcal{S}_7\mathcal{S}_9 + 2\mathcal{S}_9^2. \tag{3.163}$$

There are a base-dependent number T of tensor and $V = 7$ vector multiplets. Finally, we use $S_{\text{SU}(2)_1}^b = \{s_7 = 0\}$, $S_{\text{SU}(2)_2}^b = \{s_9 = 0\}$, (3.162), the charged spectrum in table 23 and (3.163) to confirm that all 6D anomalies in (A.1) are canceled.

We conclude with the list of all codimension three singularities of the WSF of $X_{F_{12}}$ and the corresponding Yukawa points in table 24.

3.5.3 Polyhedron F_{14} : $G_{F_{14}} = \text{SU}(3) \times \text{SU}(2)^2 \times \text{U}(1)$

Consider the elliptically fibered Calabi-Yau manifold $X_{F_{14}}$ with base B and general elliptic fiber given by the elliptic curve \mathcal{E} in $\mathbb{P}_{F_{14}}$. In figure 22 the toric data of $\mathbb{P}_{F_{14}}$ is summarized in terms of its polyhedron F_{14} , a choice of homogeneous coordinates as well as its dual polyhedron F_3 with all monomials (shown in the patch $e_i = 1, \forall i$) corresponding to its integral points. Note that $\mathbb{P}_{F_{14}}$ is the blow-up of \mathbb{P}^2 defined by the blow-up map

$$u \rightarrow e_1e_2e_3e_4e_5u, \quad w \rightarrow e_1e_4^2e_5^3w, \quad v \rightarrow e_2e_3^2v. \tag{3.164}$$

| Representation | Multiplicity | Fiber | Locus |
|---|---|-----------|---|
| $(\mathbf{2}, \mathbf{2})_{(1/2, 1/2)}$ | $\mathcal{S}_7 \cdot \mathcal{S}_9$ | | $V(I_{(1)}) := \{s_9 = s_7 = 0\}$ |
| $(\mathbf{1}, \mathbf{2})_{(-1, -1/2)}$ | $\mathcal{S}_9 \cdot (2[K_B^{-1}] - \mathcal{S}_7)$ | | $V(I_{(2)}) := \{s_9 = s_5 = 0\}$ |
| $(\mathbf{2}, \mathbf{1})_{(-1/2, -1)}$ | $\mathcal{S}_7 \cdot (2[K_B^{-1}] - \mathcal{S}_9)$ | | $V(I_{(3)}) := \{s_7 = s_2 = 0\}$ |
| $(\mathbf{1}, \mathbf{1})_{(1,0)}$ | $(2[K_B^{-1}] - \mathcal{S}_7)$ $(3[K_B^{-1}] - \mathcal{S}_9 - \mathcal{S}_7)$ | | $V(I_{(4)}) := \{s_5 = s_1 = 0\}$ |
| $(\mathbf{1}, \mathbf{1})_{(0,1)}$ | $(2[K_B^{-1}] - \mathcal{S}_9)$ $(3[K_B^{-1}] - \mathcal{S}_9 - \mathcal{S}_7)$ | | $V(I_{(5)}) := \{s_2 = s_1 = 0\}$ |
| $(\mathbf{1}, \mathbf{2})_{(0, -1/2)}$ | $\mathcal{S}_9 \cdot (6[K_B^{-1}] - 2\mathcal{S}_9 - \mathcal{S}_7)$ | | $V(I_{(6)}) := \{s_9 = 0$ $s_5 s_2^2 - s_6 s_2 s_1 + s_7 s_1^2 = 0\}$ |
| $(\mathbf{2}, \mathbf{1})_{(-1/2, 0)}$ | $\mathcal{S}_7 \cdot (6[K_B^{-1}] - \mathcal{S}_9 - 2\mathcal{S}_7)$ | | $V(I_{(7)}) := \{s_7 = 0$ $s_9 s_1^2 + s_2 s_5^2 - s_6 s_5 s_1 = 0\}$ |
| $(\mathbf{1}, \mathbf{1})_{(1,1)}$ | $2[K_B^{-1}] \cdot (3[K_B^{-1}] - \mathcal{S}_7)$ $-\mathcal{S}_9 \cdot (2[K_B^{-1}] - \mathcal{S}_9)$ | | $V(I_{(8)}) := \{s_9 s_2 - s_7 s_5 = 0$ $s_9 s_1 - s_5 s_6 = 0\} \setminus V(I_{(2)})$ |
| $(\mathbf{3}, \mathbf{1})_{(0,0)}$ | $1 + \mathcal{S}_7 \cdot \frac{(\mathcal{S}_7 - [K_B^{-1}])}{2}$ | Figure 21 | $s_7 = 0$ |
| $(\mathbf{1}, \mathbf{3})_{(0,0)}$ | $1 + \mathcal{S}_9 \cdot \frac{(\mathcal{S}_9 - [K_B^{-1}])}{2}$ | Figure 21 | $s_9 = 0$ |

Table 23. Charged matter representations under $SU(2)^2 \times U(1)^2$ and corresponding codimension two fibers of $X_{F_{12}}$. The adjoint matter is included for completeness.

| Yukawa | Locus |
|--|--|
| $(\mathbf{1}, \mathbf{1})_{(1,0)} \cdot (\mathbf{1}, \mathbf{1})_{(0,1)} \cdot \overline{(\mathbf{1}, \mathbf{1})_{(1,1)}}$ | $s_2 = s_5 = s_1 = 0$ |
| $(\mathbf{1}, \mathbf{2})_{(0,-1/2)} \cdot (\mathbf{1}, \mathbf{2})_{(-1,-1/2)} \cdot (\mathbf{1}, \mathbf{1})_{(1,1)}$ | $s_9 = s_5 = 0, s_6 s_2 - s_7 s_1 = 0$ |
| $(\mathbf{2}, \mathbf{1})_{(-1/2,-1)} \cdot (\mathbf{2}, \mathbf{1})_{(-1/2,0)} \cdot (\mathbf{1}, \mathbf{1})_{(1,1)}$ | $s_7 = s_2 = 0, s_1 s_9 - s_5 s_6 = 0$ |
| $(\mathbf{2}, \mathbf{2})_{(1/2,1/2)} \cdot (\mathbf{2}, \mathbf{1})_{(-1/2,0)} \cdot (\mathbf{1}, \mathbf{2})_{(0,-1/2)}$ | $s_9 = s_7 = 0, s_2 s_5 - s_6 s_1 = 0$ |
| $(\mathbf{2}, \mathbf{2})_{(1/2,1/2)} \cdot (\mathbf{1}, \mathbf{2})_{(-1,-1/2)} \cdot \overline{(\mathbf{2}, \mathbf{1})_{(-1/2,0)}}$ | $s_9 = s_7 = s_5 = 0$ |
| $(\mathbf{2}, \mathbf{2})_{(1/2,1/2)} \cdot \overline{(\mathbf{1}, \mathbf{2})_{(0,-1/2)}} \cdot (\mathbf{2}, \mathbf{1})_{(-1/2,-1)}$ | $s_9 = s_7 = s_2 = 0$ |
| $(\mathbf{2}, \mathbf{2})_{(1/2,1/2)} \cdot (\mathbf{2}, \mathbf{2})_{(1/2,1/2)} \cdot (\mathbf{1}, \mathbf{1})_{(1,1)}$ | $s_6 = s_7 = s_9 = 0$ |
| $(\mathbf{1}, \mathbf{2})_{(0,-1/2)} \cdot (\mathbf{1}, \mathbf{2})_{(0,-1/2)} \cdot (\mathbf{1}, \mathbf{1})_{(0,1)}$ | $s_1 = s_2 = s_9 = 0$ |
| $(\mathbf{2}, \mathbf{1})_{(-1/2,0)} \cdot (\mathbf{2}, \mathbf{1})_{(-1/2,0)} \cdot (\mathbf{1}, \mathbf{1})_{(1,0)}$ | $s_1 = s_5 = s_7 = 0$ |

Table 24. Codimension three loci and corresponding Yukawa couplings for $X_{F_{12}}$.

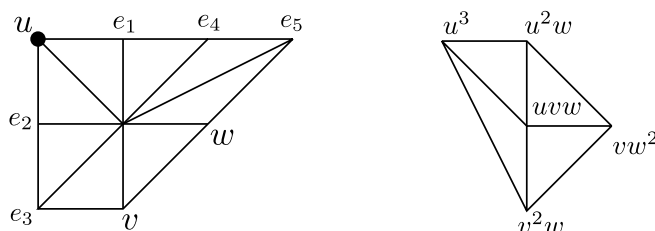


Figure 22. The toric diagram of polyhedron F_{14} and its dual. The zero section is indicated by the dot.

The homogeneous coordinates $[u : v : w : e_1 : e_2 : e_3 : e_4 : e_5]$ on $\mathbb{P}_{F_{14}}$ take values in the line bundles associated to the following divisors:

| Section | Divisor class |
|---------|---|
| u | $H - E_1 - E_2 + \mathcal{S}_9 + [K_B]$ |
| v | $H - E_2 - E_3 + \mathcal{S}_9 - \mathcal{S}_7$ |
| w | $H - E_1 - E_4 - E_5$ |
| e_1 | $E_1 - E_4$ |
| e_2 | $E_2 - E_3$ |
| e_3 | E_3 |
| e_4 | $E_4 - E_5$ |
| e_5 | E_5 |

(3.165)

The Stanley-Reisner ideal of $\mathbb{P}_{F_{14}}$ is given by

$$\begin{aligned}
 SR = \{ & ue_4, ue_5, uw, uv, ue_3, e_1e_5, e_1w, e_1v, e_1e_3, e_1e_2, \\
 & e_4w, e_4v, e_4e_3, e_4e_2, e_5v, e_5e_3, e_5e_2, we_3, we_2, ve_2 \}.
 \end{aligned}$$

(3.166)

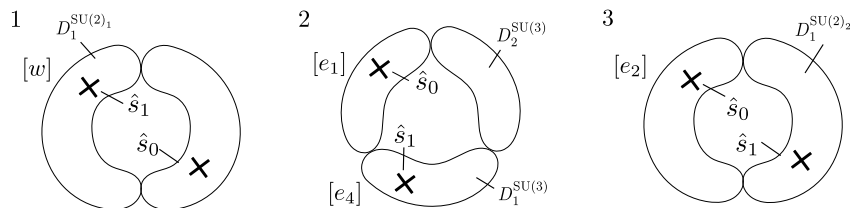


Figure 23. Codimension one fibers of $X_{F_{14}}$. The crosses denote the intersections with the two sections.

In order to find the hypersurface equation for $X_{F_{14}}$ we either use (2.23) or specialize (3.4) as $s_2 = s_3 = s_4 = s_8 = s_{10} = 0$ and apply the map (3.164). It reads

$$p_{F_{14}} = s_1 e_1^2 e_2^2 e_3 e_4 u^3 + s_5 e_1^2 e_2 e_4^2 e_5^2 u^2 w + s_6 e_1 e_2 e_3 e_4 e_5 u v w + s_7 e_2 e_3^2 v^2 w + s_9 e_1 e_4^2 e_5^3 v w^2, \quad (3.167)$$

where the classes of the sections s_i are given in (3.8).

There are four rational sections of the fibration of $X_{F_{14}}$, two of which being linearly independent. The coordinates of the sections are

$$\begin{aligned} \hat{s}_0 &= X_{F_{14}} \cap \{u = 0\} : [0 : 1 : 1 : s_7 : -s_9 : 1 : 1 : 1], \\ \hat{s}_1 &= X_{F_{14}} \cap \{e_5 = 0\} : [1 : 1 : s_1 : 1 : 1 : 1 : -s_7 : 0], \\ X_{F_{14}} \cap \{e_3 = 0\} &: [1 : s_5 : 1 : 1 : -s_9 : 0 : 1 : 1], \\ X_{F_{14}} \cap \{v = 0\} &: [1 : 0 : s_1 : 1 : 1 : -s_5 : 1 : 1], \end{aligned} \quad (3.168)$$

where we choose \hat{s}_0 as the zero section and \hat{s}_1 as the generator of the MW-group of $X_{F_{14}}$.

As a prerequisite for the analysis of the singularities of $X_{F_{14}}$, we compute its Weierstrass form (2.1). This is obtained by applying Nagell's algorithm to (3.167). The WS-coordinates of the section \hat{s}_1 are given by (B.8) after setting $s_2 = s_3 = s_4 = s_8 = 0$. Similarly, we obtain the functions f and g using the specialization $s_2 = s_3 = s_4 = s_8 = s_{10} = 0$ from the general expressions (B.1) and (B.2), respectively. To find all codimension one singularities of the WSF of $X_{F_{14}}$ we calculate the discriminant from f and g . The discriminant Δ factorizes as follows: we find two I_2 -singularities over the divisors $\mathcal{S}_{\text{SU}(2)_1}^b = \{s_1 = 0\} \cap B$ and $\mathcal{S}_{\text{SU}(2)_2}^b = \{s_9 = 0\} \cap B$ in B and one I_3 -singularity over the divisor $\mathcal{S}_{\text{SU}(3)}^b = \{s_7 = 0\} \cap B$ in B . The constraint (3.167) factorizes along these divisors as

$$\begin{aligned} \text{SU}(2)_1 : \quad p_{F_{14}}|_{s_1=0} &= w \cdot q_2, \\ \text{SU}(3) : \quad p_{F_{14}}|_{s_7=0} &= e_1 e_4 \cdot q_3, \\ \text{SU}(2)_2 : \quad p_{F_{14}}|_{s_9=0} &= e_2 \cdot q_3', \end{aligned} \quad (3.169)$$

where q_2 , q_3 and q_3' are the polynomials that remain after factoring out w , $e_1 e_4$ and e_2 . The fibers at these three codimension one loci are depicted in figure 23. In summary, the gauge group of $X_{F_{14}}$ is given by

$$G_{F_{14}} = \text{SU}(3) \times \text{SU}(2)^2 \times \text{U}(1). \quad (3.170)$$

The divisor classes of the Cartan divisors are calculated in a similar way as in section 3.3.3. They read

$$D_1^{\text{SU}(2)_1} = [w], \quad D_1^{\text{SU}(3)} = [e_4], \quad D_2^{\text{SU}(3)} = [s_7] - [e_1] - [e_4], \quad D_1^{\text{SU}(2)_2} = [s_9] - [e_2]. \quad (3.171)$$

Using these results, we compute the Shioda map (2.5) of the section \hat{s}_1 as

$$\sigma(\hat{s}_1) = S_1 - S_0 + [K_B] + \frac{1}{2}D_1^{\text{SU}(2)_1} + \frac{1}{3}(2D_1^{\text{SU}(3)} + D_2^{\text{SU}(3)}) + \frac{1}{2}D_1^{\text{SU}(2)_2}. \quad (3.172)$$

Here S_0, S_1 denote the divisor classes of the sections \hat{s}_0, \hat{s}_1 , respectively, and we have employed

$$\pi(S_1 \cdot S_0) = 0, \quad (3.173)$$

which follows from (3.168) as well as

$$S_1 \cdot C_{-\alpha_1}^{\text{SU}(2)_1} = 1, \quad S_1 \cdot C_{-\alpha_i}^{\text{SU}(3)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad S_1 \cdot C_{-\alpha_1}^{\text{SU}(2)_2} = 1, \quad (3.174)$$

which can be deduced from figure 23. Employing (3.172), we compute the height pairing (2.6), using these results and the intersection (2.7), as

$$b_{11} = -\frac{1}{2}[K_B] - \frac{1}{6}\mathcal{S}_7. \quad (3.175)$$

Next, we determine the spectrum of charged matter by investigating the codimension two singularities of the WSF of $X_{F_{14}}$. As before, all matter representations under the gauge group $G_{F_{14}}$ are determined by application of the techniques introduced in section 2.1. Again all the non-trivial representations under the non-Abelian part of $G_{F_{14}}$ easily follow from the discriminant Δ , while the charged singlets require the primary decomposition of the locus (2.14) for the section \hat{s}_1 . We find nine codimension two singularities in $X_{F_{14}}$. Six of these lead to the matter representations and the corresponding codimension two fibers in $X_{F_{14}}$ given in the first and third column of table 25, respectively. The remaining loci, namely $s_1 = s_6^2 - 4s_5s_7 = 0$, $s_9 = s_6^2 - 4s_5s_7 = 0$ and $s_6 = s_7 = 0$, support two type *III* and one type *IV* fiber, respectively, and thus do not support further representations. The adjoint representations of $G_{F_{14}}$ are shown in the last three rows of table 25 for completeness.

We note that the fiber corresponding to the representation $(\mathbf{2}, \mathbf{1}, \mathbf{2})_0$ is non-split, cf. section 3.4.2 for a more detailed discussion. We have indicated the fibers that are exchanged by codimension three monodromies by dashed lines in table 25.

The multiplicities of the charged hyper multiplets are presented in the second column of table 25. They are computed directly from all complete intersections $V(I_{(k)})$, $k = 1, \dots, 5$.

The matter spectrum of $X_{F_{14}}$ is completed by the number of neutral hyper multiplets, which can be computed from (2.11) using the Euler number (C.1) of $X_{F_{14}}$. It is given by

$$H_{\text{neut}} = 17 + 11[K_B^{-1}]^2 - 9[K_B^{-1}]\mathcal{S}_7 + 3\mathcal{S}_7^2 - 6[K_B^{-1}]\mathcal{S}_9 + 2\mathcal{S}_7\mathcal{S}_9 + 2\mathcal{S}_9^2. \quad (3.176)$$

| Representation | Multiplicity | Fiber | Locus |
|---|---|-----------|--|
| $(\mathbf{2}, \mathbf{1}, \mathbf{1})_{1/2}$ | $(3[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_9) \times (2[K_B^{-1}] - \mathcal{S}_7)$ | | $V(I_{(1)}) := \{s_1 = s_5 = 0\}$ |
| $(\mathbf{1}, \mathbf{3}, \mathbf{1})_{-1/3}$ | $\mathcal{S}_7(3[K_B^{-1}] - \mathcal{S}_7)$ | | $V(I_{(2)}) := \{s_7 = 0, s_5 s_6 - s_1 s_9 = 0\}$ |
| $(\mathbf{1}, \mathbf{1}, \mathbf{2})_{1/2}$ | $\mathcal{S}_9(2[K_B^{-1}] - \mathcal{S}_7)$ | | $V(I_{(3)}) := \{s_5 = s_9 = 0\}$ |
| $(\mathbf{2}, \mathbf{3}, \mathbf{1})_{1/6}$ | $\mathcal{S}_7(3[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_9)$ | | $V(I_{(4)}) := \{s_1 = s_7 = 0\}$ |
| $(\mathbf{2}, \mathbf{1}, \mathbf{2})_0$ | $\mathcal{S}_9(3[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_9)$ | | $V(I_{(5)}) := \{s_1 = s_9 = 0\}$ |
| $(\mathbf{1}, \mathbf{3}, \mathbf{2})_{1/6}$ | $\mathcal{S}_7 \mathcal{S}_9$ | | $V(I_{(6)}) := \{s_7 = s_9 = 0\}$ |
| $(\mathbf{3}, \mathbf{1}, \mathbf{1})_0$ | $1 + \frac{1}{2}(2[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_9) \times (3[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_9)$ | Figure 23 | $s_1 = 0$ |
| $(\mathbf{1}, \mathbf{8}, \mathbf{1})_0$ | $1 + \mathcal{S}_7 \frac{(\mathcal{S}_7 - [K_B^{-1}])}{2}$ | Figure 23 | $s_7 = 0$ |
| $(\mathbf{1}, \mathbf{1}, \mathbf{3})_0$ | $1 + \mathcal{S}_9 \frac{(\mathcal{S}_9 - [K_B^{-1}])}{2}$ | Figure 23 | $s_9 = 0$ |

Table 25. Charged matter representations under $SU(3) \times SU(2)^2 \times U(1)$ and corresponding codimension two fibers of $X_{F_{14}}$. The adjoint matter is included for completeness.

| Yukawa | Locus |
|--|-----------------------|
| $(\mathbf{2}, \mathbf{1}, \mathbf{1})_{1/2} \cdot (\mathbf{1}, \mathbf{3}, \mathbf{1})_{-1/3} \cdot \overline{(\mathbf{2}, \mathbf{3}, \mathbf{1})_{1/6}}$ | $s_1 = s_5 = s_7 = 0$ |
| $(\mathbf{2}, \mathbf{1}, \mathbf{1})_{1/2} \cdot \overline{(\mathbf{1}, \mathbf{1}, \mathbf{2})_{1/2}} \cdot (\mathbf{2}, \mathbf{1}, \mathbf{2})_0$ | $s_1 = s_5 = s_9 = 0$ |
| $(\mathbf{1}, \mathbf{3}, \mathbf{1})_{-1/3} \cdot (\mathbf{1}, \mathbf{1}, \mathbf{2})_{1/2} \cdot \overline{(\mathbf{1}, \mathbf{3}, \mathbf{2})_{1/6}}$ | $s_5 = s_7 = s_9 = 0$ |
| $(\mathbf{2}, \mathbf{3}, \mathbf{1})_{1/6} \cdot (\mathbf{2}, \mathbf{1}, \mathbf{2})_0 \cdot \overline{(\mathbf{1}, \mathbf{3}, \mathbf{2})_{1/6}}$ | $s_1 = s_7 = s_9 = 0$ |
| $(\mathbf{2}, \mathbf{3}, \mathbf{1})_{1/6} \cdot (\mathbf{2}, \mathbf{3}, \mathbf{1})_{1/6} \cdot (\mathbf{1}, \mathbf{3}, \mathbf{1})_{-1/3}$ | $s_1 = s_6 = s_7 = 0$ |
| $(\mathbf{1}, \mathbf{3}, \mathbf{2})_{1/6} \cdot (\mathbf{1}, \mathbf{3}, \mathbf{2})_{1/6} \cdot (\mathbf{1}, \mathbf{3}, \mathbf{1})_{-1/3}$ | $s_6 = s_7 = s_9 = 0$ |

Table 26. Codimension three loci and corresponding Yukawa couplings for $X_{F_{14}}$.

We note that there are a base-dependent number T of tensor and $V = 15$ vector multiplets. As a consistency check we confirm cancelation of all 6D anomalies following appendix A, employing the divisors $\mathcal{S}_{\text{SU}(2)_1}^b$, $\mathcal{S}_{\text{SU}(2)_2}^b$, $\mathcal{S}_{\text{SU}(3)}^b$, (3.175), the spectrum in table 25 and (3.176).

Finally, we list all codimension three singularities of the WSF of $X_{F_{14}}$ and the corresponding Yukawa points of an F-theory compactification to 4D in table 26.

3.6 Fibrations with gauge groups of rank 5 and 6 and MW-torsion

In this section we study the toric hypersurface fibrations constructed from the fiber polyhedra F_{13} , F_{15} and F_{16} . These are the three toric hypersurface fibrations that have non-trivial Mordell-Weil torsion and give rise to non-simply connected gauge groups in F-theory.

The Calabi-Yau manifold $X_{F_{13}}$ has Mordell-Weil group \mathbb{Z}_2 , $X_{F_{15}}$ has Mordell-Weil group $\mathbb{Z} \oplus \mathbb{Z}_2$ and the fibration $X_{F_{16}}$ has Mordell-Weil group \mathbb{Z}_3 [41, 46]. We confirm these findings by explicitly working out the WSF of these toric hypersurface fibrations, which are shown to precisely take the standard form of WSF's with these MW-torsion groups, cf. [53].

The influence of the MW-torsion on the spectrum of F-theory was discussed recently in [46]. There, the models considered in this section were also studied, but under the assumption of a holomorphic zero section. Here, we relax this condition which results in additional gauge groups and matter representations. This has interesting consequences for the phenomenology of these models, because we find that the gauge groups and matter representations of $X_{F_{13}}$ and $X_{F_{15}}$ are completed precisely into the ones of the *Pati-Salam* and *trinification* model, respectively.

3.6.1 Polyhedron F_{13} : $G_{F_{13}} = (\text{SU}(4) \times \text{SU}(2)^2)/\mathbb{Z}_2$

Consider the elliptically fibered Calabi-Yau manifold $X_{F_{13}}$ with base B and general fiber given by the elliptic curve \mathcal{E} in $\mathbb{P}_{F_{13}}$. The toric diagram of the fiber polyhedron F_{13} as well as a choice of homogeneous coordinates and its dual polyhedron are depicted in figure 24, where we have set $e_i = 1, \forall i$, in the monomials that are associated to the integral points of F_4 by (2.23).

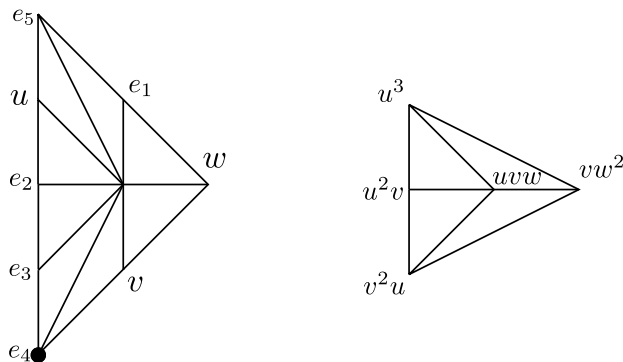


Figure 24. The toric diagram of polyhedron F_{13} and its dual. The zero section is indicated by the dot.

We note that $\mathbb{P}_{F_{13}}$ is the blow-up of \mathbb{P}^2 at five non-generic points, that is defined by

$$u \rightarrow e_1 e_2 e_3 e_4 e_5^2 u, \quad w \rightarrow e_1 e_5 w, \quad v \rightarrow e_2 e_3^2 e_4^3 v. \tag{3.177}$$

The homogeneous coordinates on the fiber after this blow-up are $[u : v : w : e_1 : e_2 : e_3 : e_4 : e_5]$ and take values in the line bundles associated to the divisor classes given by:

| Section | Divisor class |
|---------|---|
| u | $H - E_1 - E_2 - E_5 + \mathcal{S}_9 + [K_B]$ |
| v | $H - E_2 - E_3 - E_4 + \mathcal{S}_9 - \mathcal{S}_7$ |
| w | $H - E_1$ |
| e_1 | $E_1 - E_5$ |
| e_2 | $E_2 - E_3$ |
| e_3 | $E_3 - E_4$ |
| e_4 | E_4 |
| e_5 | E_5 |

(3.178)

The Stanley-Reisner ideal of $\mathbb{P}_{F_{13}}$ follows from figure 26 as

$$SR = \{ue_1, uw, vw, ue_4, ue_3, e_5w, e_5v, e_5e_4, e_5e_3, e_5e_2, e_1v, e_1e_4, e_1e_3, e_1e_2, we_4, we_3, we_2, ve_3, ve_2, e_4e_2\}. \tag{3.179}$$

We find the hypersurface equation for $X_{F_{13}}$ in the total space of the fibration (3.1) with $F_i \equiv F_{13}$ using (2.23) or directly from (3.4) by setting $s_4 = s_5 = s_7 = s_8 = s_{10} = 0$ and by applying the map (3.177). We obtain

$$p_{F_{13}} = s_1 e_1^2 e_2^2 e_3 e_4^4 u^3 + s_2 e_1 e_2^2 e_3^2 e_4^2 e_5^2 u^2 v + s_3 e_2^2 e_3^3 e_4^4 u v^2 + s_6 e_1 e_2 e_3 e_4 e_5 u v w + s_9 e_1 v w^2, \tag{3.180}$$

where the divisor classes of the s_i are given in (3.8).

There are two seemingly rational sections of the fibration of $X_{F_{13}}$. However, there is one torsional relation between them, which reveals that the MW-group is pure torsion,

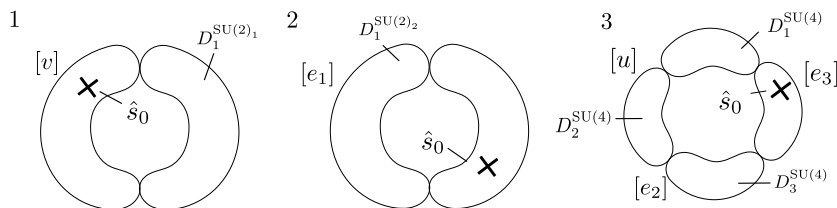


Figure 25. Codimension one fibers of $X_{F_{13}}$. The crosses denote the intersections with the zero section.

namely \mathbb{Z}_2 [41]. The coordinates of our choice for the zero section \hat{s}_0 and the section of order two are

$$\begin{aligned} \hat{s}_0 = X_{F_{13}} \cap \{e_4 = 0\} &: [1 : s_1 : 1 : 1 : 1 : -s_9 : 0 : 1], \\ X_{F_{13}} \cap \{e_5 = 0\} &: [s_9 : 1 : 1 : -s_3 : 1 : 1 : 1 : 0]. \end{aligned} \tag{3.181}$$

The presence of MW-torsion restricts the matter spectrum realized in F-theory [53]. Indeed, the torsion acts on the gauge group, turning it into a non-simply connected group, which reduces its weight lattice, i.e. the realized representations. In [46] this has recently been understood in terms of a geometric k -fractional refinement of the coweight lattice. In particular, it has been argued that the MW-torsion \mathbb{Z}_2 of $X_{F_{13}}$ forbids the presence of fundamental matter in this model. We will confirm these findings in the following explicit analysis.

We begin by computing the Weierstrass form (2.1) of (3.180). As an intermediate step we use the birational map of X_{F_5} to the Tate form given in [39] in the limit $s_5 = s_7 = s_8 = 0$. We obtain the local Tate coefficients (B.14) from which we readily compute the functions f and g , that are given in (B.16). We note that the same WSF arises from the global Tate model given (B.15), which precisely agrees with the Tate form of a model with \mathbb{Z}_2 MW-torsion as argued in [53], confirming the presence of \mathbb{Z}_2 MW-torsion in $X_{F_{13}}$.

Using these results, we readily compute the discriminant Δ , which allows us to find all codimension one singularities of the WSF of $X_{F_{13}}$. We find two I_2 -singularities over the divisors $\mathcal{S}_{\text{SU}(2)_1}^b = \{s_1 = 0\} \cap B$ and $\mathcal{S}_{\text{SU}(2)_2}^b = \{s_3 = 0\} \cap B$ in B as well as an I_4 -singularity over the divisor $\mathcal{S}_{\text{SU}(4)}^b = \{s_9 = 0\} \cap B$ in B . Along these divisors the constraint (3.180) factorizes as

$$\begin{aligned} \text{SU}(2)_1 &: p_{F_{13}}|_{s_1=0} = v \cdot q_2, \\ \text{SU}(2)_2 &: p_{F_{13}}|_{s_3=0} = e_1 \cdot q_3, \\ \text{SU}(4) &: p_{F_{13}}|_{s_9=0} = ue_2e_3 \cdot q'_2, \end{aligned} \tag{3.182}$$

where q_2 , q_3 and q'_2 are the polynomials that remain after factoring out v , e_1 and ue_2e_3 . The corresponding fibers are depicted in figure 25 and give rise to two $\text{SU}(2)$ and one $\text{SU}(4)$ gauge groups. There is another potential codimension one singularity of the WSF of $X_{F_{13}}$, where the fiber of $X_{F_{13}}$ splits into two \mathbb{P}^1 . However, as it is shown in [46], the torsional MW-group identifies these two \mathbb{P}^1 's, so that the fiber in the quotient space is a single singular \mathbb{P}^1 . Thus, there is no additional gauge symmetry and the gauge group of

$X_{F_{13}}$ is

$$G_{F_{13}} = (\mathrm{SU}(4) \times \mathrm{SU}(2)^2)/\mathbb{Z}_2. \quad (3.183)$$

We note that this is precisely the Pati-Salam group. The action of the MW-torsion on the gauge group $G_{F_{13}}$ is worked out in appendix B. To this end, we show that the WS-coordinates of the generator of the \mathbb{Z}_2 MW-torsion, given in (B.17), pass through the WS-coordinates of the singularities in the fiber at all codimension one loci $s_1 = 0$, $s_3 = 0$ and $s_9 = 0$ in (3.182).

As before, cf. section 3.3.3, we calculate the classes of the Cartan divisors of $X_{F_{13}}$ as

$$\begin{aligned} D_1^{\mathrm{SU}(2)_1} &= [s_1] - [v], & D_1^{\mathrm{SU}(2)_2} &= [e_1], \\ D_1^{\mathrm{SU}(4)} &= [s_9] - [u] - [e_2] - [e_3], & D_2^{\mathrm{SU}(4)} &= [u], & D_3^{\mathrm{SU}(4)} &= [e_2]. \end{aligned} \quad (3.184)$$

Next, we calculate the charged matter spectrum of $X_{F_{13}}$, which requires the analysis of all its codimension two singularities. We directly read off from the discriminant of $X_{F_{13}}$ the loci of six codimension two singularities. Four of these lead to the matter representations in the first column of table 27 that are determined, using the techniques discussed in section 2.1, from the corresponding codimension two fibers in $X_{F_{13}}$ given in the third column of the same table. The remaining loci $s_1 = s_6^2 - 4s_2s_9 = 0$ and $s_3 = s_6^2 - 4s_2s_9 = 0$ are both of type *III*, that we first encountered in section 3.3.3, and, thus, do not support additional matter representations. The three adjoint representations in the last three rows of table 27 are shown for completeness.

We find three singularities which support the bi-fundamental representations and one singularity leading to an anti-symmetric representation of $\mathrm{SU}(4)$, but no fundamental representation. This has been anticipated before, due to the action of MW-torsion on the gauge group $G_{F_{13}}$ given in (3.183). In addition, we find that the fibers at the loci $V(I_{(1)})$ and $V(I_{(4)})$, that correspond to the $(\mathbf{2}, \mathbf{2}, \mathbf{1})$ and the $(\mathbf{1}, \mathbf{1}, \mathbf{6})$ representation, are non-split, cf. section 3.4.2 for more details. The \mathbb{P}^1 's drawn with a dashed line are interchanged by codimension three monodromies.

The spectrum of matter of $X_{F_{13}}$ is completed by the number of neutral hyper multiplets, which is computed from (2.11) using the Euler number given in (C.1) of $X_{F_{13}}$. It reads

$$H_{\mathrm{neutral}} = 17 + 11[K_B^{-1}]^2 - 4[K_B^{-1}]\mathcal{S}_7 + 2\mathcal{S}_7^2 - 10[K_B^{-1}]\mathcal{S}_9 + 4\mathcal{S}_9^2. \quad (3.185)$$

The number T of tensor multiplets is given by the base-dependent expression (2.17) and we have $V = 27$ vector multiplets. In order to check that all 6D anomalies are canceled we use the divisors $S_{\mathrm{SU}(2)_1}^b$, $S_{\mathrm{SU}(2)_2}^b$ and $S_{\mathrm{SU}(4)}^b$ as well as the charged spectrum in table 27 and (3.185). Indeed, we find that all 6D anomalies in (A.1) are canceled.

We conclude this section with the list of all geometrically allowed Yukawa couplings of the charged matter spectrum of $X_{F_{13}}$, that is given in table 28.

3.6.2 Polyhedron F_{15} : $G_{F_{15}} = \mathrm{SU}(2)^4/\mathbb{Z}_2 \times \mathrm{U}(1)$

We construct an elliptically fibered Calabi-Yau manifold $X_{F_{15}}$ over a base B and with general fiber given by the elliptic curve \mathcal{E} in $\mathbb{P}_{F_{15}}$. The toric data of $\mathbb{P}_{F_{15}}$ is encoded in

| Representation | Multiplicity | Fiber | Locus |
|---|--|-----------|-----------------------------------|
| $(\mathbf{2}, \mathbf{2}, \mathbf{1})$ | $(3[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_9) \times ([K_B^{-1}] + \mathcal{S}_7 - \mathcal{S}_9)$ | | $V(I_{(1)}) := \{s_1 = s_3 = 0\}$ |
| $(\mathbf{2}, \mathbf{1}, \mathbf{4})$ | $(3[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_9)\mathcal{S}_9$ | | $V(I_{(2)}) := \{s_1 = s_9 = 0\}$ |
| $(\mathbf{1}, \mathbf{2}, \mathbf{4})$ | $([K_B^{-1}] + \mathcal{S}_7 - \mathcal{S}_9)\mathcal{S}_9$ | | $V(I_{(3)}) := \{s_3 = s_9 = 0\}$ |
| $(\mathbf{1}, \mathbf{1}, \mathbf{6})$ | $\mathcal{S}_9[K_B^{-1}]$ | | $V(I_{(4)}) := \{s_6 = s_9 = 0\}$ |
| $(\mathbf{3}, \mathbf{1}, \mathbf{1})$ | $1 + \frac{((2[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_9))}{2} \times (3[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_9)$ | Figure 25 | $s_1 = 0$ |
| $(\mathbf{1}, \mathbf{3}, \mathbf{1})$ | $1 + \frac{(\mathcal{S}_7 - \mathcal{S}_9)}{2} \times ([K_B^{-1}] + \mathcal{S}_7 - \mathcal{S}_9)$ | Figure 25 | $s_3 = 0$ |
| $(\mathbf{1}, \mathbf{1}, \mathbf{15})$ | $1 + \mathcal{S}_9 \frac{(\mathcal{S}_9 - [K_B^{-1}])}{2}$ | Figure 25 | $s_9 = 0$ |

Table 27. Charged matter representations under $(\text{SU}(4) \times \text{SU}(2)^2)/\mathbb{Z}_2$ and corresponding codimension two fibers of $X_{F_{13}}$. The adjoint matter is included for completeness.

| Yukawa | Locus |
|---|-----------------------|
| $(\mathbf{2}, \mathbf{2}, \mathbf{1}) \cdot \overline{(\mathbf{2}, \mathbf{1}, \mathbf{4})} \cdot (\mathbf{1}, \mathbf{2}, \mathbf{4})$ | $s_1 = s_3 = s_9 = 0$ |
| $\overline{(\mathbf{1}, \mathbf{1}, \mathbf{6})} \cdot (\mathbf{2}, \mathbf{1}, \mathbf{4}) \cdot (\mathbf{2}, \mathbf{1}, \mathbf{4})$ | $s_1 = s_6 = s_9 = 0$ |
| $\overline{(\mathbf{1}, \mathbf{1}, \mathbf{6})} \cdot (\mathbf{1}, \mathbf{2}, \mathbf{4}) \cdot (\mathbf{1}, \mathbf{2}, \mathbf{4})$ | $s_3 = s_6 = s_9 = 0$ |

Table 28. Codimension three loci and corresponding Yukawa coupling for $X_{F_{13}}$.

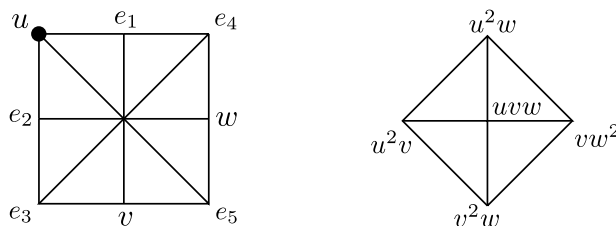


Figure 26. The toric diagram of polyhedron F_{15} and its dual. The zero section is indicated by the dot.

figure 26, that shows the polyhedron F_{15} , our convention for projective coordinates as well as the dual polyhedron F_2 , with the monomials (in the patch $e_i = 1, \forall i$) associated to its integral points. We note that $\mathbb{P}_{F_{15}}$ is obtained from \mathbb{P}^2 by the five non-generic blow-ups defined by

$$u \rightarrow e_1 e_2 e_3 e_4 u, \quad w \rightarrow e_1 e_4^2 e_5 w, \quad v \rightarrow e_2 e_3^2 e_5 v. \tag{3.186}$$

The homogeneous coordinates on the fiber after this blow-up are $[u : v : w : e_1 : e_2 : e_3 : e_4 : e_5]$. Their divisor classes are given by:

| Section | Divisor class |
|---------|---|
| u | $H - E_1 - E_2 + \mathcal{S}_9 + [K_B]$ |
| v | $H - E_2 - E_3 - E_5 + \mathcal{S}_9 - \mathcal{S}_7$ |
| w | $H - E_1 - E_4 - E_5$ |
| e_1 | $E_1 - E_4$ |
| e_2 | $E_2 - E_3$ |
| e_3 | E_3 |
| e_4 | E_4 |
| e_5 | E_5 |

(3.187)

The Stanley-Reisner ideal of $\mathbb{P}_{F_{15}}$ is can be read off from figure 26 as

$$SR = \{ue_4, uw, ue_5, uv, ue_3, e_1w, e_1e_5, e_1v, e_1e_3, e_1e_2, e_4e_5, e_4v, e_4e_3, e_4e_2, \\ vv, we_3, we_2, e_5e_3, e_5e_2, ve_2\}. \tag{3.188}$$

We use (2.23) to find the hypersurface equation for $X_{F_{15}}$ in the ambient space given by the fibration (3.1) with $\mathbb{P}_{F_i} \equiv \mathbb{P}_{F_{15}}$. Alternatively, we can set $s_1 = s_3 = s_4 = s_8 = s_{10} = 0$ in (3.4) and use the map (3.186). Either ways, we obtain

$$p_{F_{15}} = s_2 e_1 e_2^2 e_3^2 u^2 v + s_5 e_1^2 e_2 e_4^2 u^2 w + s_6 e_1 e_2 e_3 e_4 e_5 uvw + s_7 e_2 e_3^2 e_5^2 v^2 w + s_9 e_1 e_4^2 e_5^2 vw^2, \tag{3.189}$$

where the sections s_i assume values in the line bundles associated to the divisor classes in (3.8).

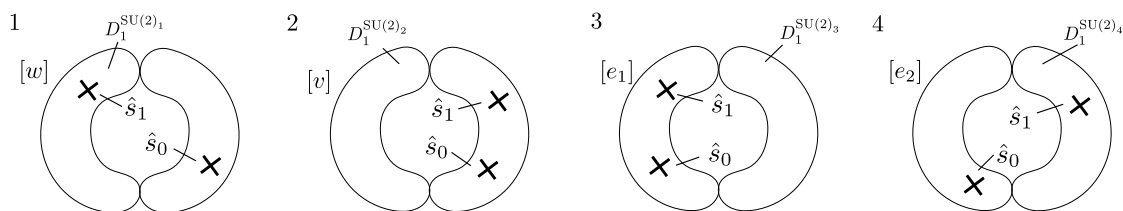


Figure 27. Codimension one fibers of $X_{F_{15}}$. The crosses denote the intersections with the two sections.

The fibration $X_{F_{15}}$ has four seemingly rational points with one linear and one torsional relation between, showing that the MW-group is $\mathbb{Z} \oplus \mathbb{Z}_2$ [41]. Their coordinates are

$$\begin{aligned}
 \hat{s}_0 &= X_{F_{15}} \cap \{u = 0\} : [0 : 1 : 1 : s_7 : -s_9 : 1 : 1 : 1], \\
 \hat{s}_1 &= X_{F_{15}} \cap \{e_4 = 0\} : [1 : 1 : s_2 : -s_7 : 1 : 1 : 0 : 1], \\
 X_{F_{15}} \cap \{e_5 = 0\} &: [1 : s_5 : -s_2 : 1 : 1 : 1 : 1 : 0], \\
 X_{F_{15}} \cap \{e_3 = 0\} &: [1 : s_5 : 1 : 1 : -s_9 : 0 : 1 : 1],
 \end{aligned} \tag{3.190}$$

where we choose \hat{s}_0 as the zero section and \hat{s}_1 as the generator of the free part of the MW-group.

Next, we compute the Weierstrass form (2.1) of (3.189). Again, we use the birational map from X_{F_5} in [39] to first obtain the local Tate coefficients (B.21), which determine the WSF (B.23). The equivalent global Tate model in (B.22) is precisely of the form of an elliptic fibration with $\mathbb{Z} \oplus \mathbb{Z}_2$ MW-group, that has been studied in [53]. The WS-coordinates of the section \hat{s}_1 are given by (B.8) after setting $s_1 = s_3 = s_4 = s_8 = 0$ and the torsion point is given in (B.24).

These results allow us to compute the discriminant Δ of $X_{F_{15}}$. We find four I_2 -singularities over the divisors $\mathcal{S}_{\text{SU}(2)_1}^b = \{s_2 = 0\} \cap B$, $\mathcal{S}_{\text{SU}(2)_2}^b = \{s_5 = 0\} \cap B$, $\mathcal{S}_{\text{SU}(2)_3}^b = \{s_7 = 0\} \cap B$ and $\mathcal{S}_{\text{SU}(2)_4}^b = \{s_9 = 0\} \cap B$ in B . Along these divisors the constraint (3.189) factorizes as

$$\begin{aligned}
 \text{SU}(2)_1 : \quad p_{F_{15}}|_{s_2=0} &= w \cdot q_2, \\
 \text{SU}(2)_2 : \quad p_{F_{15}}|_{s_5=0} &= v \cdot q'_2, \\
 \text{SU}(2)_3 : \quad p_{F_{15}}|_{s_7=0} &= e_1 \cdot q_3, \\
 \text{SU}(2)_4 : \quad p_{F_{15}}|_{s_9=0} &= e_2 \cdot q'_3,
 \end{aligned} \tag{3.191}$$

where q_2 , q'_2 , q_3 and q'_3 are the polynomials that remain after factoring out w , v , e_1 and e_2 , respectively. The corresponding codimension one fibers in $X_{F_{15}}$ are shown in figure 27. In summary, the gauge group of $X_{F_{15}}$ is

$$G_{F_{15}} = (\text{SU}(2)^4)/\mathbb{Z}_2 \times \text{U}(1). \tag{3.192}$$

As before we confirm the action of the MW-torsion on the non-Abelian factors in appendix B by explicitly working out the WS-coordinates (B.24) of the generator of the \mathbb{Z}_2 MW-torsion.

In order to calculate the Cartan divisors of $X_{F_{15}}$ we use a similar logic as in section 3.3.3. We obtain the following divisor classes:

$$D_1^{\text{SU}(2)_1} = [w], \quad D_1^{\text{SU}(2)_2} = [v], \quad D_1^{\text{SU}(2)_3} = [s_7] - [e_1], \quad D_1^{\text{SU}(2)_4} = [s_9] - [e_2]. \quad (3.193)$$

With these results at hand, we compute the Shioda map (2.5) of the section \hat{s}_1 as

$$\sigma(\hat{s}_1) = S_1 - S_0 + [K_B] + \frac{1}{2}D_1^{\text{SU}(2)_1} + \frac{1}{2}D_1^{\text{SU}(2)_4}. \quad (3.194)$$

Here S_0, S_1 denote the divisor classes of the sections \hat{s}_0, \hat{s}_1 , respectively, and we used

$$\pi(S_1 \cdot S_0) = 0, \quad (3.195)$$

which directly follows from (3.190) as well as

$$S_1 \cdot C_{-\alpha_1}^{\text{SU}(2)_1} = 1, \quad S_1 \cdot C_{-\alpha_1}^{\text{SU}(2)_2} = 0, \quad S_1 \cdot C_{-\alpha_1}^{\text{SU}(2)_3} = 0, \quad S_1 \cdot C_{-\alpha_1}^{\text{SU}(2)_4} = 1, \quad (3.196)$$

which can be read off from figure 27. Using (3.194), we compute the height pairing (2.6),

$$b_{11} = -[K_B], \quad (3.197)$$

where we used (2.7) and (3.195).

Next, we turn to the analysis of the codimension two singularities of the WSF of $X_{F_{15}}$ and the determination of the charged matter spectrum. As before, all representations are determined from the codimension two fibers in $X_{F_{15}}$ following the procedure presented in section 2.1. All codimension two singularities are easily seen from the discriminant Δ . We find nine codimension two singularities. Seven of these lead to the matter representations and the corresponding codimension two fibers in $X_{F_{15}}$ given in the first and third column of table 29, respectively. At the remaining two loci $s_2 = s_6^2 - 4s_5s_7 = 0$ and $s_9 = s_8^2 - 4s_5s_7 = 0$, the fiber is of Type *III*, cf. section 3.3.3, which means that there are no additional matter representations. Again, we observe the absence of fundamental matter which is consistent with the action of the MW-torsion on the gauge group in (3.192). The spectrum of charged matter is completed by the matter in the adjoint representations $(\mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{1})_0, (\mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1})_0, (\mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{1})_0$ and $(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3})_0$ given in the last four rows of table 29.

We emphasize that the representations $(\mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{2})_0$ and $(\mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1})_0$ at the loci $V(I_{(3)})$ and $V(I_{(4)})$, respectively, arise from non-split codimension two fibers. The dashed nodes in table 29 are interchanged by a codimension three monodromy.

The total matter spectrum of $X_{F_{15}}$ is completed by the number of neutral hyper multiplets, which is computed from (2.11) using the Euler number $\chi(X_{F_{15}})$ given in (C.1). It reads

$$H_{\text{neut}} = 17 + 7[K_B^{-1}]^2 - 4[K_B^{-1}]\mathcal{S}_7 + 2\mathcal{S}_7^2 - 4[K_B^{-1}]\mathcal{S}_9 + 2\mathcal{S}_9^2. \quad (3.198)$$

The number T of tensor multiplets is base-dependent, cf. (2.17), and we have $V = 24$ vector multiplets. Finally, we use this together with the divisors $S_{\text{SU}(2)_I}^b, I = 1, \dots, 4$, (3.197), the charged spectrum in table 29 and (3.198) to check that all 6D anomalies in (A.1) are canceled.

Finally, we present our analysis of codimension three singularities of the WSF of $X_{F_{15}}$ and the corresponding Yukawa points in a compactification to 4D in table 30.

| Representation | Multiplicity | Fiber | Locus |
|--|---|-----------|--|
| $(\mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1})_{1/2}$ | $(2[K_B^{-1}] - \mathcal{S}_7) \times (2[K_B^{-1}] - \mathcal{S}_9)$ | | $V(I_{(1)}) := \{s_2 = s_5 = 0\}$ |
| $(\mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1})_{1/2}$ | $(2[K_B^{-1}] - \mathcal{S}_9)\mathcal{S}_7$ | | $V(I_{(2)}) := \{s_2 = s_7 = 0\}$ |
| $(\mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{2})_0$ | $\mathcal{S}_9(2[K_B^{-1}] - \mathcal{S}_9)$ | | $V(I_{(3)}) := \{s_2 = s_9 = 0\}$ |
| $(\mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1})_0$ | $(2[K_B^{-1}] - \mathcal{S}_7)\mathcal{S}_7$ | | $V(I_{(4)}) := \{s_5 = s_7 = 0\}$ |
| $(\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2})_{1/2}$ | $\mathcal{S}_7\mathcal{S}_9$ | | $V(I_{(5)}) := \{s_7 = s_9 = 0\}$ |
| $(\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{2})_{1/2}$ | $(2[K_B^{-1}] - \mathcal{S}_7)\mathcal{S}_9$ | | $V(I_{(6)}) := \{s_5 = s_9 = 0\}$ |
| $(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})_1$ | $2[K_B^{-1}]^2$ | | $V(I_{(7)}) := \{s_6 = 0, s_5s_7 - s_2s_9 = 0\}$ |
| $(\mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{1})_0$ | $1 + \frac{([K_B^{-1}] - \mathcal{S}_9)}{2}(2[K_B^{-1}] - \mathcal{S}_9)$ | Figure 27 | $s_2 = 0$ |
| $(\mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1})_0$ | $1 + \frac{([K_B^{-1}] - \mathcal{S}_7)}{2}(2[K_B^{-1}] - \mathcal{S}_7)$ | Figure 27 | $s_5 = 0$ |
| $(\mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{1})_0$ | $1 + \mathcal{S}_7 \frac{(\mathcal{S}_7 - [K_B^{-1}])}{2}$ | Figure 27 | $s_7 = 0$ |
| $(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3})_0$ | $1 + \mathcal{S}_9 \frac{(\mathcal{S}_9 - [K_B^{-1}])}{2}$ | Figure 27 | $s_9 = 0$ |

Table 29. Charged matter representations under $SU(2)^4 \times U(1)/\mathbb{Z}_2$ and corresponding codimension two fibers of $X_{F_{15}}$. The adjoint matter is included for completeness.

| Yukawa | Locus |
|---|-----------------------|
| $(\overline{2, 2, 1, 1})_{1/2} \cdot (\overline{2, 1, 2, 1})_{1/2} \cdot (\overline{1, 2, 2, 1})_0$ | $s_2 = s_5 = s_7 = 0$ |
| $(\overline{2, 2, 1, 1})_{1/2} \cdot (\overline{2, 1, 1, 2})_0 \cdot (\overline{1, 2, 1, 2})_{1/2}$ | $s_2 = s_5 = s_9 = 0$ |
| $(\overline{2, 1, 2, 1})_{1/2} \cdot (\overline{2, 1, 1, 2})_0 \cdot (\overline{1, 1, 2, 2})_{1/2}$ | $s_2 = s_7 = s_9 = 0$ |
| $(\overline{1, 2, 2, 1})_0 \cdot (\overline{1, 1, 2, 2})_{1/2} \cdot (\overline{1, 2, 1, 2})_{1/2}$ | $s_5 = s_7 = s_9 = 0$ |
| $(\overline{2, 2, 1, 1})_{1/2} \cdot (\overline{2, 2, 1, 1})_{1/2} \cdot (\overline{1, 1, 1, 1})_1$ | $s_2 = s_5 = s_6 = 0$ |
| $(\overline{2, 1, 2, 1})_{1/2} \cdot (\overline{2, 1, 2, 1})_{1/2} \cdot (\overline{1, 1, 1, 1})_1$ | $s_2 = s_6 = s_7 = 0$ |
| $(\overline{1, 1, 2, 2})_{1/2} \cdot (\overline{1, 1, 2, 2})_{1/2} \cdot (\overline{1, 1, 1, 1})_1$ | $s_6 = s_7 = s_9 = 0$ |
| $(\overline{1, 2, 1, 2})_{1/2} \cdot (\overline{1, 2, 1, 2})_{1/2} \cdot (\overline{1, 1, 1, 1})_1$ | $s_5 = s_6 = s_9 = 0$ |

Table 30. Codimension three loci and corresponding Yukawa points for polyhedron F_{15} .

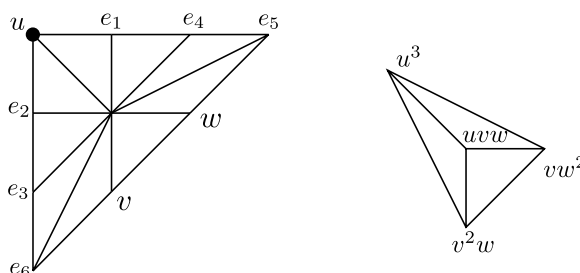


Figure 28. The toric diagram of polyhedron F_{16} and its dual. The zero section is indicated by the dot.

3.6.3 Polyhedron F_{16} : $G_{F_{16}} = \text{SU}(3)^3 / \mathbb{Z}_3$

Consider the elliptically fibered Calabi-Yau manifold $X_{F_{16}}$ with base B and general fiber given by the elliptic curve \mathcal{E} in $\mathbb{P}_{F_{16}}$. The toric data of $\mathbb{P}_{F_{16}}$ is summarized in figure 28, where the polyhedron F_{16} , a choice of projective coordinates as well as its dual polyhedron F_1 are depicted. The monomials associated to the integral points of F_1 are presented in the patch $e_i = 1, \forall i$. The toric variety $\mathbb{P}_{F_{16}}$ is the six-fold blow-up of \mathbb{P}^2 at non-generic points, that is defined as

$$u \rightarrow e_1 e_2 e_3 e_4 e_5 e_6 u, \quad w \rightarrow e_1 e_4^2 e_5^3 w, \quad v \rightarrow e_2 e_3^2 e_6^3 v. \tag{3.199}$$

After this blow-up the projective coordinates on the fiber are $[u : v : w : e_1 : e_2 : e_3 : e_4 : e_5 : e_6]$ and take values in the line bundles associated to the following divisor classes:

| Section | Divisor class |
|---------|---|
| u | $H - E_1 - E_2 + \mathcal{S}_9 + [K_B]$ |
| v | $H - E_2 - E_3 - E_6 + \mathcal{S}_9 - \mathcal{S}_7$ |
| w | $H - E_1 - E_4 - E_5$ |
| e_1 | $E_1 - E_4$ |
| e_2 | $E_2 - E_3$ |
| e_3 | $E_3 - E_6$ |
| e_4 | $E_4 - E_5$ |
| e_5 | E_5 |
| e_6 | E_6 |

(3.200)

The Stanley-Reisner ideal of $\mathbb{P}_{F_{16}}$ can be seen from figure 28 to be given by

$$SR = \{ue_4, ue_5, uw, uv, ue_6, ue_3, e_1e_5, e_1w, e_1v, e_1e_6, e_1e_3, e_1e_2, e_4w, e_4v, e_4e_6, e_4e_3, e_4e_2, e_5v, e_5e_6, e_5e_3, e_5e_2, we_6, we_3, we_2, ve_3, ve_2, e_6e_2\}. \quad (3.201)$$

We obtain the hypersurface equation for $X_{F_{16}}$ either employing (2.23) or by setting $s_2 = s_3 = s_4 = s_5 = s_8 = s_{10} = 0$ in (3.4) and applying the map (3.199). It reads

$$p_{F_{16}} = s_1e_1^2e_2^2e_3e_4u^3 + s_6e_1e_2e_3e_4e_5e_6uvw + s_7e_2e_3^2e_6^3v^2w + s_9e_1e_4^2e_5^3vw^2. \quad (3.202)$$

where the divisor classes of the sections s_i are given in (3.8).

There are three rational sections of the fibration of $X_{F_{16}}$ with two torsional relations between them which shows that the MW-group is \mathbb{Z}_3 [41]. The coordinates of the sections are

$$\begin{aligned} \hat{s}_0 = X_{F_{16}} \cap \{u = 0\} &: [0 : 1 : 1 : s_7 : -s_9 : 1 : 1 : 1 : 1], \\ X_{F_{16}} \cap \{e_5 = 0\} &: [1 : 1 : s_1 : 1 : 1 : 1 : -s_7 : 0 : 1], \\ X_{F_{16}} \cap \{e_6 = 0\} &: [1 : s_1 : 1 : 1 : 1 : 1 : -s_9 : 1 : 1 : 0], \end{aligned} \quad (3.203)$$

where we choose \hat{s}_0 as the zero section.

In order to compute the WSF (2.1) of (3.202), we first compute the Tate form using the birational map from X_{F_5} [39] in the limit $s_2 = s_3 = s_5 = s_8 = 0$. The global Tate coefficients are given in (B.28), which is precisely of the form of an elliptic fibration with MW-group \mathbb{Z}_3 [53]. The WSF is given in (B.29) and the WS-coordinates of the torsional section are given in (B.30).

We readily compute the discriminant Δ , which allows us to find all codimension one singularities of the WSF of $X_{F_{16}}$. We find three I_3 -singularities over the divisors $\mathcal{S}_{\text{SU}(3)_1}^b = \{s_1 = 0\} \cap B$, $\mathcal{S}_{\text{SU}(3)_2}^b = \{s_7 = 0\} \cap B$ and $\mathcal{S}_{\text{SU}(3)_3}^b = \{s_9 = 0\} \cap B$ in B . The hypersurface constraint (3.202) factorizes along these divisors as

$$\begin{aligned} \text{SU}(3)_1 &: p_{F_{16}}|_{s_1=0} = vw \cdot q_1, \\ \text{SU}(3)_2 &: p_{F_{16}}|_{s_7=0} = e_1e_4 \cdot q_3, \\ \text{SU}(3)_3 &: p_{F_{16}}|_{s_9=0} = e_2e_3 \cdot q'_3, \end{aligned} \quad (3.204)$$

where q_1 , q_3 and q'_3 are homogeneous polynomials in $[u : v : w]$ that remain after factoring out vw , e_1e_4 and e_2e_3 . The corresponding fibers are depicted in figure 29. In summary, the gauge group of $X_{F_{16}}$ is

$$G_{F_{16}} = \text{SU}(3)^3 / \mathbb{Z}_3. \quad (3.205)$$

We note that this is precisely the gauge group of the trinification model. Here we confirmed the action of the MW-torsion on the non-Abelian factors in appendix B by explicitly working out the Weierstrass coordinates (B.30) of the generator of the \mathbb{Z}_3 -torsion.

We calculate the classes of the Cartan divisors in the same fashion as in section 3.3.3. Using (3.204) we obtain the classes

$$\begin{aligned} D_1^{\text{SU}(3)_1} &= [v], & D_2^{\text{SU}(3)_1} &= [w], & D_1^{\text{SU}(3)_2} &= [e_4], \\ D_2^{\text{SU}(3)_2} &= [s_7] - [e_1] - [e_4], & D_1^{\text{SU}(3)_3} &= [e_3], & D_2^{\text{SU}(3)_3} &= [s_9] - [e_2] - [e_3]. \end{aligned} \quad (3.206)$$

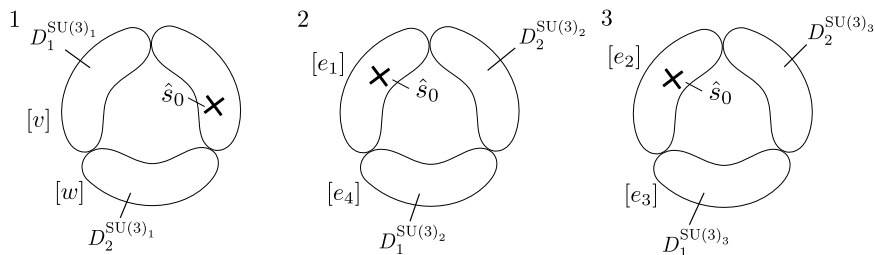


Figure 29. Codimension one fibers of $X_{F_{16}}$.

The determination of the charged matter spectrum requires the knowledge of the codimension two singularities of the WSF of $X_{F_{16}}$. Again, we then extract the corresponding representation data by application of the general recipe outlined in section 2.1. By investigation of the discriminant of $X_{F_{16}}$ we readily find three codimension two singularities that lead to bi-fundamental representations. These and their corresponding codimension two fibers in $X_{F_{16}}$ are listed in the first and third column of table 31, respectively.

There are three additional codimension two singularities at $s_6 = s_9 = 0$, $s_6 = s_7 = 0$ and $s_1 = s_6 = 0$, that, however, yield fibers of Type IV in $X_{F_{16}}$, that do not support any additional matter.

Again, we do not find fundamental matter, confirming the restrictions imposed on the spectrum of $X_{F_{16}}$ by the MW-torsion. The spectrum of charged matter is completed by the matter in the adjoint representations $(\mathbf{8}, \mathbf{1}, \mathbf{1})$, $(\mathbf{1}, \mathbf{8}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{1}, \mathbf{8})$ given in the last three rows of table 31. We recall that they do not originate from codimension two fibers of $X_{F_{16}}$, but are present if the divisors S_G^b are higher genus curves in B . The multiplicity of charged hyper multiplets in the adjoint is given by (2.8).

We complete the matter spectrum of $X_{F_{16}}$ by the number of neutral hyper multiplets, which is computed employing the Euler number (C.1) of $X_{F_{16}}$ from (2.11). It reads

$$H_{\text{neut}} = 18 + 11[K_B^{-1}]^2 - 9[K_B^{-1}]\mathcal{S}_7 + 3\mathcal{S}_7^2 - 9[K_B^{-1}]\mathcal{S}_9 + 3\mathcal{S}_7\mathcal{S}_9 + 3\mathcal{S}_9^2. \quad (3.207)$$

The base-dependent number T of tensor multiplets is given by (2.17) and we have $V = 24$ vector multiplets. Finally, we use this together with the divisors $S_{\text{SU}(3)_I}^b$ for $I = 1, 2, 3$, the charged spectrum in table 31 and (3.207) to confirm cancelation of all 6D anomalies (A.1).

By analyzing the codimension three singularities of the WSF of $X_{F_{16}}$, we finally calculate all Yukawa couplings of the charged matter spectrum of $X_{F_{16}}$, that are given in table 32.

4 The toric Higgs branch of F-theory

In section 3 we discussed in great detail the geometric derivations of the gauge groups and matter spectra of all genus-one fibrations X_{F_i} based on the 16 polyhedra. Here we show that the effective SUGRA theories obtained from these fibrations are not isolated from each other but connected by means of the Higgs mechanism. This section is devoted to the study of those transitions and the resulting network of theories summarized in figure 1.

| Representation | Multiplicity | Fiber | Locus |
|--|--|-----------|-----------------------------------|
| $(\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1})$ | $\mathcal{S}_7(3[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_9)$ | | $V(I_{(1)}) := \{s_1 = s_7 = 0\}$ |
| $(\mathbf{3}, \mathbf{1}, \bar{\mathbf{3}})$ | $\mathcal{S}_9(3[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_9)$ | | $V(I_{(2)}) := \{s_1 = s_9 = 0\}$ |
| $(\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}})$ | $\mathcal{S}_7\mathcal{S}_9$ | | $V(I_{(3)}) := \{s_7 = s_9 = 0\}$ |
| $(\mathbf{8}, \mathbf{1}, \mathbf{1})$ | $1 + \frac{(2[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_9)}{2} \times (3[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_9)$ | Figure 29 | $s_1 = 0$ |
| $(\mathbf{1}, \mathbf{8}, \mathbf{1})$ | $1 + \mathcal{S}_7 \frac{(\mathcal{S}_7 - [K_B^{-1}])}{2}$ | Figure 29 | $s_7 = 0$ |
| $(\mathbf{1}, \mathbf{1}, \mathbf{8})$ | $1 + \mathcal{S}_9 \frac{(\mathcal{S}_9 - [K_B^{-1}])}{2}$ | Figure 29 | $s_9 = 0$ |

Table 31. Charged matter representations under $SU(3)^3/\mathbb{Z}_3$ and corresponding codimension two fibers of $X_{F_{16}}$. The adjoint matter is included for completeness.

| Yukawa | Locus |
|--|-----------------------|
| $(\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1}) \cdot (\bar{\mathbf{3}}, \mathbf{1}, \bar{\mathbf{3}}) \cdot (\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}})$ | $s_1 = s_7 = s_9 = 0$ |
| $(\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1}) \cdot (\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1}) \cdot (\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1})$ | $s_1 = s_6 = s_7 = 0$ |
| $(\mathbf{3}, \mathbf{1}, \bar{\mathbf{3}}) \cdot (\mathbf{3}, \mathbf{1}, \bar{\mathbf{3}}) \cdot (\mathbf{3}, \mathbf{1}, \bar{\mathbf{3}})$ | $s_1 = s_6 = s_8 = 0$ |
| $(\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}}) \cdot (\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}}) \cdot (\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}})$ | $s_6 = s_7 = s_9 = 0$ |

Table 32. Codimension three loci and corresponding Yukawa points for F_{16} .

As we have noted earlier, this network is nothing but the field theoretic realization of the network of extremal transitions relating the X_{F_i} that are induced by blowing up/down in the toric ambient varieties \mathbb{P}_{F_i} of their genus-one fibers. This network, to which we refer to as the toric *Higgsing diagram*, is a powerful consistency check for the results of section 3

and exhibits some remarkable features:

- The Higgsing diagram is symmetric with respect to the horizontal axis corresponding to the self dual polyhedra, all of which have a gauge group rank equal to three.
- The rank of the gauge groups of a polyhedron and its dual sum up to six.
- Every toric hypersurface fibration can be reached upon a chain of Higgsings starting from one of the three manifolds $X_{F_{13}}$, $X_{F_{15}}$ and $X_{F_{16}}$ which exhibit non trivial MW-torsion.
- Both the analysis on the geometrical side (see sections 3.2.1, 3.2.2 and 3.2.3) and the Higgsings (see section 4.2) lead to the conclusion that the MW-torsion in the X_{F_i} with fibers in F_{13} , F_{15} and F_{16} , manifests itself as discrete gauge symmetries in the X_{F_i} with fibers in their respective dual polyhedra F_1 , F_2 and F_4 .

In the following we discuss the above features and the Higgsing diagram in more detail. In order to illustrate the relevant features of the Higgsing we focus on a particular sub-branch of the Higgsing diagram which we depict in figure 30. This includes the transition of the effective theory derived from F-theory on X_{F_9} to that on X_{F_5} . This transition is convenient in order to discuss certain (unphysical) redefinitions of the divisor classes that are sometimes needed in order to match the field theoretic results with the geometrical computation. In this example we also describe the matching between the gauge group generators before and after the Higgsing, and how this information can be inferred from the toric diagram (see section 4.1). After these redefinitions we obtain a perfect match of the massless spectrum of the effective theory after Higgsing with that obtained from the geometrical computation. While our results apply for any generic two-dimensional base B , we also discuss the Higgsing for the specific case of a \mathbb{P}^2 base. Here we comment on specific boundary strata of the moduli space where a specific Higgsing might not be possible but different equivalent Higgsings are.

In section 4.2 we focus on the theories corresponding to the genus-one fibrations X_{F_1} , X_{F_2} and X_{F_4} . The field theoretical Higgsings imply the presence of discrete gauge symmetries \mathbb{Z}_3 , \mathbb{Z}_2 and \mathbb{Z}_4 which confirms the results of the geometrical computations. In section 4.3 we discuss the full chain of Higgs transitions. There we summarize the relevant redefinitions of the gauge group generators and divisor classes needed to match the spectrum obtained in the geometrical computations of section 3.

4.1 Toric Higgsing: an example

We are interested in Higgs transitions relating two supersymmetric vacua in a 6D $\mathcal{N} = 1$ SUGRA theory. This requires that the vacuum expectation value (VEV) of the Higgs field triggering this transition must preserve supersymmetry. This is guaranteed by imposing flatness of the D-term potential along the direction of the VEV. As it has been observed in [92], for the case of a Higgs field that is not in the adjoint representation, at least two hyper multiplets with identical quantum numbers have to acquire a VEV in order to achieve a D-flat direction.

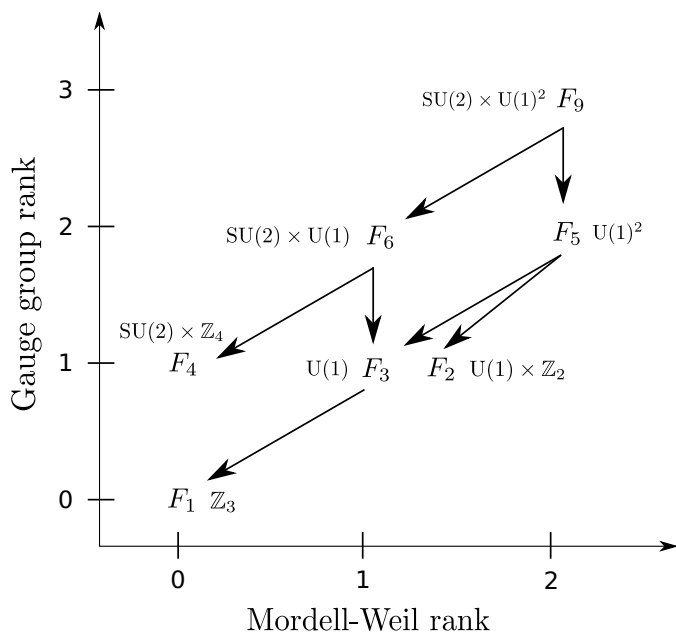


Figure 30. A subbranch of the Higgsing chain which we use to illustrate certain features relevant for all Higgsings. We use the Higgsing from $X_{F_9} \rightarrow X_{F_5}$ to demonstrate the type of bundle and charge redefinitions which are needed in order to match a Higgsed model with the geometrical computations. The Higgsings to X_{F_1} , X_{F_2} and X_{F_4} are used to confirm the presence of discrete gauge symmetries.

Furthermore, the Higgs mechanism of interest here has to relate two toric hypersurface fibrations. Transitions of this type have a clear geometric interpretation in terms of the toric diagrams. This can be seen as follows. First recall that the coefficients s_i which appear in the hypersurface constraint of X_{F_k} correspond to integral points in the dual polyhedron F_k^* , see (2.23). As we have seen in section 3, many X_{F_k} exhibit hyper multiplets at codimension two loci of the form $\{s_i = s_j = 0\}$, where s_i and s_j are neighboring vertices in the dual polyhedron F_k^* connected by an edge. If the Higgs fields are of this type, they lead to a *toric Higgsing*, i.e. the resulting theory is associated to a new toric hypersurface fibration $X_{F_{k'}}$, $k' \neq k$.²⁷ Here the polyhedron $F_{k'}$ is obtained from F_k by blowing up the dual polyhedron F_k^* precisely at the edge connecting the vertices corresponding to s_i, s_j , respectively, and taking its dual. In F_k , this corresponds to the blow-down associated to removing the corner that is dual to the aforementioned edge in F_k^* . Note that, since the Higgs fields in the toric breaking are never in the adjoint representation, the toric Higgsing is not rank preserving.

To exemplify this, consider the Calabi-Yau manifold X_{F_9} . There the possible toric Higgs fields are, cf. section 3.4.3,

$$\mathbf{1}_{(1,2)} : \{s_3 = s_7 = 0\}, \quad \mathbf{1}_{(1,0)} : \{s_1 = s_5 = 0\}, \quad (4.1)$$

$$\mathbf{2}_{(-1,-1/2)} : \{s_5 = s_9 = 0\}, \quad \mathbf{2}_{(1,3/2)} : \{s_7 = s_9 = 0\}. \quad (4.2)$$

²⁷Non-toric Higgsings on the other hand are transitions for which the resulting fibration can not anymore be described by one of the 16 polyhedra.

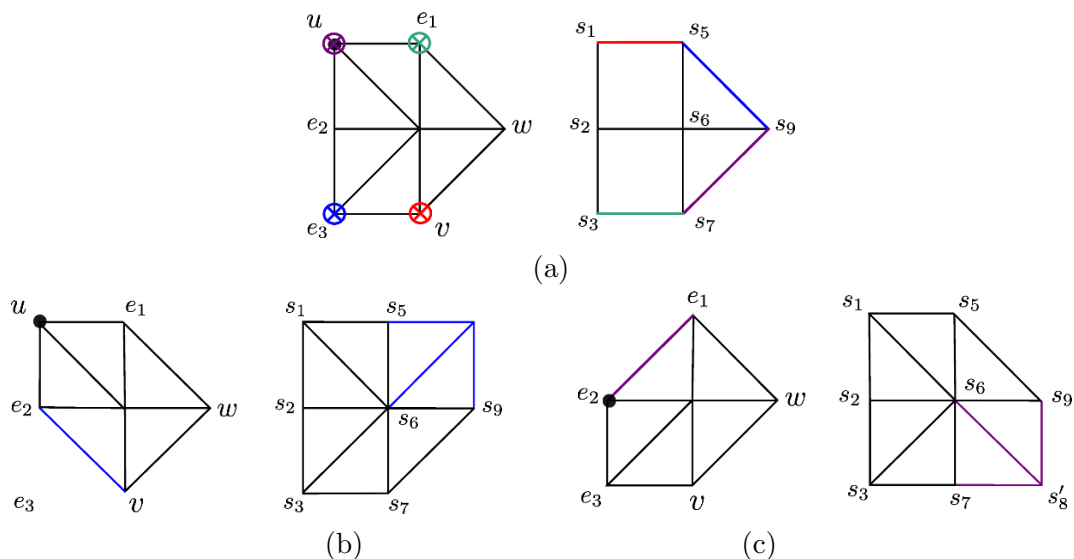


Figure 31. (a) Possible toric Higgsings in X_{F_9} . The transitions to X_{F_5} are achieved by Higgs fields in the representations $\mathbf{2}_{(-1, -1/2)}$ (b) or $\mathbf{2}_{(1, 3/2)}$ (c).

Considering the group theoretical breaking that could be induced by these fields, we see that taking the Higgs fields in any of the singlet representations in (4.1), for instance $\mathbf{1}_{(1, 2)}$ leads to $SU(2) \times U(1)$, which coincides with the gauge group of the fibration X_{F_6} , while a VEV in the $\mathbf{2}_{(-1, -1/2)}$ or $\mathbf{2}_{(1, 3/2)}$ (see (4.2)) leads to $U(1)^2$, i.e. the gauge group expected for X_{F_5} . In figure 31 (a) we depict the polyhedron which is dual to F_9 on the right and highlight in different colors the edges corresponding to the fields in (4.1) and (4.2). They have to be blown up, i.e. subdivided by a new ray, for each of the possible toric Higgsings. In the actual polyhedron of F_9 , we indicate the vertices that are dual to these edges and get cut off in the toric Higgsings. In the following we consider the Higgsing from X_{F_9} to X_{F_5} in more detail.

In the case of Higgs fields in the representation $\mathbf{2}_{(-1, -1/2)}$, the Higgsing corresponds to a removal of the lower left corner in the polyhedron F_9 and to a blow-up at its dual edge, which is the edge between s_5 and s_9 , in its dual polyhedron (see figure 31 (b)). After that, we obtain the toric diagram of F_5 as given in section 3.3.2. Similarly, if we pick VEVs in the $\mathbf{2}_{(1, 3/2)}$ representation, we observe that after the Higgsing, the resulting polyhedron and its dual both have to be reflected along the horizontal axis (see figure 31 (c)) in order to recover figure 31 (b). Thus, the obtained effective theories after these two Higgsings are physically equivalent.

However, we note that the geometrical computations leading to the spectrum of X_{F_5} have been made for the polyhedron given in figure 31 (b). Thus, whenever we perform a Higgsing with the Higgs in the representation $\mathbf{2}_{(1, 3/2)}$ (leading to figure 31 (c)) we have to transform certain divisor classes in order to match the multiplicities of hypermultiplets resulting from the Higgsing with those found geometrically. More general, the ‘‘Higgsed’’ polyhedron of F_9 can be brought to the *canonical* form (i.e. the one used for the computation of the matter spectrum and multiplicities) by means of an $SL(2, \mathbb{Z})$ transformation,

which acts simultaneously on the polyhedron and its dual. This transformation determines how to transform the divisor classes in order to recover precisely the effective theory obtained by the geometric computation on X_{F_5} , as we demonstrate next.

4.1.1 Matching the charged spectrum

In order to match the charged spectrum in the Higgsed theory arising from F-theory on X_{F_9} with that on X_{F_5} , we first have to relate the generators of the gauge groups before and after the Higgsing. In X_{F_9} , the U(1) generators are (see (3.118))

$$\sigma(\hat{s}_1) = [e_3] - [u] - [K_B^{-1}], \quad \sigma(\hat{s}_2) = [e_1] - [u] - [K_B^{-1}] - \mathcal{S}_7 + \frac{1}{2}D_1, \quad (4.3)$$

with D_1 being the class of the SU(2) Cartan divisor given in (3.121) as $D_1 = \mathcal{S}_9 - [e_2]$. The Shioda maps in X_{F_5} are given by (see (3.75))

$$\sigma(\hat{s}'_1) = [e_1] - [e_2] - [K_B^{-1}], \quad \sigma(\hat{s}'_2) = [u] - [e_2] - [K_B^{-1}] - \mathcal{S}_9. \quad (4.4)$$

Let us consider first the canonical Higgsing induced by VEVs in the $\mathbf{2}_{(-1,-1/2)}$. As shown in figure 31 this corresponds to blowing down the divisor $e_3 = 0$. We see that after setting its divisor class $[e_3] = 0$, the following relations hold

$$\sigma(\hat{s}'_1) = \sigma(\hat{s}_2) - \sigma(\hat{s}_1) + \frac{1}{2}D_1 - [K_B^{-1}] + \mathcal{S}_7 - \mathcal{S}_9, \quad \sigma(\hat{s}'_2) = -\sigma(\hat{s}_1) + D_1 - 2([K_B^{-1}] + \mathcal{S}_9). \quad (4.5)$$

Since the vertical divisors $[K_B^{-1}]$, \mathcal{S}_7 and \mathcal{S}_9 do not contribute to the U(1) charges, these equations allow us to make contact with the charges in X_{F_5} (which we denote by Q'_1 and Q'_2). Indeed, recalling that Dynkin labels and U(1)-charges are computed according to (2.12) and (2.13), respectively, we translate (4.5) into the charge relation²⁸

$$Q'_1 = Q_2 - Q_1 + T^3, \quad Q'_2 = -Q_1 + 2T^3, \quad (4.6)$$

with Q_1 , Q_2 being the U(1)-charges and $T^3 = \frac{1}{2}D_1$ the Cartan generator of the SU(2) in X_{F_9} .

For the Higgsing with fields in the $\mathbf{2}_{(1,3/2)}$ representation we can proceed in a similar manner. In this case, according to figure 31, one has to set the divisor class $[u] = 0$. In addition, one must take into account that the toric diagram is reflected with respect to the canonical one (compare figure 31 (c) with figure 8, including the location of the zero section). This implies that the classes of $[e_3]$, $[v]$ and $[e_1]$ in X_{F_9} get mapped to $[u]$, $[e_1]$ and $[v]$ in X_{F_5} , respectively. Hence, the Shioda maps (4.3) for X_{F_9} , written in terms of divisor classes on X_{F_5} , read

$$\sigma(\hat{s}_1) = [u] - [K_B^{-1}], \quad \sigma(\hat{s}_2) = [v] - 3[K_B^{-1}] + \mathcal{S}_7 + \frac{1}{2}D_1. \quad (4.7)$$

Writing both (4.4) and (4.7), in terms of the exceptional divisors E_1 , E_2 as well as the hyperplane class H using (3.8), we find the following relations among them

$$\sigma(\hat{s}'_1) = \sigma(\hat{s}_2) - \sigma(\hat{s}_1) + \frac{1}{2}D_1 - \mathcal{S}_9, \quad \sigma(\hat{s}'_2) = \sigma(\hat{s}_1) + D_1 - 2\mathcal{S}_9, \quad (4.8)$$

²⁸Note here the importance of the fact that the choices of zero sections in X_{F_5} , cf. figure 8, and X_{F_9} after Higgsing, cf. figure 31 (b), are different. This corresponds to an $\text{SL}(2, \mathbb{Z})$ -transformation of the U(1)-charges.

| | VEV: $\mathbf{2}_{(-1,-1/2)}$ $Q'_1 = (Q_2 - Q_1 + T^3)$ $Q'_2 = (-Q_1 + 2T^3)$ | VEV: $\mathbf{2}_{(1,3/2)}$ $Q'_1 = (Q_2 - Q_1 + T^3)$ $Q'_2 = (Q_1 + 2T^3)$ |
|--------------------------|---|--|
| $\mathbf{1}_{(1,2)}$ | $\mathbf{1}_{(1,-1)}$ | $\mathbf{1}_{(1,1)}$ |
| $\mathbf{1}_{(1,0)}$ | $\mathbf{1}_{(-1,-1)}$ | $\mathbf{1}_{(-1,1)}$ |
| $\mathbf{1}_{(0,1)}$ | $\mathbf{1}_{(1,0)}$ | $\mathbf{1}_{(1,0)}$ |
| $\mathbf{1}_{(1,1)}$ | $\mathbf{1}_{(0,-1)}$ | $\mathbf{1}_{(0,1)}$ |
| $\mathbf{2}_{(-1,-1/2)}$ | $\mathbf{1}_{(0,0)} + \mathbf{1}_{(1,2)}$ | $\mathbf{1}_{(1,0)} + \mathbf{1}_{(0,-2)}$ |
| $\mathbf{2}_{(1,3/2)}$ | $\mathbf{1}_{(0,-2)} + \mathbf{1}_{(1,0)}$ | $\mathbf{1}_{(1,2)} + \mathbf{1}_{(0,0)}$ |
| $\mathbf{2}_{(0,-1/2)}$ | $\mathbf{1}_{(-1,-1)} + \mathbf{1}_{(0,1)}$ | $\mathbf{1}_{(0,1)} + \mathbf{1}_{(-1,-1)}$ |
| $\mathbf{3}_{(0,0)}$ | $\mathbf{1}_{(-1,-2)} + \mathbf{1}_{(1,2)} + \mathbf{1}_{(0,0)}$ | $\mathbf{1}_{(1,2)} + \mathbf{1}_{(-1,-2)} + \mathbf{1}_{(0,0)}$ |

Table 33. Possible state decompositions from X_{F_5} to those of X_{F_5} for different Higgses.

from which it follows that the U(1)-charge redefinition in this case is given by

$$Q'_1 = Q_2 - Q_1 + T^3, \quad Q'_2 = Q_1 + 2T^3. \quad (4.9)$$

The charge formulas in both cases agree with the field theory expectations i.e. there is a complete gauge singlet in the decomposition of the Higgs field into representations of the residual gauge symmetry. The decomposition of the states for both cases is given in table 33. We observe that, indeed, all charged states in X_{F_5} have been reproduced.

In order to match their multiplicities as well, we first recall the basic fact that a hyper multiplet consists of two half-hypers transforming in representations conjugate to each other. Thus, in order to compute the resulting multiplicities after Higgsing, one has to take into account all states transforming under the representation of interest together with their complex conjugates. For example, note that in the branching induced by $\mathbf{2}_{(-1,-1/2)}$, the states $\mathbf{2}_{(0,-1/2)}$ and $\mathbf{1}_{(1,1)}$ decompose as

$$\mathbf{2}_{(0,-1/2)} \rightarrow \mathbf{1}_{(-1,-1)} + \mathbf{1}_{(0,1)}, \quad \mathbf{1}_{(1,1)} \rightarrow \mathbf{1}_{(0,-1)}, \quad (4.10)$$

thus, the multiplicity of hyper multiplets in the representation $\mathbf{1}_{(0,1)}$ after the Higgsing must be computed as the sum of the multiplicities of $\mathbf{2}_{(0,-1/2)}$ and $\mathbf{1}_{(1,1)}$ in X_{F_5} . Similarly, note that every Higgs doublet decomposes into a neutral and a charged singlet. In computing the multiplicity of such charged singlets after Higgsing, one has to take into account that two of these are absorbed as longitudinal components of the massive W bosons from the broken SU(2).

In the case of the Higgsing induced by a VEV in the $\mathbf{2}_{(-1,-1/2)}$ representation of X_{F_5} , we can directly compare the resulting multiplicities with the geometric result of section 3.3.2. However, in the case where we turn on VEVs for the fields in the representation $\mathbf{2}_{(1,3/2)}$, the multiplicities only match after performing a redefinition of divisor classes. Indeed, we note

that on the dual polyhedron, the reflection²⁹ relating figure 31 (c) with figure 8 enforces an exchange of the sections s_5 and s_1 in X_{F_9} with s_7 and s_3 in X_{F_5} , respectively. From (3.8) we see that this effectively amounts to a shift in the bundles \mathcal{S}_7 and \mathcal{S}_9 from X_{F_9} to X_{F_5} , which is given by

$$\mathcal{S}_7 \rightarrow 2[K_B^{-1}] - \mathcal{S}_7, \quad \mathcal{S}_9 \rightarrow \mathcal{S}_9. \quad (4.11)$$

Using the shift (4.11) for the second Higgsing, we find that in both cases (either VEVs in $\mathbf{2}_{(-1,-1/2)}$ or $\mathbf{2}_{(1,3/2)}$), there is a perfect agreement with the spectrum in table (9) of the toric hypersurface fibration X_{F_5} .

4.1.2 Matching of the neutral spectrum: Higgsing & Euler numbers

So far we have matched only the charged spectrum of the Higgsed theory of X_{F_9} with that computed geometrically on X_{F_5} . In this section we work out the counting of complex structure moduli, that determine the number of neutral hyper multiplets by (2.11), before and after Higgsing. We show that the mismatch of the Hodge numbers $h^{(2,1)}(X_{F_5}) - h^{(2,1)}(X_{F_9}) \geq 0$ precisely agrees with the amount of massless neutral singlets contributed from the Higgs multiplets.

First, let us discuss the geometric side of the matching. For a given Calabi-Yau manifold X , the amount of complex structure moduli can be inferred from its Euler number $\chi(X)$ as

$$h^{(2,1)}(X) = h^{(1,1)}(X) - \frac{\chi(X)}{2}, \quad (4.12)$$

with $h^{(1,1)}(X)$ given by

$$h^{(1,1)}(X) = 1 + \text{rk}(G_X) + h^{(1,1)}(B), \quad (4.13)$$

where $\text{rk}(G_X)$ is the rank of the total gauge group G_X of X . Thus, in an extremal transition from a toric hypersurface fibration X_{F_i} to X_{F_j} , with the same base B , the change in $h^{(2,1)}$ reads

$$h^{(2,1)}(X_{F_j}) - h^{(2,1)}(X_{F_i}) = \text{rk}(G_{F_j}) - \text{rk}(G_{F_i}) + \frac{\chi(X_{F_i}) - \chi(X_{F_j})}{2}. \quad (4.14)$$

For the specific Higgsing $X_{F_9} \rightarrow X_{F_5}$, their Euler numbers are given according to (C.1) as

$$\begin{aligned} \chi(X_{F_9}) &= -24[K_B^{-1}]^2 + 4[K_B^{-1}](2\mathcal{S}_7 + 3\mathcal{S}_9) - 4(\mathcal{S}_7^2 + \mathcal{S}_9^2), \\ \chi(X_{F_5}) &= -24[K_B^{-1}]^2 + 8[K_B^{-1}](\mathcal{S}_7 + \mathcal{S}_9) - 2(2\mathcal{S}_7^2 + 2\mathcal{S}_9^2 - \mathcal{S}_7\mathcal{S}_9), \end{aligned} \quad (4.15)$$

so that the difference in their Hodge numbers $h^{(2,1)}$ is given by

$$h^{(2,1)}(X_{F_5}) - h^{(2,1)}(X_{F_9}) = \mathcal{S}_9(2[K_B^{-1}] - \mathcal{S}_7) - 1. \quad (4.16)$$

Here the -1 is the contribution from the change in the rank of the gauge group.

²⁹Another possibility to bring the polyhedron F_9 ‘‘Higgsed’’ by $\mathbf{2}_{(1,3/2)}$ back to the canonical form of F_5 , is to rotate it by 90 degrees clockwise. In this case, the bundle redefinitions are $\mathcal{S}_7 \rightarrow 2[K_B^{-1}] - \mathcal{S}_9$, $\mathcal{S}_9 \rightarrow \mathcal{S}_7$.

Next, in the corresponding field theories we consider the canonical Higgsing, that is induced by a Higgs in the representation $\mathbf{2}_{(-1,-1/2)}$. The multiplicity of hyper multiplets in the representation $\mathbf{2}_{(-1,-1/2)}$ is $\mathcal{S}_9(2[K_B^{-1}] - \mathcal{S}_7)$, cf. table 17. This is also the number of new uncharged singlets produced in the Higgsing $X_{F_9} \rightarrow X_{F_5}$. However, out of those neutral singlets, exactly one gets massive, as can be seen from the D- term potential.³⁰ This is also closely related to Goldstone’s theorem: as three vectors have been lifted (an entire SU(2) is broken), three hyper multiplets must be removed from the massless spectrum. Two of these hyper multiplets are charged, as we discussed in the previous section, while the third one must be neutral, since it provides the longitudinal component of a massive U(1). In fact, only the simultaneous removal of three vectors and three hyper multiplets makes it possible for the purely gravitational anomaly in (A.1) to cancel after the Higgs mechanism.

Thus, there are precisely $\mathcal{S}_9(2[K_B^{-1}] - \mathcal{S}_7) - 1$ massless singlets after the Higgsing, which precisely agrees with (4.16). In other words, we observe that the neutral massless hyper multiplets resulting from the Higgs mechanism become the complex structure moduli that were gained in the transition $X_{F_9} \rightarrow X_{F_5}$. In this work, we explicitly confirm the matching of the complex structure moduli for all toric Higgs transitions between two toric hypersurface fibrations .

4.1.3 Allowed regions for base \mathbb{P}^2

While we considered in the previous sections the Higgsings for fibrations over an arbitrary two dimensional base, we regard it appropriate to devote some time on a concrete example with base \mathbb{P}^2 in order to discuss some subtleties that might arise.

Choosing the base fixes the range of allowed divisor classes for \mathcal{S}_7 and \mathcal{S}_9 [39]. Expanding these divisors and the canonical class $K_{\mathbb{P}^2}$ in terms of the hyperplane class H_B on \mathbb{P}^2 , we have

$$\mathcal{S}_7 = n_7 H_B, \quad \mathcal{S}_9 = n_9 H_B, \quad K_{\mathbb{P}^2} = -3H_B, \quad (4.17)$$

with n_7 and n_9 being positive integral coefficients.

The effectiveness condition on all divisor classes $[s_i]$ in (3.8) that occur for a given manifold X_{F_i} imposes constraints on the allowed values for n_7 and n_9 . These allowed values depend on the choice of the fiber. For the case of X_{F_9} and X_{F_5} the allowed regions are depicted in figure 32. A choice of a point in this diagram constitutes a consistent fibration. Note that the allowed region for X_{F_5} is fully contained inside that of X_{F_9} . Thus, there are compactifications X_{F_9} inside the blue region in figure 32, for which the transition to X_{F_5} is not possible because the effectiveness condition for some coefficient s_i in X_{F_5} would be violated.

Indeed, this can also be seen from the field theory perspective, as for the points outside the allowed region for X_{F_5} , the multiplicity of Higgses in the representation $\mathbf{2}_{(-1,-\frac{1}{2})}$, that reads

$$\mathcal{S}_9(2[K_B^{-1}] - \mathcal{S}_7) = n_9(6 - n_7), \quad (4.18)$$

³⁰In the case of an adjoint Higgs, all neutral hyper multiplets remain massless as the D-term is exactly zero.

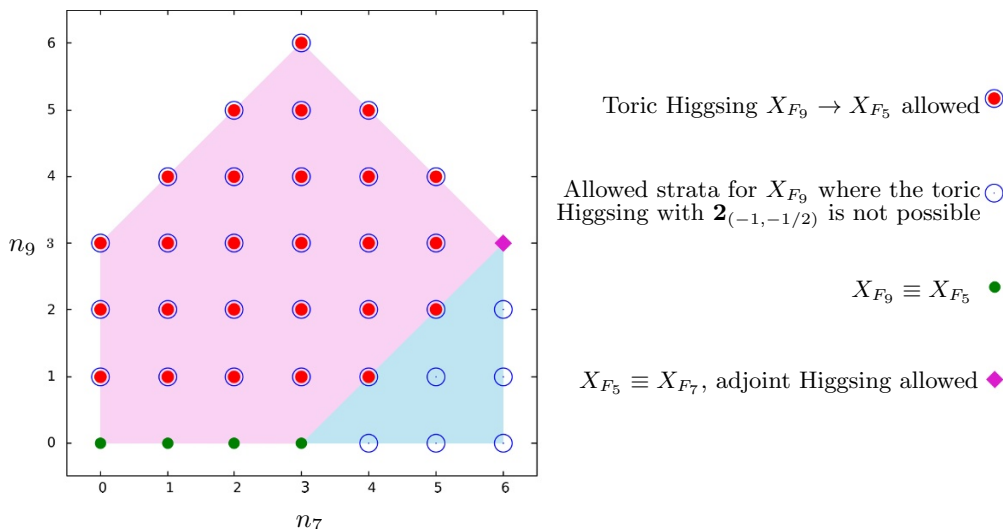


Figure 32. Allowed regions of \mathcal{S}_7 and \mathcal{S}_9 for X_{F_9} (blue and purple regions) and X_{F_5} (purple region) with base $B = \mathbb{P}^2$.

is smaller than two. Recall that in order to have a D-flat potential we need at least two Higgs fields in the same representation to acquire a VEV.³¹ We also observe that the Higgs mechanism is possible for all points (n_7, n_9) in the interior of the allowed region of X_{F_5} . However, for certain points on the boundary of the allowed region for X_{F_5} we see that the amount of doublets does not suffice for a supersymmetry preserving Higgsing. These points are

- $0 \leq n_7 \leq 3, n_9 = 0$: here we see that s_9 belongs to the trivial bundle. Since in X_{F_9} the locus of the $SU(2)$ singularity is precisely $\{s_9 = 0\}$, cf. (3.119), it is removed and the gauge group of X_{F_9} at these points equals that of X_{F_5} , namely $U(1) \times U(1)$. In addition, we see that the spectra of X_{F_5} and X_{F_9} in tables 9 and 17, respectively, match perfectly. Hence we are at points where the strata of the moduli spaces of the two theories overlap and a transition among them is trivial.
- $n_7 = 6, n_9 = 3$: at this point there are no states in X_{F_9} which transform in the $\mathbf{2}_{(-1, -\frac{1}{2})}$, so that the toric Higgsing is again not possible. Note also that since s_9 does not belong to the trivial bundle, the $SU(2)$ factor is part of the gauge symmetry of the effective theory. However, in the hypersurface constraint for X_{F_5} , the sections s_1, s_5 and s_8 transform in the trivial bundle. Hence, at this particular point one can shift the toric coordinates u, v and w in order to globally set the section $s_1 = 0$ [42], resulting in a non-toric $U(1)$. This shows that the effective theory of X_{F_5} coincides with that of X_{F_7} precisely at the point $n_7 = 6, n_9 = 3$. Since the rank of the gauge groups of X_{F_9} and X_{F_7} coincide, no toric Higgsing is possible. However, on X_{F_9} , one sees that there is one hyper multiplet in the adjoint of $SU(2)$. In fact one can use this field to induce an adjoint (non-toric) Higgsing which leads precisely to the effective

³¹Note that in this region, a different Higgsing with $\mathbf{2}_{(1, \frac{3}{2})}$ is possible.

theory of X_{F_5} at $n_7 = 6, n_9 = 3$. Indeed, we have explicitly computed that at this particular stratum in moduli space the numbers of complex structure moduli in X_{F_7} and X_{F_9} coincide. Similarly, we have confirmed that the entire matter spectrum in X_{F_7} is reproduced (with the correct multiplicities) after the adjoint breaking from X_{F_9} . We omit all the details and just state the corresponding U(1) redefinitions, in terms of the generators in X_{F_9} :

$$Q'_1 = -3T_3 - 2Q_1 + Q_2, \quad Q'_2 = -T_3 - 2Q_1 + Q_2, \quad Q'_3 = -2T_3 - Q_1. \quad (4.19)$$

Looking at the Calabi-Yau constraint for X_{F_9} we also observe that, by a shift in the toric coordinates, we can set $s_1 = 0$. Hence, the hypersurface constraint for X_{F_7} only contains the additional monomial $s_8 w^2 u$ which is absent in the one of X_{F_9} . At $n_7 = 6, n_9 = 3$ the coefficient s_8 is just a constant, cf (3.8), i.e. one degree of freedom. On the field theory side, this degree of freedom corresponds precisely to the single adjoint Higgs on X_{F_9} .

4.2 Higgsings to theories with discrete gauge symmetries

From the analysis carried out in section 3, we observe that the presence of discrete gauge symmetries is exclusive to the polyhedra F_1, F_2 and F_4 . On the field theory side, we can use the Higgs mechanism to track the discrete symmetries as well, since these correspond to surviving remnants of broken U(1) symmetries. In fact, one can use the Higgsing diagram to show that the only Higgs mechanism for which the U(1)s are broken to a discrete subgroup, are those leading precisely to F_1, F_2 and F_4 , as expected geometrically. In this section we want to discuss in some detail those Higgsings leading to the toric hypersurface fibrations with discrete symmetries. To this end, we focus on the possible Higgs branches of X_{F_6}, X_{F_5} and X_{F_3} .

In many of the transitions considered here, the charge of the Higgs fields does not allow us to directly infer whether or not there is a non-trivial discrete symmetry. Consider for example the Higgsings from $X_{F_9} \rightarrow X_{F_6}$, with the toric Higgses given in (4.1). One possibility is to have VEV fields in the representation $\mathbf{1}_{(1,2)}$. In principle one might think that, given the charge of the Higgses, there is a discrete remnant of the second U(1). However, this leftover symmetry is trivial, since there is an $SL(2, \mathbb{Z})$ transformation which maps the charge of the Higgses from $(1, 2)$ to $(1, 0)$. As discussed in section 4.1, the field $\mathbf{1}_{(1,0)}$ allows for a geometrically equivalent breaking, where it becomes clear that no discrete symmetries are present in X_{F_6} as there the U(1) charge is minimal.

More general, if we break a $U(1)^n$ gauge symmetry by the VEV of a field ϕ , there is no discrete symmetry left provided the existence of an $SL(n, \mathbb{Z})$ which makes its charge minimal. In other words, after the $SL(n, \mathbb{Z})$ transformation of the VEV field, the charge (q_1, q_2, \dots, q_n) takes the form $(1, 0, \dots, 0)$. In this new basis it is obvious that no discrete gauge group is left after the breaking. In other cases a seeming discrete group can be embedded into an unbroken U(1) or the center of a non-Abelian gauge group, showing again the absence of a discrete group.

| | VEV: $\mathbf{2}_{-\frac{3}{2}}$ $Q' = (3T^3 - Q)$ | VEV: $\mathbf{1}_2$ $Q' = Q \bmod 2$ |
|-----------------------------|---|---|
| $\mathbf{2}_{-\frac{3}{2}}$ | $\mathbf{1}_0 + \mathbf{1}_{-3}$ | $\mathbf{2}_{\frac{1}{2}}$ |
| $\mathbf{2}_{\frac{1}{2}}$ | $\mathbf{1}_1 + \mathbf{1}_{-2}$ | $\mathbf{2}_{\frac{1}{2}}$ |
| $\mathbf{1}_2$ | $\mathbf{1}_{-2}$ | $\mathbf{1}_0$ |
| $\mathbf{1}_1$ | $\mathbf{1}_{-1}$ | $\mathbf{1}_1$ |
| $\mathbf{3}_0$ | $\mathbf{1}_0 + \mathbf{1}_3 + \mathbf{1}_{-3}$ | $\mathbf{3}_0$ |

Table 34. Branching of representations under the possible Higgsings of X_{F_6} to X_{F_3} and X_{F_4} .

X_{F_6} Higgs branches

From the spectrum in X_{F_6} given in table 11 we see that there are two possible toric Higgsings, depending on whether the Higgs fields are taken in the representation $\mathbf{2}_{-3/2}$ or $\mathbf{1}_2$. In the first case the $SU(2) \times U(1)$ symmetry in X_{F_6} is broken to a single $U(1)$, so that this Higgs branch leads to X_{F_3} . In the second case the $SU(2)$ symmetry remains unbroken, as expected from a Higgsing to X_{F_4} . The splitting of the states in either cases proceeds according to table 34. Using these branchings of the representations on X_{F_6} into representations of G_{F_3} , we can compute the multiplicities of the multiplets after Higgsing using table 11. They read

$$\begin{aligned}
 \mathbf{1}_3 : & \quad (\mathcal{S}_9 - \mathcal{S}_7) ([K_B^{-1}] - \mathcal{S}_7 + \mathcal{S}_9) \\
 \mathbf{1}_2 : & \quad ([K_B^{-1}] - \mathcal{S}_9 + \mathcal{S}_7) (6[K_B^{-1}] - 2\mathcal{S}_9 + \mathcal{S}_7) \\
 \mathbf{1}_1 : & \quad 12[K_B^{-1}]^2 + [K_B^{-1}] (8\mathcal{S}_7 - \mathcal{S}_9) - 4\mathcal{S}_7^2 + \mathcal{S}_7\mathcal{S}_9 - \mathcal{S}_9^2.
 \end{aligned}
 \tag{4.20}$$

The above multiplicities agree with our geometrical result for the spectrum of X_{F_3} , see table 7. Note also that naively, due to the non-primitive $U(1)$ -charge of the Higgs field $\mathbf{2}_{-\frac{3}{2}}$, we expect a surviving discrete \mathbb{Z}_3 symmetry. However, this symmetry is contained in the surviving $U(1)$ symmetry in X_{F_3} , i.e. there is no discrete gauge group on X_{F_3} as expected geometrically.

In contrast, we see that the $U(1)$ -charge of the VEV $\mathbf{1}_2$ triggering the transition $X_{F_6} \rightarrow X_{F_4}$ is *non-minimal*. Thus, we expect a discrete gauge symmetry to be left unbroken, in addition to the $SU(2)$ gauge factor. The decomposition of representations for this Higgsing is given in table 34. There we see that the doublet after the Higgsing carries a half integral charge with respect to the discrete gauge factor. Hence, one has to rescale all charges by a factor of two, so that all charges become integral. Thus, we see that the resulting discrete symmetry is in fact \mathbb{Z}_4 . Note that in this case, the discrete factor is of physical relevance, since we can not embed it into the local gauge group of X_{F_4} . This implies that in a four dimensional theory, there will be gauge invariant couplings which are absent due to selection rules imposed by the \mathbb{Z}_4 -symmetry.

| | VEV: $\mathbf{1}_{(-1,-2)}$ $Q'_1 = (2Q_1 - Q_2)$ | VEV: $\mathbf{1}_{(-1,1)}$ $Q'_1 = (Q_1 + Q_2)$ | VEV: $\mathbf{1}_{(0,2)}$ $Q'_1 = (Q_1), Q_{\mathbb{Z}_2} = Q_2 \bmod 2$ |
|------------------------|--|--|---|
| $\mathbf{1}_{(1,-1)}$ | $\mathbf{1}_3$ | $\mathbf{1}_0$ | $\mathbf{1}_{(1,-)}$ |
| $\mathbf{1}_{(1,0)}$ | $\mathbf{1}_2$ | $\mathbf{1}_1$ | $\mathbf{1}_{(1,+)}$ |
| $\mathbf{1}_{(-1,-2)}$ | $\mathbf{1}_0$ | $\mathbf{1}_3$ | $\mathbf{1}_{(1,+)}$ |
| $\mathbf{1}_{(-1,-1)}$ | $\mathbf{1}_1$ | $\mathbf{1}_2$ | $\mathbf{1}_{(1,-)}$ |
| $\mathbf{1}_{(0,2)}$ | $\mathbf{1}_2$ | $\mathbf{1}_2$ | $\mathbf{1}_{(0,+)}$ |
| $\mathbf{1}_{(0,1)}$ | $\mathbf{1}_1$ | $\mathbf{1}_1$ | $\mathbf{1}_{(0,-)}$ |

Table 35. Branching of representations under the possible Higgsings of X_{F_5} to X_{F_3} (first two columns) and X_{F_2} (third column).

The multiplicities of charged states following from tables 34 and 11 are given by

$$\begin{aligned}
 \mathbf{2}_1 : & \quad ([K_B^{-1}] - \mathcal{S}_7 + \mathcal{S}_9) (6[K_B^{-1}] - 2\mathcal{S}_9 + 2\mathcal{S}_7) , \\
 \mathbf{1}_2 : & \quad 6[K_B^{-1}]^2 + [K_B^{-1}] (13\mathcal{S}_7 - 5\mathcal{S}_9) - 3\mathcal{S}_7^2 - 2\mathcal{S}_7\mathcal{S}_9 + \mathcal{S}_9^2 , \\
 \mathbf{3}_0 : & \quad 1 + ([K_B^{-1}] - \mathcal{S}_7 + \mathcal{S}_9) \frac{(-\mathcal{S}_7 + \mathcal{S}_9)}{2} .
 \end{aligned} \tag{4.21}$$

This precisely agrees with the geometrically obtained spectrum of X_{F_4} in table 5. We emphasize that charges of the matter states w.r.t to the \mathbb{Z}_4 , that we have obtained by Higgsing X_{F_5} , precisely coincide with those computed by intersections with the four-section in (3.48).

X_{F_5} Higgs branches

In X_{F_5} there are two possible toric Higgsing to X_{F_3} . The Higgs fields in that case are in the representations $\mathbf{1}_{(-1,-2)}$ or $\mathbf{1}_{(1,-1)}$, cf. table 9. The branching of the representations of X_{F_5} into representations of G_{F_3} are shown in table 35. The resulting spectrum matches that in (4.20), cf. table 7, up to redefinitions of \mathcal{S}_7 and \mathcal{S}_9 , that correspond to the transformations needed in order to bring the resulting polyhedron to its canonical form in figure 7.

Similarly, the polyhedron allows for an additional toric Higgsing from X_{F_5} to X_{F_2} which is triggered by VEVs in the representation $\mathbf{1}_{(0,-2)}$. This leaves the first U(1) unbroken, together with a remnant discrete \mathbb{Z}_2 symmetry from the second U(1). The multiplicities of the charged matter in X_{F_2} that are deduced by Higgsing from X_{F_5} are obtained from the group-theoretical branchings shown in table 35 and the spectrum of X_{F_5} in table 9. They read

$$\begin{aligned}
 \mathbf{1}_{(1,-)} : & \quad 6[K_B^{-1}]^2 + 4[K_B^{-1}] (\mathcal{S}_9 - \mathcal{S}_7) + 2\mathcal{S}_7^2 - 2\mathcal{S}_9^2 , \\
 \mathbf{1}_{(1,+)} : & \quad 6[K_B^{-1}]^2 + 4[K_B^{-1}] (\mathcal{S}_7 - \mathcal{S}_9) - 2\mathcal{S}_7^2 + 2\mathcal{S}_9^2 , \\
 \mathbf{1}_{(0,-)} : & \quad 6[K_B^{-1}]^2 + 4[K_B^{-1}] (\mathcal{S}_7 + \mathcal{S}_9) - 2\mathcal{S}_7^2 - 2\mathcal{S}_9^2 ,
 \end{aligned} \tag{4.22}$$

which precisely matches the geometrical result in table 3.

| | |
|----------------|--|
| | VEV: $\mathbf{1}_3$ $Q'_{\mathbb{Z}_3} = q \bmod 3$ |
| $\mathbf{1}_3$ | $\mathbf{1}_0$ |
| $\mathbf{1}_2$ | $\mathbf{1}_2$ |
| $\mathbf{1}_1$ | $\mathbf{1}_1$ |

Table 36. Branching of representations under the toric Higgsing of X_{F_3} to X_{F_1} .

X_{F_3} Higgs branches

In the spectrum of X_{F_3} , the singlet $\mathbf{1}_3$ allows for a toric Higgsing. In this transition, the $U(1)$ symmetry gets broken to a \mathbb{Z}_3 subgroup. The decomposition of representations of X_{F_3} for this Higgsing is shown in table 36.

As mentioned before, 6D hyper multiplets in the representation $\mathbf{1}_2$ under the discrete group \mathbb{Z}_3 are equivalent to hyper multiplets in the representation $\mathbf{1}_1$. Hence, there is only one type of charged hyper multiplet in X_{F_1} . This is in agreement with the geometrical computation (see section 3.2.1), where one sees a single codimension two locus supporting an I_2 -fiber. From the Higgsing we can read of the multiplicity of this charged state as

$$\mathbf{1}_1 : \quad 3 \left(6[K_B^{-1}]^2 - \mathcal{S}_7^2 + \mathcal{S}_7\mathcal{S}_9 - \mathcal{S}_9^2 + [K_B^{-1}](\mathcal{S}_7 + \mathcal{S}_9) \right). \quad (4.23)$$

4.3 The complete Higgsing chain

Having summarized the relevant features of the toric Higgsing procedure, we devote this section to a complete account on all possible toric Higgsings, that are summarized in appendix D, tables 37–39. In these tables we indicate in the first column, which toric hypersurface fibrations $X_{F_i} \rightarrow X_{F_j}$, $i \neq j$, are to be related. Then, in the second column, we state the possible toric Higgsings and which fields are to be identified as the Higgs fields that acquire a VEV in the transition. We note that all toric Higgsings between the same two toric hypersurface fibrations are physically equivalent. In the third column the $U(1)$ -generators on X_{F_j} are expressed in terms of the $U(1)$ -generators and Cartan generators on X_{F_i} . We have checked explicitly in all Higgsings that all matter charges of the fibration X_{F_j} determined in section 3 are obtained. In addition, in some Higgsings, the unbroken non-Abelian gauge group factors are interchanged in the Higgsing process. In these cases, the change of the order of non-Abelian factors is indicated in the third column of the tables in appendix D. In addition in most of the cases, an $SL(2, \mathbb{Z})$ -transformation on the “Higgsed” polyhedron is necessary in order to bring it into the canonical form used for the geometric computations in section 3. These transformations determine a unique redefinition of the divisor classes, similar as in (4.11), that is necessary in order to compare the matter multiplicities of the representations obtained after the Higgsing with the ones obtained by inspecting the geometry of X_{F_j} . The relevant redefinitions are shown in the last column of the tables in appendix D.

The decompositions of the representations on X_{F_i} under the group G_{F_i} into representations of the unbroken gauge group G_{F_j} after Higgsing can be found in appendix E for all

canonical toric Higgsings. We have checked in all cases that the matter spectra obtained by Higgsing the theory on X_{F_i} to the one corresponding to X_{F_j} agree with those of section 3, which provides another non-trivial check of the geometric analysis presented there.

We conclude this section with one final observation. As highlighted before, we observe that for every transition between two toric hypersurface fibrations $X_{F_j} \rightarrow X_{F_k}$ there exists a dual transition between $X_{F_k^*} \rightarrow X_{F_j^*}$. This symmetry of the Higgs diagram in figure 1 can be directly understood from the interpretation of the toric Higgsing on the level of the fiber polyhedron and its dual as we will explain in the following.

As pointed out in section 4.1, a toric Higgsing acts exactly as a blow-down in the original polyhedron F_j and a blow-up F_j^* in its dual polyhedron:

$$(F_j, F_j^*) \xrightarrow{\text{Higgs}} (F_j \xrightarrow{\text{blow-down}} F_k, F_j^* \xrightarrow{\text{blow-up}} F_k^*), \quad (4.24)$$

with $j > k$. Next we consider the inverse of the above process: we take F_k^* as the starting polyhedron with F_k as its dual. From the diagram (4.24), we know that there exists a blow-up map from F_j^* to F_k^* . However, now we take its inverse map as the blow-down from F_k^* to F_j^* . The same can analogously be done for the dual polyhedron F_k whose blow-up map is obtained from the inverse of the blow-down map in (4.24). Consequently, we arrive at the following map for the dual Higgs transition

$$(F_k^*, F_k) \xrightarrow{\text{Dual Higgs}} (F_k^* \xrightarrow{\text{blow-down}} F_j^*, F_k \xrightarrow{\text{blow-up}} F_j). \quad (4.25)$$

The above relation holds for every toric Higgsing and hence shows, that every Higgs transition has indeed a dual counterpart. However in general we observe more equivalent transitions between higher polyhedra than in their dual counterparts. In the geometry this reflects the fact, that in polyhedra with a larger area, there are more ways to embed subpolyhedra. On the field theory side this corresponds to less representations, that can be used for the Higgsing. An example is the transition $X_{F_{15}} \rightarrow X_{F_{12}}$ which can be equally realized by cutting any of the four vertices in the square of F_{15} . However the dual transition $F_5 \rightarrow F_2$ can only be achieved by deleting the unique node and hence there is only one Higgsing possible.

5 Conclusions

In this work we have analyzed F-theory compactifications on all toric hypersurface fibrations X_{F_i} . In these manifolds the genus-one fibers are given as a hypersurface in any of the 2D toric varieties associated to the reflexive polyhedra F_i depicted in figure 2. For each of these 16 Calabi-Yau manifolds we have computed the full MW-group (cf. with the results of [41] for the toric MW-group) and determined all codimension one, two and three singularities and the corresponding reducible fibers in the crepant resolutions X_{F_i} . Our work presents the first complete analysis of all these aspects. In the course of our study, we have encountered some codimension two fibers which are non-split and others whose complete splitting is visible only by computing their associated prime ideals. In addition, we have computed the Hodge numbers of the X_{F_i} in the case of Calabi-Yau threefolds. All

these geometric results determine the gauge groups, matter representations and Yukawa couplings of the effective SUGRA theories of F-theory on these manifolds. We have shown that these effective theories are anomaly-free in 6D, which proves in turn the completeness of our analysis of codimension one and two singularities of all these models [93].

The gauge groups we have found range from rank zero to six with up to three U(1)-factors corresponding to a rank three MW-group of rational sections. The Calabi-Yau manifolds X_{F_1} , X_{F_2} and X_{F_4} , that are constructed as fibrations of the cubic in \mathbb{P}^2 , the biquadric in $\mathbb{P}^1 \times \mathbb{P}^1$ and the quartic³² in $\mathbb{P}^2(1, 1, 2)$, respectively, do not have a section and lead to F-theory models with discrete gauge groups \mathbb{Z}_3 , \mathbb{Z}_2 and \mathbb{Z}_4 . We have established a relationship between the order of the multi-section in these Calabi-Yau manifolds and the order of the discrete gauge group. Furthermore, we have shown the existence (and computed the multiplicity) of I_2 -singularities that support matter charged only under these discrete groups. We have also a proposal for a “Shioda-map” of multi-sections, that allowed us to consistently compute the charges of all matter fields under these discrete gauge groups. In addition, by an explicit computation of the respective generators of their rank one MW-groups, we have demonstrated that X_{F_2} and X_{F_3} (which is constructed as a fibration of the elliptic curve in dP_1) both yield effective theories with one U(1)-gauge field. Most notably, we have found the first F-theory realization of charged singlets with U(1)-charge $q = 3$ in X_{F_3} . Furthermore, the non-trivial consistency conditions imposed on the U(1)-charge lattice of a model in quantum gravity [89] have been checked.

We emphasize that the non-toric nature of the generator of the MW-group of the fibration X_{F_3} was key to obtaining this U(1)-charge. We expect that the presence of non-toric sections can lead to more exotic U(1)-charge assignments of matter than those that occur in toric cases. Such a situation can be desirable as these exotic charge assignments could serve to control the phenomenology of particle physics models constructed in F-theory.

Besides these geometrical advances, we have shown that those extremal transitions between all toric hypersurface fibrations X_{F_i} , which are induced by toric blow-downs in the toric varieties \mathbb{P}_{F_i} , can be described by a corresponding Higgs mechanism in the effective theories of F-theory on the X_{F_i} . Although this correspondence between geometry and physics is expected to hold in general, also in chiral F-theory compactifications to 4D,³³ we considered here the 6D case.

We have explicitly worked out the full network of all toric Higgs transitions, shown in figure 1, in the 6D SUGRA theories of these F-theory models. Cutting off a vertex in a polyhedron F_i corresponds to a blow down in \mathbb{P}_{F_i} , which implies the removal of a corresponding divisor in X_{F_i} . After this blow-down, a new monomial can be added to the hypersurface constraint of X_{F_i} , resulting in a new Calabi-Yau manifold $X_{F_i'}$. In the effective theory of F-theory on X_{F_i} , this transition corresponds to giving a VEV to a particular matter field along a D-flat direction and a consequent breakdown of the gauge symmetry of the theory. For all extremal transitions, we have identified the relevant matter field that has to acquire a VEV and matched the effective theory after Higgsing with the

³²As remarked already, this case has been subject of recent attention in [54–56].

³³As 4D chirality is induced by G_4 -flux in F-theory, this requires the understanding of the behavior of G_4 -flux during extremal transitions in Calabi-Yau fourfolds, which is discussed in [94]. The results of [95] will be crucial for carrying out any quantitative analysis in this context.

one obtained geometrically from F-theory on $X_{F_{i'}}$. As an explicit example, for a specific choice of the base $B = \mathbb{P}^2$, we have described the transition $X_{F_9} \rightarrow X_{F_5}$ for all strata in moduli space, that are labeled by (n_7, n_9) , cf. figure 32. There we have found that a D-flat Higgsing in the effective theory is only possible for those points (n_7, n_9) , that are allowed for both X_{F_9} and X_{F_5} . In this context, we have also commented on the different gauge groups that appear on the boundary of the allowed region for (n_7, n_9) in X_{F_5} .

We have found that the full toric Higgs network in figure 1 is beautifully mirror symmetric under the exchange of a polyhedron F_i and its dual F_i^* .³⁴ This gave rise to a number of interesting observations: the toric hypersurface fibrations X_{F_i} and $X_{F_i^*}$ always have the same amount of U(1)-symmetries and the ranks of their gauge groups, G_{F_i} and $G_{F_i^*}$, always sum up to six. Indeed, this sum rule of the rank of the gauge group is a direct consequence of the sum rule for the volumes of F_i and its dual F_i^* .³⁵ The duality between X_{F_i} and $X_{F_i^*}$ is realized also on the level of the Higgs transitions, i.e. for every toric Higgs transition $X_{F_i} \rightarrow X_{F_{i'}}$, there is a dual Higgs transition $X_{F_{i'}} \rightarrow X_{F_i^*}$. In addition, we have observed that this duality maps discrete symmetries in F-theory to Mordell-Weil torsion.

Finally, let us highlight some features of toric hypersurface fibrations, which make them attractive for particle physics applications. First, recall that the presence of discrete symmetries is a desirable feature in field theoretic constructions beyond the standard model.³⁶ So far, all efforts towards embedding the standard model in F-theory have been based on compactifications with a zero section, which are typically free of discrete symmetries.³⁷ Since discrete symmetries arise naturally in genus one-fibrations with multi-sections, this type of compactifications constitutes a promising new arena for engineering semi-realistic particle physics models.

In addition, we have found concrete toric hypersurface fibrations, that directly realize the gauge group and representations of the Standard Model ($X_{F_{11}}$), the Pati-Salam model ($X_{F_{13}}$) and the trinification model ($X_{F_{16}}$). Even more interestingly, we have found that the Standard Model can be obtained via toric Higgsings from the Pati-Salam or the trinification model, both of these models being at the same time the two theories with the maximal non-Abelian gauge groups among all toric hypersurface fibrations.

Outlook

For future directions it would be interesting to use the effective theories we have obtained for particle physics applications. Since the Pati-Salam and trinification model are the two maximal toric enhancements of the Standard Model, as we have seen, they are natural candidates for toric unified model building in F-theory. It would be fascinating to work out the phenomenological implications of this observation.

³⁴We note that a similar observation has been made for elliptically fibered toric $K3$ surfaces in [96].

³⁵We thank Albrecht Klemm for explaining this fact to us.

³⁶The reader is referred to [97–101] for a selection of discrete symmetries which have been invoked in order to forbid certain unappealing operators in supersymmetric models.

³⁷Instead of discrete symmetries, the phenomenology of these models is kept under control by virtue of additional U(1) symmetries. Discrete symmetries can arise by a manual breakdown of these additional U(1)s. This possibility has been studied in e.g [102, 103].

Clearly, the results of this paper most directly apply to six dimensional or to non-chiral four dimensional compactifications of F-theory. Thus, a natural and most straightforward extension of this work, that would also be crucial for phenomenological applications, is the inclusion and construction of G_4 -flux as well as the computation of the chiral indices of all the matter representations found in all toric hypersurface fibrations X_{F_i} , following the recipe and techniques described in [42].

The beautiful realization of mirror symmetry in the Higgs network and related observations like the sum rule for the gauge groups and the connection between Mordell-Weil torsion and discrete symmetries are topics of further investigation. It would be exciting to understand all these observations by unveiling a common structure underlying all toric hypersurface fibrations, for example a master gauge group into which all gauge groups G_{F_i} could be embedded. This might require a departure from the toric framework.

There has been a lot of recent progress in obtaining matter representations in F-theory using deformations or, in physical terms, the Higgs mechanism [104, 105]. It would be very interesting to see how the results about the Higgs network of the toric hypersurface fibrations obtained here can be worked out using the deformation techniques employed there.

Finally, it would be interesting to use the toric hypersurface fibrations studied here also for compactifications of M-theory to engineer 3D $\mathcal{N} = 2$ gauge theories and to study their Coulomb-branches and phase structures, see [106–109] (and also the seminal works [110, 111]) for recent detailed analyses of the phase structure of 3D $SU(N)$ -gauge theories for all $N \leq 5$.

Acknowledgments

We would like to thank Lara Anderson, Ron Donagi, Thomas Grimm, Hans Jockers, Jan Keitel, Albrecht Klemm, Craig Lawrie, Dave Morrison, Sakura Schäfer-Nameki, Wati Taylor, Timo Weigand and especially Mirjam Cvetič for valuable discussions and correspondence. D.K. thanks the Bethe Center for Theoretical Physics Bonn and the Theory Division of CERN for hospitality during completion of the project. This work is supported by the DOE grant DE-SC0007901 (D.K., H.P.), the NSF String Vacuum Project Grant No. NSF PHY05-51164 (H.P.) and the Dean's Funds for Faculty Working Group (D.K.). The work of D.M., P.O. and J.R. is partially supported by a scholarship of the Bonn-Cologne Graduate School BCGS, the SFB-Transregio TR33 The Dark Universe (Deutsche Forschungsgemeinschaft) and the European Union 7th network program Unification in the LHC era (PITN-GA-2009-237920).

A Anomaly cancellation conditions in 6D

In this appendix we summarize the consistency relations that have to be obeyed by an anomaly-free 6D SUGRA theory. We follow the conventions and notations of [24, 39, 80], to which we also refer for further details.

There are three qualitatively different types of anomalies, the pure gravitational anomalies, the mixed gauge-gravitational anomalies and the pure gauge anomalies. De-

pending on the number of gauge group factors, mixed anomalies between different gauge group factors are present. A theory is referred to as anomaly-free if all one-loop anomalies are canceled by the contributions from the anomalous variations of Green-Schwarz (GS) counter-terms.

For an effective SUGRA theory in 6D, the anomaly cancellation conditions read:

$$\begin{aligned}
 \text{tr}R^4 : H - V + 29T = 273, \quad (\text{tr}R^2)^2 : 9 - T = a \cdot a & \quad (\text{Pure gravitational}) \\
 \text{tr}F_\kappa^2 \text{tr}R^2 : -\frac{1}{6} \left(A_{adj_\kappa} - \sum_{\mathbf{R}} x_{\mathbf{R}} A_{\mathbf{R}} \right) = a \cdot \left(\frac{b_\kappa}{\lambda_\kappa} \right) & \quad (\text{Non-Abelian-gravitational}) \\
 F_m F_n \text{tr}R^2 : -\frac{1}{6} \sum_{\underline{q}} x_{q_m, q_n} q_m q_n = a \cdot b_{mn} & \quad (\text{Abelian-gravitational}) \\
 \text{tr}F_\kappa^4 : B_{adj_\kappa} - \sum_{\mathbf{R}} x_{\mathbf{R}} B_{\mathbf{R}} = 0, & \quad (\text{Pure non-Abelian}) \\
 \text{tr}F_\kappa^2 \text{tr}F_\kappa^2 : \frac{1}{3} \left(\sum_{\mathbf{R}} x_{\mathbf{R}} C_{\mathbf{R}} - C_{adj_\kappa} \right) = \left(\frac{b_\kappa}{\lambda_\kappa} \right)^2 & \\
 F_m F_n F_k F_l : \sum_{\underline{q}} x_{q_m, q_n, q_k, q_l} q_m q_n q_k q_l = b_{(mn} \cdot b_{kl)} & \quad (\text{Pure Abelian}) \\
 F_m F_n \text{tr}F_\kappa^2 : \sum_{\mathbf{R}, q_m, q_n} x_{\mathbf{R}, q_m, q_n} q_m q_n A_{\mathbf{R}} = \left(\frac{b_\kappa}{\lambda_\kappa} \right) \cdot b_{mn} & \quad (\text{Non-Abelian-Abelian}) \\
 F_m \text{tr}F_\kappa^3 : \sum_{\mathbf{R}, q_m} x_{\mathbf{R}, q_m} q_i E_{\mathbf{R}} = 0. & \quad (\text{A.1})
 \end{aligned}$$

Here, we have given the terms in the 6D anomaly polynomial, whose coefficients are the respective anomalies. The Ricci tensor is denoted by R and the field strengths of the non-Abelian and Abelian gauge field for the gauge group factor G_κ and the m^{th} $U(1)$ are denoted by F_κ and F_m , respectively. The overall number of hyper, vector and tensor multiplets is denoted by H , V and T , respectively and the variables ‘ x .’ denote the multiplicities of certain charged hyper multiplets: $x_{\mathbf{R}}$, $x_{\mathbf{R}, q_m}$ and $x_{\mathbf{R}, q_m, q_n}$ are the number of hyper multiplets in the representation \mathbf{R} , in the representation \mathbf{R} with charge q_m under $U(1)_m$ and in the representation \mathbf{R} with charges $\{q_m, q_n\}$ under $U(1)_m \times U(1)_n$, respectively; x_{q_m, q_n} and x_{q_m, q_n, q_k, q_l} denote the number of matter hyper multiplets with charges (q_m, q_n) and (q_m, q_n, q_k, q_l) under $U(1)_m \times U(1)_n$ and $U(1)_m \times U(1)_n \times U(1)_k \times U(1)_l$, respectively.

In the contributions from the GS counter-terms, a , b_κ and b_{mn} are the anomaly coefficients. These transform as vectors of $SO(1, T)$, and are determined by the underlying microscopic theory. In our F-theory compactification these coefficients can be readily interpreted in terms of geometrical objects. We have

$$a = [K_B], \quad b_\kappa = \mathcal{S}_{G_\kappa}^b, \quad b_{mn} = -\pi(\sigma(\hat{s}_n) \cdot \sigma(\hat{s}_m)), \quad (\text{A.2})$$

where K_B is the canonical divisor of B , $\mathcal{S}_{G_\kappa}^b$ is the divisor on B defined in (2.2) supporting the non-Abelian group G_κ and $\pi(\sigma(\hat{s}_n) \cdot \sigma(\hat{s}_m))$ is the Néron-Tate height pairing defined in (2.6). Under these identifications, the inner product in (A.1) is replaced by the intersection pairing on the base B .

In addition, in the anomalies (A.1), we have made use of the following group theoretical relations between traces in different representations \mathbf{R} :

$$\mathrm{tr}_{\mathbf{R}} F_{\kappa}^2 = A_{\mathbf{R}} \mathrm{tr} F_{\kappa}^2, \quad \mathrm{tr}_{\mathbf{R}} F_{\kappa}^3 = E_{\mathbf{R}} \mathrm{tr} F_{\kappa}^3, \quad \mathrm{tr}_{\mathbf{R}} F_{\kappa}^4 = B_{\mathbf{R}} \mathrm{tr} F_{\kappa}^4 + C_{\mathbf{R}} (\mathrm{tr} F_{\kappa}^2)^2. \quad (\text{A.3})$$

Here ‘tr’ denotes the trace with respect to the fundamental representation, while $\mathrm{tr}_{\mathbf{R}}$ is the trace for a given representation \mathbf{R} . For $\kappa = \mathrm{SU}(N)$ with $N > 3$, the group theory factors in (A.3) assume the following values:³⁸

| Representation | Dimension | $A_{\mathbf{R}}$ | $B_{\mathbf{R}}$ | $C_{\mathbf{R}}$ | $E_{\mathbf{R}}$ |
|----------------|--------------|------------------|------------------|------------------|------------------|
| Fundamental | N | 1 | 1 | 0 | 1 |
| Adjoint | $N^2 - 1$ | $2N$ | $2N$ | 6 | 0 |
| Antisymmetric | $N(N - 1)/2$ | $N - 2$ | $N - 8$ | 3 | $N - 4$ |

(A.4)

For the specific case of $\mathrm{SU}(2)$ and $\mathrm{SU}(3)$, the coefficients $A_{\mathbf{R}}$ coincide with those given in the table. In contrast to that, the coefficients $B_{\mathbf{R}}$ and $E_{\mathbf{R}}$ are equal to zero in both cases. The actual coefficient $C_{\mathbf{R}}$ can be computed using the values for $B_{\mathbf{R}}$ and $C_{\mathbf{R}}$ in the above table, as $C_{\mathbf{R}} + \frac{1}{2}B_{\mathbf{R}}$, for $N = 2, 3$. Finally, the coefficient λ_{κ} in (A.1) corresponds to the group normalization constant defined by $\lambda_{\kappa} = 2c_{\kappa}/E_{\mathrm{adj}_{\kappa}}$, where c_{κ} is the dual Coxeter number for the group G_{κ} and $E_{\mathrm{adj}_{\kappa}}$ is $E_{\mathbf{R}}$ for the adjoint representation. For $G_{\kappa} = \mathrm{SU}(N)$, we have $\lambda_{\kappa} = 1$.

B Additional data on toric hypersurface fibrations

In this appendix we provide the explicit expressions for f and g of the WSF of the Jacobian fibrations of X_{F_1} , X_{F_2} and X_{F_4} . Additionally, we present the explicit WS-coordinates of the rational sections of X_{F_3} , X_{F_5} and X_{F_7} . The functions f , g as well as the WS-coordinates of the rational sections of all other toric hypersurface fibrations X_{F_i} can be obtained by specializing the ones presented here. Finally, we derive the Tate form, the WSF and the WS-coordinates of the generators of the MW-torsion of the toric hypersurface fibrations $X_{F_{13}}$, $X_{F_{15}}$ and $X_{F_{16}}$.

WSF of $J(X_{F_1})$

Here we explicitly write out the polynomials f and g of the WSF of the Jacobian fibration $J(X_{F_1})$. The discriminant Δ is calculated straightforwardly from these quantities but is omitted here due to the length of its explicit form. The functions f , g in the WSF of (3.4) read

$$\begin{aligned}
 f = \frac{1}{48} & \left(-(s_6^2 - 4(s_5s_7 + s_3s_8 + s_2s_9))^2 + 24(-s_6(s_{10}s_2s_3 - 9s_1s_{10}s_4 + s_4s_5s_8 \right. \\
 & \quad + s_2s_7s_8 + s_3s_5s_9 + s_1s_7s_9) + 2(s_{10}s_3^2s_5 + s_1s_7^2s_8 + s_2s_3s_8s_9 + s_1s_3s_9^2 \\
 & \quad + s_7(s_{10}s_2^2 - 3s_1s_{10}s_3 + s_3s_5s_8 + s_2s_5s_9) \\
 & \quad \left. + s_4(-3s_{10}s_2s_5 + s_2s_8^2 + (s_5^2 - 3s_1s_8)s_9) \right) \quad (\text{B.1})
 \end{aligned}$$

³⁸See [80] for further details.

$$\begin{aligned}
 g = & \frac{1}{864} ((s_6^2 - 4(s_5s_7 + s_3s_8 + s_2s_9))^3 - 36(s_6^2 - 4(s_5s_7 + s_3s_8 + s_2s_9)) \\
 & \times (-s_6(s_{10}s_2s_3 - 9s_1s_{10}s_4 + s_4s_5s_8 + s_2s_7s_8 + s_3s_5s_9 + s_1s_7s_9) \\
 & + 2(s_{10}s_3^2s_5 + s_1s_7^2s_8 + s_2s_3s_8s_9 + s_1s_3s_9^2 + s_7(s_{10}s_2^2 - 3s_1s_{10}s_3 + s_3s_5s_8 + s_2s_5s_9) \\
 & + s_4(-3s_{10}s_2s_5 + s_2s_8^2 + (s_5^2 - 3s_1s_8)s_9))) + 216((s_{10}s_2s_3 - 9s_1s_{10}s_4 + s_4s_5s_8 \\
 & + s_2s_7s_8 + s_3s_5s_9 + s_1s_7s_9)^2 + 4(-s_1s_{10}^2s_3^3 - s_1^2s_{10}s_7^3 - s_4^2(27s_1^2s_{10}^2 + s_{10}s_5^3 \\
 & + s_1(-9s_{10}s_5s_8 + s_8^3)) + s_{10}s_3^2(-s_2s_5 + s_1s_6)s_9 - s_1s_3^2s_8s_9^2 \\
 & - s_7^2(s_{10}(s_2^2s_5 - 2s_1s_3s_5 - s_1s_2s_6) + s_1s_8(s_3s_8 + s_2s_9)) \\
 & - s_3s_7(s_{10}(-s_2s_5s_6 + s_1s_6^2 + s_2^2s_8 + s_3(s_5^2 - 2s_1s_8) + s_1s_2s_9) \\
 & + s_9(s_2s_5s_8 - s_1s_6s_8 + s_1s_5s_9)) + s_4(-s_{10}^2(s_2^3 - 9s_1s_2s_3) \\
 & + s_{10}(s_6(-s_2s_5s_6 + s_1s_6^2 + s_2^2s_8) + s_3(s_5^2s_6 - s_2s_5s_8 - 3s_1s_6s_8)) \\
 & + (s_{10}(2s_2^2s_5 + 3s_1s_3s_5 - 3s_1s_2s_6) + s_8(-s_3s_5^2 + s_2s_5s_6 - s_1s_6^2 - s_2^2s_8 + 2s_1s_3s_8))s_9 \\
 & + (-s_2s_5^2 + s_1s_5s_6 + 2s_1s_2s_8)s_9^2 - s_1^2s_9^3 + s_7(s_{10}(2s_2s_5^2 - 3s_1s_5s_6 + 3s_1s_2s_8 + 9s_1^2s_9) \\
 & - s_8(s_2s_5s_8 - s_1s_6s_8 + s_1s_5s_9)))))) \tag{B.2}
 \end{aligned}$$

WSF of $J(X_{F_2})$ and the cubic form of the biquadric

First, we present the explicit expressions for f and g in the WSF of the Jacobian fibration $J(X_{F_2})$, where we omit the expression of the discriminant Δ . The functions f, g in the WSF of (3.12) read

$$\begin{aligned}
 f = & \frac{1}{48} [-(-4b_1b_{10} + b_6^2 - 4(b_5b_7 + b_3b_8 + b_2b_9))^2 \\
 & + 24(-b_6(b_{10}b_2b_5 + b_2b_7b_8 + b_3b_5b_9 + b_1b_7b_9) \\
 & + 2(b_{10}(b_1b_5b_7 + b_2^2b_8 + b_3(b_5^2 - 4b_1b_8) + b_1b_2b_9) + b_7(b_1b_7b_8 + b_2b_5b_9) \\
 & + b_3(b_5b_7b_8 + b_2b_8b_9 + b_1b_9^2))] , \tag{B.3}
 \end{aligned}$$

and

$$\begin{aligned}
 g = & \frac{1}{864} [(-4b_1b_{10} + b_6^2 - 4(b_5b_7 + b_3b_8 + b_2b_9))^3 \\
 & - 36(-4b_1b_{10} + b_6^2 - 4(b_5b_7 + b_3b_8 + b_2b_9)) \\
 & \times (-b_6(b_{10}b_2b_5 + b_2b_7b_8 + b_3b_5b_9 + b_1b_7b_9) \\
 & + 2(b_{10}(b_1b_5b_7 + b_2^2b_8 + b_3(b_5^2 - 4b_1b_8) + b_1b_2b_9) \\
 & + b_7(b_1b_7b_8 + b_2b_5b_9) + b_3(b_5b_7b_8 + b_2b_8b_9 + b_1b_9^2))) \\
 & + 216((b_{10}b_2b_5 + b_2b_7b_8 + b_3b_5b_9 + b_1b_7b_9)^2 \\
 & - 4(b_2b_3b_5b_7b_8b_9 + b_1^2b_{10}(-4b_{10}b_3b_8 + b_7^2b_8 + b_3b_9^2) + b_{10}(b_3^2b_5^2b_8 + b_2^2b_5b_7b_8 \\
 & + b_2b_3(-b_5b_6b_8 + b_2b_8^2 + b_5^2b_9)) + b_1(b_{10}^2(b_3b_5^2 + b_2^2b_8) + b_2b_7^2b_8b_9 + b_3^2b_8b_9^2 \\
 & + b_3b_7(b_7b_8^2 - b_6b_8b_9 + b_5b_9^2) \\
 & + b_{10}(-4b_3^2b_8^2 + b_3b_6(b_6b_8 - b_5b_9) + b_2b_7(-b_6b_8 + b_5b_9)))))] . \tag{B.4}
 \end{aligned}$$

Second, the explicit expressions of the \tilde{s}_i in (3.27) obtained by mapping X_{F_2} to X_{F_5} read

$$\tilde{s}_1 = b_1 ,$$

$$\begin{aligned}
 \tilde{s}_2 &= \frac{1}{b_8} \left(b_2 b_8 - b_1 b_9 - b_1 \sqrt{-4b_{10}b_8 + b_9^2} \right), \\
 \tilde{s}_3 &= \frac{1}{b_8^2} \left(-2b_1 b_{10} b_8 + 2b_3 b_8^2 - b_2 b_8 b_9 + b_1 b_9^2 \right. \\
 &\quad \left. - b_2 b_8 \sqrt{-4b_{10}b_8 + b_9^2} + b_1 b_9 \sqrt{-4b_{10}b_8 + b_9^2} \right), \\
 \tilde{s}_5 &= b_5, \\
 \tilde{s}_6 &= \frac{1}{b_8} \left(b_6 b_8 - b_5 b_9 - b_5 \sqrt{-4b_{10}b_8 + b_9^2} \right), \\
 \tilde{s}_7 &= \frac{1}{b_8^2} \left(-2b_{10} b_5 b_8 + 2b_7 b_8^2 - b_6 b_8 b_9 + b_5 b_9^2 \right. \\
 &\quad \left. - b_6 b_8 \sqrt{-4b_{10}b_8 + b_9^2} + b_5 b_9 \sqrt{-4b_{10}b_8 + b_9^2} \right), \\
 \tilde{s}_8 &= b_8, \\
 \tilde{s}_9 &= -\sqrt{-4b_{10}b_8 + b_9^2}.
 \end{aligned} \tag{B.5}$$

WSF of $J(X_{F_4})$

The explicit expressions for f and g in the WSF of the Jacobian fibration $J(X_{F_4})$ associated to X_{F_4} with hypersurface equation (3.17) read

$$\begin{aligned}
 f_4 &= \frac{1}{48} [-24d_9(-2d_5d_6^2 + d_4d_6d_7 - 2d_3d_6d_8 + d_2d_7d_8 \\
 &\quad - 2d_1d_8^2 - 2d_2d_4d_9 + 8d_1d_5d_9) - (d_7^2 - 4(d_6d_8 + d_3d_9))^2],
 \end{aligned} \tag{B.6}$$

and

$$\begin{aligned}
 g_4 &= \frac{1}{864} [36d_9(-2d_5d_6^2 + d_4d_6d_7 - 2d_3d_6d_8 \\
 &\quad + d_2d_7d_8 - 2d_1d_8^2 - 2d_2d_4d_9 + 8d_1d_5d_9)(d_7^2 - 4(d_6d_8 + d_3d_9)) \\
 &\quad + (d_7^2 - 4(d_6d_8 + d_3d_9))^3 + 216d_9^2[4d_2d_5d_6d_7 - 4d_1d_5d_7^2 \\
 &\quad + d_2^2d_8^2 + d_4(-2d_2d_6d_8 + 4d_1d_7d_8) \\
 &\quad - 4d_2^2d_5d_9 + d_4^2(d_6^2 - 4d_1d_9) - 4d_3(d_5d_6^2 + d_1d_8^2 - 4d_1d_5d_9)].
 \end{aligned} \tag{B.7}$$

WS-coordinates of the non-toric section of X_{F_3}

As we have shown in section 3.3.1, there is one additional rational section, besides the toric section \hat{s}_0 , of the fibration of X_{F_3} . The section \hat{s}_1 has coordinates $[x_1 : y_1 : z_1]$ in the WSF that are given by

$$\begin{aligned}
 x_1 &= \frac{1}{12} (12s_1^2s_9^6 + 4(2s_2(s_5^2 - 3s_1s_8) - 3s_1s_5s_6)s_9^5 + ((s_6^2 - 4s_5s_7)s_5^2 + 12(s_2^2 + 2s_1s_3)s_8^2 \\
 &\quad - 4(4s_3s_5^2 + s_2s_6s_5 - 3s_1(s_6^2 + 2s_5s_7))s_8)s_9^4 - 2s_8(-4(s_6s_7 + 3s_4s_8)s_5^2 \\
 &\quad + (s_6^3 - 10s_3s_8s_6 + 4s_2s_7s_8)s_5 + 2s_8(9s_1s_6s_7 + 6s_1s_4s_8 + s_2(s_6^2 + 6s_3s_8)))s_9^3 \\
 &\quad + s_8^2(s_6^4 - 2s_5s_7s_6^2 - 8s_5^2s_7^2 + 12(s_3^2 + 2s_2s_4)s_8^2 - 4(9s_4s_5s_6 - s_7(5s_2s_6 + 6s_1s_7)
 \end{aligned}$$

$$\begin{aligned}
 & + s_3(s_6^2 + 2s_5s_7))s_8)s_9^2 - 2s_8^3(12s_3s_4s_8^2 + 2(s_7(s_3s_6 + 4s_2s_7) - 3s_4(s_6^2 + 2s_5s_7))s_8 \\
 & + s_6s_7(s_6^2 - 4s_5s_7))s_9 + s_8^4((s_6^2 - 4s_5s_7)s_7^2 + 4(2s_3s_7 - 3s_4s_6)s_8s_7 + 12s_4^2s_8^2)), \\
 y_1 = & \frac{1}{2}(2s_1^3s_9^3 + s_1(2s_2(s_5^2 - 3s_1s_8) - 3s_1s_5s_6)s_9^3 + ((s_3s_5^2 - s_2s_6s_5 + s_1(s_6^2 - s_5s_7))s_5^2 \\
 & + 6s_1(s_2^2 + s_1s_3)s_8^2 + (-2s_2^2s_5^2 + 2s_1s_2s_6s_5 + s_1(3s_1(s_6^2 + 2s_5s_7) - 4s_3s_5^2))s_8)s_9^7 \\
 & - s_8(2(s_2^3 + 6s_1s_3s_2 + 3s_1^2s_4)s_8^2 - (s_5s_6s_2^2 + (6s_3s_5^2 - 4s_1(s_6^2 + 2s_5s_7))s_2 \\
 & + s_1(6s_4s_5^2 + 2s_3s_6s_5 - 9s_1s_6s_7))s_8 + s_5(3s_4s_5^3 + 2s_3s_6s_5^2 - 3s_2s_7s_5^2 - 2s_2s_6^2s_5 \\
 & + s_1s_6s_7s_5 + 2s_1s_6^3))s_9^6 + s_8^2(s_1s_6^4 - s_2s_5s_6^3 + s_3s_5^2s_6^2 + 7s_1s_5s_7s_6^2 + 9s_4s_5^3s_6 \\
 & - 8s_2s_5^2s_7s_6 + s_1s_5^2s_7^2 + 6(s_3(s_2^2 + s_1s_3) + 2s_1s_2s_4)s_8^2 - s_3s_5^3s_7 + (-4s_3^2s_5^2 \\
 & - 8s_2s_4s_5^2 - 6s_1s_4s_6s_5 + s_2^2s_6^2 + 6s_1^2s_7^2 + 2s_2(s_2s_5 + 7s_1s_6)s_7 + s_3(2s_1(s_6^2 + 2s_5s_7) \\
 & - 6s_2s_5s_6))s_8)s_9^5 - s_8^3(s_8(6s_2s_8 - 5s_5s_6)s_3^2 - 5s_6s_7(s_5^2 - 2s_1s_8)s_3 \\
 & + 5s_7(s_6s_8s_2^2 - s_5(s_6^2 + s_5s_7)s_2 + 2s_1s_7s_8s_2 + s_1s_6(s_6^2 + 2s_5s_7)) \\
 & + s_4(5(2s_6^2 + s_5s_7)s_5^2 - 10(s_3s_5 + s_2s_6)s_8s_5 + 6(s_2^2 + 2s_1s_3)s_8^2))s_9^4 \\
 & + s_8^4(2(s_3^3 + 6s_2s_4s_3 + 3s_1s_4^2)s_8^2 - (6s_4^2s_5^2 + s_3^2s_6^2 - 4(s_2^2 + 2s_1s_3)s_7^2 \\
 & + 2s_3(s_3s_5 - 3s_2s_6)s_4 + 2s_4(s_2s_6^2 + 7s_3s_5s_6 - 3s_1s_7s_6 + 2s_2s_5s_7))s_8 \\
 & + 5(s_4s_5s_6(s_6^2 + 2s_5s_7) + s_7(s_7(2s_1s_6^2 - s_2s_5s_6 + s_1s_5s_7) - s_3s_5(s_6^2 + s_5s_7))))s_9^3 \\
 & - s_8^5(3s_8(2s_2s_8 - 3s_5s_6)s_4^2 + (s_6^4 + (7s_5s_7 - 4s_3s_8)s_6^2 + 2s_2s_7s_8s_6 + s_5^2s_7^2 \\
 & - 8s_3s_5s_7s_8 + 6s_8(s_8s_3^2 + s_1s_7^2))s_4 + s_7(s_6s_8s_3^2 - (s_6^3 + 8s_5s_7s_6 - 6s_2s_7s_8)s_3 \\
 & + s_7(9s_1s_6s_7 + s_2(s_6^2 - s_5s_7))))s_9^2 + s_8^6(3s_8(-s_6^2 - 2s_5s_7 + 2s_3s_8)s_4^2 \\
 & + s_7(2s_6^3 + s_5s_7s_6 - 2s_3s_8s_6 + 4s_2s_7s_8)s_4 + s_7^2(2s_8s_3^2 - 2s_6^2s_3 - 3s_5s_7s_3 + 3s_1s_7^2 \\
 & + 2s_2s_6s_7))s_9 + s_8^7(-2s_8^2s_4^3 + 3s_6s_7s_8s_4^2 + s_7^2(-s_6^2 + s_5s_7 - 2s_3s_8)s_4 \\
 & + s_7^3(s_3s_6 - s_2s_7))), \\
 z_1 = & s_7s_8^2 + s_9(s_5s_9 - s_6s_8). \tag{B.8}
 \end{aligned}$$

WS-coordinates of the two rational sections of X_{F_5}

In addition to \hat{s}_0 , there are two rational sections of the fibration of X_{F_5} . The WS-coordinates have been worked out first in [38, 39]. We reproduce these results here for convenience.

The section \hat{s}_1 has coordinates $[x_1 : y_1 : z_1]$ in the WSF given by

$$\begin{aligned}
 x_1 &= \frac{1}{12}(s_6^2 - 4s_5s_7 + 8s_3s_8 - 4s_2s_9), \\
 y_1 &= \frac{1}{2}(s_3s_6s_8 - s_2s_7s_8 - s_3s_5s_9 + s_1s_7s_9), \\
 z_1 &= 1.
 \end{aligned} \tag{B.9}$$

The section \hat{s}_2 has coordinates $[x_2 : y_2 : z_2]$ in the WSF given by

$$\begin{aligned}
 x_2 &= \frac{1}{12}(12s_7^2s_8^2 + s_9^2(s_6^2 + 8s_3s_8 - 4s_2s_9) + 4s_7s_9(-3s_6s_8 + 2s_5s_9)), \\
 y_2 &= \frac{1}{2}(2s_7^3s_8^3 + s_3s_9^3(-s_6s_8 + s_5s_9) + s_7^2s_8s_9(-3s_6s_8 + 2s_5s_9) \\
 &\quad + s_7s_9^2(s_6^2s_8 + 2s_3s_8^2 - s_5s_6s_9 - s_2s_8s_9 + s_1s_9^2)), \\
 z_2 &= s_9.
 \end{aligned} \tag{B.10}$$

WS-coordinates of the three rational sections of X_{F_7}

There are three rational sections of the fibration of X_{F_7} besides \hat{s}_0 [41, 45]. The following results have been obtained using the birational map in [39] in the special case $s_1 = 0$.

The coordinates of \hat{s}_1 in the WSF, denoted by $[x_1 : y_1 : z_1]$, are

$$\begin{aligned}
 x_1 &= \frac{1}{12}(s_6^2s_7^2 - 4s_5s_7^3 + 8s_3s_7^2s_8 - 12s_3s_6s_7s_9 + 8s_2s_7^2s_9 + 12s_3^2s_9^2), \\
 y_1 &= \frac{1}{2}(-2s_3^3s_9^3 + s_2s_7^3(-s_7s_8 + s_6s_9) + s_3^2s_7s_9(-2s_7s_8 + 3s_6s_9) \\
 &\quad + s_3s_7^2(s_6s_7s_8 - s_6^2s_9 + s_9(s_5s_7 - 2s_2s_9))), \\
 z_1 &= s_7.
 \end{aligned} \tag{B.11}$$

Similarly, the coordinates of \hat{s}_2 in WSF, denoted by $[x_2 : y_2 : z_2]$, read

$$\begin{aligned}
 x_2 &= \frac{1}{12}(s_6^2 - 4s_5s_7 - 4s_3s_8 + 8s_2s_9), \\
 y_2 &= \frac{1}{2}(s_2s_7s_8 + s_3s_5s_9 - s_2s_6s_9), \\
 z_2 &= 1.
 \end{aligned} \tag{B.12}$$

Finally, the coordinates of \hat{s}_3 in the WSF, that we denote by $[x_3 : y_3 : z_3]$, are

$$\begin{aligned}
 x_3 &= \frac{1}{12}(12s_7^2s_8^2 + s_9^2(s_6^2 + 8s_3s_8 - 4s_2s_9) + 4s_7s_9(-3s_6s_8 + 2s_5s_9)), \\
 y_3 &= \frac{1}{2}(-2s_7^3s_8^3 + s_7^2s_8s_9(3s_6s_8 - 2s_5s_9) + s_3s_9^3(s_6s_8 - s_5s_9) \\
 &\quad + s_7s_9^2(-s_6^2s_8 + s_5s_6s_9 + s_8(-2s_3s_8 + s_2s_9))), \\
 z_3 &= s_9.
 \end{aligned} \tag{B.13}$$

Tate form, WSF and the MW-torsion of $X_{F_{13}}$

In this appendix we determine two Tate forms and the WSF of $X_{F_{13}}$. We explicitly derive the WS-coordinates of its torsional section and use this to show, that the MW-torsion acts on all codimension one singularities in $X_{F_{13}}$, i.e. on all non-Abelian gauge group factors in $G_{F_{13}}$.

We directly employ the birational map in [39] for $s_5 = s_7 = s_8 = 0$ to obtain a Tate form for the hypersurface constraint (3.180) of $X_{F_{13}}$. The Tate coefficients we naively obtain read

$$\begin{aligned}
 a_1 &= \frac{s_6^2 + 2s_2s_9}{s_6}, & a_2 &= -\frac{s_2s_9(-s_6^2 + s_2s_9)}{s_6^2}, \\
 a_3 &= s_2s_6s_9, & a_4 &= s_1s_3s_9^2, & a_6 &= s_1s_2s_3s_9^3.
 \end{aligned} \tag{B.14}$$

Due to the poles at $s_6 = 0$, this Tate model is clearly globally ill-defined. However, there exists an equivalent Tate model with WSF identical to the one of (B.14). It reads

$$y^2 + s_6xyz = x(x^2 + s_1s_9^2s_3z^4 - s_2s_9xz^2). \quad (\text{B.15})$$

This is precisely of the form of an elliptic curve with \mathbb{Z}_2 MW-torsion given in [53], after the shift $y \rightarrow y - \frac{1}{2}s_6xz$. Thus, the MW-group is indeed \mathbb{Z}_2 , in agreement with the results of [41, 46].

The following analysis is presented in the patch $z = 1$ without loss of generality. The Weierstrass equation of (B.15) takes the form

$$y^2 = \frac{1}{864}(s_6^2 - 4s_2s_9 - 12x)(s_6^4 - 8s_2s_6^2s_9 + 16s_2^2s_9^2 - 72s_1s_3s_9^2 - 6s_6^2x + 24s_2s_9x - 72x^2). \quad (\text{B.16})$$

The coordinates of the section of order two in (B.15) are

$$\left\{ x = \frac{1}{12}(s_6^2 - 4s_2s_9), y = 0 \right\}. \quad (\text{B.17})$$

At the loci of codimension one singularities, $s_1 = 0$ and $s_3 = 0$, the WSF takes the form

$$y^2 = \frac{1}{864}(s_6^2 - 4s_2s_9 - 12x)^2(s_6^2 - 4s_2s_9 + 6x), \quad (\text{B.18})$$

and on the locus $s_9 = 0$ it reads

$$y^2 = \frac{1}{864}(s_6^2 - 12x)^2(s_6^2 + 6x), \quad (\text{B.19})$$

showing that the A_4 - and both A_2 -singularities at codimension one in the fibration are located exactly at the point of order two. This implies that the \mathbb{Z}_2 associated to the torsional section acts on all non-Abelian gauge group factors, rendering the gauge group

$$G_{F_{13}} = (\text{SU}(4) \times \text{SU}(2)^2)/\mathbb{Z}_2. \quad (\text{B.20})$$

Tate form, WSF and the MW-torsion of $X_{F_{15}}$

Here, we determine two Tate forms and the WSF of $X_{F_{15}}$. The explicit WS-coordinates of its torsional section allow us to show that the MW-torsion acts on all codimension one singularities in $X_{F_{15}}$, i.e. on all non-Abelian gauge group factors in $G_{F_{15}}$.

We apply the birational map in [39] for $s_1=s_3=s_8=0$ to (3.189) to obtain the Tate coefficients

$$a_1 = \frac{s_6^2 - 2s_5s_7 + 2s_2s_9}{s_6}, \quad a_2 = \frac{s_2s_6^2s_9 - (s_5s_7 - s_2s_9)^2}{s_6^2}, \quad a_3 = s_2s_6s_9, \quad a_4 = a_6 = 0. \quad (\text{B.21})$$

Clearly this Tate form has poles at $s_6 = 0$ and is ill-defined. Fortunately, there exists an equivalent Tate model that has the same Weierstrass equation as (B.21). It reads

$$y^2 + s_6xyz = x(x^2 - (s_7s_5 + s_2s_9)xz^2 + s_2s_7s_5s_9z^4). \quad (\text{B.22})$$

Now we see that this elliptic curve is precisely of the form of the elliptic curve with a $\mathbb{Z}_2 \oplus \mathbb{Z}$ MW-group studied in [53]. This result agrees with the findings in [41, 46]. We note that the Tate coefficients (B.21) can be parametrized as $a_1 = \gamma_1$, $a_2 = -(\gamma_2 + \delta_2)$, $a_4 = \gamma_2 \delta_2$ according to [46], where we obtain $\gamma_1 = s_6$, $\gamma_2 = s_3 s_8$ and $\delta_2 = s_2 s_9$.

Let us work in the patch $z = 1$ without loss of generality. The WSF of (B.21), (B.22) reads

$$y^2 = \frac{1}{864}(s_6^2 - 4s_5 s_7 - 4s_2 s_9 - 12x)(s_6^4 - 8s_5 s_6^2 s_7 + 16s_5^2 s_7^2 - 8s_2 s_6^2 s_9 - 40s_2 s_5 s_7 s_9 + 16s_2^2 s_9^2 - 6s_6^2 x + 24s_5 s_7 x + 24s_2 s_9 x - 72x^2), \quad (\text{B.23})$$

in which the coordinates of the point of order two are

$$\left\{ x = \frac{1}{12}(s_6^2 - 4s_5 s_7 - 4s_2 s_9), y = 0 \right\}. \quad (\text{B.24})$$

At $s_2 = 0$ and $s_9 = 0$, the location of two $\text{SU}(2)$ singularities, the WSF simplifies to

$$y^2 = \frac{1}{864}(s_6^2 - 4s_5 s_7 - 12x)^2 (s_6^2 - 4s_5 s_7 + 6x), \quad (\text{B.25})$$

and at $s_5 = 0$ and $s_7 = 0$, the location of the two other $\text{SU}(2)$'s, it reads

$$y^2 = \frac{1}{864}(s_6^2 - 4s_2 s_9 - 12x)^2 (s_6^2 - 4s_2 s_9 + 6x). \quad (\text{B.26})$$

Thus, the section of order two goes through all A_2 -singularities, which implies that the \mathbb{Z}_2 acts on all non-Abelian gauge group factors, i.e.

$$G_{F_{15}} = (\text{SU}(2)^4) / \mathbb{Z}_2 \times \text{U}(1). \quad (\text{B.27})$$

Tate form, WSF and the MW-torsion of $X_{F_{16}}$

Here, we compute a Tate form and the WSF of $X_{F_{16}}$. Using the explicit WS-coordinates of the \mathbb{Z}_3 MW-generator, we show that the MW-torsion acts on the entire gauge group $G_{F_{16}}$.

The Tate form of the hypersurface equation (3.202) is obtained employing the birational map of [39] for $s_2 = s_3 = s_5 = s_8 = 0$:

$$y^2 + s_6 x y z - s_1 s_7 s_9 z^3 = x^3. \quad (\text{B.28})$$

This is the normal form of an elliptic curve with \mathbb{Z}_3 torsion [53], in agreement with [41, 46].

We work in the patch $z = 1$ in the following, without loss of generality. The WSF reads

$$y^2 - \left(\frac{s_1 s_7 s_9}{2} \right)^2 = \frac{1}{864}(s_6^2 - 12x)(s_6^4 + 36s_1 s_6 s_7 s_9 - 6s_6^2 x - 72x^2), \quad (\text{B.29})$$

from which we obtain the WS-coordinates of the order three section as

$$\left\{ x = \frac{1}{12} s_6^2, y = \frac{s_1 s_7 s_9}{2} \right\}. \quad (\text{B.30})$$

This section passes through the A_3 singularities as we see from (B.29) at $s_1 = 0$, $s_7 = 0$, $s_9 = 0$:

$$y^2 = \frac{1}{864}(s_6^2 - 12x)^2 (s_6^2 + 6x). \quad (\text{B.31})$$

Thus, the \mathbb{Z}_3 acts on all $\text{SU}(3)$'s and the gauge group is

$$G_{F_{16}} = \text{SU}(3)^3 / \mathbb{Z}_3. \quad (\text{B.32})$$

C Euler numbers of the Calabi-Yau threefolds X_{F_i}

In this section we present the explicit expressions for the Euler numbers of all Calabi-Yau threefolds X_{F_i} that are constructed as toric hypersurface fibrations over an arbitrary two-fold base B with their fibrations parametrized by two divisors \mathcal{S}_7 and \mathcal{S}_9 , see section 3.1.1.

The Euler numbers are computed using the presentation of the vertical cohomology ring of X_{F_i} as a quotient ring in its divisors, see [48, 112], and the adjunction formula. For a detailed explanation in an F-theory context and many explicit examples, we refer the reader to [42].

The following table contains our results for the Euler numbers of X_{F_i} , where we denote the first Chern class of the base B by c_1 , implicitly invoke Poincaré duality between divisors and forms and suppress the integral over B :

| Manifold | Euler number $\chi(X_{F_i})$ |
|--------------|---|
| X_{F_1} | $-6(4c_1^2 - c_1\mathcal{S}_7 + \mathcal{S}_7^2 - c_1\mathcal{S}_9 - \mathcal{S}_7\mathcal{S}_9 + \mathcal{S}_9^2)$ |
| X_{F_2} | $-4(6c_1^2 - 2c_1\mathcal{S}_7 + \mathcal{S}_7^2 - 2c_1\mathcal{S}_9 + \mathcal{S}_9^2)$ |
| X_{F_3} | $-2(12c_1^2 - 3c_1\mathcal{S}_7 + 3\mathcal{S}_7^2 - 4c_1\mathcal{S}_9 - 2\mathcal{S}_7\mathcal{S}_9 + 2\mathcal{S}_9^2)$ |
| X_{F_4} | $-4(6c_1^2 - 2c_1\mathcal{S}_7 + 3\mathcal{S}_7^2 - 2c_1\mathcal{S}_9 - 2\mathcal{S}_7\mathcal{S}_9 + \mathcal{S}_9^2)$ |
| X_{F_5} | $-2(12c_1^2 - 4c_1\mathcal{S}_7 + 2\mathcal{S}_7^2 - 4c_1\mathcal{S}_9 - \mathcal{S}_7\mathcal{S}_9 + 2\mathcal{S}_9^2)$ |
| X_{F_6} | $-2(12c_1^2 - 4c_1\mathcal{S}_7 + 4\mathcal{S}_7^2 - 4c_1\mathcal{S}_9 - 3\mathcal{S}_7\mathcal{S}_9 + 2\mathcal{S}_9^2)$ |
| X_{F_7} | $-4(4c_1^2 + \mathcal{S}_7^2 - \mathcal{S}_7\mathcal{S}_9 + \mathcal{S}_9^2 - c_1(\mathcal{S}_7 + \mathcal{S}_9))$ |
| X_{F_8} | $-2(12c_1^2 - 5c_1\mathcal{S}_7 + 3\mathcal{S}_7^2 - 4c_1\mathcal{S}_9 - 2\mathcal{S}_7\mathcal{S}_9 + 2\mathcal{S}_9^2)$ |
| X_{F_9} | $-4(6c_1^2 - 2c_1\mathcal{S}_7 + \mathcal{S}_7^2 - 3c_1\mathcal{S}_9 + \mathcal{S}_9^2)$ |
| $X_{F_{10}}$ | $-6(4c_1^2 - 2c_1\mathcal{S}_7 + 2\mathcal{S}_7^2 - c_1\mathcal{S}_9 - 2\mathcal{S}_7\mathcal{S}_9 + \mathcal{S}_9^2)$ |
| $X_{F_{11}}$ | $-2(12c_1^2 - 4c_1\mathcal{S}_7 + 2\mathcal{S}_7^2 - 7c_1\mathcal{S}_9 - \mathcal{S}_7\mathcal{S}_9 + 3\mathcal{S}_9^2)$ |
| $X_{F_{12}}$ | $-2(12c_1^2 - 6c_1\mathcal{S}_7 + 2\mathcal{S}_7^2 - 6c_1\mathcal{S}_9 + \mathcal{S}_7\mathcal{S}_9 + 2\mathcal{S}_9^2)$ |
| $X_{F_{13}}$ | $-4(6c_1^2 - 2c_1\mathcal{S}_7 + \mathcal{S}_7^2 - 5c_1\mathcal{S}_9 + 2\mathcal{S}_9^2)$ |
| $X_{F_{14}}$ | $-2(12c_1^2 - 9c_1\mathcal{S}_7 + 3\mathcal{S}_7^2 - 6c_1\mathcal{S}_9 + 2\mathcal{S}_7\mathcal{S}_9 + 2\mathcal{S}_9^2)$ |
| $X_{F_{15}}$ | $-4(4c_1^2 - 2c_1\mathcal{S}_7 + \mathcal{S}_7^2 - 2c_1\mathcal{S}_9 + \mathcal{S}_9^2)$ |
| $X_{F_{16}}$ | $-6(4c_1^2 - 3c_1\mathcal{S}_7 + \mathcal{S}_7^2 - 3c_1\mathcal{S}_9 + \mathcal{S}_7\mathcal{S}_9 + \mathcal{S}_9^2)$ |

(C.1)

D The full Higgs chain of toric hypersurface fibrations

The complete Higgs chain of all toric Higgs transition $X_{F_i} \rightarrow X_j$, the relevant hyper multiplets acquiring VEVs, the U(1)-generators and the necessary redefinitions of divisor classes are shown in tables 37, 38 and 39.

| Higgs transition | VEV | U(1) Generators | Divisor Class Matching |
|-------------------------------------|--|--|--|
| $X_{F_{16}} \rightarrow X_{F_{14}}$ | $(\mathbf{3}, \mathbf{1}, \bar{\mathbf{3}})$ | $Q = -(T_1^8 + T_3^8)$ | trivial |
| | $(\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}})$ | $Q = -(T_2^8 + T_3^8)$ | $\mathcal{S}_7 \rightarrow \mathcal{S}_9$ $\mathcal{S}_9 \rightarrow 3[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_9$ |
| | $(\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1})$ | $Q = -(T_1^8 + T_2^8)$ | $\mathcal{S}_7 \rightarrow \mathcal{S}_9$ $\mathcal{S}_9 \rightarrow \mathcal{S}_7$ |
| $X_{F_{15}} \rightarrow X_{F_{12}}$ | $(\mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1})_{(1/2)}$ | $Q'_1 = T_2^3 + Q$ $Q'_2 = T_1^3 + Q$ | trivial |
| | $(\mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1})_{(1/2)}$ | $Q'_1 = T_3^3 + Q$ $Q'_2 = T_1^3 + Q$ | $\mathcal{S}_7 \rightarrow 2[K_B^{-1}] - \mathcal{S}_7$ $\mathcal{S}_9 \rightarrow \mathcal{S}_7$ |
| | $(\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2})_{(1/2)}$ | $Q'_1 = T_4^3 + Q$ $Q'_2 = T_3^3 + Q$ | $\mathcal{S}_7 \rightarrow 2[K_B^{-1}] - \mathcal{S}_9$ $\mathcal{S}_9 \rightarrow 2[K_B^{-1}] - \mathcal{S}_7$ |
| | $(\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{2})_{(1/2)}$ | $Q'_1 = T_4^3 + Q$ $Q'_2 = T_2^3 + Q$ | $\mathcal{S}_7 \rightarrow \mathcal{S}_9$ $\mathcal{S}_9 \rightarrow 2[K_B^{-1}] - \mathcal{S}_7$ |
| $X_{F_{14}} \rightarrow X_{F_{12}}$ | $(\mathbf{2}, \mathbf{3}, \mathbf{1})_{1/6}$ | $Q'_1 = T^8 + 2Q$ $Q'_2 = 2T^8 - T^3 + Q$ | trivial |
| | $(\mathbf{1}, \mathbf{3}, \mathbf{2})_{1/6}$ | $Q'_1 = 2T^8 - T^3 + Q$ $Q'_1 = T^8 + 2Q$ | $\mathcal{S}_7 \rightarrow \mathcal{S}_9$ $\mathcal{S}_9 \rightarrow 3[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_9$ |
| $X_{F_{14}} \rightarrow X_{F_{11}}$ | $(\mathbf{1}, \mathbf{1}, \mathbf{2})_{1/2}$ | $Q' = Q_1 - T^3$ | $\mathcal{S}_7 \rightarrow \mathcal{S}_9$ $\mathcal{S}_9 \rightarrow 2[K_B^{-1}] - \mathcal{S}_7$ |
| | $(\mathbf{2}, \mathbf{1}, \mathbf{1})_{1/2}$ | $Q' = Q - T^3$ | $\mathcal{S}_7 \rightarrow \mathcal{S}_9$ $\mathcal{S}_9 \rightarrow 2[K_B^{-1}] - \mathcal{S}_7$ |
| $X_{F_{13}} \rightarrow X_{F_{11}}$ | $(\mathbf{2}, \mathbf{1}, \mathbf{4})$ | $Q' = T^{15} - T^3$ | trivial |
| | $(\mathbf{1}, \mathbf{2}, \mathbf{4})$ | $Q' = T^{15} - T^3$ | $\mathcal{S}_7 \rightarrow 2[K_B^{-1}] - \mathcal{S}_7$ $\mathcal{S}_9 \rightarrow \mathcal{S}_9$ |
| $X_{F_{12}} \rightarrow X_{F_9}$ | $(\mathbf{2}, \mathbf{1})_{(-1/2, -1)}$ | $Q'_1 = Q_1 - T^3$ $Q'_2 = Q_2 - 2T^3$ | trivial |
| | $(\mathbf{1}, \mathbf{2})_{(-1, -1/2)}$ | $Q'_1 = 2T^3 - Q_1$ $Q'_2 = T^3 - Q_2$ | $\mathcal{S}_7 \rightarrow \mathcal{S}_9$ $\mathcal{S}_9 \rightarrow 2[K_B^{-1}] - \mathcal{S}_7$ |
| $X_{F_{12}} \rightarrow X_{F_8}$ | $(\mathbf{1}, \mathbf{1})_{(1,0)}$ | $Q' = Q_2$ | $\mathcal{S}_7 \rightarrow \mathcal{S}_7$ $\mathcal{S}_9 \rightarrow [K_B^{-1}] - \mathcal{S}_7 + \mathcal{S}_9$ |
| | $(\mathbf{1}, \mathbf{1})_{(0,1)}$ | $Q' = Q_1, SU(2)'_1 = SU(2)^2$ $SU(2)'_2 = SU(2)^1$ | $\mathcal{S}_7 \rightarrow [K_B^{-1}] - \mathcal{S}_7 + \mathcal{S}_9$ $\mathcal{S}_9 \rightarrow \mathcal{S}_7$ |

Table 37. Data of toric Higgs transitions for F-theory compactified on $X_{F_{16}} - X_{F_{12}}$.

| Higgs transition | VEV | U(1) Generators | Divisor Class Matching |
|-------------------------------------|---|--|---|
| $X_{F_{12}} \rightarrow X_{F_7}$ | $(\mathbf{2}, \mathbf{2})_{(\frac{1}{2}, \frac{1}{2})}$ | $Q'_1 = 2Q_2 - Q_1 + T_1^3$ $Q'_2 = Q_1 - Q_2 - T_1^3 - T_2^3$ $Q'_3 = Q_2 + 2T_1^3 + T_2^3$ | $\mathcal{S}_7 \rightarrow \mathcal{S}_9$ $\mathcal{S}_9 \rightarrow 2[K_B^{-1}] - \mathcal{S}_7$ |
| $X_{F_{11}} \rightarrow X_{F_{10}}$ | $(\mathbf{1}, \mathbf{1})_{-1}$ | - | $\mathcal{S}_7 \rightarrow \mathcal{S}_7$ $\mathcal{S}_9 \rightarrow [K_B^{-1}] - \mathcal{S}_7 + \mathcal{S}_9$ |
| $X_{F_{11}} \rightarrow X_{F_9}$ | $(\mathbf{3}, \mathbf{2})_{-\frac{1}{6}}$ | $Q'_1 = Q - 2T_1^8 + T_2^3$ $Q'_2 = 2T_2^3 - 3T_1^8$ | trivial |
| $X_{F_{11}} \rightarrow X_{F_8}$ | $(\mathbf{3}, \mathbf{1})_{\frac{1}{3}}$ | $Q' = 2T^8 - Q$ | $\mathcal{S}_7 \rightarrow \mathcal{S}_9$ $\mathcal{S}_9 \rightarrow \mathcal{S}_7$ |
| $X_{F_{10}} \rightarrow X_{F_6}$ | $(\mathbf{3}, \mathbf{2})$ | $Q' = T^8 - 2T^3$ | trivial |
| $X_{F_9} \rightarrow X_{F_6}$ | $\mathbf{1}_{(1,2)}$ | $Q' = 2Q_1 - Q_2$ | $\mathcal{S}_7 \rightarrow 2[K_B^{-1}] - \mathcal{S}_7$ $\mathcal{S}_9 \rightarrow [K_B^{-1}] - \mathcal{S}_7 + \mathcal{S}_9$ |
| | $\mathbf{1}_{(1,0)}$ | $Q' = Q_2$ | $\mathcal{S}_7 \rightarrow \mathcal{S}_7$ $\mathcal{S}_9 \rightarrow [K_B^{-1}] - \mathcal{S}_7 + \mathcal{S}_9$ |
| $X_{F_9} \rightarrow X_{F_5}$ | $\mathbf{2}_{(-1, -\frac{1}{2})}$ | $Q'_1 = Q_2 - Q_1 + T^3$ $Q'_2 = -Q_1 + 2T^3$ | trivial |
| | $\mathbf{2}_{(1, \frac{3}{2})}$ | $Q'_1 = Q_2 - Q_1 + T^3$ $Q'_2 = Q_1 + 2T^3$ | $\mathcal{S}_7 \rightarrow 2[K_B^{-1}] - \mathcal{S}_7$ $\mathcal{S}_9 \rightarrow \mathcal{S}_9$ |
| $X_{F_8} \rightarrow X_{F_6}$ | $(\mathbf{2}, \mathbf{1})_1$ | $Q' = (Q + 2T_1^3)$ | trivial |
| $X_{F_8} \rightarrow X_{F_5}$ | $(\mathbf{2}, \mathbf{2})_{1/2}$ | $Q'_1 = Q - T_2^1$ $Q'_2 = 2(T_3^1 - T_3^2)$ | trivial |
| $X_{F_7} \rightarrow X_{F_5}$ | $\mathbf{1}_{(0, -1, 0)}$ | $Q'_1 = -Q_1 + Q_3$ $Q'_2 = Q_3$ | trivial |
| | $\mathbf{1}_{(1, 1, 0)}$ | $Q'_1 = Q_1 - Q_2$ $Q'_2 = Q_3$ | $\mathcal{S}_7 \rightarrow [K_B^{-1}] - \mathcal{S}_7 + \mathcal{S}_9$ $\mathcal{S}_9 \rightarrow \mathcal{S}_9$ |
| | $\mathbf{1}_{(2, 1, 1)}$ | $Q'_1 = Q_2 - Q_3$ $Q'_2 = Q_1 - Q_2 - Q_3$ | $\mathcal{S}_7 \rightarrow 2[K_B^{-1}] - \mathcal{S}_7$ $\mathcal{S}_9 \rightarrow [K_B^{-1}] - \mathcal{S}_7 + \mathcal{S}_9$ |
| | $\mathbf{1}_{(0, 1, 1)}$ | $Q'_1 = -Q_1$ $Q'_2 = Q_2 - Q_3$ | $\mathcal{S}_7 \rightarrow \mathcal{S}_9$ $\mathcal{S}_9 \rightarrow [K_B^{-1}] - \mathcal{S}_7 + \mathcal{S}_9$ |
| | $\mathbf{1}_{(-2, -1, -2)}$ | $Q'_1 = Q_1 - Q_3$ $Q'_2 = Q_1 - 2Q_2$ | $\mathcal{S}_7 \rightarrow 2[K_B^{-1}] - \mathcal{S}_9$ $\mathcal{S}_9 \rightarrow 2[K_B^{-1}] - \mathcal{S}_7$ |
| | $\mathbf{1}_{(1, 1, 2)}$ | $Q'_1 = Q_1 - Q_2$ $Q'_2 = Q_1 + Q_2 - 2Q_3$ | $\mathcal{S}_7 \rightarrow [K_B^{-1}] - \mathcal{S}_7 + \mathcal{S}_9$ $\mathcal{S}_9 \rightarrow 2[K_B^{-1}] - \mathcal{S}_7$ |

Table 38. Data of toric Higgs transitions for F-theory compactified on $X_{F_{12}} - X_{F_7}$.

| Higgs transition | VEV | U(1) Generators | Divisor Class Matching |
|-------------------------------|-----------------------------|---|--|
| $X_{F_6} \rightarrow X_{F_4}$ | $\mathbf{1}_2$ | $Q'_{\mathbb{Z}_4} = 2Q \pmod{4}$ | trivial |
| $X_{F_6} \rightarrow X_{F_3}$ | $\mathbf{2}_{-\frac{3}{2}}$ | $Q' = 3T^3 - Q$ | trivial |
| $X_{F_5} \rightarrow X_{F_3}$ | $\mathbf{1}_{(-1,-2)}$ | $Q'_1 = 2Q_1 - Q_2$ | $\mathcal{S}_7 \rightarrow \mathcal{S}_9$ $\mathcal{S}_9 \rightarrow \mathcal{S}_7$ |
| | $\mathbf{1}_{(-1,1)}$ | $Q'_1 = Q_1 + Q_2$ | trivial |
| $X_{F_5} \rightarrow X_{F_2}$ | $\mathbf{1}_{(0,-2)}$ | $Q'_1 = Q_1$ $Q_{\mathbb{Z}_2} = Q_2 \pmod{2}$ | trivial |
| $X_{F_3} \rightarrow X_{F_1}$ | $\mathbf{1}_3$ | $Q'_{\mathbb{Z}_3} = Q \pmod{3}$ | trivial |

Table 39. Data of toric Higgs transitions for F-theory compactified on $X_{F_6} - X_{F_3}$.

E Group theoretical decomposition of representations

In tables 40, 41, 42, 43, 44 and 45 we show the explicit decompositions of representations of the group G_{F_i} into representations of the unbroken group G_{F_j} for each Higgs transition $X_{F_i} \rightarrow X_{F_j}$.

| Breaking | Starting Multiplets | Target Multiplets |
|--|--|---|
| $X_{F_{16}} \rightarrow X_{F_{14}}$ VEV: $(\mathbf{3}, \mathbf{1}, \bar{\mathbf{3}})$ | $(\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1})$ | $(\mathbf{2}, \bar{\mathbf{3}}, \mathbf{1})_{-1/6} + (\mathbf{1}, \bar{\mathbf{3}}, \mathbf{1})_{1/3}$ |
| | $(\mathbf{3}, \mathbf{1}, \bar{\mathbf{3}})$ | $(\mathbf{2}, \mathbf{1}, \mathbf{2})_0 + (\mathbf{2}, \mathbf{1}, \mathbf{1})_{1/2}$ $+ (\mathbf{2}, \mathbf{1}, \mathbf{1})_{-1/2} + (\mathbf{1}, \mathbf{1}, \mathbf{1})_0$ |
| | $(\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}})$ | $(\mathbf{1}, \mathbf{3}, \mathbf{2})_{1/6} + (\mathbf{1}, \mathbf{3}, \mathbf{1})_{-1/3}$ |
| | $(\mathbf{8}, \mathbf{1}, \mathbf{1})$ | $(\mathbf{3}, \mathbf{1}, \mathbf{1})_0 + (\mathbf{2}, \mathbf{1}, \mathbf{1})_{1/2}$ $+ (\mathbf{2}, \mathbf{1}, \mathbf{1})_{-1/2} + (\mathbf{1}, \mathbf{1}, \mathbf{1})_0$ |
| | $(\mathbf{1}, \mathbf{8}, \mathbf{1})$ | $(\mathbf{1}, \mathbf{8}, \mathbf{1})_0$ |
| | $(\mathbf{1}, \mathbf{1}, \mathbf{8})$ | $(\mathbf{1}, \mathbf{1}, \mathbf{3})_0 + (\mathbf{1}, \mathbf{1}, \mathbf{2})_{1/2}$ $+ (\mathbf{1}, \mathbf{1}, \mathbf{2})_{-1/2} + (\mathbf{1}, \mathbf{1}, \mathbf{1})_0$ |

Table 40. Group theoretical decompositions of representation in toric Higgsings of $X_{F_{16}}$.

| Breaking | Starting Multiplets | Target Multiplets |
|--|--|--|
| $X_{F_{15}} \rightarrow X_{F_{12}}$ VEV: $(\mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1})_{(1/2)}$ | $(\mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1})_{\frac{1}{2}}$ $(\mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1})_{\frac{1}{2}}$ $(\mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{2})_0$ $(\mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1})_0$ $(\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2})_{\frac{1}{2}}$ $(\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{2})_{\frac{1}{2}}$ $(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})_1$ $(\mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{1})_0$ $(\mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1})_0$ $(\mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{1})_0$ $(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3})_0$ | $(\mathbf{1}, \mathbf{1})_{(1,1)} + (\mathbf{1}, \mathbf{1})_{(1,0)}$ $+ (\mathbf{1}, \mathbf{1})_{(0,1)} + (\mathbf{1}, \mathbf{1})_{(0,0)}$ $(\mathbf{2}, \mathbf{1})_{(\frac{1}{2},1)} + (\mathbf{2}, \mathbf{1})_{(\frac{1}{2},0)}$ $(\mathbf{1}, \mathbf{2})_{(0,\frac{1}{2})} + (\mathbf{1}, \mathbf{2})_{(0,-\frac{1}{2})}$ $(\mathbf{2}, \mathbf{1})_{(\frac{1}{2},0)} + (\mathbf{2}, \mathbf{1})_{(-\frac{1}{2},0)}$ $(\mathbf{2}, \mathbf{2})_{(\frac{1}{2},\frac{1}{2})}$ $(\mathbf{1}, \mathbf{2})_{(1,\frac{1}{2})} + (\mathbf{1}, \mathbf{2})_{(0,\frac{1}{2})}$ $(\mathbf{1}, \mathbf{1})_{(1,1)}$ $(\mathbf{1}, \mathbf{1})_{(0,1)} + (\mathbf{1}, \mathbf{1})_{(0,-1)} + (\mathbf{1}, \mathbf{1})_{(0,0)}$ $(\mathbf{1}, \mathbf{1})_{(1,0)} + (\mathbf{1}, \mathbf{1})_{(-1,0)} + (\mathbf{1}, \mathbf{1})_{(0,0)}$ $(\mathbf{3}, \mathbf{1})_{(0,0)}$ $(\mathbf{1}, \mathbf{3})_{(0,0)}$ |
| $X_{F_{14}} \rightarrow X_{F_{12}}$ VEV: $(\mathbf{2}, \mathbf{3}, \mathbf{1})_{1/6}$ | $(\mathbf{2}, \mathbf{1}, \mathbf{1})_{1/2}$ $(\mathbf{1}, \mathbf{3}, \mathbf{1})_{1/3}$ $(\mathbf{1}, \mathbf{1}, \mathbf{2})_{1/2}$ $(\mathbf{2}, \mathbf{3}, \mathbf{1})_{1/6}$ $(\mathbf{2}, \mathbf{1}, \mathbf{2})_0$ $(\mathbf{1}, \mathbf{3}, \mathbf{2})_{1/6}$ $(\mathbf{3}, \mathbf{1}, \mathbf{1})_0$ $(\mathbf{1}, \mathbf{8}, \mathbf{1})_0$ $(\mathbf{1}, \mathbf{1}, \mathbf{3})_0$ | $(\mathbf{1}, \mathbf{1})_{(1,0)} + (\mathbf{1}, \mathbf{1})_{(1,1)}$ $(\mathbf{2}, \mathbf{1})_{(-1/2,0)} + (\mathbf{1}, \mathbf{1})_{(-1,-1)}$ $(\mathbf{1}, \mathbf{2})_{(1,1/2)}$ $(\mathbf{2}, \mathbf{1})_{(1/2,0)} + (\mathbf{2}, \mathbf{1})_{(1/2,1)}$ $+ (\mathbf{1}, \mathbf{1})_{(0,-1)} + (\mathbf{1}, \mathbf{1})_{(0,0)}$ $(\mathbf{1}, \mathbf{2})_{(0,-1/2)} + (\mathbf{1}, \mathbf{2})_{(0,1/2)}$ $(\mathbf{2}, \mathbf{2})_{(1/2,1/2)} + (\mathbf{1}, \mathbf{2})_{(0,-1/2)}$ $(\mathbf{1}, \mathbf{1})_{(0,0)} + (\mathbf{1}, \mathbf{1})_{(0,1)} + (\mathbf{1}, \mathbf{1})_{(0,-1)}$ $(\mathbf{3}, \mathbf{1})_{(0,0)} + (\mathbf{1}, \mathbf{1})_{(0,0)}$ $+ (\mathbf{2}, \mathbf{1})_{(1/2,1)} + (\mathbf{2}, \mathbf{1})_{(-1/2,-1)}$ $(\mathbf{1}, \mathbf{3})_{(0,0)}$ |
| $X_{F_{14}} \rightarrow X_{F_{11}}$ VEV: $(\mathbf{1}, \mathbf{1}, \mathbf{2})_{1/2}$ | $(\mathbf{2}, \mathbf{1}, \mathbf{1})_{1/2}$ $(\mathbf{1}, \mathbf{3}, \mathbf{1})_{1/3}$ $(\mathbf{1}, \mathbf{1}, \mathbf{2})_{1/2}$ $(\mathbf{2}, \mathbf{3}, \mathbf{1})_{1/2}$ $(\mathbf{2}, \mathbf{1}, \mathbf{2})_0$ $(\mathbf{1}, \mathbf{3}, \mathbf{2})_{1/6}$ $(\mathbf{1}, \mathbf{1}, \mathbf{3})_0$ | $(\mathbf{2}, \mathbf{1})_{1/2}$ $(\mathbf{1}, \mathbf{3})_{1/3}$ $(\mathbf{1}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{1})_1$ $(\mathbf{2}, \mathbf{3})_{1/6}$ $(\mathbf{1}, \mathbf{2})_{1/2} + (\mathbf{1}, \mathbf{2})_{-1/2}$ $(\mathbf{3}, \mathbf{1})_{2/3} + (\mathbf{3}, \mathbf{1})_{-1/3}$ $(\mathbf{1}, \mathbf{1})_1 + (\mathbf{1}, \mathbf{1})_{-1}$ |

Table 41. Group theoretical decompositions of representation in toric Higgsings of $X_{F_{15}} - X_{F_{14}}$.

| Breaking | Starting Multiplets | Target Multiplets |
|--|--|--|
| $X_{F_{13}} \rightarrow X_{F_{11}}$ VEV: $(\mathbf{2}, \mathbf{1}, \mathbf{4})$ | $(\mathbf{2}, \mathbf{2}, \mathbf{1})$ $(\mathbf{2}, \mathbf{1}, \mathbf{4})$ $(\mathbf{1}, \mathbf{1}, \mathbf{6})$ $(\mathbf{1}, \mathbf{2}, \mathbf{4})$ $(\mathbf{1}, \mathbf{1}, \mathbf{15})$ $(\mathbf{1}, \mathbf{3}, \mathbf{1})$ $(\mathbf{3}, \mathbf{1}, \mathbf{1})$ | $(\mathbf{2}, \mathbf{1})_{-1/2} + (\mathbf{2}, \mathbf{1})_{1/2}$ $(\mathbf{1}, \mathbf{1})_{-1} + (\mathbf{1}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{3})_{-1/3} + (\mathbf{1}, \mathbf{3})_{2/3}$ $(\mathbf{1}, \mathbf{3})_{-1/3} + (\mathbf{1}, \bar{\mathbf{3}})_{1/3}$ $(\mathbf{2}, \mathbf{3})_{1/6} + (\mathbf{2}, \mathbf{1})_{-1/2}$ $(\mathbf{1}, \mathbf{8})_0 + (\mathbf{1}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{3})_{2/3} + (\mathbf{1}, \bar{\mathbf{3}})_{-2/3}$ $(\mathbf{3}, \mathbf{1})_0$ $(\mathbf{1}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{1})_{-1} + (\mathbf{1}, \mathbf{1})_1$ |
| $X_{F_{12}} \rightarrow X_{F_9}$ VEV: $(\mathbf{2}, \mathbf{1})_{(-1/2, -1)}$ | $(\mathbf{2}, \mathbf{2})_{(1/2, 1/2)}$ $(\mathbf{1}, \mathbf{2})_{(-1, -1/2)}$ $(\mathbf{2}, \mathbf{1})_{(-1/2, -1)}$ $(\mathbf{1}, \mathbf{1})_{(1, 0)}$ $(\mathbf{1}, \mathbf{1})_{(0, 1)}$ $(\mathbf{1}, \mathbf{2})_{(0, -1/2)}$ $(\mathbf{2}, \mathbf{1})_{(-1/2, 0)}$ $(\mathbf{1}, \mathbf{1})_{(1, 1)}$ $(\mathbf{1}, \mathbf{3})_{(0, 0)}$ $(\mathbf{3}, \mathbf{1})_{(0, 0)}$ | $\mathbf{2}_{(0, -1/2)} + \mathbf{2}_{(1, 3/2)}$ $\mathbf{2}_{(-1, -1/2)}$ $\mathbf{1}_{(-1, -2)} + \mathbf{1}_{(0, 0)}$ $\mathbf{1}_{(1, 0)}$ $\mathbf{1}_{(0, 1)}$ $\mathbf{2}_{(0, -1/2)}$ $\mathbf{1}_{(-1, -1)} + \mathbf{1}_{(0, 1)}$ $\mathbf{1}_{(1, 1)}$ $\mathbf{3}_{(0, 0)}$ $\mathbf{1}_{(-1, -2)} + \mathbf{1}_{(1, 2)} + \mathbf{1}_{(0, 0)}$ |
| $X_{F_{12}} \rightarrow X_{F_8}$ VEV: $(\mathbf{1}, \mathbf{1})_{(1, 0)}$ | $(\mathbf{2}, \mathbf{2})_{(1/2, 1/2)}$ $(\mathbf{1}, \mathbf{2})_{(-1, -1/2)}$ $(\mathbf{2}, \mathbf{1})_{(-1/2, -1)}$ $(\mathbf{1}, \mathbf{1})_{(1, 0)}$ $(\mathbf{1}, \mathbf{1})_{(0, 1)}$ $(\mathbf{1}, \mathbf{2})_{(0, -1/2)}$ $(\mathbf{2}, \mathbf{1})_{(-1/2, 0)}$ $(\mathbf{1}, \mathbf{1})_{(1, 1)}$ $(\mathbf{1}, \mathbf{3})_{(0, 0)}$ $(\mathbf{3}, \mathbf{1})_{(0, 0)}$ | $(\mathbf{2}, \mathbf{2})_{1/2}$ $(\mathbf{1}, \mathbf{2})_{-1/2}$ $(\mathbf{2}, \mathbf{1})_{-1}$ $(\mathbf{1}, \mathbf{1})_0$ $(\mathbf{1}, \mathbf{1})_1$ $(\mathbf{1}, \mathbf{2})_{-1/2}$ $(\mathbf{2}, \mathbf{1})_0$ $(\mathbf{1}, \mathbf{1})_1$ $(\mathbf{1}, \mathbf{3})_0$ $(\mathbf{3}, \mathbf{1})_0$ |

Table 42. Group theoretical decompositions of representation in toric Higgsings of $X_{F_{13}} - X_{F_{12}}$.

| Breaking | Starting Multiplets | Target Multiplets |
|--|--|--|
| $X_{F_{12}} \rightarrow X_{F_7}$ VEV: $(\mathbf{2}, \mathbf{2})_{(\frac{1}{2}, \frac{1}{2})}$ | $(\mathbf{2}, \mathbf{2})_{(1/2, 1/2)}$ $(\mathbf{1}, \mathbf{2})_{(-1, -1/2)}$ $(\mathbf{2}, \mathbf{1})_{(-1/2, -1)}$ $(\mathbf{1}, \mathbf{1})_{(1, 0)}$ $(\mathbf{1}, \mathbf{1})_{(0, 1)}$ $(\mathbf{1}, \mathbf{2})_{(0, -1/2)}$ $(\mathbf{2}, \mathbf{1})_{(-1/2, 0)}$ $(\mathbf{1}, \mathbf{1})_{(1, 1)}$ $(\mathbf{1}, \mathbf{3})_{(0, 0)}$ $(\mathbf{3}, \mathbf{1})_{(0, 0)}$ | $\mathbf{1}_{(-1, 0, -1)} + \mathbf{1}_{(0, 1, 1)}$ $\mathbf{1}_{(-1, -1, -2)} + \mathbf{1}_{(0, 0, 0)}$ $\mathbf{1}_{(0, 0, 1)} + \mathbf{1}_{(0, -1, 0)}$ $\mathbf{1}_{(1, 0, 0)} + \mathbf{1}_{(-2, -1, -2)}$ $\mathbf{1}_{(1, 1, 0)}$ $\mathbf{1}_{(2, 1, 1)}$ $\mathbf{1}_{(1, 1, 1)} + \mathbf{1}_{(1, 0, 0)}$ $\mathbf{1}_{(0, 0, 1)} + \mathbf{1}_{(1, 1, 1)}$ $\mathbf{1}_{(1, 0, 1)}$ $\mathbf{1}_{(0, 1, 1)} + \mathbf{1}_{(0, -1, -1)}$ $\mathbf{1}_{(1, 1, 2)} + \mathbf{1}_{(-1, -1, -2)}$ |
| $X_{F_{11}} \rightarrow X_{F_{10}}$ | trivial | |
| $X_{F_{11}} \rightarrow X_{F_9}$ VEV: $(\mathbf{3}, \mathbf{2})_{-\frac{1}{6}}$ | $(\mathbf{3}, \mathbf{2})_{-\frac{1}{6}}$ $(\mathbf{1}, \mathbf{2})_{\frac{1}{2}}$ $(\mathbf{3}, \mathbf{1})_{-\frac{2}{3}}$ $(\mathbf{3}, \mathbf{1})_{\frac{1}{3}}$ $(\mathbf{1}, \mathbf{1})_{-1}$ $(\mathbf{8}, \mathbf{1})_0$ $(\mathbf{1}, \mathbf{3})_0$ | $\mathbf{1}_{(0, 0)} + \mathbf{1}_{(1, 2)} + \mathbf{2}_{(0, \frac{1}{2})} + \mathbf{2}_{(-1, -\frac{3}{2})}$ $\mathbf{1}_{(1, 1)} + \mathbf{1}_{(0, -1)}$ $\mathbf{1}_{(0, 1)} + \mathbf{2}_{(-1, -\frac{1}{2})}$ $\mathbf{1}_{(1, 1)} + \mathbf{2}_{(0, -\frac{1}{2})}$ $\mathbf{1}_{(1, 0)}$ $\mathbf{2}_{(1, \frac{3}{2})} + \mathbf{2}_{(-1, -\frac{3}{2})} + \mathbf{1}_{(0, 0)} + \mathbf{3}_{(0, 0)}$ $\mathbf{1}_{(0, 0)} + \mathbf{1}_{(1, 2)} + \mathbf{1}_{(-1, -2)}$ |
| $X_{F_{11}} \rightarrow X_{F_8}$ VEV: $(\mathbf{3}, \mathbf{1})_{\frac{1}{3}}$ | $(\mathbf{3}, \mathbf{2})_{-\frac{1}{6}}$ $(\mathbf{1}, \mathbf{2})_{\frac{1}{2}}$ $(\mathbf{3}, \mathbf{1})_{-\frac{2}{3}}$ $(\mathbf{3}, \mathbf{1})_{\frac{1}{3}}$ $(\mathbf{1}, \mathbf{1})_{-1}$ $(\mathbf{8}, \mathbf{1})_0$ $(\mathbf{1}, \mathbf{3})_0$ | $(\mathbf{2}, \mathbf{2})_{1/2} + (\mathbf{1}, \mathbf{2})_{-1/2}$ $(\mathbf{1}, \mathbf{2})_{-\frac{1}{2}}$ $(\mathbf{1}, \mathbf{1})_{-1} + (\mathbf{2}, \mathbf{1})_0$ $(\mathbf{2}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{1})_0$ $(\mathbf{1}, \mathbf{1})_1$ $(\mathbf{3}, \mathbf{1})_0 + (\mathbf{2}, \mathbf{1})_1 + (\mathbf{2}, \mathbf{1})_{-1} + (\mathbf{1}, \mathbf{1})_0$ $(\mathbf{1}, \mathbf{3})_0$ |
| $X_{F_{10}} \rightarrow X_{F_6}$ VEV: $(\mathbf{3}, \mathbf{2})$ | $(\mathbf{3}, \mathbf{2})$ $(\mathbf{1}, \mathbf{2})$ $(\mathbf{3}, \mathbf{1})$ $(\mathbf{1}, \mathbf{3})$ $(\mathbf{8}, \mathbf{1})$ | $\mathbf{2}_{-1/2} + \mathbf{2}_{3/2} + \mathbf{1}_{-2} + \mathbf{1}_0$ $\mathbf{1}_1 + \mathbf{1}_{-1}$ $\mathbf{1}_{1/2} + \mathbf{1}_{-1}$ $\mathbf{1}_0 + \mathbf{1}_2 + \mathbf{1}_{-2}$ $\mathbf{3}_0 + \mathbf{2}_{3/2} + \mathbf{2}_{-3/2} + \mathbf{1}_0$ |

Table 43. Group theoretical decompositions of representation in toric Higgsings of $X_{F_{12}} - X_{F_{10}}$.

| Breaking | Starting Multiplets | Target Multiplets |
|--|--|---|
| $X_{F_9} \rightarrow X_{F_6}$ VEV: $\mathbf{1}_{(1,2)}$ | $\mathbf{1}_{(1,2)}$ $\mathbf{1}_{(1,0)}$ $\mathbf{1}_{(0,1)}$ $\mathbf{1}_{(1,1)}$ $\mathbf{2}_{(-1,-1/2)}$ $\mathbf{2}_{(1,-3/2)}$ $\mathbf{2}_{(0,-1/2)}$ $\mathbf{3}_{(0,0)}$ | $\mathbf{1}_0$ $\mathbf{1}_2$ $\mathbf{1}_{-1}$ $\mathbf{1}_1$ $\mathbf{2}_{-3/2}$ $\mathbf{2}_{1/2}$ $\mathbf{2}_{-1/2}$ $\mathbf{3}_0$ |
| $X_{F_9} \rightarrow X_{F_5}$ VEV: $\mathbf{2}_{(-1,-1/2)}$ | $\mathbf{1}_{(1,2)}$ $\mathbf{1}_{(1,0)}$ $\mathbf{1}_{(0,1)}$ $\mathbf{1}_{(1,1)}$ $\mathbf{2}_{(-1,-1/2)}$ $\mathbf{2}_{(1,-3/2)}$ $\mathbf{2}_{(0,-1/2)}$ $\mathbf{3}_{(0,0)}$ | $\mathbf{1}_0$ $\mathbf{1}_{(-1,-1)}$ $\mathbf{1}_{(1,0)}$ $\mathbf{1}_{(0,-1)}$ $\mathbf{1}_{(0,0)} + \mathbf{1}_{(1,2)}$ $\mathbf{1}_{(0,-2)} + \mathbf{1}_{(1,0)}$ $\mathbf{1}_{(-1,-1)} + \mathbf{1}_{(0,1)}$ $\mathbf{1}_{(-1,-2)} + \mathbf{1}_{(1,2)} + \mathbf{1}_{(0,0)}$ |
| $X_{F_8} \rightarrow X_{F_6}$ VEV: $(\mathbf{2}, \mathbf{1})_1$ | $(\mathbf{2}, \mathbf{2})_{1/2}$ $(\mathbf{1}, \mathbf{2})_{1/2}$ $(\mathbf{2}, \mathbf{1})_1$ $(\mathbf{2}, \mathbf{1})_0$ $(\mathbf{1}, \mathbf{1})_1$ $(\mathbf{3}, \mathbf{1})_0$ $(\mathbf{1}, \mathbf{3})_0$ | $\mathbf{2}_{\frac{1}{2}} + \mathbf{2}_{-\frac{3}{2}}$ $\mathbf{2}_{\frac{1}{2}}$ $\mathbf{1}_0 + \mathbf{1}_2$ $\mathbf{1}_1 + \mathbf{1}_{-1}$ $\mathbf{1}_1$ $\mathbf{1}_0 + \mathbf{1}_2 + \mathbf{1}_2$ $(\mathbf{1}, \mathbf{3})_0$ |
| $X_{F_8} \rightarrow X_{F_5}$ VEV: $(\mathbf{2}, \mathbf{2})_{\frac{1}{2}}$ | $(\mathbf{2}, \mathbf{2})_{1/2}$ $(\mathbf{1}, \mathbf{2})_{1/2}$ $(\mathbf{2}, \mathbf{1})_1$ $(\mathbf{2}, \mathbf{1})_0$ $(\mathbf{1}, \mathbf{1})_1$ $(\mathbf{3}, \mathbf{1})_0$ $(\mathbf{1}, \mathbf{3})_0$ | $\mathbf{1}_{(-1,0)} + \mathbf{1}_{(-1,-2)} + \mathbf{1}_{(0,2)} + \mathbf{1}_{(0,0)}$ $\mathbf{1}_{(0,-1)} + \mathbf{1}_{(1,1)}$ $\mathbf{1}_{(1,1)} + \mathbf{1}_{(1,-1)}$ $\mathbf{1}_{(0,1)} + \mathbf{1}_{(0,-1)}$ $\mathbf{1}_{(-1,-1)}$ $\mathbf{1}_{(0,2)} + \mathbf{1}_{(0,-2)} + \mathbf{1}_{(0,0)}$ $\mathbf{1}_{(1,2)} + \mathbf{1}_{(-1,-2)} + \mathbf{1}_{(0,0)}$ |

Table 44. Group theoretical decompositions of representation in toric Higgsings of $X_{F_9} - X_{F_8}$.

| Breaking | Starting Multiplets | Target Multiplets |
|---|--|--|
| $X_{F_7} \rightarrow X_{F_5}$ VEV: $\mathbf{1}_{(0,-1,0)}$ | $\mathbf{1}_{(1,1,0)}$ $\mathbf{1}_{(0,-1,0)}$ $\mathbf{1}_{(2,1,1)}$ $\mathbf{1}_{(0,1,1)}$ $\mathbf{1}_{(-2,-1,-2)}$ $\mathbf{1}_{(1,1,2)}$ $\mathbf{1}_{(1,0,0)}$ $\mathbf{1}_{(0,0,1)}$ $\mathbf{1}_{(1,0,1)}$ $\mathbf{1}_{(1,1,1)}$ | $\mathbf{1}_{(-1,0)}$ $\mathbf{1}_{(0,0)}$ $\mathbf{1}_{(-1,1)}$ $\mathbf{1}_{(1,1)}$ $\mathbf{1}_{(0,-2)}$ $\mathbf{1}_{(1,2)}$ $\mathbf{1}_{(-1,0)}$ $\mathbf{1}_{(1,1)}$ $\mathbf{1}_{(0,1)}$ $\mathbf{1}_{(0,1)}$ |
| $X_{F_6} \rightarrow X_{F_4}$ VEV: $\mathbf{1}_2$ | $\mathbf{2}_{-\frac{3}{2}}$ $\mathbf{2}_{\frac{1}{2}}$ $\mathbf{1}_2$ $\mathbf{1}_1$ $\mathbf{3}_0$ | $\mathbf{2}_{\frac{1}{2}}$ $\mathbf{2}_{\frac{1}{2}}$ $\mathbf{1}_0$ $\mathbf{1}_1$ $\mathbf{3}_0$ |
| $X_{F_6} \rightarrow X_{F_3}$ VEV: $\mathbf{2}_{-\frac{3}{2}}$ | $\mathbf{2}_{-\frac{3}{2}}$ $\mathbf{2}_{\frac{1}{2}}$ $\mathbf{1}_2$ $\mathbf{1}_1$ $\mathbf{3}_0$ | $\mathbf{1}_0 + \mathbf{1}_{-2}$ $\mathbf{1}_1 + \mathbf{1}_{-2}$ $\mathbf{1}_{-2}$ $\mathbf{1}_{-1}$ $\mathbf{1}_0 + \mathbf{1}_3 + \mathbf{1}_{-3}$ |
| $X_{F_5} \rightarrow X_{F_3}$ VEV: $\mathbf{1}_{(-1,-2)}$ | $\mathbf{1}_{(1,-1)}$ $\mathbf{1}_{(1,0)}$ $\mathbf{1}_{(-1,-2)}$ $\mathbf{1}_{(-1,-1)}$ $\mathbf{1}_{(0,2)}$ $\mathbf{1}_{(0,1)}$ | $\mathbf{1}_3$ $\mathbf{1}_2$ $\mathbf{1}_0$ $\mathbf{1}_1$ $\mathbf{1}_2$ $\mathbf{1}_1$ |
| $X_{F_5} \rightarrow X_{F_2}$ VEV: $\mathbf{1}_{(0,2)}$ | $\mathbf{1}_{(1,-1)}$ $\mathbf{1}_{(1,0)}$ $\mathbf{1}_{(-1,-2)}$ $\mathbf{1}_{(-1,-1)}$ $\mathbf{1}_{(0,2)}$ $\mathbf{1}_{(0,1)}$ | $\mathbf{1}_{(1,-)}$ $\mathbf{1}_{(1,+)}$ $\mathbf{1}_{(1,+)}$ $\mathbf{1}_{(1,-)}$ $\mathbf{1}_{(0,+)}$ $\mathbf{1}_{(0,-)}$ |
| $X_{F_3} \rightarrow X_{F_1}$ VEV: $\mathbf{1}_3$ | $\mathbf{1}_3$ $\mathbf{1}_2$ $\mathbf{1}_1$ | $\mathbf{1}_0$ $\mathbf{1}_2$ $\mathbf{1}_1$ |

Table 45. Group theoretical decompositions of representation in toric Higgsings of $X_{F_7} - X_{F_3}$.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] C. Vafa, *Evidence for F-theory*, *Nucl. Phys. B* **469** (1996) 403 [[hep-th/9602022](#)] [[INSPIRE](#)].
- [2] D.R. Morrison and C. Vafa, *Compactifications of F-theory on Calabi-Yau threefolds. 1*, *Nucl. Phys. B* **473** (1996) 74 [[hep-th/9602114](#)] [[INSPIRE](#)].
- [3] D.R. Morrison and C. Vafa, *Compactifications of F-theory on Calabi-Yau threefolds. 2*, *Nucl. Phys. B* **476** (1996) 437 [[hep-th/9603161](#)] [[INSPIRE](#)].
- [4] R. Donagi and M. Wijnholt, *Model building with F-theory*, *Adv. Theor. Math. Phys.* **15** (2011) 1237 [[arXiv:0802.2969](#)] [[INSPIRE](#)].
- [5] C. Beasley, J.J. Heckman and C. Vafa, *GUTs and exceptional branes in F-theory — I*, *JHEP* **01** (2009) 058 [[arXiv:0802.3391](#)] [[INSPIRE](#)].
- [6] H. Hayashi, R. Tatar, Y. Toda, T. Watari and M. Yamazaki, *New aspects of heterotic-F theory duality*, *Nucl. Phys. B* **806** (2009) 224 [[arXiv:0805.1057](#)] [[INSPIRE](#)].
- [7] C. Beasley, J.J. Heckman and C. Vafa, *GUTs and exceptional branes in F-theory – II: experimental predictions*, *JHEP* **01** (2009) 059 [[arXiv:0806.0102](#)] [[INSPIRE](#)].
- [8] K. Hori and D. Tong, *Aspects of non-abelian gauge dynamics in two-dimensional $N = (2, 2)$ theories*, *JHEP* **05** (2007) 079 [[hep-th/0609032](#)] [[INSPIRE](#)].
- [9] H. Jockers, V. Kumar, J.M. Lapan, D.R. Morrison and M. Romo, *Nonabelian 2D gauge theories for determinantal Calabi-Yau varieties*, *JHEP* **11** (2012) 166 [[arXiv:1205.3192](#)] [[INSPIRE](#)].
- [10] J. Marsano, N. Saulina and S. Schäfer-Nameki, *Monodromies, fluxes and compact three-generation F-theory GUTs*, *JHEP* **08** (2009) 046 [[arXiv:0906.4672](#)] [[INSPIRE](#)].
- [11] R. Blumenhagen, T.W. Grimm, B. Jurke and T. Weigand, *Global F-theory GUTs*, *Nucl. Phys. B* **829** (2010) 325 [[arXiv:0908.1784](#)] [[INSPIRE](#)].
- [12] J.J. Heckman, D.R. Morrison and C. Vafa, *On the classification of 6D SCFTs and Generalized ADE Orbifolds*, *JHEP* **05** (2014) 028 [[arXiv:1312.5746](#)] [[INSPIRE](#)].
- [13] M. Del Zotto, J.J. Heckman, A. Tomasiello and C. Vafa, *6d conformal matter*, [arXiv:1407.6359](#) [[INSPIRE](#)].
- [14] J.J. Heckman, *More on the matter of 6D SCFTs*, [arXiv:1408.0006](#) [[INSPIRE](#)].
- [15] M.R. Douglas and W. Taylor, *The landscape of intersecting brane models*, *JHEP* **01** (2007) 031 [[hep-th/0606109](#)] [[INSPIRE](#)].
- [16] M. Cvetič, J. Halverson, D. Klevers and P. Song, *On finiteness of Type IIB compactifications: magnetized branes on elliptic Calabi-Yau threefolds*, *JHEP* **06** (2014) 138 [[arXiv:1403.4943](#)] [[INSPIRE](#)].
- [17] A. Grassi and V. Perduca, *Weierstrass models of elliptic toric K3 hypersurfaces and symplectic cuts*, *Adv. Theor. Math. Phys.* **17** (2013) 741 [[arXiv:1201.0930](#)] [[INSPIRE](#)].
- [18] A.P. Braun, Y. Kimura and T. Watari, *On the classification of elliptic fibrations modulo isomorphism on K3 surfaces with large Picard number*, [arXiv:1312.4421](#) [[INSPIRE](#)].

- [19] M.R. Douglas, D.S. Park and C. Schnell, *The Cremmer-Scherk mechanism in f-theory compactifications on K3 manifolds*, *JHEP* **05** (2014) 135 [[arXiv:1403.1595](#)] [[INSPIRE](#)].
- [20] V. Kumar, D.R. Morrison and W. Taylor, *Mapping 6D $N = 1$ supergravities to F-theory*, *JHEP* **02** (2010) 099 [[arXiv:0911.3393](#)] [[INSPIRE](#)].
- [21] V. Kumar, D.R. Morrison and W. Taylor, *Global aspects of the space of 6D $N = 1$ supergravities*, *JHEP* **11** (2010) 118 [[arXiv:1008.1062](#)] [[INSPIRE](#)].
- [22] T.W. Grimm and W. Taylor, *Structure in 6D and 4D $N = 1$ supergravity theories from F-theory*, *JHEP* **10** (2012) 105 [[arXiv:1204.3092](#)] [[INSPIRE](#)].
- [23] L.B. Anderson and W. Taylor, *Geometric constraints in dual F-theory and heterotic string compactifications*, *JHEP* **08** (2014) 025 [[arXiv:1405.2074](#)] [[INSPIRE](#)].
- [24] D.S. Park and W. Taylor, *Constraints on 6D supergravity theories with abelian gauge symmetry*, *JHEP* **01** (2012) 141 [[arXiv:1110.5916](#)] [[INSPIRE](#)].
- [25] V. Kumar, D.S. Park and W. Taylor, *6D supergravity without tensor multiplets*, *JHEP* **04** (2011) 080 [[arXiv:1011.0726](#)] [[INSPIRE](#)].
- [26] F. Bonetti, T.W. Grimm and T.G. Pugh, *Non-supersymmetric F-theory compactifications on Spin(7) manifolds*, *JHEP* **01** (2014) 112 [[arXiv:1307.5858](#)] [[INSPIRE](#)].
- [27] F. Bonetti, T.W. Grimm, E. Palti and T.G. Pugh, *F-theory on Spin(7) manifolds: weak-coupling limit*, *JHEP* **02** (2014) 076 [[arXiv:1309.2287](#)] [[INSPIRE](#)].
- [28] D.R. Morrison and W. Taylor, *Classifying bases for 6D F-theory models*, *Central Eur. J. Phys.* **10** (2012) 1072 [[arXiv:1201.1943](#)] [[INSPIRE](#)].
- [29] D.R. Morrison and W. Taylor, *Toric bases for 6D F-theory models*, *Fortsch. Phys.* **60** (2012) 1187 [[arXiv:1204.0283](#)] [[INSPIRE](#)].
- [30] G. Martini and W. Taylor, *6D F-theory models and elliptically fibered Calabi-Yau threefolds over semi-toric base surfaces*, [arXiv:1404.6300](#) [[INSPIRE](#)].
- [31] T.W. Grimm and T. Weigand, *On abelian gauge symmetries and proton decay in global F-theory GUTs*, *Phys. Rev. D* **82** (2010) 086009 [[arXiv:1006.0226](#)] [[INSPIRE](#)].
- [32] S. Krause, C. Mayrhofer and T. Weigand, *G_4 flux, chiral matter and singularity resolution in F-theory compactifications*, *Nucl. Phys. B* **858** (2012) 1 [[arXiv:1109.3454](#)] [[INSPIRE](#)].
- [33] T.W. Grimm and H. Hayashi, *F-theory fluxes, chirality and Chern-Simons theories*, *JHEP* **03** (2012) 027 [[arXiv:1111.1232](#)] [[INSPIRE](#)].
- [34] D.S. Park, *Anomaly equations and intersection theory*, *JHEP* **01** (2012) 093 [[arXiv:1111.2351](#)] [[INSPIRE](#)].
- [35] M. Cvetič, T.W. Grimm and D. Klevers, *Anomaly cancellation and abelian gauge symmetries in F-theory*, *JHEP* **02** (2013) 101 [[arXiv:1210.6034](#)] [[INSPIRE](#)].
- [36] C. Mayrhofer, E. Palti and T. Weigand, *$U(1)$ symmetries in F-theory GUTs with multiple sections*, *JHEP* **03** (2013) 098 [[arXiv:1211.6742](#)] [[INSPIRE](#)].
- [37] V. Braun, T.W. Grimm and J. Keitel, *New Global F-theory GUTs with $U(1)$ symmetries*, *JHEP* **09** (2013) 154 [[arXiv:1302.1854](#)] [[INSPIRE](#)].
- [38] J. Borchmann, C. Mayrhofer, E. Palti and T. Weigand, *Elliptic fibrations for $SU(5) \times U(1) \times U(1)$ F-theory vacua*, *Phys. Rev. D* **88** (2013) 046005 [[arXiv:1303.5054](#)] [[INSPIRE](#)].

- [39] M. Cvetič, D. Klevers and H. Piragua, *F-theory compactifications with multiple U(1)-factors: constructing elliptic fibrations with rational sections*, *JHEP* **06** (2013) 067 [[arXiv:1303.6970](#)] [[INSPIRE](#)].
- [40] T.W. Grimm, A. Kapfer and J. Keitel, *Effective action of 6D F-theory with U(1) factors: rational sections make Chern-Simons terms jump*, *JHEP* **07** (2013) 115 [[arXiv:1305.1929](#)] [[INSPIRE](#)].
- [41] V. Braun, T.W. Grimm and J. Keitel, *Geometric engineering in toric F-theory and GUTs with U(1) gauge factors*, *JHEP* **12** (2013) 069 [[arXiv:1306.0577](#)] [[INSPIRE](#)].
- [42] M. Cvetič, A. Grassi, D. Klevers and H. Piragua, *Chiral four-dimensional F-theory compactifications with SU(5) and multiple U(1)-factors*, *JHEP* **04** (2014) 010 [[arXiv:1306.3987](#)] [[INSPIRE](#)].
- [43] J. Borchmann, C. Mayrhofer, E. Palti and T. Weigand, *SU(5) tops with multiple U(1)s in F-theory*, *Nucl. Phys. B* **882** (2014) 1 [[arXiv:1307.2902](#)] [[INSPIRE](#)].
- [44] M. Cvetič, D. Klevers and H. Piragua, *F-theory compactifications with multiple U(1)-factors: addendum*, *JHEP* **12** (2013) 056 [[arXiv:1307.6425](#)] [[INSPIRE](#)].
- [45] M. Cvetič, D. Klevers, H. Piragua and P. Song, *Elliptic fibrations with rank three Mordell-Weil group: F-theory with U(1) × U(1) × U(1) gauge symmetry*, *JHEP* **03** (2014) 021 [[arXiv:1310.0463](#)] [[INSPIRE](#)].
- [46] C. Mayrhofer, D.R. Morrison, O. Till and T. Weigand, *Mordell-Weil torsion and the global structure of gauge groups in F-theory*, *JHEP* **1410** (2014) 16 [[arXiv:1405.3656](#)] [[INSPIRE](#)].
- [47] G. Aldazabal, A. Font, L.E. Ibáñez and A.M. Uranga, *New branches of string compactifications and their F-theory duals*, *Nucl. Phys. B* **492** (1997) 119 [[hep-th/9607121](#)] [[INSPIRE](#)].
- [48] A. Klemm, P. Mayr and C. Vafa, *BPS states of exceptional noncritical strings*, *Nucl. Phys. Proc. Suppl.* **58** (1997) 177 [[hep-th/9607139](#)] [[INSPIRE](#)].
- [49] A. Klemm, M. Kreuzer, E. Riegler and E. Scheidegger, *Topological string amplitudes, complete intersection Calabi-Yau spaces and threshold corrections*, *JHEP* **05** (2005) 023 [[hep-th/0410018](#)] [[INSPIRE](#)].
- [50] M. Esole, J. Fullwood and S.-T. Yau, *D₅ elliptic fibrations: non-Kodaira fibers and new orientifold limits of F-theory*, [arXiv:1110.6177](#) [[INSPIRE](#)].
- [51] T.W. Grimm, M. Kerstan, E. Palti and T. Weigand, *Massive abelian gauge symmetries and fluxes in F-theory*, *JHEP* **12** (2011) 004 [[arXiv:1107.3842](#)] [[INSPIRE](#)].
- [52] A.P. Braun, A. Collinucci and R. Valandro, *The fate of U(1)'s at strong coupling in F-theory*, *JHEP* **07** (2014) 028 [[arXiv:1402.4054](#)] [[INSPIRE](#)].
- [53] P.S. Aspinwall and D.R. Morrison, *Nonsimply connected gauge groups and rational points on elliptic curves*, *JHEP* **07** (1998) 012 [[hep-th/9805206](#)] [[INSPIRE](#)].
- [54] V. Braun and D.R. Morrison, *F-theory on genus-one fibrations*, *JHEP* **08** (2014) 132 [[arXiv:1401.7844](#)] [[INSPIRE](#)].
- [55] D.R. Morrison and W. Taylor, *Sections, multisections and U(1) fields in F-theory*, [arXiv:1404.1527](#) [[INSPIRE](#)].
- [56] L.B. Anderson, I. García-Etxebarria, T.W. Grimm and J. Keitel, *Physics of F-theory compactifications without section*, *JHEP* **12** (2014) 156 [[arXiv:1406.5180](#)] [[INSPIRE](#)].

- [57] M.-X. Huang, A. Klemm and M. Poretschkin, *Refined stable pair invariants for E -, M - and $[p, q]$ -strings*, *JHEP* **11** (2013) 112 [[arXiv:1308.0619](#)] [[INSPIRE](#)].
- [58] M.-x. Huang, A. Klemm, J. Reuter and M. Schiereck, *Quantum geometry of del Pezzo surfaces in the Nekrasov-Shatashvili limit*, [arXiv:1401.4723](#) [[INSPIRE](#)].
- [59] K. Kodaira, *On compact analytic surfaces: II*, *Ann. Math.* **77** (1963) 563.
- [60] J. Tate, *Algorithm for determining the type of a singular fiber in an elliptic pencil*, in *Modular functions of one variable IV*, B.J. Birch and W. Kuyk, Springer, Germany (1975).
- [61] M. Bershadsky et al., *Geometric singularities and enhanced gauge symmetries*, *Nucl. Phys. B* **481** (1996) 215 [[hep-th/9605200](#)] [[INSPIRE](#)].
- [62] S. Katz, D.R. Morrison, S. Schäfer-Nameki and J. Sully, *Tate's algorithm and F -theory*, *JHEP* **08** (2011) 094 [[arXiv:1106.3854](#)] [[INSPIRE](#)].
- [63] C. Lawrie and S. Schäfer-Nameki, *The Tate form on steroids: resolution and higher codimension fibers*, *JHEP* **04** (2013) 061 [[arXiv:1212.2949](#)] [[INSPIRE](#)].
- [64] P. Candelas and A. Font, *Duality between the webs of heterotic and type-II vacua*, *Nucl. Phys. B* **511** (1998) 295 [[hep-th/9603170](#)] [[INSPIRE](#)].
- [65] V. Bouchard and H. Skarke, *Affine Kac-Moody algebras, CHL strings and the classification of tops*, *Adv. Theor. Math. Phys.* **7** (2003) 205 [[hep-th/0303218](#)] [[INSPIRE](#)].
- [66] M. Kuntzler and S. Schäfer-Nameki, *Tate trees for elliptic fibrations with rank one Mordell-Weil group*, [arXiv:1406.5174](#) [[INSPIRE](#)].
- [67] L. Lin and T. Weigand, *Towards the standard model in F -theory*, [arXiv:1406.6071](#) [[INSPIRE](#)].
- [68] L.M. Krauss and F. Wilczek, *Discrete gauge symmetry in continuum theories*, *Phys. Rev. Lett.* **62** (1989) 1221 [[INSPIRE](#)].
- [69] T. Banks, *Effective lagrangian description of discrete gauge symmetries*, *Nucl. Phys. B* **323** (1989) 90 [[INSPIRE](#)].
- [70] L.E. Ibanez and G.G. Ross, *Should discrete symmetries be anomaly free?*, [CERN-TH-6000-91](#) (1991) [[INSPIRE](#)].
- [71] J.H. Silverman, *The arithmetic of elliptic curves*, Graduate texts in mathematics volume 106, Springer, Germany (2009).
- [72] S. Lang and A. Neron, *Rational points of abelian varieties over function fields*, *Am. J. Math.* **81** (1959) 95.
- [73] B. Mazur, *Modular curves and the eisenstein ideal*, *Inst. Hautes Études Sci. Publ.* **47** (1977) 33.
- [74] B. Mazur and D. Goldfeld, *Rational isogenies of prime degree*, *Inv. Math.* **44** (1978) 129.
- [75] N. Nakayama, *On Weierstrass models*, in *Algebraic geometry and commutative algebra II*, H. Hijikata and M. Nagata eds., Kinokuniya (1988).
- [76] S.Y. An et al., *Jacobians of genus one curves*, *J. Number Theory* **90** (2001) 304.
- [77] D.R. Morrison and D.S. Park, *F -theory and the Mordell-Weil group of elliptically-fibered Calabi-Yau threefolds*, *JHEP* **10** (2012) 128 [[arXiv:1208.2695](#)] [[INSPIRE](#)].

- [78] T.W. Grimm, *The $N = 1$ effective action of F-theory compactifications*, *Nucl. Phys. B* **845** (2011) 48 [[arXiv:1008.4133](#)] [[INSPIRE](#)].
- [79] F. Denef, *Les Houches lectures on constructing string vacua*, [arXiv:0803.1194](#) [[INSPIRE](#)].
- [80] W. Taylor, *TASI lectures on supergravity and string vacua in various dimensions*, [arXiv:1104.2051](#) [[INSPIRE](#)].
- [81] E. Witten, *Phase transitions in M-theory and F-theory*, *Nucl. Phys. B* **471** (1996) 195 [[hep-th/9603150](#)] [[INSPIRE](#)].
- [82] S.H. Katz, D.R. Morrison and M.R. Plesser, *Enhanced gauge symmetry in type-II string theory*, *Nucl. Phys. B* **477** (1996) 105 [[hep-th/9601108](#)] [[INSPIRE](#)].
- [83] R. Hartshorne, *Algebraic geometry*, Springer, Germany (1977).
- [84] W. Fulton, *Introduction to toric varieties*, Princeton University Press, Princeton U.S.A. (1993).
- [85] D.A. Cox, J.B. Little and H.K. Schenck, *Toric varieties*, American Mathematical Society, U.S.A. (2011).
- [86] D.A. Cox, *The homogeneous coordinate ring of a toric variety, revised version*, [alg-geom/9210008](#) [[INSPIRE](#)].
- [87] V.V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties*, *J. Alg. Geom.* **3** (1994) 493 [[alg-geom/9310003](#)] [[INSPIRE](#)].
- [88] W. Decker, G.M. Greuel, G. Pfister and H. Schönemann, *SINGULAR 3-1-6 — A computer algebra system for polynomial computations*, <http://www.singular.uni-kl.de> (2012).
- [89] T. Banks and N. Seiberg, *Symmetries and strings in field theory and gravity*, *Phys. Rev. D* **83** (2011) 084019 [[arXiv:1011.5120](#)] [[INSPIRE](#)].
- [90] H. Hayashi, C. Lawrie, D.R. Morrison and S. Schäfer-Nameki, *Box graphs and singular fibers*, *JHEP* **05** (2014) 048 [[arXiv:1402.2653](#)] [[INSPIRE](#)].
- [91] R. Friedman, J. Morgan and E. Witten, *Vector bundles and F-theory*, *Commun. Math. Phys.* **187** (1997) 679 [[hep-th/9701162](#)] [[INSPIRE](#)].
- [92] G. Honecker and M. Trapletti, *Merging heterotic orbifolds and K3 compactifications with line bundles*, *JHEP* **01** (2007) 051 [[hep-th/0612030](#)] [[INSPIRE](#)].
- [93] A. Grassi and D.R. Morrison, *Anomalies and the Euler characteristic of elliptic Calabi-Yau threefolds*, *Commun. Num. Theor. Phys.* **6** (2012) 51 [[arXiv:1109.0042](#)] [[INSPIRE](#)].
- [94] K. Intriligator, H. Jockers, P. Mayr, D.R. Morrison and M.R. Plesser, *Conifold transitions in M-theory on Calabi-Yau Fourfolds with background fluxes*, *Adv. Theor. Math. Phys.* **17** (2013) 601 [[arXiv:1203.6662](#)] [[INSPIRE](#)].
- [95] N.C. Bizet, A. Klemm and D.V. Lopes, *Landscaping with fluxes and the E8 Yukawa point in F-theory*, [arXiv:1404.7645](#) [[INSPIRE](#)].
- [96] P. Berglund and P. Mayr, *Heterotic string/F theory duality from mirror symmetry*, *Adv. Theor. Math. Phys.* **2** (1999) 1307 [[hep-th/9811217](#)] [[INSPIRE](#)].
- [97] G.R. Farrar and P. Fayet, *Phenomenology of the production, decay and detection of new hadronic states associated with supersymmetry*, *Phys. Lett. B* **76** (1978) 575 [[INSPIRE](#)].
- [98] S. Dimopoulos, S. Raby and F. Wilczek, *Proton decay in supersymmetric models*, *Phys. Lett. B* **112** (1982) 133 [[INSPIRE](#)].

- [99] L.E. Ibáñez and G.G. Ross, *Discrete gauge symmetries and the origin of baryon and lepton number conservation in supersymmetric versions of the standard model*, *Nucl. Phys. B* **368** (1992) 3 [[INSPIRE](#)].
- [100] H.K. Dreiner, C. Luhn and M. Thormeier, *What is the discrete gauge symmetry of the MSSM?*, *Phys. Rev. D* **73** (2006) 075007 [[hep-ph/0512163](#)] [[INSPIRE](#)].
- [101] H.M. Lee et al., *A unique Z_4^R symmetry for the MSSM*, *Phys. Lett. B* **694** (2011) 491 [[arXiv:1009.0905](#)] [[INSPIRE](#)].
- [102] E. Dudas and E. Palti, *Froggatt-Nielsen models from E_8 in F-theory GUTs*, *JHEP* **01** (2010) 127 [[arXiv:0912.0853](#)] [[INSPIRE](#)].
- [103] S. Krippendorf, D.K. Mayorga Pena, P.-K. Oehlmann and F. Ruehle, *Rational F-theory GUTs without exotics*, *JHEP* **07** (2014) 013 [[arXiv:1401.5084](#)] [[INSPIRE](#)].
- [104] A. Grassi, J. Halverson and J.L. Shaneson, *Matter from geometry without resolution*, *JHEP* **10** (2013) 205 [[arXiv:1306.1832](#)] [[INSPIRE](#)].
- [105] A. Grassi, J. Halverson and J.L. Shaneson, *Non-abelian gauge symmetry and the Higgs mechanism in F-theory*, [arXiv:1402.5962](#) [[INSPIRE](#)].
- [106] H. Hayashi, C. Lawrie and S. Schäfer-Nameki, *Phases, flops and F-theory: SU(5) gauge theories*, *JHEP* **10** (2013) 046 [[arXiv:1304.1678](#)] [[INSPIRE](#)].
- [107] M. Esole, S.-H. Shao and S.-T. Yau, *Singularities and gauge theory phases*, [arXiv:1402.6331](#) [[INSPIRE](#)].
- [108] M. Esole, S.-H. Shao and S.-T. Yau, *Singularities and gauge theory phases II*, [arXiv:1407.1867](#) [[INSPIRE](#)].
- [109] A.P. Braun and S. Schäfer-Nameki, *Box graphs and resolutions I*, [arXiv:1407.3520](#) [[INSPIRE](#)].
- [110] M. Esole and S.-T. Yau, *Small resolutions of SU(5)-models in F-theory*, *Adv. Theor. Math. Phys.* **17** (2013) 1195 [[arXiv:1107.0733](#)] [[INSPIRE](#)].
- [111] J. Marsano and S. Schäfer-Nameki, *Yukawas, G-flux and spectral covers from resolved Calabi-Yau's*, *JHEP* **11** (2011) 098 [[arXiv:1108.1794](#)] [[INSPIRE](#)].
- [112] P. Mayr, *Mirror symmetry, $N = 1$ superpotentials and tensionless strings on Calabi-Yau four folds*, *Nucl. Phys. B* **494** (1997) 489 [[hep-th/9610162](#)] [[INSPIRE](#)].