Neutrino Physics

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Abstract
I give a theoretical overview of some basic properties of massive neutrinos in these lectures. Particular attention is paid to the origin of neutrino masses, the pattern of lepton flavor mixing, the feature of leptonic CP violation and the electromagnetic properties of massive neutrinos. I highlight the TeV seesaw mechanisms as a possible bridge between neutrino physics and collider physics in the era characterized by the Large Hadron Collider.

1 Finite Neutrino Masses

It is well known that the mass of an elementary particle represents its inertial energy when it exists at rest. Hence a massless particle has no way to exist at rest — instead, it must always move at the speed of light. A massive fermion (either lepton or quark) must exist in both left-handed and right-handed states, since the field operators responsible for the non-vanishing mass of a fermion have to be bilinear products of the spinor fields which flip the fermion's handedness or chirality.

The standard model (SM) of electroweak interactions contains three neutrinos $(\nu_e, \nu_\mu, \nu_\tau)$ which are purely left-handed and massless. In the SM the masslessness of the photon is guaranteed by the electromagnetic $U(1)_{\rm Q}$ gauge symmetry. Although the masslessness of three neutrinos corresponds to the lepton number conservation [1], the latter is an accidental symmetry rather than a fundamental symmetry of the SM. Hence many physicists strongly believed that neutrinos should be massive even long before some incontrovertible experimental evidence for massive neutrinos were accumulated. A good reason for this belief is that neutrinos are more natural to be massive than to be massless in some grand unified theories, such as the $\rm SO(10)$ theory, which try to unify electromagnetic, weak and strong interactions as well as leptons and quarks.

If neutrinos are massive and their masses are non-degenerate, it will in general be impossible to find a flavor basis in which the coincidence between flavor and mass eigenstates holds both for charged leptons (e, μ, τ) and for neutrinos $(\nu_e, \nu_\mu, \nu_\tau)$. In other words, the phenomenon of flavor mixing is naturally expected to appear between three charged leptons and three massive neutrinos, just like the phenomenon of flavor mixing between three up-type quarks (u, c, t) and three down-type quarks (d, s, b). If there exist irremovable complex phases in the Yukawa interactions, CP violation will naturally appear both in the quark sector and in the lepton sector.

1.1 Some preliminaries

To write out the mass term for three known neutrinos, let us make a minimal extension of the SM by introducing three right-handed neutrinos. Then we totally have six neutrino fields [2]:

$$\nu_{\rm L} = \begin{pmatrix} \nu_{e{\rm L}} \\ \nu_{\mu{\rm L}} \\ \nu_{\tau{\rm L}} \end{pmatrix} \ , \quad N_{\rm R} = \begin{pmatrix} N_{1{\rm R}} \\ N_{2{\rm R}} \\ N_{3{\rm R}} \end{pmatrix} \ , \tag{1}$$

[1] It is actually the $B-L$ symmetry that makes neutrinos exactly massless in the SM, where B = baryon number and L = lepton number. The reason is simply that a neutrino and an antineutrino have different values of $B-L$. Thus the naive argument for massless neutrinos is valid to all orders in perturbation and non-perturbation theories, if $B-L$ is an exact symmetry.

[2] The left- and right-handed components of a fermion field $\psi(x)$ are denoted as $\psi_{\rm L}(x) = P_{\rm L}\psi(x)$ and $\psi_{\rm R}(x) = P_{\rm R}\psi(x)$, respectively, where $P_{\rm L} \equiv (1-\gamma_5)/2$ and $P_{\rm R} \equiv (1+\gamma_5)/2$ are the chiral projection operators. Note, however, that $\nu_{\rm L} = P_{\rm L}\nu_{\rm L}$ and $N_{\rm R} = P_{\rm R}N_{\rm R}$ are in general independent of each other.

where only the left-handed fields take part in the electroweak interactions. The charge-conjugate counterparts of $\nu_{\rm L}$ and $N_{\rm R}$ are defined as

$$(\nu_{\rm L})^c \equiv \mathcal{C}\overline{\nu_{\rm L}}^T \; , \quad (N_{\rm R})^c \equiv \mathcal{C}\overline{N_{\rm R}}^T \; ; \tag{2}$$

and accordingly,

$$\overline{(\nu_{\rm L})^c} = (\nu_{\rm L})^T \mathcal{C} \; , \quad \overline{(N_{\rm R})^c} = (N_{\rm R})^T \mathcal{C} \; , \tag{3}$$

where $\mathcal{C}$ denotes the charge-conjugation matrix and satisfies the conditions

$$\mathcal{C}\gamma_\mu^T \mathcal{C}^{-1} = -\gamma_\mu \; , \quad \mathcal{C}\gamma_5^T \mathcal{C}^{-1} = \gamma_5 \; , \quad \mathcal{C}^{-1} = \mathcal{C}^\dagger = \mathcal{C}^T = -\mathcal{C} \; . \tag{4}$$

It is easy to check that $P_{\rm L}(N_{\rm R})^c = (N_{\rm R})^c$ and $P_{\rm R}(\nu_{\rm L})^c = (\nu_{\rm L})^c$ hold; namely, $(\nu_{\rm L})^c = (\nu^c)_{\rm R}$ and $(N_{\rm R})^c = (N^c)_{\rm L}$ hold. Hence $(\nu_{\rm L})^c$ and $(N_{\rm R})^c$ are right- and left-handed fields, respectively. One may then use the neutrino fields $\nu_{\rm L}$, $N_{\rm R}$ and their charge-conjugate partners to write out the gauge-invariant and Lorentz-invariant neutrino mass terms.

In the SM the weak charged-current interactions of three active neutrinos are given by

$$\mathcal{L}_{\rm cc} = \frac{g}{\sqrt{2}} \overline{(e \; \mu \; \tau)_{\rm L}} \, \gamma^\mu \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}_{\rm L} W_\mu^- + {\rm h.c.} \; . \tag{5}$$

Without loss of generality, we choose the basis in which the mass eigenstates of three charged leptons are identified with their flavor eigenstates. If neutrinos have non-zero and non-degenerate masses, their flavor and mass eigenstates are in general not identical in the chosen basis. This mismatch signifies lepton flavor mixing.

1.2 Dirac neutrino masses

A Dirac neutrino is described by a four-component Dirac spinor $\nu = \nu_{\rm L} + N_{\rm R}$, whose left-handed and right-handed components are just $\nu_{\rm L}$ and $N_{\rm R}$. The Dirac neutrino mass term comes from the Yukawa interactions

$$-\mathcal{L}_{\rm Dirac} = \overline{\ell_{\rm L}} Y_\nu \tilde{H} N_{\rm R} + {\rm h.c.} \; , \tag{6}$$

where $\tilde{H} \equiv i\sigma_2 H^*$ with H being the SM Higgs doublet, and $\ell_{\rm L}$ denotes the left-handed lepton doublet. After spontaneous gauge symmetry breaking (i.e., $SU(2)_{\rm L} \times U(1)_{\rm Y} \to U(1)_{\rm Q}$), we obtain

$$-\mathcal{L}'_{\rm Dirac} = \overline{\nu_{\rm L}} M_{\rm D} N_{\rm R} + {\rm h.c.} \; , \tag{7}$$

where $M_{\rm D} = Y_\nu \langle H \rangle$ with $\langle H \rangle \simeq 174$ GeV being the vacuum expectation value of H. This mass matrix can be diagonalized by a bi-unitary transformation: $V^\dagger M_{\rm D} U = \widehat{M}_\nu \equiv {\rm Diag}\{m_1, m_2, m_3\}$ with m_i being the neutrino masses (for $i = 1, 2, 3$). After this diagonalization,

$$-\mathcal{L}'_{\rm Dirac} = \overline{\nu'_{\rm L}} \widehat{M}_\nu N'_{\rm R} + {\rm h.c.} \; , \tag{8}$$

where $\nu'_{\rm L} = V^\dagger \nu_{\rm L}$ and $N'_{\rm R} = U^\dagger N_{\rm R}$. Then the four-component Dirac spinor

$$\nu' = \nu'_{\rm L} + N'_{\rm R} = \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \; , \tag{9}$$

which automatically satisfies $P_{\rm L}\nu' = \nu'_{\rm L}$ and $P_{\rm R}\nu' = N'_{\rm R}$, describes the mass eigenstates of three Dirac neutrinos. In other words,

$$-\mathcal{L}'_{\rm Dirac} = \overline{\nu'} \widehat{M}_\nu \nu' = \sum_{i=1}^{3} m_i \overline{\nu_i} \nu_i \; . \tag{10}$$

The kinetic term of Dirac neutrinos reads

$$\mathcal{L}_{\rm kinetic} = i\overline{\nu_{\rm L}}\gamma_\mu\partial^\mu\nu_{\rm L} + i\overline{N_{\rm R}}\gamma_\mu\partial^\mu N_{\rm R} = i\overline{\nu'}\gamma_\mu\partial^\mu\nu' = i\sum_{k=1}^{3}\overline{\nu_k}\gamma_\mu\partial^\mu\nu_k \; , \tag{11}$$

where $V^\dagger V = VV^\dagger = \mathbf{1}$ and $U^\dagger U = UU^\dagger = \mathbf{1}$ have been used.

Now we rewrite the weak charged-current interactions of three neutrinos in Eq. (5) in terms of their mass eigenstates $\nu'_{\rm L} = V^\dagger\nu_{\rm L}$ in the chosen basis where the flavor and mass eigenstates of three charged leptons are identical:

$$\mathcal{L}_{\rm cc} = \frac{g}{\sqrt{2}}\overline{(e\;\mu\;\tau)_{\rm L}}\;\gamma^\mu V \begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix}_{\rm L} W^-_\mu + {\rm h.c.}\;. \tag{12}$$

The 3×3 unitary matrix V, which actually links the neutrino mass eigenstates (ν_1, ν_2, ν_3) to the neutrino flavor eigenstates $(\nu_e, \nu_\mu, \nu_\tau)$, just measures the phenomenon of neutrino mixing.

A salient feature of massive Dirac neutrinos is lepton number conservation. To see why massive Dirac neutrinos are lepton-number-conserving, we make the global phase transformations

$$l(x) \to e^{i\Phi}l(x)\;, \quad \nu'_{\rm L}(x) \to e^{i\Phi}\nu'_{\rm L}(x)\;, \quad N'_{\rm R}(x) \to e^{i\Phi}N'_{\rm R}(x)\;, \tag{13}$$

where l denotes the column vector of e, μ and τ fields, and Φ is an arbitrary spacetime-independent phase. As the mass term $\mathcal{L}'_{\rm Dirac}$, the kinetic term $\mathcal{L}_{\rm kinetic}$ and the charged-current interaction term $\mathcal{L}_{\rm cc}$ are all invariant under these transformations, the lepton number must be conserved for massive Dirac neutrinos. It is evident that lepton flavors are violated, unless $M_{\rm D}$ is diagonal or equivalently V is the identity matrix. In other words, lepton flavor mixing leads to lepton flavor violation, or vice versa.

For example, the decay mode $\pi^- \to \mu^- + \overline{\nu}_\mu$ preserves both the lepton number and lepton flavors. In contrast, $\mu^+ \to e^+ + \gamma$ preserves the lepton number but violates the lepton flavors. The observed phenomena of neutrino oscillations verify the existence of neutrino flavor violation. Note that the $0\nu2\beta$ decay $(A, Z) \to (A, Z+2) + 2e^-$ violates the lepton number. This process cannot take place if neutrinos are massive Dirac particles, but it may naturally happen if neutrinos are massive Majorana particles.

1.3 Majorana neutrino masses

The left-handed neutrino field $\nu_{\rm L}$ and its charge-conjugate counterpart $(\nu_{\rm L})^c$ can in principle form a neutrino mass term, as $(\nu_{\rm L})^c$ is actually right-handed. But this Majorana mass term is forbidden by the $SU(2)_{\rm L}\times U(1)_{\rm Y}$ gauge symmetry in the SM, which contains only one $SU(2)_{\rm L}$ Higgs doublet and preserves lepton number conservation. We shall show later that the introduction of an $SU(2)_{\rm L}$ Higgs triplet into the SM can accommodate such a neutrino mass term with gauge invariance. Here we ignore the details of the Higgs triplet models and focus on the Majorana neutrino mass term itself:

$$-\mathcal{L}'_{\rm Majorana} = \frac{1}{2}\overline{\nu_{\rm L}}M_{\rm L}(\nu_{\rm L})^c + {\rm h.c.}\;. \tag{14}$$

Note that the mass matrix $M_{\rm L}$ must be symmetric. Because the mass term is a Lorentz scalar whose transpose keeps unchanged, we have

$$\overline{\nu_L}M_{\rm L}(\nu_L)^c = \left[\overline{\nu_L}M_{\rm L}(\nu_L)^c\right]^T = -\overline{\nu_L}\mathcal{C}^T M_{\rm L}^T\overline{\nu_L}^T = \overline{\nu_L}M_{\rm L}^T(\nu_L)^c\;, \tag{15}$$

where a minus sign appears when interchanging two fermion field operators, and $\mathcal{C}^T = -\mathcal{C}$ has been used. Hence $M_{\rm L}^T = M_{\rm L}$ holds. This symmetric mass matrix can be diagonalized by the transformation $V^\dagger M_{\rm L}V^* = \widehat{M}_\nu \equiv {\rm Diag}\{m_1, m_2, m_3\}$, where V is a unitary matrix. After this, Eq. (14) becomes

$$-\mathcal{L}'_{\rm Majorana} = \frac{1}{2}\overline{\nu'_{\rm L}}\widehat{M}_\nu(\nu'_{\rm L})^c + {\rm h.c.}\;, \tag{16}$$

where $\nu'_{\rm L} = V^\dagger \nu_{\rm L}$ and $(\nu'_{\rm L})^c = \mathcal{C}\overline{\nu'_{\rm L}}^T$. Then the Majorana field

$$\nu' = \nu'_{\rm L} + (\nu'_{\rm L})^c = \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} , \tag{17}$$

which certainly satisfies the Majorana condition $(\nu')^c = \nu'$, describes the mass eigenstates of three Majorana neutrinos. In other words,

$$-\mathcal{L}'_{\rm Majorana} = \frac{1}{2}\overline{\nu'}\widehat{M}_\nu \nu' = \frac{1}{2}\sum_{i=1}^{3} m_i \overline{\nu_i}\nu_i \ . \tag{18}$$

The kinetic term of Majorana neutrinos reads

$$\mathcal{L}_{\rm kinetic} = i\overline{\nu_{\rm L}}\gamma_\mu \partial^\mu \nu_{\rm L} = i\overline{\nu'_{\rm L}}\gamma_\mu \partial^\mu \nu'_{\rm L} = \frac{i}{2}\overline{\nu'}\gamma_\mu \partial^\mu \nu' = \frac{i}{2}\sum_{k=1}^{3} \overline{\nu_k}\gamma_\mu \partial^\mu \nu_k \ , \tag{19}$$

where we have used a generic relationship $\overline{(\psi_{\rm L})^c}\gamma_\mu \partial^\mu (\psi_{\rm L})^c = \overline{\psi_{\rm L}}\gamma_\mu \partial^\mu \psi_{\rm L}$. This relationship can easily be proved by taking account of $\partial^\mu \left[\overline{(\psi_{\rm L})^c}\gamma_\mu (\psi_{\rm L})^c\right] = 0$; i.e., we have

$$\begin{aligned} \overline{(\psi_{\rm L})^c}\gamma_\mu \partial^\mu (\psi_{\rm L})^c &= -\partial^\mu \overline{(\psi_{\rm L})^c}\gamma_\mu (\psi_{\rm L})^c = -\left[\partial^\mu \overline{(\psi_{\rm L})^c}\gamma_\mu (\psi_{\rm L})^c\right]^T \\ &= \left(\mathcal{C}\overline{\psi_{\rm L}}^T\right)^T \gamma_\mu^T \partial^\mu \left[(\psi_{\rm L})^T \mathcal{C}\right]^T = \overline{\psi_{\rm L}}\gamma_\mu \partial^\mu \psi_{\rm L}, \end{aligned} \tag{20}$$

where $\mathcal{C}^T \gamma_\mu^T \mathcal{C}^T = \gamma_\mu$, which may be read off from Eq. (4), has been used.

It is worth pointing out that the factor $1/2$ in $\mathcal{L}'_{\rm Majorana}$ allows us to get the Dirac equation of massive Majorana neutrinos analogous to that of massive Dirac neutrinos. To see this point more clearly, let us consider the Lagrangian of free Majorana neutrinos (i.e., their kinetic and mass terms):

$$\begin{aligned} \mathcal{L}_\nu &= i\overline{\nu_{\rm L}}\gamma_\mu \partial^\mu \nu_{\rm L} - \left[\frac{1}{2}\overline{\nu_{\rm L}}M_{\rm L}(\nu_{\rm L})^c + {\rm h.c.}\right] = i\overline{\nu'_{\rm L}}\gamma_\mu \partial^\mu \nu'_{\rm L} - \left[\frac{1}{2}\overline{\nu'_{\rm L}}\widehat{M}_\nu (\nu'_{\rm L})^c + {\rm h.c.}\right] \\ &= \frac{1}{2}\left(i\overline{\nu'}\gamma_\mu \partial^\mu \nu' - \overline{\nu'}\widehat{M}_\nu \nu'\right) = -\frac{1}{2}\left(i\partial^\mu \overline{\nu'}\gamma_\mu \nu' + \overline{\nu'}\widehat{M}_\nu \nu'\right) \ , \end{aligned} \tag{21}$$

where $\partial^\mu (\overline{\nu'}\gamma_\mu \nu') = 0$ has been used. Then we substitute $\mathcal{L}_\nu$ into the Euler-Lagrange equation

$$\partial^\mu \frac{\partial \mathcal{L}_\nu}{\partial\left(\partial^\mu \overline{\nu'}\right)} - \frac{\partial \mathcal{L}_\nu}{\partial \overline{\nu'}} = 0 \tag{22}$$

and obtain the Dirac equation

$$i\gamma_\mu \partial^\mu \nu' - \widehat{M}_\nu \nu' = 0 \ . \tag{23}$$

More explicitly, $i\gamma_\mu \partial^\mu \nu_k - m_k \nu_k = 0$ holds (for $k = 1, 2, 3$). That is why the factor $1/2$ in $\mathcal{L}'_{\rm Majorana}$ makes sense.

The weak charged-current interactions of three neutrinos in Eq. (5) can now be rewritten in terms of their mass eigenstates $\nu'_{\rm L} = V^\dagger \nu_{\rm L}$. In the chosen basis where the flavor and mass eigenstates of three charged leptons are identical, the expression of $\mathcal{L}_{cc}$ for Majorana neutrinos is the same as that given in Eq. (12) for Dirac neutrinos. The unitary matrix V is just the 3×3 Majorana neutrino mixing matrix, which contains two more irremovable CP-violating phases than the 3×3 Dirac neutrino mixing matrix (see section 4 for detailed discussions).

The most salient feature of massive Majorana neutrinos is lepton number violation. Let us make the global phase transformations

$$l(x) \to e^{i\Phi} l(x) \; , \quad \nu'_{\rm L}(x) \to e^{i\Phi} \nu'_{\rm L}(x) \; , \tag{24}$$

where l stands for the column vector of e, μ and τ fields, and Φ is an arbitrary spacetime-independent phase. One can immediately see that the kinetic term $\mathcal{L}_{\rm kinetic}$ and the charged-current interaction term $\mathcal{L}_{\rm cc}$ are invariant under these transformations, but the mass term $\mathcal{L}'_{\rm Majorana}$ is not invariant because of both $\overline{\nu'_{\rm L}} \to e^{-i\Phi}\overline{\nu'_{\rm L}}$ and $(\nu'_{\rm L})^c \to e^{-i\Phi}(\nu'_{\rm L})^c$. The lepton number is therefore violated for massive Majorana neutrinos. Similar to the case of Dirac neutrinos, the lepton flavor violation of Majorana neutrinos is also described by V.

The $0\nu 2\beta$ decay $(A, Z) \to (A, Z+2) + 2e^-$ is a clean signature of the Majorana nature of massive neutrinos. This lepton-number-violating process can occur when there exists neutrino-antineutrino mixing induced by the Majorana mass term (i.e., the neutrino mass eigenstates are self-conjugate, $\overline{\nu}_i = \nu_i$). The effective mass of the $0\nu 2\beta$ decay is defined as

$$\langle m \rangle_{ee} \equiv \left| \sum_i m_i V_{ei}^2 \right| \; , \tag{25}$$

where m_i comes from the helicity suppression factor m_i/E for the ν_i exchange between two beta decays with E being the energy of the virtual ν_i neutrino. Current experimental data only yield an upper bound $\langle m \rangle_{ee} < 0.23$ eV (or < 0.85 eV as a more conservative bound) at the 2σ level.

1.4 Hybrid neutrino mass terms

Similar to Eq. (14), the right-handed neutrino field $N_{\rm R}$ and its charge-conjugate counterpart $(N_{\rm R})^c$ can also form a Majorana mass term. Hence it is possible to write out the following hybrid neutrino mass terms in terms of $\nu_{\rm L}$, $N_{\rm R}$, $(\nu_{\rm L})^c$ and $(N_{\rm R})^c$ fields:

$$\begin{aligned} -\mathcal{L}'_{\rm hybrid} &= \overline{\nu_{\rm L}} M_{\rm D} N_{\rm R} + \frac{1}{2}\overline{\nu_{\rm L}} M_{\rm L} (\nu_{\rm L})^c + \frac{1}{2}\overline{(N_{\rm R})^c} M_{\rm R} N_{\rm R} + {\rm h.c.} \\ &= \frac{1}{2}\begin{bmatrix}\overline{\nu_{\rm L}} & \overline{(N_{\rm R})^c}\end{bmatrix} \begin{pmatrix} M_{\rm L} & M_{\rm D} \\ M_{\rm D}^T & M_{\rm R}\end{pmatrix} \begin{bmatrix} (\nu_{\rm L})^c \\ N_{\rm R}\end{bmatrix} + {\rm h.c.} \; , \end{aligned} \tag{26}$$

where $M_{\rm L}$ and $M_{\rm R}$ are symmetric mass matrices because the corresponding mass terms are of the Majorana type, and the relationship

$$\overline{(N_{\rm R})^c} M_{\rm D}^T (\nu_{\rm L})^c = \left[(N_{\rm R})^T \mathcal{C} M_{\rm D}^T \mathcal{C} \overline{\nu_{\rm L}}^T \right]^T = \overline{\nu_{\rm L}} M_{\rm D} N_{\rm R} \tag{27}$$

has been used. The overall 6×6 mass matrix in Eq. (26) is also symmetric, and thus it can be diagonalized by a 6×6 unitary matrix through the transformation

$$\begin{pmatrix} V & R \\ S & U \end{pmatrix}^\dagger \begin{pmatrix} M_{\rm L} & M_{\rm D} \\ M_{\rm D}^T & M_{\rm R}\end{pmatrix} \begin{pmatrix} V & R \\ S & U \end{pmatrix}^* = \begin{pmatrix} \widehat{M}_\nu & \mathbf{0} \\ \mathbf{0} & \widehat{M}_N \end{pmatrix} \; , \tag{28}$$

where we have defined $\widehat{M}_\nu \equiv {\rm Diag}\{m_1, m_2, m_3\}$, $\widehat{M}_N \equiv {\rm Diag}\{M_1, M_2, M_3\}$, and the 3×3 matrices V, R, S and U satisfy the unitarity conditions

$$\begin{aligned} VV^\dagger + RR^\dagger = SS^\dagger + UU^\dagger &= \mathbf{1} \; , \\ V^\dagger V + S^\dagger S = R^\dagger R + U^\dagger U &= \mathbf{1} \; , \\ VS^\dagger + RU^\dagger = V^\dagger R + S^\dagger U &= \mathbf{0} \; . \end{aligned} \tag{29}$$

After this diagonalization, Eq. (26) becomes

$$-\mathcal{L}'_{\rm hybrid} = \frac{1}{2} \begin{bmatrix} \overline{\nu'_{\rm L}} & \overline{(N'_{\rm R})^c} \end{bmatrix} \begin{pmatrix} \widehat{M}_\nu & \mathbf{0} \\ \mathbf{0} & \widehat{M}_N \end{pmatrix} \begin{bmatrix} (\nu'_{\rm L})^c \\ N'_{\rm R} \end{bmatrix} + {\rm h.c.} \; , \tag{30}$$

where $\nu'_{\rm L} = V^\dagger \nu_{\rm L} + S^\dagger (N_{\rm R})^c$ and $N'_{\rm R} = R^T (\nu_{\rm L})^c + U^T N_{\rm R}$ together with $(\nu'_{\rm L})^c = \mathcal{C}\overline{\nu'_{\rm L}}^T$ and $(N'_{\rm R})^c = \mathcal{C}\overline{N'_{\rm R}}^T$. Then the Majorana field

$$\nu' = \begin{bmatrix} \nu'_{\rm L} \\ (N'_{\rm R})^c \end{bmatrix} + \begin{bmatrix} (\nu'_{\rm L})^c \\ N'_{\rm R} \end{bmatrix} = \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ N_1 \\ N_2 \\ N_3 \end{pmatrix} \tag{31}$$

satisfies the Majorana condition $(\nu')^c = \nu'$ and describes the mass eigenstates of six Majorana neutrinos. In other words,

$$-\mathcal{L}'_{\rm hybrid} = \frac{1}{2} \overline{\nu'} \begin{pmatrix} \widehat{M}_\nu & \mathbf{0} \\ \mathbf{0} & \widehat{M}_N \end{pmatrix} \nu' = \frac{1}{2} \sum_{i=1}^{3} \left(m_i \overline{\nu_i} \nu_i + M_i \overline{N_i} N_i \right) \; . \tag{32}$$

Because of $\nu_{\rm L} = V\nu'_{\rm L} + R(N'_{\rm R})^c$ and $N_{\rm R} = S^*(\nu'_{\rm L})^c + U^* N'_{\rm R}$, we immediately have $(\nu_{\rm L})^c = V^*(\nu'_{\rm L})^c + R^* N'_{\rm R}$ and $(N_{\rm R})^c = S\nu'_{\rm L} + U(N'_{\rm R})^c$. Given the generic relations $\overline{(\psi_{\rm L})^c}\gamma_\mu \partial^\mu (\psi_{\rm L})^c = \overline{\psi_{\rm L}}\gamma_\mu \partial^\mu \psi_{\rm L}$ and $\overline{(\psi_{\rm R})^c}\gamma_\mu \partial^\mu (\psi_{\rm R})^c = \overline{\psi_{\rm R}}\gamma_\mu \partial^\mu \psi_{\rm R}$ for an arbitrary fermion field ψ, the kinetic term of Majorana neutrinos under consideration turns out to be

$$\begin{aligned} \mathcal{L}_{\rm kinetic} &= i\overline{\nu_{\rm L}}\gamma_\mu \partial^\mu \nu_{\rm L} + i\overline{N_{\rm R}}\gamma_\mu \partial^\mu N_{\rm R} = i\overline{\nu'_{\rm L}}\gamma_\mu \partial^\mu \nu'_{\rm L} + i\overline{N'_{\rm R}}\gamma_\mu \partial^\mu N'_{\rm R} = \frac{i}{2}\overline{\nu'}\gamma_\mu \partial^\mu \nu' \\ &= \frac{i}{2} \sum_{k=1}^{3} \left(\overline{\nu_k}\gamma_\mu \partial^\mu \nu_k + \overline{N_k}\gamma_\mu \partial^\mu N_k \right) \; , \end{aligned} \tag{33}$$

where the unitarity conditions given in Eq. (29) have been used.

The weak charged-current interactions of active neutrinos in Eq. (5) can now be rewritten in terms of the mass eigenstates of six Majorana neutrinos via $\nu_{\rm L} = V\nu'_{\rm L} + R(N'_{\rm R})^c$. In the chosen basis where the flavor and mass eigenstates of three charged leptons are identical, we have

$$\mathcal{L}_{\rm cc} = \frac{g}{\sqrt{2}} \overline{(e \; \mu \; \tau)_{\rm L}} \, \gamma^\mu \left[V \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}_{\rm L} + R \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix}_{\rm L} \right] W^-_\mu + {\rm h.c.} \; . \tag{34}$$

Note that V and R are responsible for the charged-current interactions of three known neutrinos ν_i and three new neutrinos N_i (for $i = 1, 2, 3$), respectively. Their correlation is described by $VV^\dagger + RR^\dagger = \mathbf{1}$, and thus V is not unitary unless ν_i and N_i are completely decoupled (i.e., $R = \mathbf{0}$).

As a consequence of lepton number violation, the $0\nu 2\beta$ decay $(A, Z) \rightarrow (A, Z+2) + 2e^-$ can now take place via the exchanges of both ν_i and N_i between two beta decays, whose coupling matrix elements are V_{ei} and R_{ei} respectively. The relative contributions of ν_i and N_i to this lepton-number-violating process depend not only on m_i, M_i, V_{ei} and R_{ei} but also on the relevant nuclear matrix elements which cannot be reliably evaluated. For a realistic seesaw mechanism working at the TeV scale (i.e., $M_i \sim \mathcal{O}(1)$ TeV) or at a superhigh-energy scale, however, the contribution of ν_i to the $0\nu 2\beta$ decay is in most cases dominant.

The hybrid neutrino mass terms in Eq. (26) provide us with the necessary ingredients of a dynamic mechanism to interpret why three known neutrinos have non-zero but tiny masses. The key point is that the mass scales of $M_{\rm L}$, $M_{\rm D}$ and $M_{\rm R}$ may have a strong hierarchy. First, $M_{\rm D} \sim \langle H \rangle \approx 174$ GeV is naturally characterized by the electroweak symmetry breaking scale. Second, $M_{\rm L} \ll \langle H \rangle$ satisfies 't Hooft's naturalness criterion because this Majorana mass term violates lepton number conservation. Third, $M_{\rm R} \gg \langle H \rangle$ is naturally expected since right-handed neutrinos are $SU(2)_{\rm L}$ gauge singlets and thus their mass term is not subject to the electroweak symmetry breaking scale. The hierarchy $M_{\rm R} \gg M_{\rm D} \gg M_{\rm L}$ can therefore allow us to make reliable approximations in deriving the effective mass matrix of three active neutrinos $(\nu_e, \nu_\mu, \nu_\tau)$ from Eq. (28). The latter yields

$$\begin{aligned} R\widehat{M}_N &= M_{\rm L} R^* + M_{\rm D} U^* \; , \\ S\widehat{M}_\nu &= M_{\rm D}^T V^* + M_{\rm R} S^* \; ; \end{aligned} \tag{35}$$

and

$$\begin{aligned} U\widehat{M}_N &= M_{\rm R} U^* + M_{\rm D}^T R^* \; , \\ V\widehat{M}_\nu &= M_{\rm L} V^* + M_{\rm D} S^* \; . \end{aligned} \tag{36}$$

Given $M_{\rm R} \gg M_{\rm D} \gg M_{\rm L}$, $R \sim S \sim \mathcal{O}(M_{\rm D}/M_{\rm R})$ naturally holds, implying that U and V are almost unitary up to the accuracy of $\mathcal{O}(M_{\rm D}^2/M_{\rm R}^2)$. Hence Eq. (36) leads to

$$\begin{aligned} U\widehat{M}_N U^T &= M_{\rm R} (UU^\dagger)^T + M_{\rm D}^T (R^* U^T) \approx M_{\rm R} \; , \\ V\widehat{M}_\nu V^T &= M_{\rm L} (VV^\dagger)^T + M_{\rm D} (S^* V^T) \approx M_{\rm L} + M_{\rm D} (S^* V^T) \; . \end{aligned} \tag{37}$$

$S^* V^T = M_{\rm R}^{-1} S \widehat{M}_\nu V^T - M_{\rm R}^{-1} M_{\rm D}^T (VV^\dagger)^T \approx -M_{\rm R}^{-1} M_{\rm D}^T$ can be derived from Eq. (35). We substitute this expression into Eq. (37) and then obtain

$$M_\nu \equiv V\widehat{M}_\nu V^T \approx M_{\rm L} - M_{\rm D} M_{\rm R}^{-1} M_{\rm D}^T \; . \tag{38}$$

This result, known as the type-(I+II) seesaw relation, is just the effective mass matrix of three light neutrinos. The small mass scale of M_ν is attributed to the small mass scale of $M_{\rm L}$ and the large mass scale of $M_{\rm R}$. There are two particularly interesting limits: (1) If $M_{\rm L}$ is absent from Eq. (26), one will be left with the canonical or type-I seesaw relation $M_\nu \approx -M_{\rm D} M_{\rm R}^{-1} M_{\rm D}^T$; (2) If only $M_{\rm L}$ is present in Eq. (26), one will get the type-II seesaw relation $M_\nu = M_{\rm L}$. More detailed discussions about various seesaw mechanisms and their phenomenological consequences will be presented in sections 6, 7 and 8.

2 Diagnosis of CP Violation

2.1 C, P and T transformations

We begin with a brief summary of the transformation properties of quantum fields under the discrete space-time symmetries of parity (P), charge conjugation (C) and time reversal (T). The parity transformation changes the space coordinates $\vec{x}$ into $-\vec{x}$. The charge conjugation flips the signs of internal charges of a particle, such as the electric charge and the lepton (baryon) number. The time reversal reflects the time coordinate t into $-t$.

A free Dirac spinor $\psi(t,\vec{x})$ or $\overline{\psi}(t,\vec{x})$ transforms under C, P and T as [3]

$$\begin{aligned} \psi(t,\vec{x}) &\xrightarrow{\rm C} \mathcal{C}\overline{\psi}^T(t,\vec{x}) \; , \\ \overline{\psi}(t,\vec{x}) &\xrightarrow{\rm C} -\psi^T(t,\vec{x})\mathcal{C}^{-1} \; , \end{aligned}$$

[3] For simplicity, here we have omitted a phase factor associated with each transformation. Because one is always interested in the spinor bilinears, the relevant phase factor usually plays no physical role.

Table 1: Transformation properties of the scalar-, pseudoscalar-, vector-, pseudovector- and tensor-like spinor bilinears under C, P and T. Here $\vec{x} \to -\vec{x}$ under P, CP and CPT, together with $t \to -t$ under T and CPT, is hidden and self-explaining for ψ_1 and ψ_2.

	$\overline{\psi_1}\psi_2$	$i\overline{\psi_1}\gamma_5\psi_2$	$\overline{\psi_1}\gamma_\mu\psi_2$	$\overline{\psi_1}\gamma_\mu\gamma_5\psi_2$	$\overline{\psi_1}\sigma_{\mu\nu}\psi_2$
C	$\overline{\psi_2}\psi_1$	$i\overline{\psi_2}\gamma_5\psi_1$	$-\overline{\psi_2}\gamma_\mu\psi_1$	$\overline{\psi_2}\gamma_\mu\gamma_5\psi_1$	$-\overline{\psi_2}\sigma_{\mu\nu}\psi_1$
P	$\overline{\psi_1}\psi_2$	$-i\overline{\psi_1}\gamma_5\psi_2$	$\overline{\psi_1}\gamma^\mu\psi_2$	$-\overline{\psi_1}\gamma^\mu\gamma_5\psi_2$	$\overline{\psi_1}\sigma^{\mu\nu}\psi_2$
T	$\overline{\psi_1}\psi_2$	$-i\overline{\psi_1}\gamma_5\psi_2$	$\overline{\psi_1}\gamma^\mu\psi_2$	$\overline{\psi_1}\gamma^\mu\gamma_5\psi_2$	$-\overline{\psi_1}\sigma^{\mu\nu}\psi_2$
CP	$\overline{\psi_2}\psi_1$	$-i\overline{\psi_2}\gamma_5\psi_1$	$-\overline{\psi_2}\gamma^\mu\psi_1$	$-\overline{\psi_2}\gamma^\mu\gamma_5\psi_1$	$-\overline{\psi_2}\sigma^{\mu\nu}\psi_1$
CPT	$\overline{\psi_2}\psi_1$	$i\overline{\psi_2}\gamma_5\psi_1$	$-\overline{\psi_2}\gamma_\mu\psi_1$	$-\overline{\psi_2}\gamma_\mu\gamma_5\psi_1$	$\overline{\psi_2}\sigma_{\mu\nu}\psi_1$

$$\begin{aligned} \psi(t,\vec{x}) &\xrightarrow{\rm P} \mathcal{P}\psi(t,-\vec{x}) \ , \\ \overline{\psi}(t,\vec{x}) &\xrightarrow{\rm P} \overline{\psi}(t,-\vec{x})\mathcal{P}^\dagger \ , \\ \psi(t,\vec{x}) &\xrightarrow{\rm T} \mathcal{T}\psi(-t,\vec{x}) \ , \\ \overline{\psi}(t,\vec{x}) &\xrightarrow{\rm T} \overline{\psi}(-t,\vec{x})\mathcal{T}^\dagger \ , \end{aligned} \tag{39}$$

where $\mathcal{C} = i\gamma_2\gamma_0$, $\mathcal{P} = \gamma_0$ and $\mathcal{T} = \gamma_1\gamma_3$ in the Dirac-Pauli representation. These transformation properties can simply be deduced from the requirement that the Dirac equation $i\gamma_\mu\partial^\mu\psi(t,\vec{x}) = m\psi(t,\vec{x})$ be invariant under C, P or T operation. Note that all the classical numbers (or c-numbers), such as the coupling constants and γ-matrix elements, must be complex-conjugated under T. Note also that the charge-conjugation matrix $\mathcal{C}$ satisfies the conditions given in Eq. (4). It is very important to figure out how the Dirac spinor bilinears transform under C, P and T, because both leptons and quarks are described by spinor fields and they always appear in the bilinear forms in a Lorentz-invariant Lagrangian. Let us consider the following scalar-, pseudoscalar-, vector-, pseudovector- and tensor-like spinor bilinears: $\overline{\psi_1}\psi_2$, $i\overline{\psi_1}\gamma_5\psi_2$, $\overline{\psi_1}\gamma_\mu\psi_2$, $\overline{\psi_1}\gamma_\mu\gamma_5\psi_2$ and $\overline{\psi_1}\sigma_{\mu\nu}\psi_2$, where $\sigma_{\mu\nu} \equiv i[\gamma_\mu, \gamma_\nu]/2$ is defined. One may easily verify that all these bilinears are Hermitian. Under C, P and T, for example,

$$\begin{aligned} \overline{\psi_1}\gamma_\mu\psi_2 &\xrightarrow{\rm C} -\psi_1^T\mathcal{C}^{-1}\gamma_\mu\mathcal{C}\overline{\psi_2}^T = \psi_1^T\gamma_\mu^T\overline{\psi_2}^T = -\left[\overline{\psi_2}\gamma_\mu\psi_1\right]^T = -\overline{\psi_2}\gamma_\mu\psi_1 \ , \\ \overline{\psi_1}\gamma_\mu\psi_2 &\xrightarrow{\rm P} \overline{\psi_1}\gamma_0\gamma_\mu\gamma_0\psi_2 = \overline{\psi_1}\gamma^\mu\psi_2 \ , \\ \overline{\psi_1}\gamma_\mu\psi_2 &\xrightarrow{\rm T} \overline{\psi_1}\left(\gamma_1\gamma_3\right)^\dagger\gamma_\mu^*\left(\gamma_1\gamma_3\right)\psi_2 = \overline{\psi_1}\gamma^\mu\psi_2 \ ; \end{aligned} \tag{40}$$

and thus

$$\begin{aligned} \overline{\psi_1}\gamma_\mu\psi_2 &\xrightarrow{\rm CP} -\overline{\psi_2}\gamma^\mu\psi_1 \ , \\ \overline{\psi_1}\gamma_\mu\psi_2 &\xrightarrow{\rm CPT} -\overline{\psi_2}\gamma_\mu\psi_1 \ , \end{aligned} \tag{41}$$

with $\vec{x} \to -\vec{x}$ under P and $t \to -t$ under T for ψ_1 and ψ_2. The transformation properties of five spinor bilinears under C, P, T, CP and CPT are summarized in Table 1, where one should keep in mind that all the c-numbers are complex-conjugated under T and CPT.

It is well known that CPT is a good symmetry in a local quantum field theory which is Lorentz-invariant and possesses a Hermitian Lagrangian. The latter is necessary in order to have a unitary transition operator (i.e., the S-matrix). The CPT invariance of a theory implies that CP and T must be simultaneously conserving or broken, as already examined in the quark sector of the SM via the K^0-$\bar{K}^0$ mixing system. After a slight modification of the SM by introducing the Dirac or Majorana mass term for three neutrinos, one may also look at possible sources of CP or T violation in the lepton sector.

2.2 The source of CP violation

The SM of electroweak interactions is based on the $SU(2)_{\rm L} \times U(1)_{\rm Y}$ gauge symmetry and the Higgs mechanism. The latter triggers the spontaneous symmetry breaking $SU(2)_{\rm L} \times U(1)_{\rm Y} \to U(1)_{\rm Q}$, such that three gauge bosons, three charged leptons and six quarks can all acquire masses. But this mechanism itself does not spontaneously break CP, and thus one may examine the source of CP violation in the SM either before or after spontaneous symmetry breaking.

The Lagrangian of the SM $\mathcal{L} = \mathcal{L}_{\rm G} + \mathcal{L}_{\rm H} + \mathcal{L}_{\rm F} + \mathcal{L}_{\rm Y}$ is composed of four parts: the kinetic term of the gauge fields and their self-interactions ($\mathcal{L}_{\rm G}$), the kinetic term of the Higgs doublet and its potential and interactions with the gauge fields ($\mathcal{L}_{\rm H}$), the kinetic term of the fermion fields and their interactions with the gauge fields ($\mathcal{L}_{\rm F}$), and the Yukawa interactions of the fermion fields with the Higgs doublet ($\mathcal{L}_{\rm Y}$):

$$
\begin{aligned}
\mathcal{L}_{\rm G} &= -\frac{1}{4}\left(W^{i\mu\nu}W^i_{\mu\nu} + B^{\mu\nu}B_{\mu\nu}\right) \ , \\
\mathcal{L}_{\rm H} &= \left(D^\mu H\right)^\dagger \left(D_\mu H\right) - \mu^2 H^\dagger H - \lambda \left(H^\dagger H\right)^2 \ , \\
\mathcal{L}_{\rm F} &= \overline{Q_{\rm L}} i\not{D} Q_{\rm L} + \overline{\ell_{\rm L}} i\not{D} \ell_{\rm L} + \overline{U_{\rm R}} i\not{\partial}' U_{\rm R} + \overline{D_{\rm R}} i\not{\partial}' D_{\rm R} + \overline{E_{\rm R}} i\not{\partial}' E_{\rm R} \ , \\
\mathcal{L}_{\rm Y} &= -\overline{Q_{\rm L}} Y_{\rm u} \tilde{H} U_{\rm R} - \overline{Q_{\rm L}} Y_{\rm d} H D_{\rm R} - \overline{\ell_{\rm L}} Y_l H E_{\rm R} + {\rm h.c.} \ ,
\end{aligned}
\tag{42}
$$

whose notations are self-explanatory. To accommodate massive neutrinos, the simplest way is to slightly modify the $\mathcal{L}_{\rm F}$ and $\mathcal{L}_{\rm Y}$ parts (e.g., by introducing three right-handed neutrinos into the SM and allowing for the Yukawa interactions between neutrinos and the Higgs doublet). CP violation is due to the coexistence of $\mathcal{L}_{\rm F}$ and $\mathcal{L}_{\rm Y}$.

We first show that $\mathcal{L}_{\rm G}$ is always invariant under CP. The transformation properties of gauge fields B_μ and W^i_μ under C and P are

$$
\begin{aligned}
\left[B_\mu,\ W^1_\mu,\ W^2_\mu,\ W^3_\mu\right] &\xrightarrow{\rm C} \left[-B_\mu,\ -W^1_\mu,\ +W^2_\mu,\ -W^3_\mu\right] \ , \\
\left[B_\mu,\ W^1_\mu,\ W^2_\mu,\ W^3_\mu\right] &\xrightarrow{\rm P} \left[B^\mu,\ W^{1\mu},\ W^{2\mu},\ W^{3\mu}\right] \ , \\
\left[B_\mu,\ W^1_\mu,\ W^2_\mu,\ W^3_\mu\right] &\xrightarrow{\rm CP} \left[-B^\mu,\ -W^{1\mu},\ +W^{2\mu},\ -W^{3\mu}\right]
\end{aligned}
\tag{43}
$$

with $\vec{x} \to -\vec{x}$ under P and CP for relevant fields. Then the gauge field tensors $B_{\mu\nu}$ and $W^i_{\mu\nu}$ transform under CP as follows:

$$
\left[B_{\mu\nu},\ W^1_{\mu\nu},\ W^2_{\mu\nu},\ W^3_{\mu\nu}\right] \xrightarrow{\rm CP} \left[-B^{\mu\nu},\ -W^{1\mu\nu},\ +W^{2\mu\nu},\ -W^{3\mu\nu}\right] \ . \tag{44}
$$

Hence $\mathcal{L}_{\rm G}$ is formally invariant under CP.

We proceed to show that $\mathcal{L}_{\rm H}$ is also invariant under CP. The Higgs doublet H contains two scalar components ϕ^+ and ϕ^0; i.e.,

$$
H = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \ , \quad H^\dagger = \begin{pmatrix} \phi^- & \phi^{0*} \end{pmatrix} \ . \tag{45}
$$

Therefore,

$$
H(t, \vec{x}) \xrightarrow{\rm CP} H^*(t, -\vec{x}) = \begin{pmatrix} \phi^- \\ \phi^{0*} \end{pmatrix} \ . \tag{46}
$$

It is very trivial to prove that the $H^\dagger H$ and $(H^\dagger H)^2$ terms of $\mathcal{L}_{\rm H}$ are CP-invariant. To examine how the $(D^\mu H)^\dagger (D_\mu H)$ term of $\mathcal{L}_{\rm H}$ transforms under CP, we explicitly write out

$$
D_\mu H = \left(\partial_\mu - i g \tau^k W^k_\mu - i g' Y B_\mu\right) H = \begin{pmatrix} \partial_\mu \phi^+ - i X^+_\mu \phi^0 - i Y^+_\mu \phi^+ \\ \partial_\mu \phi^0 - i X^-_\mu \phi^+ + i Y^-_\mu \phi^0 \end{pmatrix} \tag{47}
$$

with $X^\pm_\mu \equiv gW^\pm_\mu/\sqrt{2} = g(W^1_\mu \mp iW^2_\mu)/2$, $Y^\pm \equiv \pm g'YB_\mu + gW^3_\mu/2$, and $k = 1, 2, 3$. Note that

$$X^\pm_\mu \xrightarrow{\rm CP} -X^{\mp\mu} \; , \quad Y^\pm_\mu \xrightarrow{\rm CP} -Y^{\pm\mu} \; , \tag{48}$$

together with $\partial_\mu \to \partial^\mu$, $\phi^\pm \to \phi^\mp$ and $\phi^0 \to \phi^{0*}$ under CP. So it is easy to check that $(D^\mu H)^\dagger (D_\mu H)$ is also CP-invariant. Therefore, $\mathcal{L}_{\rm H}$ is formally invariant under CP.

The next step is to examine the CP invariance of $\mathcal{L}_{\rm F}$. To be more specific, we divide $\mathcal{L}_{\rm F}$ into the quark sector and the lepton sector; i.e., $\mathcal{L}_{\rm F} = \mathcal{L}_q + \mathcal{L}_l$. We only analyze the CP property of $\mathcal{L}_q$ in the following, because that of $\mathcal{L}_l$ can be analyzed in the same way. The explicit form of $\mathcal{L}_q$ reads

$$\begin{aligned}
\mathcal{L}_q = \overline{Q_{\rm L}} i\not{D} Q_{\rm L} + \overline{U_{\rm R}} i\not{\partial}' U_{\rm R} + \overline{D_{\rm R}} i\not{\partial}' D_{\rm R} = \sum_{j=1}^{3} \Big\{ & \frac{g}{2}\left[\overline{q'_j}\gamma^\mu P_{\rm L} W^1_\mu q_j + \overline{q_j}\gamma^\mu P_{\rm L} W^1_\mu q'_j\right] \\
& + \frac{g}{2}\left[i\overline{q'_j}\gamma^\mu P_{\rm L} W^2_\mu q_j - i\overline{q_j}\gamma^\mu P_{\rm L} W^2_\mu q'_j\right] \\
& + \frac{g}{2}\left[\overline{q_j}\gamma^\mu P_{\rm L} W^3_\mu q_j - \overline{q'_j}\gamma^\mu P_{\rm L} W^3_\mu q'_j\right] \\
& + i\left[\overline{q_j}\gamma^\mu P_{\rm L}\left(\partial_\mu - i\frac{g'}{6}B_\mu\right) q_j\right] \\
& + i\left[\overline{q'_j}\gamma^\mu P_{\rm L}\left(\partial_\mu - i\frac{g'}{6}B_\mu\right) q'_j\right] \\
& + i\left[\overline{q_j}\gamma^\mu P_{\rm R}\left(\partial_\mu - i\frac{2g'}{3}B_\mu\right) q_j\right] \\
& + i\left[\overline{q'_j}\gamma^\mu P_{\rm R}\left(\partial_\mu + i\frac{g'}{3}B_\mu\right) q'_j\right]\Big\} \; ,
\end{aligned} \tag{49}$$

where q_j and q'_j (for $j = 1, 2, 3$) run over (u, c, t) and (d, s, b), respectively. The transformation properties of gauge fields B_μ and W^i_μ under C and P have been given in Eq. (43). With the help of Table 1, one can see that the relevant spinor bilinears transform under C and P as follows:

$$\begin{aligned}
\overline{\psi_1}\gamma_\mu (1 \pm \gamma_5)\psi_2 &\xrightarrow{\rm C} -\overline{\psi_2}\gamma_\mu (1 \mp \gamma_5)\psi_1 \; , \\
\overline{\psi_1}\gamma_\mu (1 \pm \gamma_5)\psi_2 &\xrightarrow{\rm P} +\overline{\psi_1}\gamma^\mu (1 \mp \gamma_5)\psi_2 \; , \\
\overline{\psi_1}\gamma_\mu (1 \pm \gamma_5)\psi_2 &\xrightarrow{\rm CP} -\overline{\psi_2}\gamma^\mu (1 \pm \gamma_5)\psi_1 \; ,
\end{aligned} \tag{50}$$

with $\vec{x} \to -\vec{x}$ under P and CP for ψ_1 and ψ_2. Furthermore,

$$\begin{aligned}
\overline{\psi_1}\gamma_\mu (1 \pm \gamma_5)\partial^\mu\psi_2 &\xrightarrow{\rm C} \overline{\psi_2}\gamma_\mu (1 \mp \gamma_5)\partial^\mu\psi_1 \; , \\
\overline{\psi_1}\gamma_\mu (1 \pm \gamma_5)\partial^\mu\psi_2 &\xrightarrow{\rm P} \overline{\psi_1}\gamma^\mu (1 \mp \gamma_5)\partial_\mu\psi_2 \; , \\
\overline{\psi_1}\gamma_\mu (1 \pm \gamma_5)\partial^\mu\psi_2 &\xrightarrow{\rm CP} \overline{\psi_2}\gamma^\mu (1 \pm \gamma_5)\partial_\mu\psi_1 \; ,
\end{aligned} \tag{51}$$

with $\vec{x} \to -\vec{x}$ under P and CP for ψ_1 and ψ_2. It is straightforward to check that $\mathcal{L}_q$ in Eq. (49) is formally invariant under CP. Following the same procedure and using Eqs. (49), (50) and (51), one can easily show that $\mathcal{L}_l = \overline{\ell_{\rm L}} i\not{D}\ell_{\rm L} + \overline{E_{\rm R}} i\not{\partial}' E_{\rm R}$ is also CP-invariant. Thus we conclude that $\mathcal{L}_{\rm F}$ is invariant under CP.

The last step is to examine whether $\mathcal{L}_{\rm Y}$ is CP-conserving or not. Explicitly,

$$\begin{aligned}
-\mathcal{L}_{\rm Y} &= \overline{Q_{\rm L}} Y_{\rm u}\tilde{H} U_{\rm R} + \overline{Q_{\rm L}} Y_{\rm d} H D_{\rm R} + \overline{\ell_{\rm L}} Y_l H E_{\rm R} + {\rm h.c.} \\
&= \sum_{j,k=1}^{3} \Big\{ (Y_{\rm u})_{jk}\left[\overline{q_j} P_{\rm R} q_k \phi^{0*} - \overline{q'_j} P_{\rm R} q_k \phi^-\right]
\end{aligned}$$

$$
\begin{aligned}
&+(Y_{\rm u})^*_{jk}\left[\overline{q_k}P_{\rm L}q_j\phi^0 - \overline{q_k}P_{\rm L}q'_j\phi^+\right] \\
&+(Y_{\rm d})_{jk}\left[\overline{q_j}P_{\rm R}q'_k\phi^+ + \overline{q'_j}P_{\rm R}q'_k\phi^0\right] \\
&+(Y_{\rm d})^*_{jk}\left[\overline{q'_k}P_{\rm L}q_j\phi^- + \overline{q'_k}P_{\rm L}q'_j\phi^{0*}\right] \\
&+(Y_l)_{jk}\left[\overline{\nu_j}P_{\rm R}l_k\phi^+ + \overline{l_j}P_{\rm R}l_k\phi^0\right] \\
&+(Y_l)^*_{jk}\left[\overline{l_k}P_{\rm L}\nu_j\phi^- + \overline{l_k}P_{\rm L}l_j\phi^{0*}\right]\Big\} \; , \qquad (52)
\end{aligned}
$$

where q_j and q'_j (for $j = 1, 2, 3$) run over (u, c, t) and (d, s, b), respectively; while ν_j and l_j (for $j = 1, 2, 3$) run over $(\nu_e, \nu_\mu, \nu_\tau)$ and (e, μ, τ), respectively. Because of $\phi^\pm \to \phi^\mp$, $\phi^0 \to \phi^{0*}$ and $\overline{\psi_1}(1 \pm \gamma_5)\psi_2 \to \overline{\psi_2}(1 \mp \gamma_5)\psi_1$ under CP, we immediately arrive at

$$
\begin{aligned}
-\mathcal{L}_{\rm Y} \stackrel{\rm CP}{\longrightarrow} \sum_{j,k=1}^{3} \Big\{ &(Y_{\rm u})_{jk}\left[\overline{q_k}P_{\rm L}q_j\phi^0 - \overline{q_k}P_{\rm L}q'_j\phi^+\right] \\
&+(Y_{\rm u})^*_{jk}\left[\overline{q_j}P_{\rm R}q_k\phi^{0*} - \overline{q'_j}P_{\rm R}q_k\phi^-\right] \\
&+(Y_{\rm d})_{jk}\left[\overline{q'_k}P_{\rm L}q_j\phi^- + \overline{q'_k}P_{\rm L}q'_j\phi^{0*}\right] \\
&+(Y_{\rm d})^*_{jk}\left[\overline{q_j}P_{\rm R}q'_k\phi^+ + \overline{q'_j}P_{\rm R}q'_k\phi^0\right] \\
&+(Y_l)_{jk}\left[\overline{l_k}P_{\rm L}\nu_j\phi^- + \overline{l_k}P_{\rm L}l_j\phi^{0*}\right] \\
&+(Y_l)^*_{jk}\left[\overline{\nu_j}P_{\rm R}l_k\phi^+ + \overline{l_j}P_{\rm R}l_k\phi^0\right]\Big\} \; , \qquad (53)
\end{aligned}
$$

with $\vec{x} \to -\vec{x}$ for both scalar and spinor fields under consideration. Comparing between Eqs. (52) and (53), we see that $\mathcal{L}_{\rm Y}$ will be formally invariant under CP if the conditions

$$
(Y_{\rm u})_{jk} = (Y_{\rm u})^*_{jk} \; , \quad (Y_{\rm d})_{jk} = (Y_{\rm d})^*_{jk} \; , \quad (Y_l)_{jk} = (Y_l)^*_{jk} \qquad (54)
$$

are satisfied. In other words, the Yukawa coupling matrices $Y_{\rm u}$, $Y_{\rm d}$ and Y_l must be real to guarantee the CP invartiance of $\mathcal{L}_{\rm Y}$. Given three massless neutrinos in the SM, it is always possible to make Y_l real by redefining the phases of charged-lepton fields. But it is in general impossible to make both $Y_{\rm u}$ and $Y_{\rm d}$ real for three families of quarks, and thus CP violation can only appear in the quark sector.

Given massive neutrinos beyond the SM, $\mathcal{L}_{\rm Y}$ must be modified. The simplest way is to introduce three right-handed neutrinos and incorporate the Dirac neutrino mass term in Eq. (6) into $\mathcal{L}_{\rm Y}$. In this case one should also add the kinetic term of three right-handed neutrinos into $\mathcal{L}_{\rm F}$. It is straightforward to show that the conditions of CP invariance in the lepton sector turn out to be

$$
Y_\nu = Y^*_\nu \; , \qquad Y_l = Y^*_l \; , \qquad (55)
$$

exactly in parallel with the quark sector. If an effective Majorana mass term is introduced into $\mathcal{L}_{\rm Y}$, as shown in Eq. (14), then the conditions of CP invariance in the lepton sector become

$$
M_{\rm L} = M^*_{\rm L} \; , \quad Y_l = Y^*_l \; , \qquad (56)
$$

where $M_{\rm L}$ is the effective Majorana neutrino mass matrix. One may diagonalize both Y_ν (or $M_{\rm L}$) and Y_l to make them real and positive, but such a treatment will transfer CP violation from the Yukawa interactions to the weak charged-current interactions. Then lepton flavor mixing and CP violation are described by the 3×3 unitary matrix V given in Eq. (12), analogous to the 3×3 unitary matrix of quark flavor mixing and CP violation. In other words, the source of CP violation is the irremovable complex phase(s) in the flavor mixing matrix of quarks or leptons. That is why we claim that CP violation stems from the coexistence of $\mathcal{L}_{\rm F}$ and $\mathcal{L}_{\rm Y}$ within the SM and, in most cases, beyond the SM.

It is worth reiterating that the process of spontaneous gauge symmetry breaking in the SM does not spontaneously violate CP. After the Higgs doublet H acquires its vacuum expectation value (i.e., $\phi^+ \to 0$ and $\phi^0 \to v/\sqrt{2}$ with v being real), we obtain three massive gauge bosons $W^\pm_\mu$ and Z_μ as well as one massless gauge boson A_μ. According to their relations with W^i_μ and B_μ, it is easy to find out the transformation properties of these physical fields under CP:

$$W^\pm_\mu \xrightarrow{\rm CP} -W^{\mp\mu} \; , \quad Z_\mu \xrightarrow{\rm CP} -Z^\mu \; , \quad A_\mu \xrightarrow{\rm CP} -A^\mu \; , \tag{57}$$

with $\vec{x} \to -\vec{x}$ under P and CP for each field. In contrast, the neutral Higgs boson h is a CP-even particle. After spontaneous electroweak symmetry breaking, we are left with the quark mass matrices $M_{\rm u} = vY_{\rm u}/\sqrt{2}$ and $M_{\rm d} = vY_{\rm d}/\sqrt{2}$ or the lepton mass matrices $M_{\rm D} = vY_\nu/\sqrt{2}$ and $M_l = vY_l/\sqrt{2}$. The conditions of CP invariance given above can therefore be replaced with the corresponding mass matrices.

3 Electromagnetic Properties

3.1 Electromagnetic form factors

Although a neutrino does not possess any electric charge, it can have electromagnetic interactions via quantum loops. One may summarize such interactions by means of the following effective interaction term:

$$\mathcal{L}_{\rm EM} = \overline{\psi}\Gamma_\mu \psi A^\mu \equiv J_\mu(x) A^\mu(x) \; , \tag{58}$$

where the form of the electromagnetic current $J_\mu(x)$ is our present concern. Dirac and Majorana neutrinos couple to the photon in different ways, which are described by their respective electromagnetic form factors.

For an arbitrary Dirac particle (e.g., a Dirac neutrino), let us write down the matrix element of $J_\mu(x)$ between two one-particle states:

$$\langle \psi(p')|J_\mu(x)|\psi(p)\rangle = e^{-iqx}\langle \psi(p')|J_\mu(0)|\psi(p)\rangle = e^{-iqx}\overline{u}(\vec{p}')\Gamma_\mu(p,p')u(\vec{p}) \tag{59}$$

with $q = p - p'$. Because $J_\mu(x)$ is a Lorentz vector, the electromagnetic vertex function $\Gamma_\mu(p,p')$ must be a Lorentz vector too. The electromagnetic current conservation (or $U(1)_{\rm Q}$ gauge symmetry) requires $\partial^\mu J_\mu(x) = 0$, leading to

$$\langle \psi(p')|\partial^\mu J_\mu(x)|\psi(p)\rangle = (-iq^\mu)\, e^{-iqx}\overline{u}(\vec{p}')\Gamma_\mu(p,p')u(\vec{p}) = 0 \; . \tag{60}$$

Thus

$$q^\mu \overline{u}(\vec{p}')\Gamma_\mu(p,p')u(\vec{p}) = 0 \tag{61}$$

holds as one of the model-independent constraints on the form of $\Gamma_\mu(p,p')$. In addition, the Hermiticity of $J_\mu(x)$ or its matrix element implies

$$\begin{aligned} & e^{-iqx}\overline{u}(\vec{p}')\Gamma_\mu(p,p')u(\vec{p}) = e^{+iqx}\left[\overline{u}(\vec{p}')\Gamma_\mu(p,p')u(\vec{p})\right]^\dagger \\ & = e^{+iqx}\overline{u}(\vec{p})\left[\gamma_0\Gamma^\dagger_\mu(p,p')\gamma_0\right]u(\vec{p}') = e^{-iqx}\overline{u}(\vec{p}')\left[\gamma_0\Gamma^\dagger_\mu(p',p)\gamma_0\right]u(\vec{p}) \; , \end{aligned} \tag{62}$$

from which we immediately arrive at the second constraint on $\Gamma_\mu(p,p')$:

$$\Gamma_\mu(p,p') = \gamma_0\Gamma^\dagger_\mu(p',p)\gamma_0 \; . \tag{63}$$

Because of $p^2 = p'^2 = m^2$ with m being the fermion mass, we have $(p+p')^2 = 4m^2 - q^2$. Hence $\Gamma_\mu(p,p')$ depends only on the Lorentz-invariant quantity q^2.

A careful analysis of the Lorentz structure of $\overline{u}(\vec{p}')\Gamma_\mu(p,p')u(\vec{p})$, with the help of the Gordon-like identities and the constraints given above, shows that $\Gamma_\mu(p,p')$ may in general consist of four independent terms:

$$\Gamma_\mu(p,p') = f_{\rm Q}(q^2)\gamma_\mu + f_{\rm M}(q^2)i\sigma_{\mu\nu}q^\nu + f_{\rm E}(q^2)\sigma_{\mu\nu}q^\nu\gamma_5 + f_{\rm A}(q^2)\left(q^2\gamma_\mu - q_\mu \not{q}\right)\gamma_5 \; , \tag{64}$$

where $f_{\rm Q}(q^2)$, $f_{\rm M}(q^2)$, $f_{\rm E}(q^2)$ and $f_{\rm A}(q^2)$ are usually referred to as the charge, magnetic dipole, electric dipole and anapole form factors, respectively. In the non-relativistic limit of $\mathcal{L}_{\rm EM}$, it is easy to find that $f_{\rm Q}(0) = Q$ represents the electric charge of the particle, $f_{\rm M}(0) \equiv \mu$ denotes the magnetic dipole moment of the particle (i.e., $\mathcal{L}_{\rm EM}(f_{\rm M}) = -\mu\vec{\sigma}\cdot\vec{B}$ with $\vec{B}$ being the static magnetic field), $f_{\rm E}(0) \equiv \epsilon$ stands for the electric dipole moment of the particle (i.e., $\mathcal{L}_{\rm EM}(f_{\rm E}) = -\epsilon\vec{\sigma}\cdot\vec{E}$ with $\vec{E}$ being the static electric field), and $f_{\rm A}(0)$ corresponds to the Zeldovich anapole moment of the particle (i.e., $\mathcal{L}_{\rm EM}(f_{\rm A}) \propto f_{\rm A}(0)\vec{\sigma}\cdot[\nabla\times\vec{B} - \dot{\vec{E}}]$). One can observe that these form factors are not only Lorentz-invariant but also real (i.e., ${\rm Im}f_{\rm Q} = {\rm Im}f_{\rm M} = {\rm Im}f_{\rm E} = {\rm Im}f_{\rm A} = 0$). The latter is actually guaranteed by the Hermiticity condition in Eq. (62).

Given the form of Γ_μ in Eq. (64), it is straightforward to check the CP properties of $\mathcal{L}_{\rm EM}$ in Eq. (58). Note that the photon field transforms as $A^\mu \to -A_\mu$ under CP, and [4]

$$\begin{aligned}
\overline{\psi}\gamma_\mu\psi &\xrightarrow{\rm CP} -\overline{\psi}\gamma^\mu\psi \; , \\
\overline{\psi}\gamma_\mu\gamma_5\psi &\xrightarrow{\rm CP} -\overline{\psi}\gamma^\mu\gamma_5\psi \; , \\
\overline{\psi}\sigma_{\mu\nu}\psi &\xrightarrow{\rm CP} -\overline{\psi}\sigma^{\mu\nu}\psi \; , \\
\overline{\psi}\sigma_{\mu\nu}\gamma_5\psi &\xrightarrow{\rm CP} +\overline{\psi}\sigma^{\mu\nu}\gamma_5\psi \; .
\end{aligned} \tag{65}$$

Hence only the term proportional to $f_{\rm E}$ in $\mathcal{L}_{\rm EM}$ is CP-violating. If CP were conserved, then this term would vanish (i.e., $f_{\rm E} = 0$ would hold). Although there is no experimental hint at CP violation in the lepton sector, we expect that it should exist as in the quark sector. In any case, all four form factors are finite for a Dirac neutrino.

If neutrinos are massive Majorana particles, their electromagnetic properties will be rather different. The reason is simply that Majorana particles are their own antiparticles and thus can be described by using a smaller number of degrees of freedom. A free Majorana neutrino field ψ is by definition equal to its charge-conjugate field $\psi^c = C\overline{\psi}^T$ up to a global phase. Then

$$\overline{\psi}\Gamma_\mu\psi = \overline{\psi^c}\Gamma_\mu\psi^c = \psi^T\mathcal{C}\Gamma_\mu\mathcal{C}\overline{\psi}^T = \left(\psi^T\mathcal{C}\Gamma_\mu\mathcal{C}\overline{\psi}^T\right)^T = -\overline{\psi}\mathcal{C}^T\Gamma_\mu^T\mathcal{C}^T\psi \; , \tag{66}$$

from which one arrives at

$$\Gamma_\mu = -\mathcal{C}^T\Gamma_\mu^T\mathcal{C}^T = \mathcal{C}\Gamma_\mu^T\mathcal{C}^{-1} \; . \tag{67}$$

Substituting Eq. (64) into the right-hand side of Eq. (67) and taking account of $\mathcal{C}\gamma_\mu^T\mathcal{C}^{-1} = -\gamma_\mu$, $\mathcal{C}(\gamma_\mu\gamma_5)^T\mathcal{C}^{-1} = +\gamma_\mu\gamma_5$, $\mathcal{C}\sigma_{\mu\nu}^T\mathcal{C}^{-1} = -\sigma_{\mu\nu}$ and $\mathcal{C}(\sigma_{\mu\nu}\gamma_5)^T\mathcal{C}^{-1} = -\sigma_{\mu\nu}\gamma_5$, we obtain

$$\Gamma_\mu(p,p') = -f_{\rm Q}(q^2)\gamma_\mu - f_{\rm M}(q^2)i\sigma_{\mu\nu}q^\nu - f_{\rm E}(q^2)\sigma_{\mu\nu}q^\nu\gamma_5 + f_{\rm A}(q^2)\left(q^2\gamma_\mu - q_\mu\not{q}\right)\gamma_5 \; . \tag{68}$$

A comparison between Eqs. (64) and (68) yields

$$f_{\rm Q}(q^2) = f_{\rm M}(q^2) = f_{\rm E}(q^2) = 0 \; . \tag{69}$$

This result means that a Majorana neutrino only has the anapole form factor $f_{\rm A}(q^2)$.

[4] Taking account of $\mathcal{C}^{-1}\sigma_{\mu\nu}\mathcal{C} = -\sigma_{\mu\nu}^T$ and $\mathcal{C}^{-1}\gamma_5\mathcal{C} = \gamma_5^T$, one may easily prove that $\overline{\psi}\sigma_{\mu\nu}\gamma_5\psi$ is odd under both C and P. Thus $\overline{\psi}\sigma_{\mu\nu}\gamma_5\psi$ is CP-even.

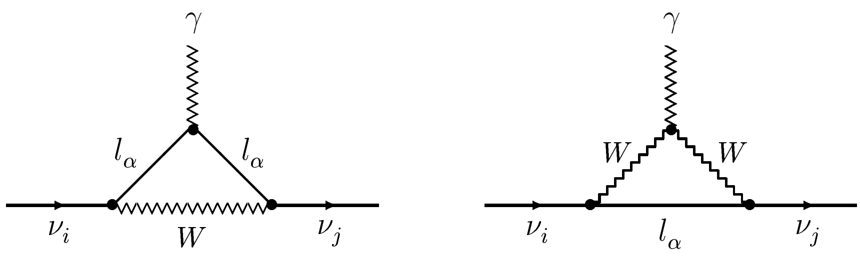

Fig. 1: One-loop Feynman diagrams contributing to the magnetic and electric dipole moments of massive Dirac neutrinos, where $\alpha = e, \mu, \tau$ and $i, j = 1, 2, 3$.

More generally, one may write out the matrix elements of the electromagnetic current $J_\mu(x)$ between two different states (i.e., the incoming and outgoing particles are different):

$$\langle \psi_j(p')|J_\mu(x)|\psi_i(p)\rangle = e^{-iqx}\overline{u_j}(\vec{p}')\Gamma^{ij}_\mu(p,p')u_i(\vec{p}) \; , \tag{70}$$

where $q = p - p'$ together with $p^2 = m_i^2$ and $p'^2 = m_j^2$ (for $i \neq j$). Here the electromagnetic vertex matrix $\Gamma_\mu(p,p')$ can be decomposed into the following Lorentz-invariant form in terms of four form factors:

$$\Gamma_\mu(p,p') = F_{\rm Q}(q^2)\left(q^2\gamma_\mu - q_\mu \not{q}\right) + F_{\rm M}(q^2) i\sigma_{\mu\nu}q^\nu + F_{\rm E}(q^2)\sigma_{\mu\nu}q^\nu\gamma_5 + F_{\rm A}(q^2)\left(q^2\gamma_\mu - q_\mu\not{q}\right)\gamma_5 \; , \tag{71}$$

where $F_{\rm Q}$, $F_{\rm M}$, $F_{\rm E}$ and $F_{\rm A}$ are all the 2×2 matrices in the space of neutrino mass eigenstates. The diagonal case (i.e., $i = j$) has been discussed above, from Eq. (59) to Eq. (69). In the off-diagonal case (i.e., $i \neq j$), the Hermiticity of $J_\mu(x)$ is no more a constraint on $\Gamma_\mu(p,p')$ for Dirac neutrinos because Eq. (62) only holds for $i = j$. It is now possible for Majorana neutrinos to have finite *transition* dipole moments, simply because Eqs. (66)—(69) do not hold when ψ_i and ψ_j represent different flavors.

We conclude that Dirac neutrinos may have both electric and magnetic dipole moments, while Majorana neutrinos have neither electric nor magnetic dipole moments. But massive Majorana neutrinos can have *transition* dipole moments which involve two different neutrino flavors in the initial and final states, so can massive Dirac neutrinos.

3.2 Magnetic and electric dipole moments

The magnetic and electric dipole moments of massive neutrinos, denoted as $\mu \equiv F_{\rm M}(0)$ and $\epsilon \equiv F_{\rm E}(0)$, are interesting in both theories and experiments because they are closely related to the dynamics of neutrino mass generation and to the characteristic of new physics.

Let us consider a minimal extension of the SM in which three right-handed neutrinos are introduced and lepton number conservation is required. In this case massive neutrinos are Dirac particles and their magnetic and electric dipole moments can be evaluated by calculating the Feynman diagrams in Fig. 1. Taking account of the smallness of both m_α^2/M_W^2 and m_i^2/M_W^2, where m_α (for $\alpha = e, \mu, \tau$) and m_i (for $i = 1, 2, 3$) stand respectively for the charged-lepton and neutrino masses, one obtains

$$\begin{aligned} \mu^{\rm D}_{ij} &= \frac{3eG_{\rm F}m_i}{32\sqrt{2}\pi^2}\left(1 + \frac{m_j}{m_i}\right) \times \sum_\alpha \left(2 - \frac{m_\alpha^2}{M_W^2}\right) V_{\alpha i}V^*_{\alpha j} \; , \\ \epsilon^{\rm D}_{ij} &= \frac{3eG_{\rm F}m_i}{32\sqrt{2}\pi^2}\left(1 - \frac{m_j}{m_i}\right) \times \sum_\alpha \left(2 - \frac{m_\alpha^2}{M_W^2}\right) V_{\alpha i}V^*_{\alpha j} \; , \end{aligned} \tag{72}$$

to an excellent degree of accuracy. Here $V_{\alpha i}$ and $V_{\alpha j}$ are the elements of the unitary lepton flavor mixing matrix V. Some discussions are in order.

(1) In the diagonal case (i.e., $i = j$), we are left with vanishing electric dipole moments (i.e., $\epsilon^{\rm D}_{ii} = 0$). The magnetic dipole moments $\mu^{\rm D}_{ii}$ are finite and proportional to the neutrino masses m_i (for $i = 1, 2, 3$):

$$\mu^{\rm D}_{ii} = \frac{3eG_{\rm F}m_i}{8\sqrt{2}\pi^2}\left(1 - \frac{1}{2}\sum_\alpha \frac{m^2_\alpha}{M^2_W}|V_{\alpha i}|^2\right) . \tag{73}$$

Hence a massless Dirac neutrino in the SM has no magnetic dipole moment. In the leading-order approximation, $\mu^{\rm D}_{ii}$ are independent of the strength of lepton flavor mixing and have tiny values

$$\mu^{\rm D}_{ii} \approx \frac{3eG_{\rm F}m_i}{8\sqrt{2}\pi^2} \approx 3\times 10^{-19}\left(\frac{m_i}{1~{\rm eV}}\right)\mu_{\rm B} , \tag{74}$$

where $\mu_{\rm B} = e\hbar/(2m_e)$ is the Bohr magneton. Given $m_i \leq 1$ eV, the magnitude of $\mu^{\rm D}_{ii}$ is far below its present experimental upper bound ($<$ a few $\times 10^{-11}\mu_{\rm B}$).

(2) In the off-diagonal case (i.e., $i \neq j$), the unitarity of V allows us to simplify Eq. (72) to

$$\mu^{\rm D}_{ij} = -\frac{3eG_{\rm F}m_i}{32\sqrt{2}\pi^2}\left(1 + \frac{m_j}{m_i}\right)\sum_\alpha \frac{m^2_\alpha}{M^2_W}V_{\alpha i}V^*_{\alpha j} ,$$
$$\epsilon^{\rm D}_{ij} = -\frac{3eG_{\rm F}m_i}{32\sqrt{2}\pi^2}\left(1 - \frac{m_j}{m_i}\right)\sum_\alpha \frac{m^2_\alpha}{M^2_W}V_{\alpha i}V^*_{\alpha j} . \tag{75}$$

We see that the magnitudes of $\mu^{\rm D}_{ij}$ and $\epsilon^{\rm D}_{ij}$ (for $i \neq j$), compared with that of $\mu^{\rm D}_{ii}$, are further suppressed due to the smallness of m^2_α/M^2_W. Similar to the expression given in Eq. (74),

$$\mu^{\rm D}_{ij} \approx -4\times 10^{-23}\left(\frac{m_i + m_j}{1~{\rm eV}}\right)\times\left(\sum_\alpha \frac{m^2_\alpha}{m^2_\tau}V_{\alpha i}V^*_{\alpha j}\right)\mu_{\rm B} ,$$
$$\epsilon^{\rm D}_{ij} \approx -4\times 10^{-23}\left(\frac{m_i - m_j}{1~{\rm eV}}\right)\times\left(\sum_\alpha \frac{m^2_\alpha}{m^2_\tau}V_{\alpha i}V^*_{\alpha j}\right)\mu_{\rm B} , \tag{76}$$

which can illustrate how small $\mu^{\rm D}_{ij}$ and $\epsilon^{\rm D}_{ij}$ are.

(3) Although Majorana neutrinos do not have intrinsic ($i = j$) magnetic and electric dipole moments, they may have finite transition ($i \neq j$) dipole moments. Because of the fact that Majorana neutrinos are their own antiparticles, their magnetic and electric dipole moments can also get contributions from two additional one-loop Feynman diagrams involving the charge-conjugate fields of ν_i, ν_j, l_α, $W^\pm$ and γ shown in Fig. 1 [5]. In this case one obtains

$$\mu^{\rm M}_{ij} = -\frac{3eG_{\rm F}i}{16\sqrt{2}\pi^2}\left(m_i + m_j\right)\times\sum_\alpha \frac{m^2_\alpha}{M^2_W}{\rm Im}\left(V_{\alpha i}V^*_{\alpha j}\right) ,$$
$$\epsilon^{\rm M}_{ij} = -\frac{3eG_{\rm F}}{16\sqrt{2}\pi^2}\left(m_i - m_j\right)\times\sum_\alpha \frac{m^2_\alpha}{M^2_W}{\rm Re}\left(V_{\alpha i}V^*_{\alpha j}\right) , \tag{77}$$

where $m_i \neq m_j$ must hold. Comparing between Eqs. (75) and (77), we observe that the magnitudes of $\mu^{\rm M}_{ij}$ and $\epsilon^{\rm M}_{ij}$ are the same order as those of $\mu^{\rm D}_{ij}$ and $\epsilon^{\rm D}_{ij}$ in most cases, although the CP-violating phases hidden in $V_{\alpha i}V^*_{\alpha j}$ are possible to give rise to significant cancellations in some cases.

(4) The fact that μ_{ij} and ϵ_{ij} are proportional to m_i or m_j can be understood in the following way. Note that both tensor- and pseudotensor-like spinor bilinears are chirality-changing operators, which link the left-handed state to the right-handed one [6]:

$$\overline{\psi}\sigma_{\mu\nu}\psi = \overline{\psi_{\rm L}}\sigma_{\mu\nu}\psi_{\rm R} + {\rm h.c.} ,$$

[5] Here we confine ourselves to a simple extension of the SM with three known neutrinos to be massive Majorana particles.

[6] That is why both magnetic and electric dipole moments must vanish for a Weyl neutrino, because it is massless and does not possess the right-handed component.

$$\overline{\psi}\sigma_{\mu\nu}\gamma_5\psi = \overline{\psi_{\rm L}}\sigma_{\mu\nu}\gamma_5\psi_{\rm R} - {\rm h.c.} \; . \tag{78}$$

Note also that the same relations hold when ψ is replaced by its charge-conjugate field ψ^c for Majorana neutrinos. Because $(\nu_i)_{\rm R}$ and $(\nu_j)_{\rm R}$ do not have any interactions with $W^\pm$ in Fig. 1, it seems that only $(\nu_i)_{\rm L}$ and $(\nu_j)_{\rm L}$ are flowing along the external fermion lines. To obtain a chirality-changing contribution from the effective (one-loop) electromagnetic vertex, one has to put a mass insertion on one of the external legs in the Feynman diagrams. As a result, the magnetic and electric dipole moments must involve m_i and m_j, the masses of ν_i and ν_j neutrinos.

(5) Is the magnetic or electric dipole moment of a neutrino always proportional to its mass? The answer is negative if new physics beyond the $SU(2)_{\rm L} \times U(1)_{\rm Y}$ gauge theory is involved. For instance, a new term proportional to the charged-lepton mass can contribute to the magnetic dipole moment of a massive Dirac neutrino in the $SU(2)_{\rm L} \times SU(2)_{\rm R} \times U(1)_{\rm Y}$ model with broken left-right symmetry. Depending on the details of this model, such a term might cancel or exceed the term proportional to the neutrino mass in the expression of the magnetic dipole moment.

Finite magnetic and electric dipole moments of massive neutrinos may produce a variety of new processes beyond the SM. For example, (a) radiative neutrino decays $\nu_i \to \nu_j + \gamma$ can happen, so can the Cherenkov radiation of neutrinos in an external electromagnetic field; (b) the elastic neutrino-electron or neutrino-nucleon scattering can be mediated by the magnetic and electric dipole moments; (c) the phenomenon of precession of the neutrino spin can occur in an external magnetic field; (d) the photon (or plasmon) can decay into a neutrino-antineutrino pair in a plasma (i.e., $\gamma^* \to \nu\overline{\nu}$). Of course, non-vanishing electromagnetic dipole moments contribute to neutrino masses too.

3.3 Radiative neutrino decays

If the electromagnetic moments of a massive neutrino ν_i are finite, it can decay into a lighter neutrino ν_j and a photon γ. The Lorentz-invariant vertex matrix of this $\nu_i \to \nu_j + \gamma$ process is in general described by $\Gamma_\mu(p,p')$ in Eq. (71). Because $q^2 = 0$ and $q_\mu \varepsilon^\mu = 0$ hold for a real photon γ, where ε^μ represents the photon polarization, the form of $\Gamma_\mu(p,p')$ can be simplified to

$$\Gamma_\mu(p,p') = \left[iF_{\rm M}(0) + F_{\rm E}(0)\gamma_5\right]\sigma_{\mu\nu}q^\nu \; . \tag{79}$$

By definition, $F^{ij}_{\rm M}(0) \equiv \mu_{ij}$ and $F^{ij}_{\rm E}(0) \equiv \epsilon_{ij}$ are just the magnetic and electric transition dipole moments between ν_i and ν_j neutrinos. Given the transition matrix element $\overline{u_j}(\vec{p}')\Gamma^{ij}_\mu(p,p')u_i(\vec{p})$, it is straightforward to calculate the decay rate. In the rest frame of the decaying neutrino ν_i,

$$\Gamma_{\nu_i \to \nu_j + \gamma} = \frac{\left(m_i^2 - m_j^2\right)^3}{8\pi m_i^3}\left(\left|\mu_{ij}\right|^2 + \left|\epsilon_{ij}\right|^2\right) \; . \tag{80}$$

This result is valid for both Dirac and Majorana neutrinos.

In the $SU(2)_{\rm L} \times U(1)_{\rm Y}$ gauge theory with three massive Dirac (or Majorana) neutrinos, the radiative decay $\nu_i \to \nu_j + \gamma$ is mediated by the one-loop Feynman diagrams (and their charge-conjugate diagrams) shown in Fig. 1. The explicit expressions of μ_{ij} and ϵ_{ij} have been given in Eq. (75) for Dirac neutrinos and in Eq. (77) for Majorana neutrinos. Hence

$$\Gamma^{\rm (D)}_{\nu_i \to \nu_j + \gamma} = \frac{\left(m_i^2 - m_j^2\right)^3}{8\pi m_i^3}\left(\left|\mu^{\rm D}_{ij}\right|^2 + \left|\epsilon^{\rm D}_{ij}\right|^2\right) = \frac{9\alpha G_{\rm F}^2 m_i^5}{2^{11}\pi^4}\left(1 - \frac{m_j^2}{m_i^2}\right)^3\left(1 + \frac{m_j^2}{m_i^2}\right) \times \left|\sum_\alpha \frac{m_\alpha^2}{M_W^2} V_{\alpha i}V^*_{\alpha j}\right|^2 \; , \tag{81}$$

for Dirac neutrinos; or

$$\Gamma^{(\mathrm{M})}_{\nu_i \to \nu_j + \gamma} = \frac{\left(m_i^2 - m_j^2\right)^3}{8\pi m_i^3} \left(|\mu^{\mathrm{M}}_{ij}|^2 + |\epsilon^{\mathrm{M}}_{ij}|^2\right) = \frac{9\alpha G_{\mathrm{F}}^2 m_i^5}{2^{10}\pi^4} \left(1 - \frac{m_j^2}{m_i^2}\right)^3 \left\{\left(1 + \frac{m_j}{m_i}\right)^2 \right.$$
$$\left. \times \left[\sum_\alpha \frac{m_\alpha^2}{M_W^2} \mathrm{Im}\left(V_{\alpha i} V^*_{\alpha j}\right)\right]^2 + \left(1 - \frac{m_j}{m_i}\right)^2 \left[\sum_\alpha \frac{m_\alpha^2}{M_W^2} \mathrm{Re}\left(V_{\alpha i} V^*_{\alpha j}\right)\right]^2 \right\} , \qquad (82)$$

for Majorana neutrinos, where $\alpha = e^2/(4\pi)$ denotes the electromagnetic fine-structure constant.

To compare $\Gamma_{\nu_i \to \nu_j + \gamma}$ with the experimental data in a simpler way, one may define an effective magnetic dipole moment

$$\mu_{\mathrm{eff}} \equiv \sqrt{\left|\mu_{ij}\right|^2 + \left|\epsilon_{ij}\right|^2} \ . \qquad (83)$$

Eq. (80) can then be expressed as

$$\Gamma_{\nu_i \to \nu_j + \gamma} = 5.3 \times \left(1 - \frac{m_j^2}{m_i^2}\right)^3 \left(\frac{m_i}{1\ \mathrm{eV}}\right)^3 \times \left(\frac{\mu_{\mathrm{eff}}}{\mu_{\mathrm{B}}}\right)^2 \mathrm{s}^{-1} \ . \qquad (84)$$

Although μ_{eff} is extremely small in some simple extensions of the SM, it could be sufficiently large in some more complicated or exotic scenarios beyond the SM, such as a class of extra-dimension models. Experimentally, radiative decays of massive neutrinos can be constrained by seeing no emission of the photons from solar ν_e and reactor $\overline{\nu}_e$ fluxes. Much stronger constraints on μ_{eff} can be obtained from the Supernova 1987A limit on the neutrino decay and from the astrophysical limit on distortions of the cosmic microwave background (CMB) radiation. A brief summary of these limits is

$$\frac{\mu_{\mathrm{eff}}}{\mu_{\mathrm{B}}} < \begin{cases} 0.9 \times 10^{-1} \left(\dfrac{\mathrm{eV}}{m_\nu}\right)^2 & \text{Reactor} \\ 0.5 \times 10^{-5} \left(\dfrac{\mathrm{eV}}{m_\nu}\right)^2 & \text{Sun} \\ 1.5 \times 10^{-8} \left(\dfrac{\mathrm{eV}}{m_\nu}\right)^2 & \text{SN 1987A} \\ 1.0 \times 10^{-11} \left(\dfrac{\mathrm{eV}}{m_\nu}\right)^{9/4} & \text{CMB} \end{cases}$$

where m_ν denotes the effective mass of the decaying neutrino (i.e., $m_\nu = m_i$).

3.4 Electromagnetic ν_e-e scattering

In practice, the most sensitive way of probing the electromagnetic dipole moments of a massive neutrino is to measure the cross section of elastic neutrino-electron (or antineutrino-electron) scattering, which can be expressed as a sum of the contribution from the SM (σ_0) and that from the electromagnetic dipole moments of massive neutrinos (σ_μ):

$$\frac{\mathrm{d}\sigma}{\mathrm{d}T} = \frac{\mathrm{d}\sigma_0}{\mathrm{d}T} + \frac{\mathrm{d}\sigma_\mu}{\mathrm{d}T} \ , \qquad (85)$$

where $T = E_e - m_e$ denotes the kinetic energy of the recoil electron in this process. We have

$$\frac{\mathrm{d}\sigma_0}{\mathrm{d}T} = \frac{G_{\mathrm{F}}^2 m_e}{2\pi} \left[g_+^2 + g_-^2 \left(1 - \frac{T}{E_\nu}\right)^2 - g_+ g_- \frac{m_e T}{E_\nu^2}\right] \qquad (86)$$

for neutrino-electron scattering, where $g_+ = 2\sin^2\theta_{\rm w} + 1$ for ν_e, $g_+ = 2\sin^2\theta_{\rm w} - 1$ for ν_μ and ν_τ, and $g_- = 2\sin^2\theta_{\rm w}$ for all flavors. Note that Eq. (86) is also valid for antineutrino-electron scattering if one simply exchanges the positions of g_+ and g_-. On the other hand,

$$\frac{{\rm d}\sigma_\mu}{{\rm d}T} = \frac{\alpha^2\pi}{m_e^2}\left(\frac{1}{T} - \frac{1}{E_\nu}\right)\left(\frac{\mu_\nu}{\mu_{\rm B}}\right)^2 \tag{87}$$

with $\mu_\nu^2 \equiv |\mu_{ii}^{\rm D}|^2 + |\epsilon_{ii}^{\rm D}|^2$ (for $i = 1$, 2 or 3), which holds for both neutrinos and antineutrinos. In obtaining Eqs. (86) and (87) one has assumed the scattered neutrino to be a Dirac particle and omitted the effects of finite neutrino masses and flavor mixing (i.e., $\nu_e = \nu_1$, $\nu_\mu = \nu_2$ and $\nu_\tau = \nu_3$ have been taken). Hence there is no interference between the contributions coming from the SM and electromagnetic dipole moments — the latter leads to a helicity flip of the neutrino but the former is always helicity-conserving. While an interference term will appear if one takes account of neutrino masses and flavor mixing, its magnitude linearly depends on the neutrino masses and thus is strongly suppressed in comparison with the pure weak and electromagnetic terms. So the incoherent sum of ${\rm d}\sigma_0/{\rm d}T$ and ${\rm d}\sigma_\mu/{\rm d}T$ in Eq. (85) is actually an excellent approximation of ${\rm d}\sigma/{\rm d}T$.

It is obvious that the two terms of ${\rm d}\sigma/{\rm d}T$ depend on the kinetic energy of the recoil electron in quite different ways. In particular, ${\rm d}\sigma_\mu/{\rm d}T$ grows rapidly with decreasing values of T. Hence a measurement of smaller T can probe smaller μ_ν in this kind of experiments. The magnitude of ${\rm d}\sigma_\mu/{\rm d}T$ becomes larger than that of ${\rm d}\sigma_0/{\rm d}T$ if the condition

$$T \leq \frac{\alpha^2\pi^2}{G_{\rm F}^2 m_e^3}\left(\frac{\mu_\nu}{\mu_{\rm B}}\right)^2 \approx 3\times 10^{22}\left(\frac{\mu_\nu}{\mu_{\rm B}}\right)^2 {\rm keV} \tag{88}$$

is roughly satisfied, as one can easily see from Eqs. (86) and (87). No distortion of the recoil electron energy spectrum of $\nu_\alpha e^-$ or $\overline{\nu}_\alpha e^-$ scattering (for $\alpha = e, \mu, \tau$) has so far been observed in any direct laboratory experiments, and thus only the upper bounds on μ_ν can be derived. For instance, an analysis of the T-spectrum in the Super-Kamiokande experiment yields $\mu_\nu < 1.1\times 10^{-10}\mu_{\rm B}$. More stringent bounds on μ_ν can hopefully be achieved in the future.

In view of current experimental data on neutrino oscillations, we know that neutrinos are actually massive. Hence the effects of finite neutrino masses and flavor mixing should be taken into account in calculating the cross section of elastic neutrino-electron or antineutrino-electron scattering. Here let us illustrate how the neutrino oscillation may affect the weak and electromagnetic terms of elastic $\overline{\nu}_e e^-$ scattering in a reactor experiment, where the antineutrinos are produced from the beta decay of fission products and detected by their elastic scattering with electrons in a detector. The antineutrino state created in this beta decay (via $W^- \to e^- + \overline{\nu}_e$) at the reactor is a superposition of three antineutrino mass eigenstates:

$$|\overline{\nu}_e(0)\rangle = \sum_{j=1}^{3} V_{ej}|\overline{\nu}_j\rangle \ . \tag{89}$$

Such a $\overline{\nu}_e$ beam propagates over the distance L to the detector,

$$|\overline{\nu}_e(L)\rangle = \sum_{j=1}^{3} e^{iq_jL}V_{ej}|\overline{\nu}_j\rangle \ , \tag{90}$$

in which $q_j = \sqrt{E_\nu^2 - m_j^2}$ is the momentum of ν_j with E_ν being the beam energy and m_j being the mass of ν_j. After taking account of the effect of neutrino oscillations, one obtains the differential cross section of elastic antineutrino-electron scattering as follows:

$$\frac{{\rm d}\sigma'}{{\rm d}T} = \frac{{\rm d}\sigma'_0}{{\rm d}T} + \frac{{\rm d}\sigma'_\mu}{{\rm d}T} \ , \tag{91}$$

where

$$\frac{\mathrm{d}\sigma_0'}{\mathrm{d}T} = \frac{G_{\mathrm{F}}^2 m_e}{2\pi}\left\{g_-^2 + \left(g_- - 1\right)^2\left(1-\frac{T}{E_\nu}\right)^2 - g_-\left(g_- - 1\right)\frac{m_e T}{E_\nu^2} + 2g_-\left|\sum_{j=1}^{3} e^{iq_jL}|V_{ej}|^2\right|^2\left[2\left(1-\frac{T}{E_\nu}\right)^2 - \frac{m_e T}{E_\nu^2}\right]\right\} \tag{92}$$

with $g_- = 2\sin^2\theta_{\mathrm{w}}$ for $\overline{\nu}_e$, and

$$\frac{\mathrm{d}\sigma_\mu'}{\mathrm{d}T} = \frac{\alpha^2\pi}{m_e^2}\sum_{k=1}^{3}\left|\sum_{j=1}^{3} e^{iq_jL}V_{ej}\frac{\epsilon_{jk}+i\mu_{jk}}{\mu_{\mathrm{B}}}\right|^2 \times \left(\frac{1}{T}-\frac{1}{E_\nu}\right) \tag{93}$$

with μ_{jk} and ϵ_{jk} being the magnetic and electric transition dipole moments between ν_j and ν_k neutrinos as defined in Eq. (79). Because different neutrino mass eigenstates are in principle distinguishable in the electromagnetic $\overline{\nu}_e e^-$ scattering, their contributions to the total cross section are incoherent. Eq. (93) shows that it is in general difficult to determine or constrain the magnitudes of μ_{jk} and ϵ_{jk} (for $j,k = 1,2,3$) from a single measurement.

4 Lepton Flavor Mixing and CP Violation

Regardless of the dynamical origin of tiny neutrino masses [7], we may discuss lepton flavor mixing by taking account of the effective mass terms of charged leptons and Majorana neutrinos at low energies [8],

$$-\mathcal{L}_{\mathrm{lepton}}' = \overline{(e\ \mu\ \tau)_{\mathrm{L}}}\ M_l \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}_{\mathrm{R}} + \frac{1}{2}\overline{(\nu_e\ \nu_\mu\ \nu_\tau)_{\mathrm{L}}}\ M_\nu \begin{pmatrix} \nu_e^c \\ \nu_\mu^c \\ \nu_\tau^c \end{pmatrix}_{\mathrm{R}} + \mathrm{h.c.}\ . \tag{94}$$

The phenomenon of lepton flavor mixing arises from a mismatch between the diagonalizations of M_l and M_ν in an arbitrary flavor basis: $V_l^\dagger M_l U_l = \mathrm{Diag}\{m_e, \mu_\mu, m_\tau\}$ and $V_\nu^\dagger M_\nu V_\nu^* = \mathrm{Diag}\{m_1, m_2, m_3\}$, where V_l, U_l and V_ν are the 3×3 unitary matrices. In the basis of mass eigenstates, it is the unitary matrix $V = V_l^\dagger V_\nu$ that will appear in the weak charged-current interactions in Eq. (12). Although the basis of $M_l = \mathrm{Diag}\{m_e, m_\mu, m_\tau\}$ with $V_l = \mathbf{1}$ and $V = V_\nu$ is often chosen in neutrino phenomenology, one should keep in mind that both the charged-lepton and neutrino sectors may in general contribute to lepton flavor mixing. In other words, both V_l and V_ν are not fully physical, and only their product $V = V_l^\dagger V_\nu$ is a physical description of lepton flavor mixing and CP violation at low energies.

4.1 Parametrizations of *V*

Flavor mixing among n different lepton families can be described by an $n\times n$ unitary matrix V, whose number of independent parameters relies on the nature of neutrinos. If neutrinos are Dirac particles, one may make use of $n(n-1)/2$ rotation angles and $(n-1)(n-2)/2$ phase angles to parametrize V. If neutrinos are Majorana particles, however, a full parametrization of V needs $n(n-1)/2$ rotation angles and the same number of phase angles [9]. The flavor mixing between charged leptons and Dirac

[7] For simplicity, here we do not consider possible non-unitarity of the 3×3 neutrino mixing matrix because its effects are either absent or very small.

[8] As for Dirac neutrinos, the corresponding mass term is the same as that given in Eq. (7). In this case the neutrino mass matrix M_ν is in general not symmetric and can be diagonalized by means of the transformation $V_\nu^\dagger M_\nu U_\nu = \mathrm{Diag}\{m_1, m_2, m_3\}$, where both V_ν and U_ν are unitary.

[9] No matter whether neutrinos are Dirac or Majorana particles, the $n\times n$ unitary flavor mixing matrix has $(n-1)^2(n-2)^2/4$ Jarlskog invariants of CP violation defined as $\mathcal{J}_{\alpha\beta}^{ij} \equiv \mathrm{Im}\left(V_{\alpha i}V_{\beta j}V_{\alpha j}^* V_{\beta i}^*\right)$.

neutrinos is completely analogous to that of quarks, for which a number of different parametrizations have been proposed and classified in the literature. Here we classify all possible parametrizations for the flavor mixing between charged leptons and Majorana neutrinos with $n = 3$. Regardless of the freedom of phase reassignments, we find that there are nine structurally different parametrizations for the 3×3 lepton flavor mixing matrix V.

The 3×3 lepton flavor mixing matrix V, which is often called the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix, can be expressed as a product of three unitary matrices O_1, O_2 and O_3. They correspond to simple rotations in the complex (1,2), (2,3) and (3,1) planes:

$$
\begin{aligned}
O_1 &= \begin{pmatrix} c_1 e^{i\alpha_1} & s_1 e^{-i\beta_1} & 0 \\ -s_1 e^{i\beta_1} & c_1 e^{-i\alpha_1} & 0 \\ 0 & 0 & e^{i\gamma_1} \end{pmatrix} , \\
O_2 &= \begin{pmatrix} e^{i\gamma_2} & 0 & 0 \\ 0 & c_2 e^{i\alpha_2} & s_2 e^{-i\beta_2} \\ 0 & -s_2 e^{i\beta_2} & c_2 e^{-i\alpha_2} \end{pmatrix} , \\
O_3 &= \begin{pmatrix} c_3 e^{i\alpha_3} & 0 & s_3 e^{-i\beta_3} \\ 0 & e^{i\gamma_3} & 0 \\ -s_3 e^{i\beta_3} & 0 & c_3 e^{-i\alpha_3} \end{pmatrix} ,
\end{aligned}
\tag{95}
$$

where $s_i \equiv \sin\theta_i$ and $c_i \equiv \cos\theta_i$ (for $i = 1, 2, 3$). Obviously $O_i O_i^\dagger = O_i^\dagger O_i = \mathbf{1}$ holds, and any two rotation matrices do not commute with each other. We find twelve different ways to arrange the product of O_1, O_2 and O_3, which can cover the whole 3×3 space and provide a full description of V. Explicitly, six of the twelve different combinations of O_i belong to the type

$$
V = O_i(\theta_i, \alpha_i, \beta_i, \gamma_i) \otimes O_j(\theta_j, \alpha_j, \beta_j, \gamma_j) \otimes O_i(\theta'_i, \alpha'_i, \beta'_i, \gamma'_i)
\tag{96}
$$

with $i \neq j$, where the complex rotation matrix O_i occurs twice; and the other six belong to the type

$$
V = O_i(\theta_i, \alpha_i, \beta_i, \gamma_i) \otimes O_j(\theta_j, \alpha_j, \beta_j, \gamma_j) \otimes O_k(\theta_k, \alpha_k, \beta_k, \gamma_k)
\tag{97}
$$

with $i \neq j \neq k$, in which the rotations take place in three different complex planes. The products $O_i O_j O_i$ and $O_i O_k O_i$ (for $i \neq k$) in Eq. (97) are correlated with each other, if the relevant phase parameters are switched off. Hence only nine of the twelve parametrizations, three from Eq. (96) and six from Eq. (97), are structurally different.

In each parametrization of V, there apparently exist nine phase parameters. Some of them or their combinations can be absorbed by redefining the relevant phases of charged-lepton and neutrino fields. If neutrinos are Dirac particles, V contains only a single irremovable CP-violating phase δ. If neutrinos are Majorana particles, however, there is no freedom to rearrange the relative phases of three Majorana neutrino fields. Hence V may in general contain three irremovable CP-violating phases in the Majorana case (δ and two Majorana phases). Both CP- and T-violating effects in neutrino oscillations depend only upon the Dirac-like phase δ.

Different parametrizations of V are mathematically equivalent, so adopting any of them does not directly point to physical significance. But it is very likely that one particular parametrization is more useful and transparent than the others in studying the neutrino phenomenology and (or) exploring the underlying dynamics responsible for lepton mass generation and CP violation. Here we highlight two particular parametrizations of the PMNS matrix V. The first one is the so-called “standard" parametrization advocated by the Particle Data Group:

$$
V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13} e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} P' ,
\tag{98}
$$

where $c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$ (for $ij = 12, 13, 23$) together with the Majorana phase matrix $P' = \text{Diag}\{e^{i\rho}, e^{i\sigma}, 1\}$. Without loss of generality, the three mixing angles $(\theta_{12}, \theta_{13}, \theta_{23})$ can all be arranged to lie in the first quadrant. Arbitrary values between 0 and 2π are allowed for three CP-violating phases (δ, ρ, σ). A remarkable merit of this parametrization is that its three mixing angles are approximately equivalent to the mixing angles of solar (θ_{12}), atmospheric (θ_{23}) and CHOOZ reactor (θ_{13}) neutrino oscillation experiments. Another useful parametrization is the Fritzsch-Xing (FX) parametrization proposed originally for quark mixing and later for lepton mixing:

$$V = \begin{pmatrix} c_l & s_l & 0 \\ -s_l & c_l & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-i\phi} & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix} \begin{pmatrix} c_\nu & -s_\nu & 0 \\ s_\nu & c_\nu & 0 \\ 0 & 0 & 1 \end{pmatrix} P' \; , \tag{99}$$

where $c_{l,\nu} \equiv \cos\theta_{l,\nu}$, $s_{l,\nu} \equiv \sin\theta_{l,\nu}$, $c \equiv \cos\theta$, $s \equiv \sin\theta$, and P' is a diagonal phase matrix containing two nontrivial CP-violating phases. Although the form of V in Eq. (99) is apparently different from that in Eq. (98), their corresponding flavor mixing angles $(\theta_l, \theta_\nu, \theta)$ and $(\theta_{12}, \theta_{13}, \theta_{23})$ have quite similar meanings in interpreting the experimental data on neutrino oscillations. In the limit $\theta_l = \theta_{13} = 0$, one easily arrives at $\theta_\nu = \theta_{12}$ and $\theta = \theta_{23}$. As a natural consequence of very small θ_l, three mixing angles of the FX parametrization can also be related to those of solar (θ_ν), atmospheric (θ) and CHOOZ reactor $(\theta_l \sin\theta)$ neutrino oscillation experiments in the leading-order approximation. A striking merit of this parametrization is that its six parameters have very simple renormalization-group equations when they run from a superhigh-energy scale to the electroweak scale or vice versa.

4.2 Democratic or tri-bimaximal mixing?

Current neutrino oscillation data indicate the essential feature of lepton flavor mixing: two mixing angles are quite large ($\theta_{12} \sim 34^\circ$ and $\theta_{23} \sim 45^\circ$) while the third one is very small ($\theta_{13} < 10^\circ$). Such a flavor mixing pattern is far beyond the original imagination of most people because it is rather different from the well-known quark mixing pattern ($\vartheta_{12} \approx 14.5^\circ$, $\vartheta_{23} \approx 2.6^\circ$, $\vartheta_{13} \approx 0.23^\circ$ and $\delta = 76.5^\circ$) described by the same parametrization of the Cabibbo-Kobayashi-Maskawa (CKM) matrix. To understand this difference, a number of constant lepton mixing patterns have been proposed as the starting point of model building. Possible flavor symmetries and their spontaneous or explicit breaking mechanisms hidden in those constant patterns might finally help us pin down the dynamics responsible for lepton mass generation and flavor mixing. To illustrate, let us first comment on the "democratic" neutrino mixing pattern and then pay more attention to the "tri-bimaximal" neutrino mixing pattern.

The "democratic" lepton flavor mixing pattern

$$U_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \tag{100}$$

was originally obtained by Fritzsch and Xing as the leading term of the 3×3 lepton mixing matrix from the breaking of flavor democracy or $S(3)_{\rm L} \times S(3)_{\rm R}$ symmetry of the charged-lepton mass matrix in the basis where the Majorana neutrino mass matrix is diagonal and possesses the $S(3)$ symmetry. Its naive predictions $\theta_{12} = 45^\circ$ and $\theta_{23} \approx 54.7^\circ$ are no more favored today, but they may receive proper corrections from the symmetry-breaking perturbations so as to fit current neutrino oscillation data.

Today's most popular constant pattern of neutrino mixing is the "tri-bimaximal" mixing matrix:

$$V_0 = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{101}$$

which looks like a twisted form of the democratic mixing pattern with the same entries. Its strange name comes from the fact that this flavor mixing pattern is actually a product of the "tri-maximal" mixing matrix and a "bi-maximal" mixing matrix:

$$V_0' = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{\omega}{\sqrt{3}} & \frac{\omega^2}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{\omega^2}{\sqrt{3}} & \frac{\omega}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} = P V_0 P' \; , \tag{102}$$

where $\omega = e^{i2\pi/3}$ denotes the complex cube-root of unity (i.e., $\omega^3 = 1$), and $P = \mathrm{Diag}\{1, \omega, \omega^2\}$ and $P' = \mathrm{Diag}\{1, 1, i\}$ are two diagonal phase matrices. V_0 or V_0' predicts $\theta_{12} = \arctan(1/\sqrt{2}) \approx 35.3°$, $\theta_{13} = 0°$ and $\theta_{23} = 45°$, consistent quite well with current neutrino oscillation data. Because the entries of U_0 or V_0 are all formed from small integers (0, 1, 2 and 3) and their square roots, it is often suggestive of certain discrete flavor symmetries in the language of group theories. That is why the democratic or tri-bimaximal neutrino mixing pattern can serve as a good starting point of model building based on a variety of flavor symmetries, such as Z_2, Z_3, S_3, S_4, A_4, D_4, D_5, Q_4, Q_6, $\Delta(27)$ and $\Sigma(81)$. In particular, a lot of interest has been paid to the derivation of V_0 with the help of the non-Abelian discrete A_4 symmetry.

Note that the democratic mixing matrix U_0 and the tri-bimaximal mixing matrix V_0 are related with each other via the following transformation:

$$V_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_0 & -\sin\theta_0 \\ 0 & \sin\theta_0 & \cos\theta_0 \end{pmatrix} U_0 \begin{pmatrix} \cos\theta_0 & -\sin\theta_0 & 0 \\ \sin\theta_0 & \cos\theta_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \; , \tag{103}$$

where $\theta_0 = \arctan(\sqrt{2} - 1)^2 \approx 9.7°$. This angle is actually a measure of the difference between the mixing angles of U_0 and V_0 (namely, $45° - 35.3° = 54.7° - 45° = 9.7°$). In this sense, we argue that it is worthwhile to explore possible flavor symmetries behind both V_0 and U_0 so as to build realistic models for neutrino mass generation and lepton flavor mixing.

Let us remark that a specific constant mixing pattern should be regarded as the leading-order approximation of the "true" lepton flavor mixing matrix, whose mixing angles should in general depend on both the ratios of charged-lepton masses and those of neutrino masses. We may at least make the following naive speculation about how to phenomenologically understand the observed pattern of lepton flavor mixing:

- Large values of θ_{12} and θ_{23} could arise from a weak hierarchy or a near degeneracy of the neutrino mass spectrum, because the strong hierarchy of charged-lepton masses implies that m_e/m_μ and m_μ/m_τ at the electroweak scale are unlikely to contribute to θ_{12} and θ_{23} in a dominant way.
- Special values of θ_{12} and θ_{23} might stem from an underlying flavor symmetry of the charged-lepton mass matrix or the neutrino mass matrix. Then the contributions of lepton mass ratios to flavor mixing angles, due to flavor symmetry breaking, are expected to serve as perturbative corrections to U_0 or V_0, or another constant mixing pattern.
- Vanishing or small θ_{13} could be a natural consequence of the explicit textures of lepton mass matrices. It might also be related to the flavor symmetry which gives rise to sizable θ_{12} and θ_{23} (e.g., in U_0 or V_0).
- Small corrections to a constant flavor mixing pattern may also result from the renormalization-group running effects of leptons and quarks, e.g., from a superhigh-energy scale to low energies or vice versa.

There are too many possibilities of linking the observed pattern of lepton flavor mixing to a certain flavor symmetry, and none of them is unique from the theoretical point of view. In this sense, flavor symmetries should not be regarded as a perfect guiding principle of model building.

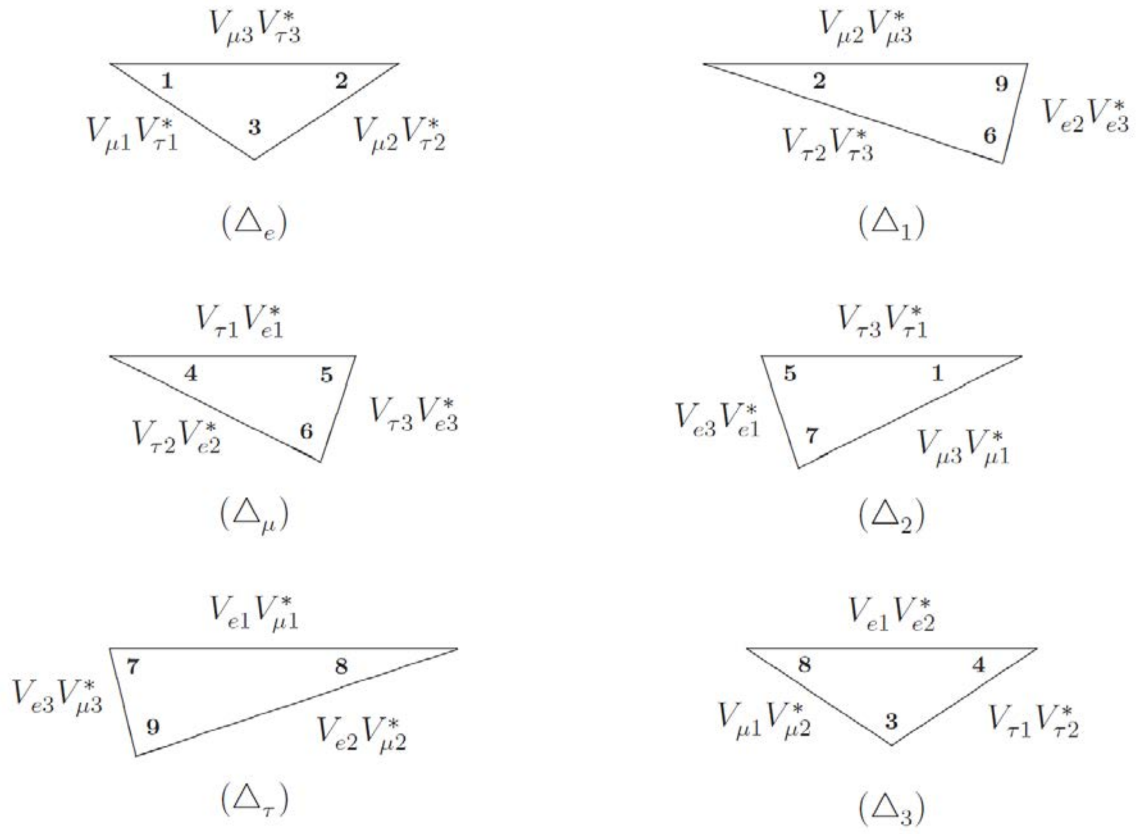

Fig. 2: Unitarity triangles of the 3×3 PMNS matrix in the complex plane. Each triangle is named by the index that does not manifest in its three sides.

4.3 Leptonic unitarity triangles

In the basis where the flavor eigenstates of charged leptons are identified with their mass eigenstates, the PMNS matrix V relates the neutrino mass eigenstates (ν_1, ν_2, ν_3) to the neutrino flavor eigenstates $(\nu_e, \nu_\mu, \nu_\tau)$:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} V_{e1} & V_{e2} & V_{e3} \\ V_{\mu 1} & V_{\mu 2} & V_{\mu 3} \\ V_{\tau 1} & V_{\tau 2} & V_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} . \tag{104}$$

The unitarity of V represents two sets of normalization and orthogonality conditions:

$$\sum_i \left(V_{\alpha i} V_{\beta i}^* \right) = \delta_{\alpha\beta} , \quad \sum_\alpha \left(V_{\alpha i} V_{\alpha j}^* \right) = \delta_{ij} , \tag{105}$$

where Greek and Latin subscripts run over (e, μ, τ) and $(1, 2, 3)$, respectively. In the complex plane the six orthogonality relations in Eq. (105) define six triangles $(\triangle_e, \triangle_\mu, \triangle_\tau)$ and $(\triangle_1, \triangle_2, \triangle_3)$ shown in Fig. 2, the so-called unitarity triangles. These six triangles have eighteen different sides and nine different inner (or outer) angles. But the unitarity of V requires that all six triangles have the same area amounting to $\mathcal{J}/2$, where $\mathcal{J}$ is the Jarlskog invariant of CP violation defined through

$$\mathrm{Im} \left(V_{\alpha i} V_{\beta j} V_{\alpha j}^* V_{\beta i}^* \right) = \mathcal{J} \sum_\gamma \epsilon_{\alpha\beta\gamma} \sum_k \epsilon_{ijk} . \tag{106}$$

One has $\mathcal{J} = c_{12} s_{12} c_{13}^2 s_{13} c_{23} s_{23}^2 \sin\delta$ in the standard parametrization of V as well as $\mathcal{J} = c_l s_l c_\nu s_\nu c s^2 \sin\phi$ in the FX parametrization of V. No matter whether neutrinos are Dirac or Majorana particles, the strength of CP or T violation in neutrino oscillations depends only upon $\mathcal{J}$.

To show why the areas of six unitarity triangles are identical with one another, let us take triangles $\triangle_\tau$ and $\triangle_3$ for example. They correspond to the orthogonality relations

$$\begin{aligned} V_{e1}V^*_{\mu 1} + V_{e2}V^*_{\mu 2} + V_{e3}V^*_{\mu 3} &= 0 \; , \\ V_{e1}V^*_{e2} + V_{\mu 1}V^*_{\mu 2} + V_{\tau 1}V^*_{\tau 2} &= 0 \; . \end{aligned} \tag{107}$$

Multiplying these two equations by $V_{\mu 2}V^*_{e2}$ and $V_{\mu 2}V^*_{\mu 1}$ respectively, we arrive at two rescaled triangles which share the side

$$V_{e1}V_{\mu 2}V^*_{e2}V^*_{\mu 1} = -|V_{e2}V_{\mu 2}|^2 - V_{e3}V_{\mu 2}V^*_{e2}V^*_{\mu 3} = -|V_{\mu 1}V_{\mu 2}|^2 - V_{\mu 2}V_{\tau 1}V^*_{\mu 1}V^*_{\tau 2} \; . \tag{108}$$

This result is consistent with the definition of $\mathcal{J}$ in Eq. (106); i.e., $\mathrm{Im}(V_{e1}V_{\mu 2}V^*_{e2}V^*_{\mu 1}) = \mathcal{J}$ and $\mathrm{Im}(V_{e3}V_{\mu 2}V^*_{e2}V^*_{\mu 3}) = \mathrm{Im}(V_{\mu 2}V_{\tau 1}V^*_{\mu 1}V^*_{\tau 2}) = -\mathcal{J}$. The latter simultaneously implies that the areas of $\triangle_\tau$ and $\triangle_3$ are equal to $\mathcal{J}/2$. One may analogously prove that all the six unitarity triangles have the same area $\mathcal{J}/2$. If CP or T were an exact symmetry, $\mathcal{J} = 0$ would hold and those unitarity triangles would collapse into lines in the complex plane. Note that the shape and area of each unitarity triangle are irrelevant to the nature of neutrinos; i.e., they are the same for Dirac and Majorana neutrinos.

Because of $V^*_{e1}V_{\mu 1} + V^*_{e2}V_{\mu 2} = -V^*_{e3}V_{\mu 3}$ or equivalently $|V_{e1}V^*_{\mu 1} + V_{e2}V^*_{\mu 2}|^2 = |V_{e3}V^*_{\mu 3}|^2$, it is easy to obtain

$$2\mathrm{Re}\left(V_{e1}V_{\mu 2}V^*_{e2}V^*_{\mu 1}\right) = |V_{e3}|^2|V_{\mu 3}|^2 - |V_{e1}|^2|V_{\mu 1}|^2 - |V_{e2}|^2|V_{\mu 2}|^2 \; . \tag{109}$$

Combining $V_{e1}V_{\mu 2}V^*_{e2}V^*_{\mu 1} = \mathrm{Re}(V_{e1}V_{\mu 2}V^*_{e2}V^*_{\mu 1}) + i\mathcal{J}$ with Eq. (109) leads us to the result

$$\begin{aligned} \mathcal{J}^2 &= |V_{e1}|^2|V_{\mu 2}|^2|V_{e2}|^2|V_{\mu 1}|^2 - \frac{1}{4}\left(|V_{e3}|^2|V_{\mu 3}|^2 - |V_{e1}|^2|V_{\mu 1}|^2 - |V_{e2}|^2|V_{\mu 2}|^2\right)^2 \\ &= |V_{e1}|^2|V_{\mu 2}|^2|V_{e2}|^2|V_{\mu 1}|^2 - \frac{1}{4}\left(1 + |V_{e1}|^2|V_{\mu 2}|^2 + |V_{e2}|^2|V_{\mu 1}|^2 \right. \\ &\qquad \left. -|V_{e1}|^2 - |V_{\mu 2}|^2 - |V_{e2}|^2 - |V_{\mu 1}|^2\right)^2 \; . \end{aligned} \tag{110}$$

As a straightforward generalization of Eq. (110), $\mathcal{J}^2$ can be expressed in terms of the moduli of any four independent matrix elements of V:

$$\begin{aligned} \mathcal{J}^2 = |V_{\alpha i}|^2|V_{\beta j}|^2|V_{\alpha j}|^2|V_{\beta i}|^2 - \frac{1}{4}\left(1 + |V_{\alpha i}|^2|V_{\beta j}|^2 + |V_{\alpha j}|^2|V_{\beta i}|^2 \right. \\ \left. -|V_{\alpha i}|^2 - |V_{\beta j}|^2 - |V_{\alpha j}|^2 - |V_{\beta i}|^2\right)^2 \; , \end{aligned} \tag{111}$$

in which $\alpha \neq \beta$ running over (e, μ, τ) and $i \neq j$ running over $(1, 2, 3)$. The implication of this result is very obvious: the information about leptonic CP violation can in principle be extracted from the measured moduli of the neutrino mixing matrix elements.

As a consequence of the unitarity of V, two interesting relations can be derived from the normalization conditions in Eq. (105):

$$\begin{aligned} |V_{e2}|^2 - |V_{\mu 1}|^2 = |V_{\mu 3}|^2 - |V_{\tau 2}|^2 = |V_{\tau 1}|^2 - |V_{e3}|^2 \equiv \Delta_{\rm L} \; , \\ |V_{e2}|^2 - |V_{\mu 3}|^2 = |V_{\mu 1}|^2 - |V_{\tau 2}|^2 = |V_{\tau 3}|^2 - |V_{e1}|^2 \equiv \Delta_{\rm R} \; . \end{aligned} \tag{112}$$

The off-diagonal asymmetries $\Delta_{\rm L}$ and $\Delta_{\rm R}$ characterize the geometrical structure of V about its V_{e1}-$V_{\mu 2}$-$V_{\tau 3}$ and V_{e3}-$V_{\mu 2}$-$V_{\tau 1}$ axes, respectively. For instance, $\Delta_{\rm L} = 1/6$ and $\Delta_{\rm R} = -1/6$ hold for the tri-bimaximal neutrino mixing pattern V_0. If $\Delta_{\rm L} = 0$ (or $\Delta_{\rm R} = 0$) held, V would be symmetric about the V_{e1}-$V_{\mu 2}$-$V_{\tau 3}$ (or V_{e3}-$V_{\mu 2}$-$V_{\tau 1}$) axis. Geometrically this would correspond to the congruence between two unitarity triangles; i.e.,

$$\Delta_{\rm L} = 0 : \; \triangle_e \cong \triangle_1 \; , \triangle_\mu \cong \triangle_2 \; , \triangle_\tau \cong \triangle_3 \; ;$$

Table 2: Some important discoveries in the developments of flavor physics.

	Discoveries of lepton flavors, quark flavors and CP violation
1897	electron (Thomson, 1897)
1919	proton (up and down quarks) (Rutherford, 1919)
1932	neutron (up and down quarks) (Chadwick, 1932)
1933	positron (Anderson, 1933)
1936	muon (Neddermeyer and Anderson, 1937)
1947	Kaon (strange quark) (Rochester and Butler, 1947)
1956	electron antineutrino (Cowan *et al.*, 1956)
1962	muon neutrino (Danby *et al.*, 1962)
1964	CP violation in s-quark decays (Christenson *et al.*, 1964)
1974	charm quark (Aubert *et al.*, 1974; Abrams *et al.*, 1974)
1975	tau (Perl *et al.*, 1975)
1977	bottom quark (Herb *et al.*, 1977)
1995	top quark (Abe *et al.*, 1995; Abachi *et al.*, 1995)
2000	tau neutrino (Kodama *et al.*, 2000)
2001	CP violation in b-quark decays (Aubert *et al.*, 2001; Abe *et al.*, 2001)

$$\Delta_{\rm R} = 0: \ \triangle_e \cong \triangle_3 \, , \triangle_\mu \cong \triangle_2 \, , \triangle_\tau \cong \triangle_1 \, . \tag{113}$$

Indeed the counterpart of $\Delta_{\rm L}$ in the quark sector is only of $\mathcal{O}(10^{-5})$; i.e., the CKM matrix is almost symmetric about its V_{ud}-V_{cs}-V_{tb} axis. An exactly symmetric flavor mixing matrix might hint at an underlying flavor symmetry, from which some deeper understanding of the fermion mass texture could be achieved.

4.4 Flavor problems in particle physics

In the subatomic world the fundamental building blocks of matter have twelve flavors: six quarks and six leptons (and their antiparticles). Table 2 is a brief list of some important discoveries in flavor physics, which can partly give people a ball-park feeling of a century of developments in particle physics. The SM of electromagnetic and weak interactions contain thirteen free parameters in its lepton and quark sectors: three charged-lepton masses, six quark masses, three quark flavor mixing angles and one CP-violating phase. If three known neutrinos are massive Majorana particles, one has to introduce nine free parameters to describe their flavor properties: three neutrino masses, three lepton flavor mixing angles and three CP-violating phases. Thus an effective theory of electroweak interactions at low energies totally consists of twenty-two flavor parameters which can only be determined from experiments. Why is the number of degrees of freedom so big in the flavor sector? What is the fundamental physics behind these parameters? Such puzzles constitute the flavor problems in particle physics.

Current experimental data on neutrino oscillations can only tell us $m_1 < m_2$. It remains unknown whether m_3 is larger than m_2 (normal hierarchy) or smaller than m_1 (inverted hierarchy). The possibility $m_1 \approx m_2 \approx m_3$ (near degeneracy) cannot be excluded at present. In contrast, three families of charged fermions have very strong mass hierarchies:

$$\begin{aligned} \frac{m_e}{m_\mu} \sim \frac{m_u}{m_c} \sim \frac{m_c}{m_t} \sim \lambda^4 \, , \\ \frac{m_\mu}{m_\tau} \sim \frac{m_d}{m_s} \sim \frac{m_s}{m_b} \sim \lambda^2 \, , \end{aligned} \tag{114}$$

where $\lambda \equiv \sin\theta_{\rm C} \approx 0.22$ with $\theta_{\rm C}$ being the Cabibbo angle of quark flavor mixing. In the standard parametrization of the CKM matrix, three quark mixing angles exhibit an impressive hierarchy:

$$\vartheta_{12} \sim \lambda \ , \quad \vartheta_{23} \sim \lambda^2 \ , \quad \vartheta_{13} \sim \lambda^4 \ . \tag{115}$$

These two kinds of hierarchies might intrinsically be related to each other, because the flavor mixing angles actually measure a mismatch between the mass and flavor eigenstates of up- and down-type quarks. For example, the relations $\vartheta_{12} \approx \sqrt{m_d/m_s}$, $\vartheta_{23} \approx \sqrt{m_d/m_b}$ and $\vartheta_{13} \approx \sqrt{m_u/m_t}$ are compatible with Eqs. (114) and (115). They can be derived from a specific pattern of up- and down-type quark mass matrices with five texture zeros. On the other hand, it seems quite difficult to find a simple way of linking two large lepton flavor mixing angles $\theta_{12} \sim \pi/6$ and $\theta_{23} \sim \pi/4$ to small m_e/m_μ and m_μ/m_τ. One might ascribe the largeness of θ_{12} and θ_{23} to a very weak hierarchy of three neutrino masses and the smallness of θ_{13} to the strong mass hierarchy in the charged-lepton sector. There are of course many possibilities of model building to understand the observed lepton flavor mixing pattern, but none of them has experimentally and theoretically been justified.

Among a number of concrete flavor puzzles that are currently facing us, the following three are particularly intriguing.

– The pole masses of three charged leptons satisfy the equality

$$\frac{m_e + m_\mu + m_\tau}{\left(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau}\right)^2} = \frac{2}{3} \tag{116}$$

to an amazingly good degree of accuracy — its error bar is only of $\mathcal{O}(10^{-5})$.

– There are two quark-lepton “complementarity" relations in flavor mixing:

$$\theta_{12} + \vartheta_{12} \approx \theta_{23} + \vartheta_{23} \approx \frac{\pi}{4} \ , \tag{117}$$

which are compatible with the present experimental data.

– Two unitarity triangles of the CKM matrix, defined by the orthogonality conditions $V_{ud}V^*_{ub} + V_{cd}V^*_{cb} + V_{td}V^*_{tb} = 0$ and $V_{tb}V^*_{ub} + V_{ts}V^*_{us} + V_{td}V^*_{ud} = 0$, are almost the right triangles. Namely, the common inner angle of these two triangles satisfies

$$\alpha \equiv \arg\left(-\frac{V_{ud}V^*_{ub}}{V_{td}V^*_{tb}}\right) \approx \frac{\pi}{2} \ , \tag{118}$$

indicated by current experimental data on quark mixing and CP violation.

Such special numerical relations might just be accidental. One or two of them might also be possible to result from a certain (underlying) flavor symmetry.

5 Running of Neutrino Mass Parameters

5.1 One-loop RGEs

The spirit of seesaw mechanisms is to attribute the small masses of three known neutrinos to the existence of some heavy degrees of freedom, such as the $SU(2)_{\rm L}$ gauge-singlet fermions, the $SU(2)_{\rm L}$ gauge-triplet scalars or the $SU(2)_{\rm L}$ gauge-triplet fermions. All of them point to the unique dimension-5 Weinberg operator in an effective theory after the corresponding heavy particles are integrated out:

$$\frac{\mathcal{L}_{\rm d=5}}{\Lambda} = \frac{1}{2}\kappa_{\alpha\beta}\overline{\ell_{\alpha\rm L}}\tilde{H}\tilde{H}^T\ell^c_{\beta\rm L} + {\rm h.c.} \ , \tag{119}$$

where Λ is the cutoff scale, $\ell_{\rm L}$ denotes the left-handed lepton doublet, $\tilde{H} \equiv i\sigma_2 H^*$ with H being the SM Higgs doublet, and κ stands for the effective neutrino coupling matrix. After spontaneous gauge symmetry breaking, $\tilde{H}$ gains its vacuum expectation value $\langle\tilde{H}\rangle = v/\sqrt{2}$ with $v \approx 246$ GeV. We are then left with the effective Majorana mass matrix $M_\nu = \kappa v^2/2$ for three light neutrinos from Eq. (119). If the dimension-5 Weinberg operator is obtained in the framework of the minimal supersymmetric standard model (MSSM), one will be left with $M_\nu = \kappa(v\sin\beta)^2/2$, where $\tan\beta$ denotes the ratio of the vacuum expectation values of two MSSM Higgs doublets.

Eq. (119) or its supersymmetric counterpart can provide a simple but generic way of generating tiny neutrino masses. There are a number of interesting possibilities of building renormalizable gauge models to realize the effective Weinberg mass operator, either radiatively or at the tree level. The latter case is just associated with the well-known seesaw mechanisms to be discussed in section 6. Here we assume that $\mathcal{L}_{\rm d=5}/\Lambda$ arises from an underlying seesaw model, whose lightest heavy particle has a mass of $\mathcal{O}(\Lambda)$. In other words, Λ characterizes the seesaw scale. Above Λ there may exist one or more energy thresholds corresponding to the masses of heavier seesaw particles. Below Λ the energy dependence of the effective neutrino coupling matrix κ is described by its renormalization-group equation (RGE). The evolution of κ from Λ down to the electroweak scale is formally independent of any details of the relevant seesaw model from which κ is derived.

At the one-loop level κ obeys the RGE

$$16\pi^2\frac{{\rm d}\kappa}{{\rm d}t} = \alpha_\kappa\kappa + C_\kappa\left[(Y_lY_l^\dagger)\kappa + \kappa(Y_lY_l^\dagger)^T\right] \tag{120}$$

where $t \equiv \ln(\mu/\Lambda)$ with μ being an arbitrary renormalization scale between the electroweak scale and the seesaw scale, and Y_l is the charged-lepton Yukawa coupling matrix. The RGE of Y_l and those of $Y_{\rm u}$ (up-type quarks) and $Y_{\rm d}$ (down-type quarks) are given by

$$\begin{aligned}
16\pi^2\frac{{\rm d}Y_l}{{\rm d}t} &= \left[\alpha_l + C_l^l(Y_lY_l^\dagger)\right]Y_l\ ,\\
16\pi^2\frac{{\rm d}Y_{\rm u}}{{\rm d}t} &= \left[\alpha_{\rm u} + C_{\rm u}^{\rm u}(Y_{\rm u}Y_{\rm u}^\dagger) + C_{\rm u}^{\rm d}(Y_{\rm d}Y_{\rm d}^\dagger)\right]Y_{\rm u}\ ,\\
16\pi^2\frac{{\rm d}Y_{\rm d}}{{\rm d}t} &= \left[\alpha_{\rm d} + C_{\rm d}^{\rm u}(Y_{\rm u}Y_{\rm u}^\dagger) + C_{\rm d}^{\rm d}(Y_{\rm d}Y_{\rm d}^\dagger)\right]Y_{\rm d}\ .
\end{aligned} \tag{121}$$

In the framework of the SM we have

$$\begin{aligned}
C_\kappa &= C_{\rm u}^{\rm d} = C_{\rm d}^{\rm u} = -\frac{3}{2}\ ,\\
C_l^l &= C_{\rm u}^{\rm u} = C_{\rm d}^{\rm d} = +\frac{3}{2}\ ,
\end{aligned} \tag{122}$$

and

$$\begin{aligned}
\alpha_\kappa &= -3g_2^2 + \lambda + 2{\rm Tr}\left[3(Y_{\rm u}Y_{\rm u}^\dagger) + 3(Y_{\rm d}Y_{\rm d}^\dagger) + (Y_lY_l^\dagger)\right]\ ,\\
\alpha_l &= -\frac{9}{4}g_1^2 - \frac{9}{4}g_2^2 + {\rm Tr}\left[3(Y_{\rm u}Y_{\rm u}^\dagger) + 3(Y_{\rm d}Y_{\rm d}^\dagger) + (Y_lY_l^\dagger)\right]\ ,\\
\alpha_{\rm u} &= -\frac{17}{20}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2 + {\rm Tr}\left[3(Y_{\rm u}Y_{\rm u}^\dagger) + 3(Y_{\rm d}Y_{\rm d}^\dagger) + (Y_lY_l^\dagger)\right]\ ,\\
\alpha_{\rm d} &= -\frac{1}{4}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2 + {\rm Tr}\left[3(Y_{\rm u}Y_{\rm u}^\dagger) + 3(Y_{\rm d}Y_{\rm d}^\dagger) + (Y_lY_l^\dagger)\right]\ ;
\end{aligned} \tag{123}$$

and in the framework of the MSSM we have

$$\begin{aligned}
C_\kappa &= C_{\rm u}^{\rm d} = C_{\rm d}^{\rm u} = +1\ ,\\
C_l^l &= C_{\rm u}^{\rm u} = C_{\rm d}^{\rm d} = +3\ ,
\end{aligned} \tag{124}$$

and

$$\alpha_\kappa = -\frac{6}{5}g_1^2 - 6g_2^2 + 6{\rm Tr}(Y_{\rm u}Y_{\rm u}^\dagger)\ ,$$

$$\alpha_l = -\frac{9}{5}g_1^2 - 3g_2^2 + {\rm Tr}\left[3(Y_{\rm d}Y_{\rm d}^\dagger) + (Y_l Y_l^\dagger)\right] \; ,$$
$$\alpha_{\rm u} = -\frac{13}{15}g_1^2 - 3g_2^2 - \frac{16}{3}g_3^2 + 3{\rm Tr}(Y_{\rm u}Y_{\rm u}^\dagger) \; ,$$
$$\alpha_{\rm d} = -\frac{7}{15}g_1^2 - 3g_2^2 - \frac{16}{3}g_3^2 + {\rm Tr}\left[3(Y_{\rm d}Y_{\rm d}^\dagger) + (Y_l Y_l^\dagger)\right] \; . \quad (125)$$

Here g_1, g_2 and g_3 are the gauge couplings and satisfy their RGEs

$$16\pi^2 \frac{{\rm d}g_i}{{\rm d}t} = b_i g_i^3 \; , \quad (126)$$

where $(b_1, b_2, b_3) = (41/10, -19/6, -7)$ in the SM or $(33/5, 1, -3)$ in the MSSM. In addition, λ is the Higgs self-coupling parameter of the SM and obeys the RGE

$$16\pi^2 \frac{{\rm d}\lambda}{{\rm d}t} = 6\lambda^2 - 3\lambda\left(\frac{3}{5}g_1^2 + 3g_2^2\right) + \frac{3}{2}\left(\frac{3}{5}g_1^2 + g_2^2\right)^2 + 3g_2^4$$
$$+4\lambda{\rm Tr}\left[3(Y_{\rm u}Y_{\rm u}^\dagger) + 3(Y_{\rm d}Y_{\rm d}^\dagger) + (Y_l Y_l^\dagger)\right]$$
$$-8{\rm Tr}\left[3(Y_{\rm u}Y_{\rm u}^\dagger)^2 + 3(Y_{\rm d}Y_{\rm d}^\dagger)^2 + (Y_l Y_l^\dagger)^2\right] \; . \quad (127)$$

The relation between λ and the Higgs mass M_h is given by $\lambda = M_h^2/(2v^2)$, where $v \approx 246$ GeV is the vacuum expectation value of the Higgs field.

The above RGEs allow us to evaluate the running behavior of κ together with those of Y_l, $Y_{\rm u}$ and $Y_{\rm d}$, from the seesaw scale to the electroweak scale or vice versa. We shall examine the evolution of neutrino masses, lepton flavor mixing angles and CP-violating phases in the following.

5.2 Running neutrino mass parameters

Without loss of any generality, we choose the flavor basis where Y_l is diagonal: $Y_l = D_l \equiv {\rm Diag}\{y_e, y_\mu, y_\tau\}$ with y_α being the eigenvalues of Y_l. In this case the effective Majorana neutrino coupling matrix κ can be diagonalized by the PMNS matrix V; i.e., $V^\dagger \kappa V^* = \widehat{\kappa} \equiv {\rm Diag}\{\kappa_1, \kappa_2, \kappa_3\}$ with κ_i being the eigenvalues of κ. Then

$$\frac{{\rm d}\kappa}{{\rm d}t} = \dot{V}\widehat{\kappa}V^T + V\dot{\widehat{\kappa}}V^T + V\widehat{\kappa}\dot{V}^T = \frac{1}{16\pi^2}\left[\alpha_\kappa V\widehat{\kappa}V^T + C_\kappa\left(D_l^2 V\widehat{\kappa}V^T + V\widehat{\kappa}V^T D_l^2\right)\right] \; , \quad (128)$$

with the help of Eq. (120). After a definition of the Hermitian matrix $S \equiv V^\dagger D_l^2 V$ and the anti-Hermitian matrix $T \equiv V^\dagger \dot{V}$, Eq. (128) leads to

$$\dot{\widehat{\kappa}} = \frac{1}{16\pi^2}\left[\alpha_\kappa \widehat{\kappa} + C_\kappa(S\widehat{\kappa} + \widehat{\kappa}S^*)\right] - T\widehat{\kappa} + \widehat{\kappa}T^*. \quad (129)$$

Because $\widehat{\kappa}$ is by definition diagonal and real, the left- and right-hand sides of Eq. (129) must be diagonal and real. We can therefore arrive at

$$\dot{\kappa}_i = \frac{1}{16\pi^2}\left(\alpha_\kappa + 2C_\kappa {\rm Re}S_{ii}\right)\kappa_i \; , \quad (130)$$

together with ${\rm Im}T_{ii} = {\rm Re}T_{ii} = {\rm Im}S_{ii} = 0$ (for $i = 1, 2, 3$). As the off-diagonal parts of Eq. (129) are vanishing, we have

$$T_{ij}\kappa_j - \kappa_i T_{ij}^* = \frac{C_\kappa}{16\pi^2}\left(S_{ij}\kappa_j + \kappa_i S_{ij}^*\right) \quad (131)$$

with $i \neq j$. Therefore,

$${\rm Re}T_{ij} = -\frac{C_\kappa}{16\pi^2}\frac{\kappa_i + \kappa_j}{\kappa_i - \kappa_j}{\rm Re}S_{ij} \; ,$$

$$\text{Im}T_{ij} = -\frac{C_\kappa}{16\pi^2}\frac{\kappa_i - \kappa_j}{\kappa_i + \kappa_j}\text{Im}S_{ij} \; . \tag{132}$$

Due to $\dot{V} = VT$, Eq. (132) actually governs the evolution of V with energies.

We proceed to define $V \equiv PUP'$, in which $P \equiv \text{Diag}\{e^{i\phi_e}, e^{i\phi_\mu}, e^{i\phi_\tau}\}$, $P' \equiv \text{Diag}\{e^{i\rho}, e^{i\sigma}, 1\}$, and U is the CKM-like matrix containing three neutrino mixing angles and one CP-violating phase. Although P does not have any physical meaning, its phases have their own RGEs. In contrast, P' serves for the Majorana phase matrix. We find

$$T' \equiv P'TP'^\dagger = P'V^\dagger \dot{V} P'^\dagger = \dot{P}'P'^\dagger + U^\dagger \dot{U} + U^\dagger P^\dagger \dot{P} U \; , \tag{133}$$

from which we can obtain six independent constraint equations:

$$\begin{aligned}
T'_{11} &= i\dot{\rho} + \sum_\alpha \left[U^*_{\alpha 1}\dot{U}_{\alpha 1} + iU_{\alpha 1}\dot{\phi}_\alpha\right] \; , \\
T'_{22} &= i\dot{\sigma} + \sum_\alpha \left[U^*_{\alpha 2}\dot{U}_{\alpha 2} + iU_{\alpha 2}\dot{\phi}_\alpha\right] \; , \\
T'_{33} &= \sum_\alpha \left[U^*_{\alpha 3}\dot{U}_{\alpha 3} + iU_{\alpha 3}\dot{\phi}_\alpha\right] \; ; \\
T'_{12} &= \sum_\alpha \left[U^*_{\alpha 1}\dot{U}_{\alpha 2} + iU_{\alpha 2}\dot{\phi}_\alpha\right] \; , \\
T'_{13} &= \sum_\alpha \left[U^*_{\alpha 1}\dot{U}_{\alpha 3} + iU_{\alpha 3}\dot{\phi}_\alpha\right] \; , \\
T'_{23} &= \sum_\alpha \left[U^*_{\alpha 2}\dot{U}_{\alpha 3} + iU_{\alpha 3}\dot{\phi}_\alpha\right] \; ,
\end{aligned} \tag{134}$$

where α runs over e, μ and τ. Note that $T_{ii} = 0$ holds and T_{ij} is given by Eq. (132). In view of $y_e \ll y_\mu \ll y_\tau$, we take $D_l^2 \approx \text{Diag}\{0, 0, y_\tau^2\}$ as an excellent approximation. Then S_{ij}, T_{ij} and T'_{ij} can all be expressed in terms of y_τ^2 and the parameters of U and P'. After a straightforward calculation, we obtain the explicit expressions of Eqs. (130) and (134) as follows:

$$\dot{\kappa}_i = \frac{\kappa_i}{16\pi^2}\left(\alpha_\kappa + 2C_\kappa y_\tau^2 |U_{\tau i}|^2\right) \; , \tag{135}$$

and

$$\begin{aligned}
\sum_\alpha \left[U^*_{\alpha 1}\left(i\dot{U}_{\alpha 1} - U_{\alpha 1}\dot{\phi}_\alpha\right)\right] &= \dot{\rho} \; , \\
\sum_\alpha \left[U^*_{\alpha 2}\left(i\dot{U}_{\alpha 2} - U_{\alpha 2}\dot{\phi}_\alpha\right)\right] &= \dot{\sigma} \; , \\
\sum_\alpha \left[U^*_{\alpha 3}\left(i\dot{U}_{\alpha 3} - U_{\alpha 3}\dot{\phi}_\alpha\right)\right] &= 0 \; , \\
\sum_\alpha \left[U^*_{\alpha 1}\left(\dot{U}_{\alpha 2} + iU_{\alpha 2}\dot{\phi}_\alpha\right)\right] &= -\frac{C_\kappa y_\tau^2}{16\pi^2}e^{i(\rho-\sigma)}\left[\zeta_{12}^{-1}\text{Re}\left(U^*_{\tau 1}U_{\tau 2}e^{i(\sigma-\rho)}\right) + i\zeta_{12}\text{Im}\left(U^*_{\tau 1}U_{\tau 2}e^{i(\sigma-\rho)}\right)\right] \\
\sum_\alpha \left[U^*_{\alpha 1}\left(\dot{U}_{\alpha 3} + iU_{\alpha 3}\dot{\phi}_\alpha\right)\right] &= -\frac{C_\kappa y_\tau^2}{16\pi^2}e^{i\rho}\left[\zeta_{13}^{-1}\text{Re}\left(U^*_{\tau 1}U_{\tau 3}e^{-i\rho}\right) + i\zeta_{13}\text{Im}\left(U^*_{\tau 1}U_{\tau 3}e^{-i\rho}\right)\right] \; , \\
\sum_\alpha \left[U^*_{\alpha 2}\left(\dot{U}_{\alpha 3} + iU_{\alpha 3}\dot{\phi}_\alpha\right)\right] &= -\frac{C_\kappa y_\tau^2}{16\pi^2}e^{i\sigma}\left[\zeta_{23}^{-1}\text{Re}\left(U^*_{\tau 2}U_{\tau 3}e^{-i\sigma}\right) + i\zeta_{23}\text{Im}\left(U^*_{\tau 2}U_{\tau 3}e^{-i\sigma}\right)\right] \; , \qquad (13
\end{aligned}$$

where $\zeta_{ij} \equiv (\kappa_i - \kappa_j)/(\kappa_i + \kappa_j)$ with $i \neq j$. One can see that those y_τ^2-associated terms only consist of the matrix elements $U_{\tau i}$ (for $i = 1, 2, 3$). If a parametrization of U assures $U_{\tau i}$ to be as simple as possible, the resultant RGEs of neutrino mixing angles and CP-violating phases will be very concise. We find that the FX parametrization advocated in Eq. (99) with

$$U = \begin{pmatrix} s_l s_\nu c + c_l c_\nu e^{-i\phi} & s_l c_\nu c - c_l s_\nu e^{-i\phi} & s_l s \\ c_l s_\nu c - s_l c_\nu e^{-i\phi} & c_l c_\nu c + s_l s_\nu e^{-i\phi} & c_l s \\ -s_\nu s & -c_\nu s & c \end{pmatrix}$$

accords with the above observation, while the "standard" parametrization in Eq. (98) does not. That is why the RGEs of neutrino mixing angles and CP-violating phases in the standard parametrization are rather complicated.

Here we take the FX form of U to derive the RGEs of neutrino mass and mixing parameters. Combining Eqs. (135), (136) and the FX form of U, we arrive at

$$\begin{aligned} \dot{\kappa}_1 &= \frac{\kappa_1}{16\pi^2}\left(\alpha_\kappa + 2C_\kappa y_\tau^2 s_\nu^2 s^2\right) , \\ \dot{\kappa}_2 &= \frac{\kappa_2}{16\pi^2}\left(\alpha_\kappa + 2C_\kappa y_\tau^2 c_\nu^2 s^2\right) , \\ \dot{\kappa}_3 &= \frac{\kappa_3}{16\pi^2}\left(\alpha_\kappa + 2C_\kappa y_\tau^2 c^2\right) , \end{aligned} \tag{137}$$

where $\alpha_\kappa \approx -3g_2^2 + 6y_t^2 + \lambda$ (SM) or $\alpha_\kappa \approx -1.2g_1^2 - 6g_2^2 + 6y_t^2$ (MSSM); and

$$\begin{aligned} \dot{\theta}_l &= \frac{C_\kappa y_\tau^2}{16\pi^2}\, c_\nu s_\nu c \left[\zeta_{13}^{-1} c_\rho c_{(\rho-\phi)} + \zeta_{13} s_\rho s_{(\rho-\phi)} - \zeta_{23}^{-1} c_\sigma c_{(\sigma-\phi)} - \zeta_{23} s_\sigma s_{(\sigma-\phi)}\right] , \\ \dot{\theta}_\nu &= \frac{C_\kappa y_\tau^2}{16\pi^2}\, c_\nu s_\nu \left[s^2\left(\zeta_{12}^{-1} c_{(\sigma-\rho)}^2 + \zeta_{12} s_{(\sigma-\rho)}^2\right) + c^2\left(\zeta_{13}^{-1} c_\rho^2 + \zeta_{13} s_\rho^2\right) - c^2\left(\zeta_{23}^{-1} c_\sigma^2 + \zeta_{23} s_\sigma^2\right)\right] , \\ \dot{\theta} &= \frac{C_\kappa y_\tau^2}{16\pi^2}\, cs \left[s_\nu^2\left(\zeta_{13}^{-1} c_\rho^2 + \zeta_{13} s_\rho^2\right) + c_\nu^2\left(\zeta_{23}^{-1} c_\sigma^2 + \zeta_{23} s_\sigma^2\right)\right] ; \end{aligned} \tag{138}$$

as well as

$$\begin{aligned} \dot{\rho} &= \frac{C_\kappa y_\tau^2}{16\pi^2}\left[\widehat{\zeta}_{12} c_\nu^2 s^2 c_{(\sigma-\rho)} s_{(\sigma-\rho)} + \widehat{\zeta}_{13}\left(s_\nu^2 s^2 - c^2\right) c_\rho s_\rho + \widehat{\zeta}_{23} c_\nu^2 s^2 c_\sigma s_\sigma\right] , \\ \dot{\sigma} &= \frac{C_\kappa y_\tau^2}{16\pi^2}\left[\widehat{\zeta}_{12} s_\nu^2 s^2 c_{(\sigma-\rho)} s_{(\sigma-\rho)} + \widehat{\zeta}_{13} s_\nu^2 s^2 c_\rho s_\rho + \widehat{\zeta}_{23}\left(c_\nu^2 s^2 - c^2\right) c_\sigma s_\sigma\right] , \\ \dot{\phi} &= \frac{C_\kappa y_\tau^2}{16\pi^2}\left[\left(c_l^2 - s_l^2\right) c_l^{-1} s_l^{-1} c_\nu s_\nu c \left(\zeta_{13}^{-1} c_\rho s_{(\rho-\phi)} - \zeta_{13} s_\rho c_{(\rho-\phi)} - \zeta_{23}^{-1} c_\sigma s_{(\sigma-\phi)} + \zeta_{23} s_\sigma c_{(\sigma-\phi)}\right)\right. \\ &\quad \left. + \widehat{\zeta}_{12} s^2 c_{(\sigma-\rho)} s_{(\sigma-\rho)} + \widehat{\zeta}_{13}\left(s_\nu^2 - c_\nu^2 c^2\right) c_\rho s_\rho + \widehat{\zeta}_{23}\left(c_\nu^2 - s_\nu^2 c^2\right) c_\sigma s_\sigma\right] , \end{aligned} \tag{139}$$

where $\widehat{\zeta}_{ij} \equiv \zeta_{ij}^{-1} - \zeta_{ij} = 4\kappa_i\kappa_j/\left(\kappa_i^2 - \kappa_j^2\right)$, $c_a \equiv \cos a$ and $s_a \equiv \sin a$ (for $a = \rho$, σ, $\sigma-\rho$, $\rho-\phi$ or $\sigma-\phi$).

Some discussions on the basic features of RGEs of three neutrino masses, three flavor mixing angles and three CP-violating phases are in order.

(a) The running behaviors of three neutrino masses m_i (or equivalently κ_i) are essentially identical and determined by α_κ, unless $\tan\beta$ is large enough in the MSSM to make the y_τ^2-associated term competitive with the α_κ term. In our phase convention, $\dot{\kappa}_i$ or $\dot{m}_i$ (for $i = 1, 2, 3$) are independent of the CP-violating phase ϕ.

(b) Among three neutrino mixing angles, only the derivative of θ_ν contains a term proportional to ζ_{12}^{-1}. Note that $\zeta_{ij}^{-1} = (m_i + m_j)^2/\Delta m_{ij}^2$ with $\Delta m_{ij}^2 \equiv m_i^2 - m_j^2$ holds. Current solar and atmospheric neutrino oscillation data yield $\Delta m_{21}^2 \approx 7.7 \times 10^{-5}$ eV^2 and $\left|\Delta m_{32}^2\right| \approx \left|\Delta m_{31}^2\right| \approx 2.4 \times 10^{-3}$ eV^2. So θ_ν is in general more sensitive to radiative corrections than θ_l and θ. The evolution of θ_ν can be suppressed through the fine-tuning of $(\sigma - \rho)$. The smallest neutrino mixing angle θ_l may get radiative corrections even if its initial value is zero, and thus it can be radiatively generated from other neutrino mixing angles and CP-violating phases.

(c) The running behavior of ϕ is quite different from those of ρ and σ, because it includes a peculiar term proportional to s_l^{-1}. This term, which dominates $\dot{\phi}$ when θ_l is sufficiently small, becomes divergent in the limit $\theta_l \rightarrow 0$. Indeed, ϕ is not well-defined if θ_l is exactly vanishing. But both θ_l and ϕ can be radiatively generated. We may require that $\dot{\phi}$ remain finite when θ_l approaches zero, implying that the following necessary condition can be extracted from the expression of $\dot{\phi}$ in Eq. (139):

$$\zeta_{13}^{-1} c_\rho s_{(\rho-\phi)} - \zeta_{13} s_\rho c_{(\rho-\phi)} - \zeta_{23}^{-1} c_\sigma s_{(\sigma-\phi)} + \zeta_{23} s_\sigma c_{(\sigma-\phi)} = 0 \; . \qquad (140)$$

Note that the initial value of θ_l, if it is exactly zero or extremely small, may immediately drive ϕ to its *quasi-fixed point*. In this case Eq. (140) can be used to understand the relationship between ϕ and two Majorana phases ρ and σ at the quasi-fixed point.

(d) The running behaviors of ρ and σ are relatively mild in comparison with that of ϕ. A remarkable feature of $\dot{\rho}$ and $\dot{\sigma}$ is that they will vanish, if both ρ and σ are initially vanishing. This observation indicates that ρ and σ cannot simultaneously be generated from ϕ via the RGEs.

6 How to Generate Neutrino Masses?

Neutrinos are assumed or required to be massless in the SM, just because the structure of the SM itself is too simple to accommodate massive neutrinos.

- Two fundamentals of the SM are the $SU(2)_{\rm L} \times U(1)_{\rm Y}$ gauge symmetry and the Lorentz invariance. Both of them are mandatory to guarantee that the SM is a consistent quantum field theory.
- The particle content of the SM is rather economical. There are no right-handed neutrinos in the SM, so a Dirac neutrino mass term is not allowed. There is only one Higgs doublet, so a gauge-invariant Majorana mass term is forbidden.
- The SM is a renormalizable quantum field theory. Hence an effective dimension-5 operator, which may give each neutrino a Majorana mass, is absent.

In other words, the SM accidently possesses the $(B-L)$ symmetry which assures three known neutrinos to be exactly massless.

But today's experiments have convincingly indicated the existence of neutrino oscillations. This quantum phenomenon can appear if and only if neutrinos are massive and lepton flavors are mixed, and thus it is a kind of new physics beyond the SM. To generate non-zero but tiny neutrino masses, one or more of the above-mentioned constraints on the SM must be abandoned or relaxed. It is intolerable to abandon the gauge symmetry and Lorentz invariance; otherwise, one would be led astray. Given the framework of the SM as a consistent field theory, its particle content can be modified and (or) its renormalizability can be abandoned to accommodate massive neutrinos. There are several ways to this goal.

6.1 Relaxing the renormalizability

In 1979, Weinberg extended the SM by introducing some higher-dimension operators in terms of the fields of the SM itself:

$$\mathcal{L}_{\rm eff} = \mathcal{L}_{\rm SM} + \frac{\mathcal{L}_{\rm d=5}}{\Lambda} + \frac{\mathcal{L}_{\rm d=6}}{\Lambda^2} + \cdots \; , \qquad (141)$$

where Λ denotes the cut-off scale of this effective theory. Within such a framework, the lowest-dimension operator that violates the lepton number (L) is the unique dimension-5 operator $HHLL/\Lambda$. After spontaneous gauge symmetry breaking, this Weinberg operator yields $m_i \sim \langle H \rangle^2/\Lambda$ for neutrino masses, which can be sufficiently small (≤ 1 eV) if Λ is not far away from the scale of grand unified theories ($\Lambda \sim 10^{13}$ GeV for $\langle H \rangle \sim 10^2$ GeV). In this sense we argue that neutrino masses can serve as a low-energy window onto new physics at superhigh energies.

6.2 A pure Dirac neutrino mass term?

Given three right-handed neutrinos, the gauge-invariant and lepton-number-conserving mass terms of charged leptons and neutrinos are

$$-\mathcal{L}_{\rm lepton} = \overline{\ell_{\rm L}} Y_l H E_{\rm R} + \overline{\ell_{\rm L}} Y_\nu \tilde{H} N_{\rm R} + {\rm h.c.} \; , \tag{142}$$

where $\tilde{H} \equiv i\sigma_2 H^*$ is defined and $\ell_{\rm L}$ denotes the left-handed lepton doublet. After spontaneous gauge symmetry breaking, we arrive at the charged-lepton mass matrix $M_l = Y_l v/\sqrt{2}$ and the Dirac neutrino mass matrix $M_\nu = Y_\nu v/\sqrt{2}$ with $v \simeq 246$ GeV. In this case, the smallness of three neutrino masses m_i (for $i = 1, 2, 3$) is attributed to the smallness of three eigenvalues of Y_ν (denoted as y_i for $i = 1, 2, 3$). Then we encounter a transparent hierarchy problem: $y_i/y_e = m_i/m_e \leq 0.5\,{\rm eV}/0.5\,{\rm MeV} \sim 10^{-6}$. Why is y_i so small? There is no explanation at all in this Dirac-mass picture.

A speculative way out is to invoke extra dimensions; namely, the smallness of Dirac neutrino masses is ascribed to the assumption that three right-handed neutrinos have access to one or more extra spatial dimensions. The idea is simply to confine the SM particles onto a brane and to allow $N_{\rm R}$ to travel in the bulk. For example, the wave-function of $N_{\rm R}$ spreads out over the extra dimension y, giving rise to a suppressed Yukawa interaction at $y = 0$ (i.e., the location of the brane):

$$\left[\overline{\ell_{\rm L}} Y_\nu \tilde{H} N_{\rm R}\right]_{y=0} \sim \frac{1}{\sqrt{L}} \left[\overline{\ell_{\rm L}} Y_\nu \tilde{H} N_{\rm R}\right]_{y=L} \; . \tag{143}$$

The magnitude of $1/\sqrt{L}$ is measured by $\Lambda/\Lambda_{\rm Planck}$, and thus it can naturally be small for an effective theory far below the Planck scale.

6.3 Seesaw mechanisms

This approach works at the tree level and reflects the essential spirit of seesaw mechanisms — tiny masses of three known neutrinos are attributed to the existence of heavy degrees of freedom and lepton number violation.

– Type-I seesaw — three heavy right-handed neutrinos are added into the SM and the lepton number is violated by their Majorana mass term:

$$-\mathcal{L}_{\rm lepton} = \overline{\ell_{\rm L}} Y_l H E_{\rm R} + \overline{\ell_{\rm L}} Y_\nu \tilde{H} N_{\rm R} + \frac{1}{2} \overline{N^{\rm c}_{\rm R}} M_{\rm R} N_{\rm R} + {\rm h.c.} \; , \tag{144}$$

where $M_{\rm R}$ is the Majorana mass matrix.

– Type-II seesaw — one heavy Higgs triplet is added into the SM and the lepton number is violated by its interactions with both the lepton doublet and the Higgs doublet:

$$-\mathcal{L}_{\rm lepton} = \overline{\ell_{\rm L}} Y_l H E_{\rm R} + \frac{1}{2} \overline{\ell_{\rm L}} Y_\Delta \Delta i\sigma_2 \ell^{\rm c}_{\rm L} - \lambda_\Delta M_\Delta H^T i\sigma_2 \Delta H + {\rm h.c.} \; , \tag{145}$$

where

$$\Delta \equiv \begin{pmatrix} \Delta^- & -\sqrt{2}\,\Delta^0 \\ \sqrt{2}\,\Delta^{--} & -\Delta^- \end{pmatrix} \tag{146}$$

denotes the $SU(2)_{\rm L}$ Higgs triplet.

– Type-III seesaw — three heavy triplet fermions are added into the SM and the lepton number is violated by their Majorana mass term:

$$-\mathcal{L}_{\rm lepton} = \overline{\ell_{\rm L}} Y_l H E_{\rm R} + \overline{\ell_{\rm L}} \sqrt{2} Y_\Sigma \Sigma^{\rm c} \tilde{H} + \frac{1}{2} {\rm Tr}\left(\overline{\Sigma} M_\Sigma \Sigma^{\rm c}\right) + {\rm h.c.} \; , \tag{147}$$

where

$$\Sigma = \begin{pmatrix} \Sigma^0/\sqrt{2} & \Sigma^+ \\ \Sigma^- & -\Sigma^0/\sqrt{2} \end{pmatrix} \tag{148}$$

denotes the $SU(2)_{\rm L}$ fermion triplet.

Of course, there are a number of variations or combinations of these three typical seesaw mechanisms in the literature.

For each of the above seesaw pictures, one may arrive at the unique dimension-5 Weinberg operator of neutrino masses after integrating out the corresponding heavy degrees of freedom:

$$\frac{\mathcal{L}_{\rm d=5}}{\Lambda} = \begin{cases} \dfrac{1}{2}\left(Y_\nu M_{\rm R}^{-1} Y_\nu^T\right)_{\alpha\beta} \overline{\ell_{\alpha \rm L}} \tilde{H} \tilde{H}^T \ell^{\rm c}_{\beta \rm L} + {\rm h.c.} \\ -\dfrac{\lambda_\Delta}{M_\Delta} \left(Y_\Delta\right)_{\alpha\beta} \overline{\ell_{\alpha \rm L}} \tilde{H} \tilde{H}^T \ell^{\rm c}_{\beta \rm L} + {\rm h.c.} \\ \dfrac{1}{2}\left(Y_\Sigma M_\Sigma^{-1} Y_\Sigma^T\right)_{\alpha\beta} \overline{\ell_{\alpha \rm L}} \tilde{H} \tilde{H}^T \ell^{\rm c}_{\beta \rm L} + {\rm h.c.} \end{cases}$$

corresponding to type-I, type-II and type-III seesaws. After spontaneous gauge symmetry breaking, $\tilde{H}$ achieves its vacuum expectation value $\langle \tilde{H} \rangle = v/\sqrt{2}$ with $v \simeq 246$ GeV. Then we are left with the effective Majorana neutrino mass term for three known neutrinos,

$$-\mathcal{L}_{\rm mass} = \frac{1}{2} \overline{\nu_{\rm L}} M_\nu \nu^{\rm c}_{\rm L} + {\rm h.c.} \; , \tag{149}$$

where the Majorana mass matrix M_ν is given by

$$M_\nu = \begin{cases} -\dfrac{1}{2} Y_\nu \dfrac{v^2}{M_{\rm R}} Y_\nu^T & ({\rm Type\ I}) \; , \\ \lambda_\Delta Y_\Delta \dfrac{v^2}{M_\Delta} & ({\rm Type\ II}) \; , \\ -\dfrac{1}{2} Y_\Sigma \dfrac{v^2}{M_\Sigma} Y_\Sigma^T & ({\rm Type\ III}) \; . \end{cases} \tag{150}$$

It becomes obvious that the smallness of M_ν can be attributed to the largeness of $M_{\rm R}$, M_Δ or M_Σ in the seesaw mechanism.

6.4 Radiative origin of neutrino masses

In a seminal paper published in 1972, Weinberg pointed out that "in theories with spontaneously broken gauge symmetries, various masses or mass differences may vanish in zeroth order as a consequence of the representation content of the fields appearing in the Lagrangian. These masses or mass differences can then be calculated as finite higher-order effects." Such a mechanism may allow us to slightly go beyond the SM and radiatively generate tiny neutrino masses. A typical example is the well-known Zee model,

$$-\mathcal{L}_{\rm lepton} = \overline{\ell_{\rm L}} Y_l H E_{\rm R} + \overline{\ell_{\rm L}} Y_S S^- i\sigma_2 l^{\rm c}_{\rm L} + \tilde{\Phi}^T F S^+ i\sigma_2 \tilde{H} + {\rm h.c.} \; , \tag{151}$$

where $S^\pm$ are charged $SU(2)_{\rm L}$ singlet scalars, Φ denotes a new $SU(2)_{\rm L}$ doublet scalar which has the same quantum number as the SM Higgs doublet H, Y_S is an anti-symmetric matrix, and F represents a mass. Without loss of generality, we choose the basis of $M_l = Y_l \langle H \rangle = {\rm Diag}\{m_e, m_\mu, m_\tau\}$. In this model neutrinos are massless at the tree level, but their masses can radiatively be generated via the one-loop corrections. Given $M_S \gg M_H \sim M_\Phi \sim F$ and $\langle \Phi \rangle \sim \langle H \rangle$, the elements of the effective mass matrix of three light Majorana neutrinos are

$$(M_\nu)_{\alpha\beta} \sim \frac{M_H}{16\pi^2} \cdot \frac{m_\alpha^2 - m_\beta^2}{M_S^2} (Y_S)_{\alpha\beta} \; , \tag{152}$$

where α and β run over e, μ and τ. The smallness of M_ν is therefore ascribed to the smallness of Y_S and $(m_\alpha^2 - m_\beta^2)/M_S^2$. Although the original version of the Zee model is disfavored by current experimental data on neutrino oscillations, its extensions or variations at the one-loop or two-loop level can survive.

7 On the Scales of Seesaw Mechanisms

As we have seen, the key point of a seesaw mechanism is to ascribe the smallness of neutrino masses to the existence of some new degrees of freedom heavier than the Fermi scale $v \simeq 246$ GeV, such as heavy Majorana neutrinos or heavy Higgs bosons. The energy scale where a seesaw mechanism works is crucial, because it is relevant to whether this mechanism is theoretically natural and experimentally testable. Between Fermi and Planck scales, there might exist two other fundamental scales: one is the scale of a grand unified theory (GUT) at which strong, weak and electromagnetic forces can be unified, and the other is the TeV scale at which the unnatural gauge hierarchy problem of the SM can be solved or at least softened by a kind of new physics.

7.1 How about a very low seesaw scale?

In reality, however, there is no direct evidence for a high or extremely high seesaw scale. Hence eV-, keV-, MeV- and GeV-scale seesaws are all possible, at least in principle, and they are technically natural in the sense that their lepton-number-violating mass terms are naturally small according to 't Hooft's naturalness criterion — "At any energy scale μ, a set of parameters $\alpha_i(\mu)$ describing a system can be small, if and only if, in the limit $\alpha_i(\mu) \to 0$ for each of these parameters, the system exhibits an enhanced symmetry." But there are several potential problems associated with low-scale seesaws: (a) a low-scale seesaw does not give any obvious connection to a theoretically well-justified fundamental physical scale (such as the Fermi scale, the TeV scale, the GUT scale or the Planck scale); (b) the neutrino Yukawa couplings in a low-scale seesaw model turn out to be tiny, giving no actual explanation of why the masses of three known neutrinos are so small; and (c) in general, a very low seesaw scale does not allow the "canonical" thermal leptogenesis mechanism to work.

7.2 Seesaw-induced hierarchy problem

Many theorists argue that the conventional seesaw scenarios are natural because their scales (i.e., the masses of heavy degrees of freedom) are close to the GUT scale. This argument is reasonable on the one hand, but it reflects the drawbacks of the conventional seesaw models on the other hand. In other words, the conventional seesaw models have no direct experimental testability and involve a potential hierarchy problem. The latter is usually spoke of when two largely different energy scales exist in a model, but there is no symmetry to stabilize the low-scale physics suffering from large corrections coming from the high-scale physics.

Such a seesaw-induced fine-tuning problem means that the SM Higgs mass is very sensitive to quantum corrections from the heavy degrees of freedom in a seesaw mechanism. For example,

$$\delta M_H^2 = \begin{cases} -\dfrac{y_i^2}{8\pi^2}\left(\Lambda^2 + M_i^2 \ln \dfrac{M_i^2}{\Lambda^2}\right) & \text{(I)} \\ \dfrac{3}{16\pi^2}\left[\lambda_3\left(\Lambda^2 + M_\Delta^2 \ln \dfrac{M_\Delta^2}{\Lambda^2}\right) + 4\lambda_\Delta^2 M_\Delta^2 \ln \dfrac{M_\Delta^2}{\Lambda^2}\right] & \text{(II)} \\ -\dfrac{3y_i^2}{8\pi^2}\left(\Lambda^2 + M_i^2 \ln \dfrac{M_i^2}{\Lambda^2}\right) & \text{(III)} \end{cases}$$

in three typical seesaw scenarios, where Λ is the regulator cut-off, y_i and M_i (for $i = 1, 2, 3$) stand respectively for the eigenvalues of Y_ν (or Y_Σ) and $M_{\rm R}$ (or M_Σ), and the contributions proportional to v^2 and M_H^2 have been omitted. The above results show a quadratic sensitivity to the new scale which is characteristic of the seesaw model, implying that a high degree of fine-tuning would be necessary to accommodate the experimental data on M_H if the seesaw scale is much larger than v (or the Yukawa couplings are not extremely fine-tuned in type-I and type-III seesaws). Taking the type-I seesaw scenario for illustration, we assume $\Lambda \sim M_i$ and require $|\delta M_H^2| \leq 0.1$ TeV^2. Then the above equation leads us to the following rough estimate:

$$M_i \sim \left[\frac{(2\pi v)^2|\delta M_H^2|}{m_i}\right]^{1/3} \leq 10^7\text{GeV}\left[\frac{0.2\text{ eV}}{m_i}\right]^{1/3}\left[\frac{|\delta M_H^2|}{0.1\text{ TeV}^2}\right]^{1/3} . \tag{153}$$

This naive result indicates that a hierarchy problem will arise if the masses of heavy Majorana neutrinos are larger than about 10^7 GeV in the type-I seesaw scheme. Because of $m_i \sim y_i^2 v^2/(2M_i)$, the bound $M_i \leq 10^7$ GeV implies $y_i \sim \sqrt{2m_i M_i}/v \leq 2.6 \times 10^{-4}$ for $m_i \sim 0.2$ eV. Such a small magnitude of y_i seems to be a bit unnatural in the sense that the conventional seesaw idea attributes the smallness of m_i to the largeness of M_i other than the smallness of y_i.

There are two possible ways out of this impasse: one is to appeal for the supersymmetry, and the other is to lower the seesaw scale. We shall follow the second way to discuss the TeV seesaw mechanisms which do not suffer from the above-mentioned hierarchy problem.

7.3 Why are the TeV seesaws interesting?

There are several reasons for people to expect some new physics at the TeV scale. This kind of new physics should be able to stabilize the Higgs-boson mass and hence the electroweak scale; in other words, it should be able to solve or soften the unnatural gauge hierarchy problem. It has also been argued that the weakly-interacting particle candidates for dark matter should weigh about one TeV or less. If the TeV scale is really a fundamental scale, may we argue that the TeV seesaws are natural? Indeed, we are reasonably motivated to speculate that possible new physics existing at the TeV scale and responsible for the electroweak symmetry breaking might also be responsible for the origin of neutrino masses. It is interesting and meaningful in this sense to investigate and balance the “naturalness" and “testability" of TeV seesaws at the energy frontier set by the LHC.

As a big bonus of the conventional (type-I) seesaw mechanism, the thermal leptogenesis mechanism provides us with an elegant dynamic picture to interpret the cosmological matter-antimatter asymmetry characterized by the observed ratio of baryon number density to photon number density, $\eta_B \equiv n_B/n_\gamma = (6.1 \pm 0.2) \times 10^{10}$. When heavy Majorana neutrino masses are down to the TeV scale, the Yukawa couplings should be reduced by more than six orders of magnitude so as to generate tiny masses for three known neutrinos via the type-I seesaw and satisfy the out-of-equilibrium condition, but the CP-violating asymmetries of heavy Majorana neutrino decays can still be enhanced by the resonant effects in order to account for η_B. This “resonant leptogenesis" scenario might work in a specific TeV seesaw model.

Is there a TeV Noah’s Ark which can naturally and simultaneously accommodate the seesaw idea, the leptogenesis picture and the collider signatures? We are most likely not so lucky and should not be too ambitious at present. In the following we shall concentrate on the TeV seesaws themselves and their possible collider signatures and low-energy consequences.

8 TeV Seesaws: Natural and Testable?

The neutrino mass terms in three typical seesaw mechanisms have been given before. Without loss of generality, we choose the basis in which the mass eigenstates of three charged leptons are identified with their flavor eigenstates.

8.1 Type-I seesaw

Given $M_{\rm D} = Y_\nu v/\sqrt{2}$, the approximate type-I seesaw formula in Eq. (150) can be rewritten as $M_\nu = -M_{\rm D} M_{\rm R}^{-1} M_{\rm D}^T$. Note that the 3×3 light neutrino mixing matrix V is not exactly unitary in this seesaw scheme, and its deviation from unitarity is of $\mathcal{O}(M_{\rm D}^2/M_{\rm R}^2)$. Let us consider two interesting possibilities. (1) $M_{\rm D} \sim \mathcal{O}(10^2)$ GeV and $M_{\rm R} \sim \mathcal{O}(10^{15})$ GeV to get $M_\nu \sim \mathcal{O}(10^{-2})$ eV. In this conventional and *natural* case, $M_{\rm D}/M_{\rm R} \sim \mathcal{O}(10^{-13})$ holds. Hence the non-unitarity of V is only at the $\mathcal{O}(10^{-26})$ level, too small to be observed. (2) $M_{\rm D} \sim \mathcal{O}(10^2)$ GeV and $M_{\rm R} \sim \mathcal{O}(10^3)$ GeV to get $M_\nu \sim \mathcal{O}(10^{-2})$ eV. In this *unnatural* case, a significant "structural cancellation" has to be imposed on the textures of $M_{\rm D}$ and $M_{\rm R}$. Because of $M_{\rm D}/M_{\rm R} \sim \mathcal{O}(0.1)$, the non-unitarity of V can reach the percent level and may lead to observable effects.

Now we discuss how to realize the above "structural cancellation" for the type-I seesaw mechanism at the TeV scale. For the sake of simplicity, we take the basis of $M_{\rm R} = {\rm Diag}\{M_1, M_2, M_3\}$ for three heavy Majorana neutrinos (N_1, N_2, N_3). It is well known that M_ν vanishes if

$$M_{\rm D} = m \begin{pmatrix} y_1 & y_2 & y_3 \\ \alpha y_1 & \alpha y_2 & \alpha y_3 \\ \beta y_1 & \beta y_2 & \beta y_3 \end{pmatrix} \; , \quad \sum_{i=1}^{3} \frac{y_i^2}{M_i} = 0 \tag{154}$$

simultaneously hold. Tiny neutrino masses can be generated from tiny corrections to the texture of $M_{\rm D}$ in Eq. (154). For example, $M'_{\rm D} = M_{\rm D} - \epsilon X_{\rm D}$ with $M_{\rm D}$ given above and ϵ being a small dimensionless parameter (i.e., $|\epsilon| \ll 1$) yields

$$M'_\nu = -M'_{\rm D} M_{\rm R}^{-1} M'^T_{\rm D} \simeq \epsilon \left(M_{\rm D} M_{\rm R}^{-1} X_{\rm D}^T + X_{\rm D} M_{\rm R}^{-1} M_{\rm D}^T \right) \; , \tag{155}$$

from which $M'_\nu \sim \mathcal{O}(10^{-2})$ eV can be obtained by adjusting the size of ϵ.

A lot of attention has recently been paid to a viable type-I seesaw model and its collider signatures at the TeV scale. At least the following lessons can be learnt:

- Two necessary conditions must be satisfied in order to test a type-I seesaw model at the LHC: (a) M_i are of $\mathcal{O}(1)$ TeV or smaller; and (b) the strength of light-heavy neutrino mixing (i.e., $M_{\rm D}/M_{\rm R}$) is large enough. Otherwise, it would be impossible to produce and detect N_i at the LHC.
- The collider signatures of N_i are essentially decoupled from the mass and mixing parameters of three light neutrinos ν_i. For instance, the small parameter ϵ in Eq. (155) has nothing to do with the ratio $M_{\rm D}/M_{\rm R}$.
- The non-unitarity of V might lead to some observable effects in neutrino oscillations and other lepton-flavor-violating or lepton-number-violating processes, if $M_{\rm D}/M_{\rm R} \leq \mathcal{O}(0.1)$ holds.
- The clean LHC signatures of heavy Majorana neutrinos are the $\Delta L = 2$ like-sign dilepton events, such as $pp \to W^{*\pm} W^{*\pm} \to \mu^\pm \mu^\pm jj$ and $pp \to W^{*\pm} \to \mu^\pm N_i \to \mu^\pm \mu^\pm jj$ (a dominant channel due to the resonant production of N_i).

Some instructive and comprehensive analyses of possible LHC events for a single heavy Majorana neutrino have recently been done, but they only serve for illustration because such a simplified type-I seesaw scenario is actually unrealistic.

8.2 Type-II seesaw

The type-II seesaw formula $M_\nu = Y_\Delta v_\Delta = \lambda_\Delta Y_\Delta v^2/M_\Delta$ has been given in Eq. (150). Note that the last term of Eq. (145) violates both L and $B - L$, and thus the smallness of λ_Δ is naturally allowed according to 't Hooft's naturalness criterion (i.e., setting $\lambda_\Delta = 0$ will increase the symmetry of $\mathcal{L}_{\rm lepton}$). Given $M_\Delta \sim \mathcal{O}(1)$ TeV, for example, this seesaw mechanism works to generate $M_\nu \sim \mathcal{O}(10^{-2})$ eV provided $\lambda_\Delta Y_\Delta \sim \mathcal{O}(10^{-12})$ holds. The neutrino mixing matrix V is exactly unitary in the type-II seesaw mechanism, simply because the heavy degrees of freedom do not mix with the light ones.

There are totally seven physical Higgs bosons in the type-II seesaw scheme: doubly-charged H^{++} and H^{--}, singly-charged H^+ and H^-, neutral A^0 (CP-odd), and neutral h^0 and H^0 (CP-even), where h^0 is the SM-like Higgs boson. Except for $M^2_{h^0}$, we get a quasi-degenerate mass spectrum for other scalars: $M^2_{H^{\pm\pm}} = M^2_\Delta \approx M^2_{H^0} \approx M^2_{H^\pm} \approx M^2_{A^0}$. As a consequence, the decay channels $H^{\pm\pm} \to W^\pm H^\pm$ and

$H^{\pm\pm} \to H^{\pm}H^{\pm}$ are kinematically forbidden. The production of $H^{\pm\pm}$ at the LHC is mainly through $q\bar{q} \to \gamma^*, Z^* \to H^{++}H^{--}$ and $q\bar{q}' \to W^* \to H^{\pm\pm}H^{\mp}$ processes, which do not rely on the small Yukawa couplings.

The typical collider signatures in this seesaw scenario are the lepton-number-violating $H^{\pm\pm} \to l^{\pm}_{\alpha} l^{\pm}_{\beta}$ decays as well as $H^+ \to l^+_{\alpha}\overline{\nu}$ and $H^- \to l^-_{\alpha}\nu$ decays. Their branching ratios

$$\mathcal{B}(H^{\pm\pm} \to l^{\pm}_{\alpha} l^{\pm}_{\beta}) = \frac{|(M_\nu)_{\alpha\beta}|^2 \left(2 - \delta_{\alpha\beta}\right)}{\sum\limits_{\rho,\sigma} |(M_\nu)_{\rho\sigma}|^2} ,$$
$$\mathcal{B}(H^+ \to l^+_{\alpha}\overline{\nu}) = \frac{\sum\limits_{\beta} |(M_\nu)_{\alpha\beta}|^2}{\sum\limits_{\rho,\sigma} |(M_\nu)_{\rho\sigma}|^2} \tag{156}$$

are closely related to the masses, flavor mixing angles and CP-violating phases of three light neutrinos, because $M_\nu = V\widehat{M}_\nu V^T$ with $\widehat{M}_\nu = \mathrm{Diag}\{m_1, m_2, m_3\}$ holds. Some detailed analyses of such decay modes together with the LHC signatures of $H^{\pm\pm}$ and $H^{\pm}$ bosons have been done in the literature.

It is worth pointing out that the following dimension-6 operator can easily be derived from the type-II seesaw mechanism,

$$\frac{\mathcal{L}_{\rm d=6}}{\Lambda^2} = -\frac{(Y_\Delta)_{\alpha\beta}\,(Y_\Delta)^\dagger_{\rho\sigma}}{4M^2_\Delta} (\overline{\ell_{\alpha\rm L}}\gamma^\mu \ell_{\sigma\rm L})(\overline{\ell_{\beta\rm L}}\gamma_\mu \ell_{\rho\rm L}) , \tag{157}$$

which has two immediate low-energy effects: the non-standard interactions of neutrinos and the lepton-flavor-violating interactions of charged leptons. An analysis of such effects provides us with some preliminary information:

- The magnitudes of non-standard interactions of neutrinos and the widths of lepton-flavor-violating tree-level decays of charged leptons are both dependent on neutrino masses m_i and flavor-mixing and CP-violating parameters of V.
- For a long-baseline neutrino oscillation experiment, the neutrino beam encounters the earth matter and the electron-type non-standard interaction contributes to the matter potential.
- At a neutrino factory, the lepton-flavor-violating processes $\mu^- \to e^-\nu_e\overline{\nu}_\mu$ and $\mu^+ \to e^+\overline{\nu}_e\nu_\mu$ could cause some wrong-sign muons at a near detector.

Current experimental constraints tell us that such low-energy effects are very small, but they might be experimentally accessible in the future precision measurements.

8.3 Type-(I+II) seesaw

The type-(I+II) seesaw mechanism can be achieved by combining the neutrino mass terms in Eqs. (144) and (145). After spontaneous gauge symmetry breaking, we are left with the overall neutrino mass term

$$-\mathcal{L}_{\rm mass} = \frac{1}{2}\,\overline{(\nu_{\rm L}\, N^{\rm c}_{\rm R})} \begin{pmatrix} M_{\rm L} & M_{\rm D} \\ M^T_{\rm D} & M_{\rm R} \end{pmatrix} \begin{pmatrix} \nu^{\rm c}_{\rm L} \\ N_{\rm R} \end{pmatrix} + \text{h.c.} , \tag{158}$$

where $M_{\rm D} = Y_\nu v/\sqrt{2}$ and $M_{\rm L} = Y_\Delta v_\Delta$ with $\langle H\rangle \equiv v/\sqrt{2}$ and $\langle \Delta \rangle \equiv v_\Delta$ corresponding to the vacuum expectation values of the neutral components of the Higgs doublet H and the Higgs triplet Δ. The 6×6 mass matrix in Eq. (158) is symmetric and can be diagonalized by the unitary transformation done in Eq. (28); i.e.,

$$\begin{pmatrix} V & R \\ S & U \end{pmatrix}^\dagger \begin{pmatrix} M_{\rm L} & M_{\rm D} \\ M^T_{\rm D} & M_{\rm R} \end{pmatrix} \begin{pmatrix} V & R \\ S & U \end{pmatrix}^* = \begin{pmatrix} \widehat{M}_\nu & \mathbf{0} \\ \mathbf{0} & \widehat{M}_N \end{pmatrix} , \tag{159}$$

where $\widehat{M}_\nu = \text{Diag}\{m_1, m_2, m_3\}$ and $\widehat{M}_N = \text{Diag}\{M_1, M_2, M_3\}$. Needless to say, $V^\dagger V + S^\dagger S = VV^\dagger + RR^\dagger = \mathbf{1}$ holds as a consequence of the unitarity of this transformation. Hence V, the flavor mixing matrix of light Majorana neutrinos, must be non-unitary if R and S are non-zero.

In the leading-order approximation, the type-(I+II) seesaw formula reads as

$$M_\nu \approx M_{\rm L} - M_{\rm D} M_{\rm R}^{-1} M_{\rm D}^T \; . \tag{160}$$

Hence type-I and type-II seesaws can be regarded as two extreme cases of the type-(I+II) seesaw. Note that two mass terms in Eq. (160) are possibly comparable in magnitude. If both of them are small, their contributions to M_ν may have significant interference effects which make it practically impossible to distinguish between type-II and type-(I+II) seesaws; but if both of them are large, their contributions to M_ν must be destructive. The latter case unnaturally requires a significant cancellation between two big quantities in order to obtain a small quantity, but it is interesting in the sense that it may give rise to possibly observable collider signatures of heavy Majorana neutrinos.

Let me briefly describe a particular type-(I+II) seesaw model and comment on its possible LHC signatures. First, we assume that both M_i and M_Δ are of $\mathcal{O}(1)$ TeV. Then the production of $H^{\pm\pm}$ and $H^\pm$ bosons at the LHC is guaranteed, and their lepton-number-violating signatures will probe the Higgs triplet sector of the type-(I+II) seesaw mechanism. On the other hand, $\mathcal{O}(M_{\rm D}/M_{\rm R}) \leq \mathcal{O}(0.1)$ is possible as a result of $\mathcal{O}(M_{\rm R}) \sim \mathcal{O}(1)$ TeV and $\mathcal{O}(M_{\rm D}) \leq \mathcal{O}(v)$, such that appreciable signatures of N_i can be achieved at the LHC. Second, the small mass scale of M_ν implies that the relation $\mathcal{O}(M_{\rm L}) \sim \mathcal{O}(M_{\rm D} M_{\rm R}^{-1} M_{\rm D}^T)$ must hold. In other words, it is the significant but incomplete cancellation between $M_{\rm L}$ and $M_{\rm D} M_{\rm R}^{-1} M_{\rm D}^T$ terms that results in the non-vanishing but tiny masses for three light neutrinos. We admit that dangerous radiative corrections to two mass terms of M_ν require a delicate fine-tuning of the cancellation at the loop level. But this scenario allows us to reconstruct $M_{\rm L}$ via the excellent approximation $M_{\rm L} = V\widehat{M}_\nu V^T + R\widehat{M}_N R^T \approx R\widehat{M}_N R^T$, such that the elements of the Yukawa coupling matrix Y_Δ read as follows:

$$(Y_\Delta)_{\alpha\beta} \;=\; \frac{(M_{\rm L})_{\alpha\beta}}{v_\Delta} \approx \sum_{i=1}^{3} \frac{R_{\alpha i} R_{\beta i} M_i}{v_\Delta} \; , \tag{161}$$

where the subscripts α and β run over e, μ and τ. This result implies that the leptonic decays of $H^{\pm\pm}$ and $H^\pm$ bosons depend on both R and M_i, which actually determine the production and decays of N_i. Thus we have established an interesting correlation between the singly- or doubly-charged Higgs bosons and the heavy Majorana neutrinos. To observe the correlative signatures of $H^\pm$, $H^{\pm\pm}$ and N_i at the LHC will serve for a direct test of this type-(I+II) seesaw model.

8.4 Type-III seesaw

The lepton mass terms in the type-III seesaw scheme have already been given in Eq. (147). After spontaneous gauge symmetry breaking, we are left with

$$\begin{aligned} -\mathcal{L}_{\rm mass} &= \frac{1}{2}\, \overline{(\nu_{\rm L}\; \Sigma^0)} \begin{pmatrix} \mathbf{0} & M_{\rm D} \\ M_{\rm D}^T & M_\Sigma \end{pmatrix} \begin{pmatrix} \nu_{\rm L}^{\rm c} \\ \Sigma^{0{\rm c}} \end{pmatrix} + {\rm h.c.} \; , \\ -\mathcal{L}'_{\rm mass} &= \overline{(e_{\rm L}\; \Psi_{\rm L})} \begin{pmatrix} M_l & \sqrt{2} M_{\rm D} \\ \mathbf{0} & M_\Sigma \end{pmatrix} \begin{pmatrix} E_{\rm R} \\ \Psi_{\rm R} \end{pmatrix} + {\rm h.c.} \; , \end{aligned} \tag{162}$$

respectively, for neutral and charged fermions, where $M_l = Y_l v/\sqrt{2}$, $M_{\rm D} = Y_\Sigma v/\sqrt{2}$, and $\Psi = \Sigma^- + \Sigma^{+{\rm c}}$. The symmetric 6×6 neutrino mass matrix can be diagonalized by the following unitary transformation:

$$\begin{pmatrix} V & R \\ S & U \end{pmatrix}^\dagger \begin{pmatrix} \mathbf{0} & M_{\rm D} \\ M_{\rm D}^T & M_\Sigma \end{pmatrix} \begin{pmatrix} V & R \\ S & U \end{pmatrix}^* = \begin{pmatrix} \widehat{M}_\nu & \mathbf{0} \\ \mathbf{0} & \widehat{M}_\Sigma \end{pmatrix} \; , \tag{163}$$

where $\widehat{M}_\nu = \text{Diag}\{m_1, m_2, m_3\}$ and $\widehat{M}_\Sigma = \text{Diag}\{M_1, M_2, M_3\}$. In the leading-order approximation, this diagonalization yields the type-III seesaw formula $M_\nu = -M_\text{D} M_\Sigma^{-1} M_\text{D}^T$, which is equivalent to the one derived from the effective dimension-5 operator in Eq. (150). Let us use one sentence to comment on the similarities and differences between type-I and type-III seesaw mechanisms: the non-unitarity of the 3×3 neutrino mixing matrix V has appeared in both cases, although the modified couplings between the Z^0 boson and three light neutrinos differ and the non-unitary flavor mixing is also present in the couplings between the Z^0 boson and three charged leptons in the type-III seesaw scenario.

At the LHC, the typical lepton-number-violating signatures of the type-III seesaw mechanism can be $pp \to \Sigma^+\Sigma^0 \to l_\alpha^+ l_\beta^+ + Z^0 W^- (\to 4j)$ and $pp \to \Sigma^-\Sigma^0 \to l_\alpha^- l_\beta^- + Z^0 W^+ (\to 4j)$ processes. A detailed analysis of such collider signatures have been done in the literature. As for the low-energy phenomenology, a consequence of this seesaw scenario is the non-unitarity of the 3×3 flavor mixing matrix N ($\approx V$) in both charged- and neutral-current interactions. Current experimental bounds on the deviation of $NN^\dagger$ from the identity matrix are at the 0.1% level, much stronger than those obtained in the type-I seesaw scheme, just because the flavor-changing processes with charged leptons are allowed at the tree level in the type-III seesaw mechanism.

8.5 Inverse and multiple seesaws

Given the naturalness and testability as two prerequisites, the double or inverse seesaw mechanism is another interesting possibility of generating tiny neutrino masses at the TeV scale. The idea of this seesaw picture is to add three heavy right-handed neutrinos N_R, three SM gauge-singlet neutrinos S_R and one Higgs singlet Φ into the SM, such that the gauge-invariant lepton mass terms can be written as

$$-\mathcal{L}_\text{lepton} = \overline{l_\text{L}} Y_l H E_\text{R} + \overline{l_\text{L}} Y_\nu \tilde{H} N_\text{R} + \overline{N_\text{R}^\text{c}} Y_S \Phi S_\text{R} + \frac{1}{2} \overline{S_\text{R}^\text{c}} \mu S_\text{R} + \text{h.c.} \,, \tag{164}$$

where the μ-term is naturally small according to 't Hooft's naturalness criterion, because it violates the lepton number. After spontaneous gauge symmetry breaking, the overall neutrino mass term turns out to be

$$-\mathcal{L}_\text{mass} = \frac{1}{2} \overline{\left(\nu_\text{L} \; N_\text{R}^\text{c} \; S_\text{R}^\text{c}\right)} \begin{pmatrix} \mathbf{0} & M_\text{D} & \mathbf{0} \\ M_\text{D}^T & \mathbf{0} & M_S \\ \mathbf{0} & M_S^T & \mu \end{pmatrix} \begin{pmatrix} \nu_\text{L}^\text{c} \\ N_\text{R} \\ S_\text{R} \end{pmatrix} , \tag{165}$$

where $M_\text{D} = Y_\nu \langle H \rangle$ and $M_S = Y_S \langle \Phi \rangle$. A diagonalization of the symmetric 9×9 matrix $\mathcal{M}$ leads us to the effective light neutrino mass matrix

$$M_\nu \approx M_\text{D} \frac{1}{M_S^T} \mu \frac{1}{M_S} M_\text{D}^T \tag{166}$$

in the leading-order approximation. Hence the smallness of M_ν can be attributed to both the smallness of μ itself and the doubly-suppressed M_D/M_S term for $M_\text{D} \ll M_S$. For example, $\mu \sim \mathcal{O}(1)$ keV and $M_\text{D}/M_S \sim \mathcal{O}(10^{-2})$ naturally give rise to a sub-eV M_ν. One has $M_\nu = \mathbf{0}$ in the limit $\mu \to \mathbf{0}$, which reflects the restoration of the slightly-broken lepton number. The heavy sector consists of three pairs of pseudo-Dirac neutrinos whose CP-conjugated Majorana components have a tiny mass splitting characterized by the order of μ.

Going beyond the canonical (type-I) and inverse seesaw mechanisms, one may build the so-called "multiple" seesaw mechanisms to further lower the seesaw scales.

9 Non-unitary Neutrino Mixing

It is worth remarking that the charged-current interactions of light and heavy Majorana neutrinos are not completely independent in either the type-I seesaw or the type-(I+II) seesaw. The standard charged-current interactions of ν_i and N_i are already given in Eq. (34), where V is just the light neutrino mixing matrix responsible for neutrino oscillations, and R describes the strength of charged-current interactions between (e, μ, τ) and (N_1, N_2, N_3). Since V and R belong to the same unitary transformation done in Eq. (28) or Eq. (159), they must be correlated with each other and their correlation signifies an important relationship between neutrino physics and collider physics.

It can be shown that V and R share nine rotation angles (θ_{i4}, θ_{i5} and θ_{i6} for $i = 1$, 2 and 3) and nine phase angles (δ_{i4}, δ_{i5} and δ_{i6} for $i = 1$, 2 and 3). To see this point clearly, let us decompose V into $V = AV_0$, where V_0 is the standard (unitary) parametrization of the 3×3 PMNS matrix in which three CP-violating phases δ_{ij} (for $ij = 12, 13, 23$) are associated with s_{ij} (i.e., $c_{ij} \equiv \cos\theta_{ij}$ and $\hat{s}_{ij} \equiv e^{i\delta_{ij}} \sin\theta_{ij}$). Because of $VV^\dagger = AA^\dagger = \mathbf{1} - RR^\dagger$, it is obvious that $V \to V_0$ in the limit of $A \to \mathbf{1}$ (or equivalently, $R \to \mathbf{0}$). Considering the fact that the non-unitarity of V must be a small effect (at most at the percent level as constrained by current neutrino oscillation data and precision electroweak data), we expect $s_{ij} \leq \mathcal{O}(0.1)$ (for $i = 1, 2, 3$ and $j = 4, 5, 6$) to hold. Then we obtain

$$R = \begin{pmatrix} \hat{s}^*_{14} & \hat{s}^*_{15} & \hat{s}^*_{16} \\ \hat{s}^*_{24} & \hat{s}^*_{25} & \hat{s}^*_{26} \\ \hat{s}^*_{34} & \hat{s}^*_{35} & \hat{s}^*_{36} \end{pmatrix} \tag{167}$$

as an excellent approximations. A striking consequence of the non-unitarity of V is the loss of universality for the Jarlskog invariants of CP violation, $J^{ij}_{\alpha\beta} \equiv \mathrm{Im}(V_{\alpha i} V_{\beta j} V^*_{\alpha j} V^*_{\beta i})$, where the Greek indices run over (e, μ, τ) and the Latin indices run over $(1, 2, 3)$. For example, the extra CP-violating phases of V are possible to give rise to a significant asymmetry between $\nu_\mu \to \nu_\tau$ and $\overline{\nu}_\mu \to \overline{\nu}_\tau$ oscillations.

The probability of $\nu_\alpha \to \nu_\beta$ oscillations in vacuum, defined as $P_{\alpha\beta}$, is given by

$$P_{\alpha\beta} = \frac{\displaystyle\sum_i |V_{\alpha i}|^2 |V_{\beta i}|^2 + 2\sum_{i<j} \mathrm{Re}\left(V_{\alpha i} V_{\beta j} V^*_{\alpha j} V^*_{\beta i}\right) \cos\Delta_{ij} - \sum_{i<j} J^{ij}_{\alpha\beta} \sin\Delta_{ij}}{\left(VV^\dagger\right)_{\alpha\alpha} \left(VV^\dagger\right)_{\beta\beta}} \; , \tag{168}$$

where $\Delta_{ij} \equiv \Delta m^2_{ij} L/(2E)$ with $\Delta m^2_{ij} \equiv m^2_i - m^2_j$, E being the neutrino beam energy and L being the baseline length. If V is exactly unitary (i.e., $A = \mathbf{1}$ and $V = V_0$), the denominator of Eq. (168) will become unity and the conventional formula of $P_{\alpha\beta}$ will be reproduced. Note that $\nu_\mu \to \nu_\tau$ and $\overline{\nu}_\mu \to \overline{\nu}_\tau$ oscillations may serve as a good tool to probe possible signatures of non-unitary CP violation. To illustrate this point, we consider a short- or medium-baseline neutrino oscillation experiment with $|\sin\Delta_{13}| \sim |\sin\Delta_{23}| \gg |\sin\Delta_{12}|$, in which the terrestrial matter effects are expected to be insignificant or negligibly small. Then the dominant CP-conserving and CP-violating terms of $P(\nu_\mu \to \nu_\tau)$ and $P(\overline{\nu}_\mu \to \overline{\nu}_\tau)$ are

$$\begin{aligned} P(\nu_\mu \to \nu_\tau) &\approx \sin^2 2\theta_{23} \sin^2 \frac{\Delta_{23}}{2} - 2\left(J^{23}_{\mu\tau} + J^{13}_{\mu\tau}\right) \sin\Delta_{23} \; , \\ P(\overline{\nu}_\mu \to \overline{\nu}_\tau) &\approx \sin^2 2\theta_{23} \sin^2 \frac{\Delta_{23}}{2} + 2\left(J^{23}_{\mu\tau} + J^{13}_{\mu\tau}\right) \sin\Delta_{23} \; , \end{aligned} \tag{169}$$

where the good approximation $\Delta_{13} \approx \Delta_{23}$ has been used in view of the experimental fact $|\Delta m^2_{13}| \approx |\Delta m^2_{23}| \gg |\Delta m^2_{12}|$, and the sub-leading and CP-conserving "zero-distance" effect has been omitted. For simplicity, I take V_0 to be the exactly tri-bimaximal mixing pattern (i.e., $\theta_{12} = \arctan(1/\sqrt{2})$, $\theta_{13} = 0$ and $\theta_{23} = \pi/4$ as well as $\delta_{12} = \delta_{13} = \delta_{23} = 0$) and then arrive at

$$2\left(J^{23}_{\mu\tau} + J^{13}_{\mu\tau}\right) \; \approx \; \sum_{l=4}^{6} s_{2l} s_{3l} \sin\left(\delta_{2l} - \delta_{3l}\right) \; . \tag{170}$$

Given $s_{2l} \sim s_{3l} \sim \mathcal{O}(0.1)$ and $(\delta_{2l} - \delta_{3l}) \sim \mathcal{O}(1)$ (for $l = 4, 5, 6$), this non-trivial CP-violating quantity can reach the percent level. When a long-baseline neutrino oscillation experiment is concerned, however, the terrestrial matter effects must be taken into account because they might fake the genuine CP-violating signals. As for $\nu_\mu \to \nu_\tau$ and $\overline{\nu}_\mu \to \overline{\nu}_\tau$ oscillations under discussion, the dominant matter effect results from the neutral-current interactions and modifies the CP-violating quantity of Eq. (170) in the following way:

$$2\left(J^{23}_{\mu\tau} + J^{13}_{\mu\tau}\right) \Longrightarrow \sum_{l=4}^{6} s_{2l} s_{3l} \left[\sin\left(\delta_{2l} - \delta_{3l}\right) + A_{\rm NC} L \cos\left(\delta_{2l} - \delta_{3l}\right)\right] \ , \tag{171}$$

where $A_{\rm NC} = G_{\rm F} N_n / \sqrt{2}$ with N_n being the background density of neutrons, and L is the baseline length. It is easy to find $A_{\rm NC} L \sim \mathcal{O}(1)$ for $L \sim 4 \times 10^3$ km.

10 Concluding Remarks

I have briefly described some basic properties of massive neutrinos in an essentially model-independent way in these lectures, which are largely based on the book by Dr. Shun Zhou and myself [1] and on a few review articles or lectures [2]— [6]. It is difficult to cite all the relevant references. I apologize for missing other people's works due to the tight page limit of these proceedings. For the same reason I am unable to write in the cosmological matter-antimatter asymmetry and the leptogenesis mechanism, although they were discussed in my lectures. Here let me just give a few remarks on the naturalness and testability of TeV seesaw mechanisms.

Although the seesaw ideas are elegant, they have to appeal for some or many new degrees of freedom in order to interpret the observed neutrino mass hierarchy and lepton flavor mixing. According to Weinberg's *third law of progress in theoretical physics*, "you may use any degrees of freedom you like to describe a physical system, but if you use the wrong ones, you will be sorry." What could be better?

Anyway, we hope that the LHC might open a new window for us to understand the origin of neutrino masses and the dynamics of lepton number violation. A TeV seesaw might work (*naturalness*?) and its heavy degrees of freedom might show up at the LHC (*testability*?). A bridge between collider physics and neutrino physics is highly anticipated and, if it exists, will lead to rich phenomenology.

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