BEAM BREAK UP

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ABSTRACT
After a brief review of the experimental evidence for the beam break up instability in electron linacs, emphasis is given to the transverse deflection which arises from radio frequency fields. Typical transverse RF modes in circular iris loaded waveguides are then described. Finally two types of beam break up are discussed : the regenerative BBU which occurs in a single accelerating section and the cumulative BBU which is a multi-section effect.

1. EXPERIMENTAL EVIDENCE

The beam break up (BBU), also known as beam blow up, is a transverse instability observed in RF electron linacs and in induction linacs. As early as 1957 the phenomenon was observed on short linacs operating with long pulses (> 1 μs) in the range of 500 mA peak. Above the current threshold the beam pulse length, as observed at the output of the accelerator, is shortened[1]. This suggests induced fields by the head of the pulse which act back on the tail to make it unstable. This mechanism which can occur in a single accelerating structure was called regenerative beam break up.

Later on, with longer linacs made of many successive accelerating structures, the same pulse shortening effect could be observed but at a much lower threshold, of the order of 10 mA peak, suggesting a cumulative effect from all the structures. This second manifestation of the beam break up has been intensively studied on the Stanford Linear Accelerator, two-miles long, since it appeared as the main limiting effect[2]. The experimental observations can be summarized as follows:

a) At any location along the accelerator the typical pictures of the beam pulses below and above threshold are illustrated in Fig. 1. The shortening is more pronounced as the current from the injector is increased

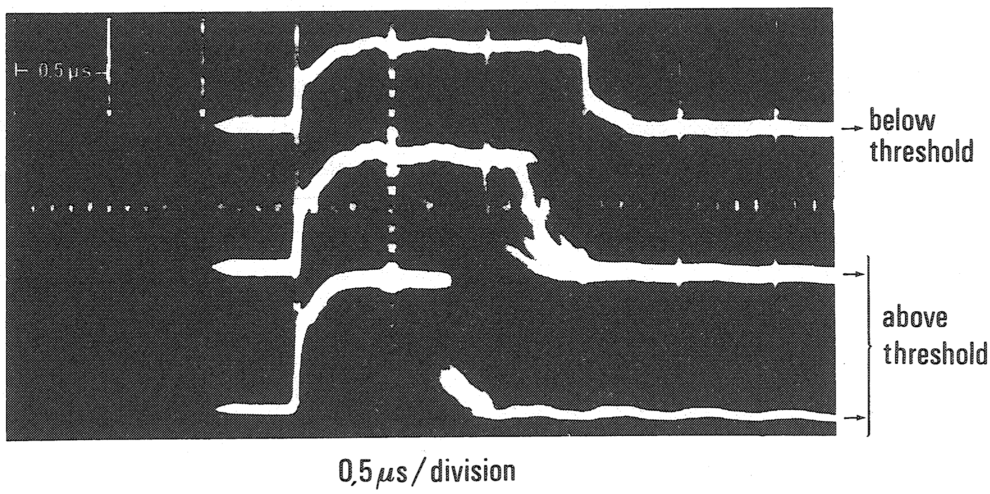

Fig. 1 Oscillograms of beam pulses below and above beam break up threshold

b) Above the threshold the amount of transmitted charge along the accelerator decreases erratically, and if the current is further increased the losses will appear earlier along the accelerator path (Fig. 2).

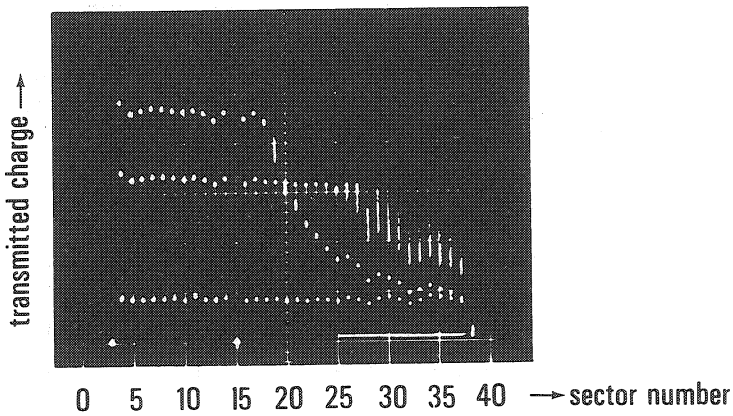

Fig. 2 Transmitted charge along the accelerator for different injected currents

c) Above the threshold the beam cross section as observed at the end of the accelerator is randomly increased in both transverse directions, suggesting a transverse instability.
d) A suitable external magnetic focusing system, as provided for instance by quadrupole magnets, can improve the BBU threshold.
e) The BBU effect strongly depends on misalignment of the accelerating structures (off-axis beam) and on the noise level from the RF power sources.

The beam break up, as described above, either regenerative or cumulative, has been identified as a beam interaction with parasitic deflecting modes which can propagate in the accelerating structures. Such single modes can have relatively high shunt impedances and high Q values; hence the beam-induced wake field has a long memory which can affect the tail of long beam pulses. In these cases the equivalent impedance of the surroundings, which causes deflection, is a narrow-band type resonator.

In more recent times, linacs are being operated with very short, high peak current, pulses. This is for instance the case at SLAC, where the SLC project (SLAC Linear Collider) requires acceleration of single, high density, RF bunches. In that case a new type of beam break up has been identified in which the head of the bunch also influences the tail[3]. However since the pulse is very short the mechanism needs a much faster induced wake field. Such a fast wake field can be generated by the equivalent low-Q, wide-band transverse impedance of the accelerating structures, corresponding to an average effect of all the narrow-band parasitic modes up to quite high frequencies. Since the frequency spectrum of very short bunches is wide, the excitation of this impedance is made possible. The effect is very similar to the head-tail effect in storage rings, although here there is no synchrotron oscillation to enhance the amplification mechanism. In a linear accelerator the head of the bunch does not allow deflection, only the tail is affected, leading to a banana shape as shown on Fig. 3 a. However this corresponds to the most simple oscillating mode (dipole mode) for the tail motion under deflecting forces. Figure 3 b for instance shows a quadrupole oscillating mode for which the center of gravity of the tail does not move. In both cases the effective transverse emittance is increased.

Fig. 3 BBU induced by short beam pulses
a) dipole mode b) quadrupole mode

According to the similarity of the banana effect with the head tail, it will not be treated further in this lecture.

2. TRANSVERSE DEFLECTION OF CHARGED PARTICLES IN RADIO-FREQUENCY FIELDS

Consider an electron travelling parallel to the axis of an accelerating structure with a velocity $v < c$. If the structure develops an electromagnetic field having transverse components, the transverse force applied to the electron is :

$$F_\perp = e\,[\mathcal{E}_\perp + v \times \mathcal{B}_\perp]\, e^{i\theta_o}$$

where $v = v\,u$, u being the unit vector along the longitudinal axis oz, $\mathcal{B}_\perp = \mu\,\mathcal{H}_\perp$ and θ_o is the initial phase difference between the particle and the longitudinal component of the wave.

First considering a travelling wave propagating along oz one has :

$$\mathcal{E}_\perp = E_\perp(x,y)\ \exp i(\omega t - \beta z)$$
$$\mathcal{H}_\perp = H_\perp(x,y)\ \exp i(\omega t - \beta z)$$

where the phase velocity is defined as $v_\phi = \omega/\beta$

Hence :

$$F_\perp = e\,[E_\perp + v\mu\,(u \times H_\perp)]\ \exp i(\omega t - \beta z + \theta_o)$$

Analysis of Maxwell's equations leads to the following identity relating the transverse components of the field to the longitudinal electric component :

$$\mu(u \times H_\perp) = -\frac{1}{v_\phi} E_\perp + \frac{i}{\omega} \nabla_\perp E_z$$

and the force now becomes

$$F_\perp = e[(1 - \frac{v}{v_\phi}) E_\perp + i\,\frac{v}{\omega} \nabla_\perp E_z]\ \exp i(\omega t - \beta z + \theta_o)$$

Assuming the particle travels in synchronism with the wave, $v = v_\phi$, and since $t = z/v$ one gets $\omega t = \omega z/v = \beta z$. Hence[4)]

$$F_\perp = \frac{e}{\beta} \nabla_\perp E_z$$

having choosen $\theta_o = -\pi/2$ which means that the particle is in phase with the transverse components of the travelling wave.

The previous expression is general and can be applied to all types of travelling waves. It shows that for a synchronous wave the combined effect from the transverse electric and magnetic fields is proportionnal to the transverse gradient of the longitudinal electric field component. Applied to classical waves one can conclude that[5]:

- for TE waves, since $E_z = 0$, there is no transverse deflection whatever the particle velocity is, provided the synchronism condition is satisfied. In other words the transverse magnetic field exactly compensates the transverse electric field for this case.

- for TM waves, synchronous with the particle, the deflecting force is finite but decreases as $v = v_\phi$ increases. For v_ϕ approaching c, it can be seen from Maxwell's equations that $\nabla_\perp E_z$ tends to zero ; hence the transverse deflection from synchronous TM waves goes to zero for ultra-relativistic particles.

As a first conclusion, one does not expect any transverse deflection from classical travelling waves, synchronous with ultra-relativistic particles. Notice that above a few MeV electrons can be considered as ultra-relativistic ($v = c$).

Let us consider now the case of classical standing-wave modes, knowing that although an accelerating structure has been designed to propagate a peculiar accelerating mode without reflection, higher-order standing-wave modes can develop locally for instance where mechanical transitions occur (change in iris diameter for tapered iris-loaded structures to keep the accelerating gradient constant). Deflecting properties of standing-wave modes can be emphazised with a very simple example. Consider for instance a cylindrical cavity in which a TM_{110} mode is excited ; the field components are[6] :

$$E_z = E_o\, J_1(kr)\, \cos\phi$$

$$H_r = -i\, \frac{E_o}{Z_o}\, [J_1(kr)/kr] \sin\phi$$

$$H_\phi = -i\, \frac{E_o}{Z_o}\, J_1'\,(kr)\, \cos\phi$$

with $Z_o = \sqrt{\mu_o/\varepsilon_o}$, $k = 2\pi/\lambda_o = \omega/c$, where λ_o is the free space wavelength.

Expanding the Bessel functions and assuming the wave is polarized in the horizontal plane, one gets near the axis :

$$E_z \simeq \left(\frac{\partial E_z}{\partial x}\right) x$$

$$H_y = -\, i\, \frac{E_o}{Z_o} = -\, \frac{i}{Z_o k} \left(\frac{\partial E_z}{\partial x}\right) .$$

A relativistic electron traversing the cavity of length L near the axis will receive a transverse momentum impulse from the H_y component :

$$\Delta p_x = -\, e \int_o^L (v\mu_o H_y)\, \frac{dz}{v} = i\, \frac{e}{\omega} \int_o^L \left(\frac{\partial E_z}{\partial x}\right) dz$$

showing that a standing wave TM mode can deflect relativistic particles. In fact this finite deflection comes from the interaction of the particle with the backward wave, which of course is not synchronous with the particle, and can only have a limited interaction length.

At this point one should mention that the off-axis particle will also interact with the longitudinal electric component of the deflecting mode, leading to an energy increment:

$$\Delta W = e \int_0^L \left(\frac{\partial E_z}{\partial x}\right) x \, dz$$

Depending on the sign of this quantity the particle can either get more energy or lose a fraction of its initial energy. In the latter case the particle gives energy to the deflecting mode.

A similar treatment in the case of standing wave TE mode would show that no deflection is expected in that case, which in fact is quite obvious since both the backward and forward waves have no E_z component.

3 DEFLECTING MODES IN CIRCULAR IRIS LOADED WAVEGUIDES

Up to now only TM and TE modes have been considered, and for which little trouble can be expected, since for relativistic particles no synchronous deflection can occur. However, these modes happen to be independent solutions of Maxwell's equations only in the case of simplified structures such as smooth waveguides and closed boxes. In practice there must be holes, for instance in a resonant cavity, for the beam passage and if one considers a TM mode in such a real cavity, the magnetic component of the field, which normally lies in a plane perpendicular to the axis, will be distorted in the neighbourhood of the holes (Fig. 4) resulting in an additional axial magnetic field component. Thus the presence of the end holes results in a mode which is no longer a pure TM mode, but a TM like mode with an associated longitudinal magnetic field. Such a mode is called a hybrid mode.

In order to satisfy Maxwell's equations it can be shown that in the general case two independant hybrid modes are found[7]; they are called HE (hybrid electric) and HM (hybrid magnetic) and they become TE and TM modes in the special case of simple boundary conditions. Hybrid modes are also very often called HEM (hybrid electro magnetic) modes in the literature.

To accelerate relativistic electrons one mainly uses travelling iris loaded waveguides[8,9] (Fig. 5) which change the phase velocity to that of light.

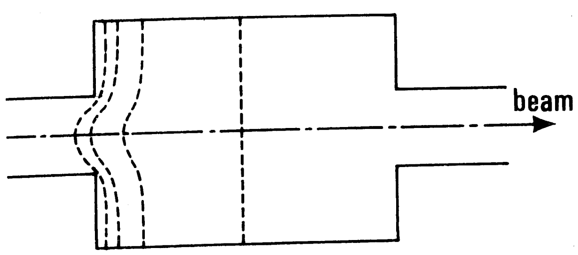

Fig. 4 Fringing field in the cut-off of a resonant cavity

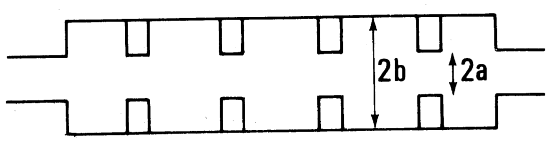

Fig. 5 Iris loaded structure

Analysis of TE and TM modes shows that they tend to become plane waves in the limiting case where $v_\phi \simeq c$; thus they are no longer independent solutions, and here again hybrid modes are necessary to satisfy completely Maxwell's equations. More generally the irises of a loaded waveguide will have an effect similar to the end holes of a cavity since they will distort the field lines and introduce additional field components.

The first hybrid deflecting mode is the HEM_{11} since it can be shown that the HEM_{01} tends to split into two independent TM_{01} and TE_{01}, the former being used for acceleration.

The general expressions for the components of this deflecting hybrid mode are complicated[7]. However in the limiting case where the phase velocity v_ϕ is equal to the light velocity c, one gets simple algebraic terms for an iris loaded structure[10] :

$$E_r = -iE_o\,[(\tfrac{1}{2}ka)^2 + (\tfrac{1}{2}kr)^2]\,\exp(i\phi)\,\exp(ikz-i\omega t)$$
$$E_\phi = E_o\,[(\tfrac{1}{2}ka)^2 - (\tfrac{1}{2}kr)^2]\,\exp(i\phi)\,\exp(ikz-i\omega t)$$
$$E_z = 2E_o\,[(\tfrac{1}{2}kr)]\,\exp(i\phi)\,\exp(ikz-i\omega t)$$
$$Z_oH_r = E_o\,[1 - \{(\tfrac{1}{2}ka)^2 - (\tfrac{1}{2}kr)^2\}]\,\exp(i\phi)\,\exp(ikz-i\omega t)$$
$$Z_oH_\phi = iE_o\,[1 - \{(\tfrac{1}{2}ka)^2 + (\tfrac{1}{2}kr)^2\}]\,\exp(i\phi)\,\exp(ikz-i\omega t)$$
$$Z_oH_z = i2E_o\,[(\tfrac{1}{2}kr)]\,\exp(i\phi)\,\exp(ikz-i\omega t)$$

where $Z_o = \mu_o c$ is the free space wave impedance, $k = 2\pi/\lambda_o$ is the free space propagation constant, and a is the iris radius.
Since the transverse force vector is given by

$$F_\perp = \frac{e}{k}\,\nabla_\perp\,E_z \qquad (k = \omega/c = \beta)$$

it can be seen that in the present case the magnetic and electric forces no longer compensate each other at light velocity. The components of the transverse force are :

$$F_r = eE_o\,\sin\phi\,\exp(ikz-i\omega t)$$
$$F_\phi = eE_o\,\cos\phi\,\exp(ikz-i\omega t)$$

Note that space harmonics also exist in order to satisfy the periodic boundary conditions due to the irises, but they do not contribute to the deflection since in general they are not in synchronism with the particle.

Hybrid modes can exhibit some curious properties ; depending on the choice of the waveguide parameters, such as 2a and 2b (see Fig. 5), the group velocity can be either positive or negative.
The group velocity is given by :

$$v_g = P/w_s$$

where w_s is the stored energy per unit length and P the time average power transmitted across the waveguide, for instance the closed surface S defined by the iris hole :

$$P = \mathcal{R}_e\,\frac{1}{2}\iint_S \left[E_\perp \times H_\perp^*\right] dS \quad .$$

Since

$$dS = r dr\, d\phi$$

one gets :

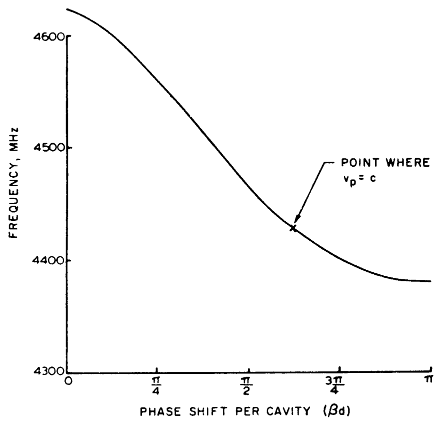

Fig. 6a Lower Passband

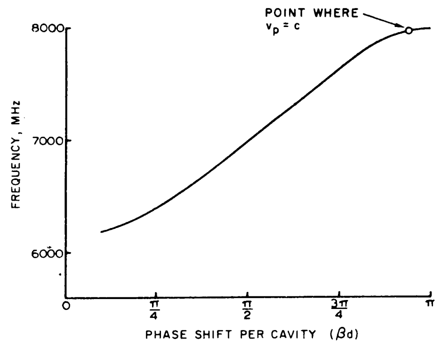

Fig. 6b Upper passband

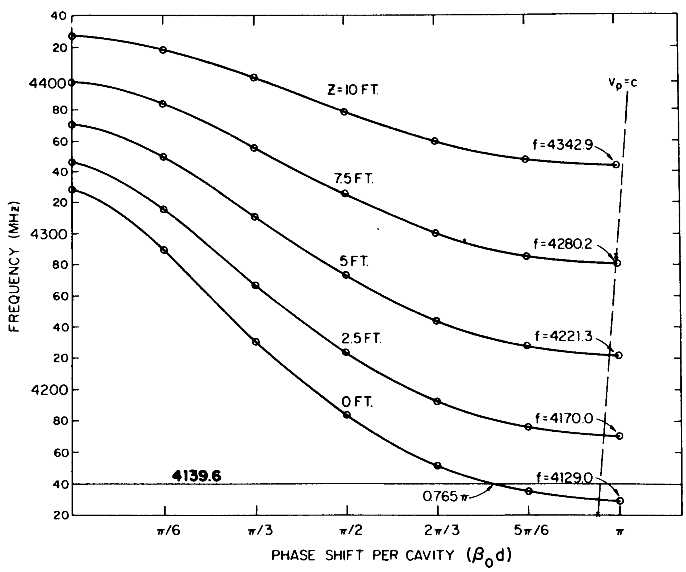

Fig. 7 Brillouin (or ω-β) diagram for the HEM_{11} mode at five locations along SLAC ten-foot accelerator section

$$P = \mathcal{R}_e \frac{1}{2} \int_0^a \int_0^{2\pi} \left(E_r H_\phi^* - E_\phi H_r^*\right) r dr\, d\phi \quad .$$

From the field components set previously for the HEM_{11} mode one gets finally :

$$P = \pi a^2 \frac{E_o^2}{Z_o} \frac{(ka)^2}{16} \left[\frac{(ka)^2}{6} - 1\right]$$

showing that the group velocity is negative if $ka < \sqrt{6}$.

For a standard iris loaded waveguide, such as the SLAC one at 3 GHz, the first hybrid mode exhibits a negative group velocity (Fig.6 a), but not the next one (Fig.6 b).

As will be seen later the first hybrid mode, which frequency is roughly 1.5 times the accelerating mode frequency, is the deflecting mode of interest for BBU because of its negative group velocity.

However, since most of the iris-loaded structures used for electron acceleration are tapered to keep the accelerating gradient constant, or quasi constant, the Brillouin diagram will show as many dispersion curves as there are different iris diameters along the structure. This is for instance illustrated in Fig. 7 for the SLAC case[6].

Figures of merit are also defined for deflecting modes, such as r, Q and r/Q. The expression for $r_\perp$ relevant in calculating the transverse deflection is given by :

$$r_\perp = \left[\frac{1}{\beta}\left(\frac{\partial E_z}{\partial r}\right)\right]^2 \Bigg/ \left|\frac{dP}{dz}\right|$$

which takes account of the fact that $E_z = 0$ on the axis but not apart from the axis ($\nabla_\perp E_z \neq 0$).

4 REGENERATIVE BEAM BREAK UP

Regenerative BBU occurs in one accelerating section and is due to the deflecting HEM_{11} wave travelling in the direction of the electron motion with a phase velocity slightly lower than the light velocity so that approximate synchronism is possible. The small difference in velocity will make the electron slip ahead. Depending on its initial phase the electron can be deflected in one or the other direction (Fig. 8).

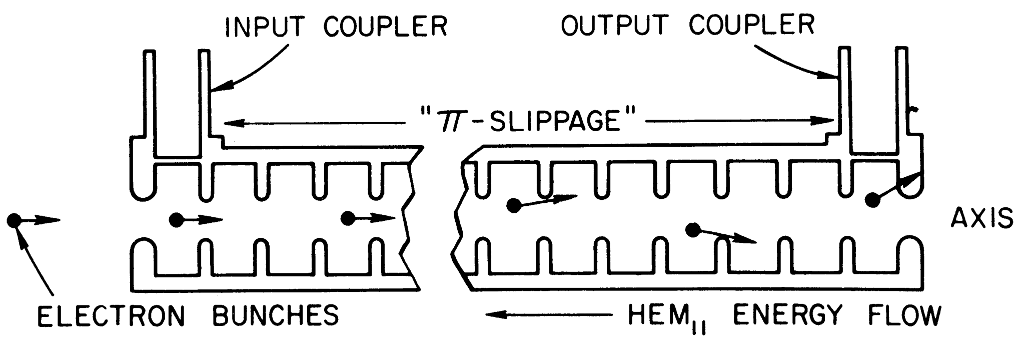

Fig. 8 Deflecting HEM_{11} mode in an iris-loaded structure

If, for instance, the electron enters the structure with an initial phase such that the transverse deflecting force has its maximum value, then the longitudinal electric component of the mode is zero, but since the electron slips ahead it will enter in a longitudinally decelerating field, off-axis, and give energy to the mode. As a consequence a noise-generated HEM_{11} wave can be amplified by the beam itself as soon as it has been brought off-axis.

This can be better understood by using a schematic representation of the Lorentz force near the axis, in a system co-moving with the HEM_{11} wave (Fig. 9).

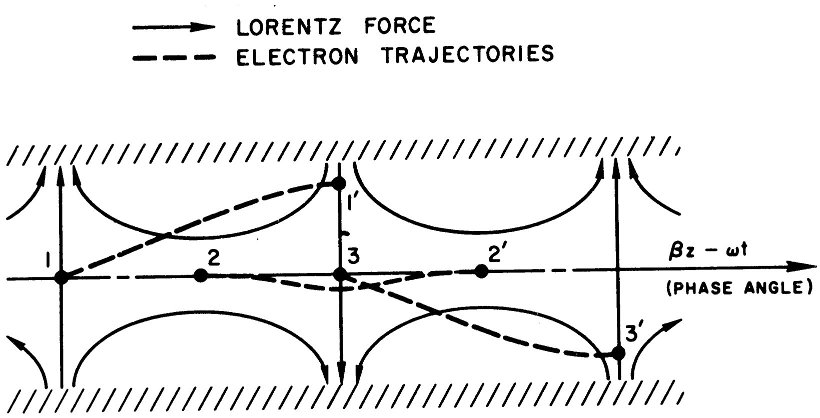

Fig. 9 Lorentz force near the axis in the HEM_{11} mode, in a system co-moving with the wave

Electrons entering the structure at phase points 1 and 3, corresponding to maximum deflecting forces, will move off axis and since they travel faster than the wave they will enter in a retarding field and thus will transfer energy to the field. If these electrons leave the structure at points 1′ and 3′ corresponding to a π-slippage they will have reached the maximum deflection. Electrons which enter the structure at intermediate phases, such as point 2 will in general also transfer a positive (or zero) amount of energy to the field. The optimum phase slip, giving a maximum deflection, depends on the initial electron phase relative to the wave, and it can be as much as 180° as seen before.

In a dispersive structure there will be in general some frequency at which the phase slip is optimized and it is near this frequency that beam break up is most likely to occur (Fig. 7).

In addition to the amplification mechanism which has been described, regeneration (or enhancement of the amplification) can occur due to the backward wave characteristic of the HEM_{11} mode ($v_g < 0$). As a matter of fact the energy deposited flowing back upstream will reinforce the original deflecting field.

Finally if the corresponding generated power exceeds the power losses into the walls, both the field and the deflection will grow exponentially leading to a transverse instability.

The starting current, or instability threshold, for the backward wave oscillation has been estimated by several authors[1,6] by equating the power generated by the beam to the power propagating into the wave and lost into the walls.

The energy given by the beam to the deflecting mode can be written as :

$$\Delta W = - \int_o^L q\, E_z\, dz = - \int_o^L q \left[\frac{\partial E_z}{\partial x}\, x\right]_b dz$$

where q is the accelerated charge, E_z the longitudinal electric component of the deflecting mode, x the beam transverse displacement relative to the axis. The index "b" means that the integral must be performed along the beam path.

The generated power, time averaged over an RF cycle, in complex notation, becomes :

$$P_b = - \frac{1}{2}\, I \mathcal{R}_e \int_o^L \left[\frac{\partial E_z^*}{\partial x}\, x\right] dz$$

where I is the beam current, L the length of the accelerating structure.

The first term in the bracket is obtained from the expressions of the field components of the HEM_{11} mode :

$$\frac{\partial E_z}{\partial x} = k\, E_o\, e^{\,i(\omega t - \beta z)}$$

where $t = z/c$ assuming the particle velocity remains constant and equal to c and where β is assumed to be slightly different from $k = \omega/c$. Expanding β near the point $\beta_o = \omega_o/c$ where the straight line $v_\phi = c$ crosses the dispersion curve (see Fig. 6 a) one gets :

$$\beta = \beta_o - (\omega - \omega_o)/|v_g| \quad .$$

Introducing the following quantity :

$$\delta\beta = (\omega - \omega_o) \left[\frac{1}{c} + \frac{1}{|v_g|}\right]$$

one gets :

$$\frac{\partial E_z}{\partial x} = k\, E_o\, e^{i\, \delta\beta\, z} \quad .$$

The second term in the bracket is obtained as follows ; since :

$$p_x = p_z \frac{dx}{dz}$$

then :

$$x(z) = \int_o^z \frac{P_x}{P_z}\, dz \quad .$$

Using the results of Section 2 for the transverse momentum :

$$p_x = \frac{ie}{kc} \int \frac{\partial E_z}{\partial x}\, dz$$

one finally gets :

$$P_b = -\frac{i}{2}\frac{e}{pc} I\, k\, E_o^2 \mathcal{R}_e \int_o^L dz \int_o^z dz' \int_o^{z'} e^{i\delta\beta(z''-z)} dz''$$

having assumed $p_z = p = cte$.
Integration of the previous expression leads to :

$$P_b = 2k(L/\pi)^3 \; (e/pc)\; I\; E_o^2\; g(\alpha)$$

with

$$\alpha = L\;\delta\beta$$

$$g(\alpha) = \frac{1}{2}\,(1-\cos\alpha - \frac{1}{2}\,\alpha\,\sin\alpha)/(\alpha/\pi)^3 \quad .$$

It is found that there is an optimum value for the phase-slip parameter α which gives the maximum effect :

$$\alpha = 2.65 \quad \longrightarrow \quad g(\alpha) = 1.04 \quad .$$

The power loss can be deduced from the definition of the transverse shunt impedance per unit length :

$$r_\perp = \frac{(1/k)^2\; (\partial E_z/\partial x)^2}{dP/dz} \quad .$$

Since :

$$Q = \frac{\omega\, w_s}{dP/dz}$$

$$w_s = P/v_g$$

where w_s is the stored energy per unit length, then :

$$P = \frac{v_g}{\omega}\,\frac{Q}{r_\perp}\,\frac{1}{k^2}\left(\frac{\partial E_z}{\partial x}\right)^2 = \frac{v_g}{\omega}\,\frac{Q}{r_\perp}\,E_o^2 \quad .$$

Equating the power loss to the generated power leads to the current threshold for regenerative BBU :

$$I_{th} = \frac{\pi}{g}\,\frac{v_g}{c}\,\frac{Q}{r_\perp\lambda_o}\,\frac{pc}{e}\left(\frac{\lambda_o}{2L}\right)^3$$

where λ_o is the free-space wavelength.

<u>Numerical application</u>

Consider a 1 meter long S-band structure operating at 2.8 GHz in the $\pi/2$ mode, with an accelerating gradient of 15 MeV/m. The estimated characteristics of the first deflecting mode are :

$$\text{frequency} = 4.25 \text{ GHz}$$

$$v_g/c = -\ .02$$

$$\frac{r_\perp\,\lambda_o}{Q} = 200\ \Omega \quad .$$

Hence, one gets for the threshold, I_{th} = 66 mA. It must be noticed that in similar structures BBU has been observed at the level of 100 mA, after several microseconds.

5 CUMULATIVE BEAM BREAK UP

The cumulative beam break up, or multisection beam break up, differs considerably from the previous one. Here each section acts like an amplifier which provides a small increase in the amplitude of the transverse deflecting wave (Fig. 10). Even though the gain per stage is very small the total gain in a long accelerator can be very large. At each amplifying cavity there is a transverse displacement modulation and a transverse momentum modulation on the beam.

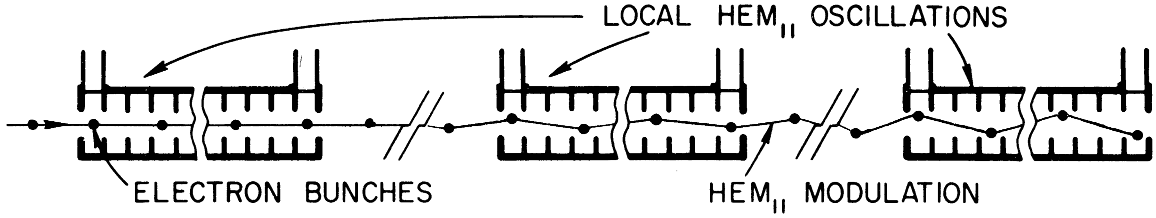

Fig. 10 Multisection HEM_{11} transverse deflection

The transverse displacement modulation excites the cavity through the interaction with the off-axis E_z field component of the HEM_{11} mode, and the resulting H_ϕ field component provides an additional momentum kick to the beam.

In the drift space between cavities the transverse momentum is converted into additional displacement.

This modulation further excites the resonant field in the downstream cavities which in turn deflect the next bunches even more until finally they scrape the accelerator walls.

The effect manifests itself at beam currents well below the threshold for regenerative BBU. It was first observed and extensively studied at SLAC[6)].

The model for cumulative BBU assumes that the effect of an entire accelerator section is equivallent to an impulse at a single point (Fig. 11). This description applies particularly to machines which use tapered sections for which the synchronous length at any frequency is very short.

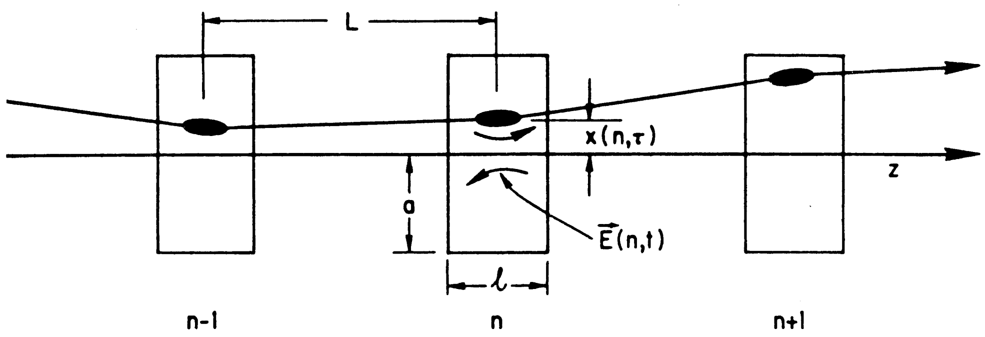

Fig. 11 Schematic representation for cumulative BBU

Hence the whole system can be considered as a transport system :

$$\begin{bmatrix} x \\ p \end{bmatrix}_n = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \Delta p_{n-1} & 1 \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}_{n-1}$$

where the matrix m_{ij} can include external focusing lenses.

Obviously cumulative BBU will depend on the magnetic focussing channel along the accelerator which gives additional transverse momentum kicks that can compensate partially the cavity deflections.

Since the information is transfered from one unit to the next by the beam itself, a slight change in the geometry of each structure, which also changes the deflecting mode frequency, will provide a detuning effect that can lower the efficiency of cumulative beam break up. This trick has been used on recent linacs and is very similar to the one which consists of tapering a single structure to lower the regenerative effect.

The standard model for cumulative BBU does not use the field configuration for the mode, but rather assumes that the mode is characterized by a vector potential A and makes use of the fact that the transverse kick is directly related to that vector potential[11] :

$$\Delta p_x = \frac{ie}{kc} \int \frac{\partial E_z}{\partial x}\, dz = e \int \frac{\partial A_z}{\partial x}\, dz \quad .$$

The deflecting mode is considered as a single cavity eigen mode and is characterized by a vector potential A_λ that obeys the Helmholtz equation :

$$\nabla^2 \mathbf{A}_\lambda + (\omega_\lambda/c)^2\, \mathbf{A}_\lambda = 0 \quad .$$

However, since a beam travels along the cavity the actual time dependant vector potential A(r,t) must obey the wave equation :

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = - \mu_o \mathbf{J} \quad .$$

Assuming the following expansion[12,13,14] :

$$\mathbf{A}(r,t) = \sigma(t)\, \mathbf{A}_\lambda(r)$$

one gets :

$$(\ddot{\sigma} + \omega_\lambda^2\, \sigma)\, \mathbf{A}_\lambda = \frac{1}{\varepsilon_o} \mathbf{J} \quad .$$

If the current is flowing in the z direction, $J_x = J_y = 0$ and the component $A_{\lambda z}$ remains.

Taking the scalar product of the previous equation with $\mathbf{A}_\lambda$ and integrating over the volume occupied by the field leadsto :

$$\ddot{\sigma} + \omega_\lambda^2\, \sigma = \frac{1}{\varepsilon_o} \frac{\int_V J_z\, A_{\lambda z}\, dV}{\int_V A_\lambda^2\, dV} \quad .$$

Assuming x is independent of z in the active region and if in addition $A_{\lambda z}$ varies linearly with x, one can write :

$$\int J_z\, A_{\lambda z}\, dV \simeq I\, x \int \frac{\partial A_{\lambda z}}{\partial x}\, dz \quad .$$

The integral in the denominator is related to the stored energy :

$$W_s = \frac{\varepsilon_o}{2} \int E_\lambda^2\, dV = \frac{\varepsilon_o \omega^2}{2} \int A_\lambda^2\, dV \quad .$$

Mixing previous formulae leads to :

$$\left[\frac{d^2}{dt^2} + \omega^2\right] \Delta p_x = \frac{e\, I\, x\, \omega^2}{2\, W_s} \left[\int \frac{\partial A_{\lambda z}}{\partial x}\, dz\right]^2 \quad .$$

The total transverse shunt impedance of the cavity is such that :

$$\frac{R_\perp}{Q} = \frac{\left|\int \frac{1}{k} \left(\frac{\partial E_z}{\partial x}\right) dz\right|^2}{\omega\, W_s} = \frac{\left|\int c \left(\frac{\partial A_{\lambda z}}{\partial x}\right) dz\right|^2}{\omega\, W_s} \quad \text{(in ohm)}.$$

Hence one finally gets :

$$\left[\frac{d^2}{dt^2} + \omega^2\right] \Delta p_x = \frac{e\, I\, x\, \omega^3}{2\, c^2} \left(\frac{R_\perp}{Q}\right) \quad .$$

Losses in the cavity can be taken into account in the bracket by adding a term $(\omega/Q)d/dt$, and defining the damping factor as $\alpha = \omega/2Q$.

Integrating the previous expression by applying the Green-function method gives the solution relating Δp_n at the n^{th} unit to x_n at the same unit :

$$\Delta p_{xn} = \frac{e\omega^2}{2c^2} \left(\frac{R_\perp}{Q}\right) \int_o^t I(t')\, x_n(t')\, e^{-\alpha(t-t')} \sin \omega(t-t')\, dt'$$

having introduced the initial conditions such that $\Delta p = 0$ and $d(\Delta p)/dt = 0$ at $t = 0$. The integral shows that at time t, in cavity number n, the effect depends on the sum of the displacements of each part of the beam which has already passed the cavity. The displacement of each part of the beam depends on the momentum kick which was given to that part of the beam in the previous cavity and can be computed from the transfer matrix. A computer code would divide the beam in elementary portions corresponding to transit times $\Delta t'$ and replace the integral by a sum. In fact since the beam in a linac is bunched these portions could be taken corresponding to the RF micro bunches. A sophisticated theory[14)] has included the bunching in the starting assumptions giving the final result in cavity number N in terms of a summation over the bunch number M.

Attempts to find analytic solutions of the equations of cumulative BBU have been made by several authors[6,11,13,14)].

It is found that the beam break up can be characterized by three regimes. The first corresponds to an exponential increase of bunch displacement with time (or RF bunch number). The second corresponds to the maximum displacement, while the third is the steady state regime.

In the exponential growth regime, for a coasting beam and no focusing the e-folding factor is given by :

$$F_e(t) = \frac{3^{3/2}}{2^{5/3}} \left(\frac{z^2 \; I_o \; c \; t \; \pi^2 \; R_\perp}{L \; V_o \; \lambda^2 \; Q} \right)^{1/3}$$

where L is the distance between the input of two successive cavities, $e\,V_o = \gamma m_o c^2$, λ the wave length of the active mode, z the position along the accelerator.

The maximum steady state displacement arising from an initially modulated beam has an e-folding factor which can be written, with no focusing :

$$F_e(c\ w) = (3)^{3/4} \left(\frac{z \; \pi \; I_o \; R_\perp}{2L \; V' \; \lambda} \right)^{1/2}$$

valid for an accelerated beam with energy much greater than the initial energy. $V' = dV/dz$ represents a uniform accelerating gradient.

In the case where a smooth focusing is introduced and if the accelerating gradient and the focusing strength are constant along the accelerator the e-folding factor becomes:

$$F'_e = F_e \left[1 - C \; k_\beta^2 \; z^2 / \; F_e^2 \right] \quad .$$

Here k_β is the betatron wave number of the focussing system and C is equal to 1/2 for the steady state case and 3/4 for the transient case.

Compensation of cumulative beam break up can be partly attained by good design of the focusing system. Improvement can also be made by minimizing the positioning errors (beam off-axis) and the noise from RF sources.

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