

Pileup Subtraction for Jet Shapes

Gregory Soyez,¹ Gavin P. Salam,^{2,3,4} Ji-Hun Kim,¹ Souvik Dutta,⁵ and Matteo Cacciari^{3,6}

¹CEA Saclay, CNRS URA 2306, IPhT, F-91191 Gif-sur-Yvette, France

²PH-TH, CERN, CH-1211 Geneva 23, Switzerland

³LPTHE, CNRS UMR 7589, UPMC Université Paris 6, Paris 75252, France

⁴Department of Physics, Princeton University, Princeton, New Jersey 08544, USA

⁵Indian Institute of Technology Bombay, Mumbai 400076, India

⁶Université Paris Diderot, Paris 75205, France

(Received 25 November 2012; published 16 April 2013)

Jets in high energy hadronic collisions often contain the fingerprints of the particles that produced them. Those fingerprints, and thus the nature of the particles that produced the jets, can be read off with the help of quantities known as jet shapes. Jet shapes are, however, severely affected by pileup, the accumulation in the detector of the residues of the many simultaneous collisions taking place in the Large Hadron Collider (LHC). We introduce a method to correct for pileup effects in jet shapes. Relative to earlier, limited approaches, the key advance resides in its full generality, achieved through a numerical determination, for each jet, of a given shape's susceptibility to pileup. The method rescues the possibility of using jet shapes in the high pileup environment of current and future LHC running, as we show with examples of quark-gluon discrimination and top tagging.

DOI: [10.1103/PhysRevLett.110.162001](https://doi.org/10.1103/PhysRevLett.110.162001)

PACS numbers: 13.85.-t, 13.87.-a

When energetic quarks or gluons (partons) fragment, they produce collimated bunches of hadrons known as jets. Jets mostly conserve the energy and direction of the originating parton; consequently, they have long been used at colliders as a stand-in for generic partons, as is the case currently at CERN's Large Hadron Collider (LHC). In recent years extensive interest has developed in going beyond this basic use. Since jets often contain the fingerprints of the particles that produced them, one can, for example, try to understand if the originating parton is a quark or a gluon, or identify rare cases where a single jet originated from multiple hard partons, perhaps from the hadronic decay of a highly boosted W , Z , or Higgs boson, top quark, or other massive object [1–3]. The “jet substructure” techniques being developed in this context will be crucial to exploit the full kinematic reach of the LHC, notably the high transverse-momentum (high- p_t) region, and to maximize the LHC's sensitivity to hadronic manifestations of new physics scenarios.

Two key classes of approach are available to probe substructure: one identifies smaller “subjets” within a larger jet and then performs selections based on the kinematics of those subjets, for example, Refs. [4–10], the other involves jet-shape observables, sensitive to the geometrical spread of the energy within the jet, e.g., Refs. [11–18]. Both classes appear to be powerful and viable experimentally (see, e.g., Refs. [19,20]), and ultimate performance in exploiting jet substructure will probably be obtained through some combination of them.

One potentially severe issue in substructure studies is that of pileup: with the LHC now operating at high instantaneous luminosities, each interesting, high- p_t proton-proton

collision is accompanied by dozens of additional pp collisions, which add substantial low- p_t noise to the event. Pileup modifies a jet's kinematics, on average shifting its p_t in proportion to the level of noise in the event and to the jet's extent, or “area” [21], in rapidity ($y = \frac{1}{2} \ln \frac{E+p_z}{E-p_z}$) and azimuth (ϕ). Two techniques are in common use to correct for this: the removal of an “offset” from the jet in proportion to the number of observed pileup events [22], and the “area-median” method, which subtracts an amount given by the product of the event's measured pileup p_t density (ρ) and the jet's measured area (A) [23–26]. While the second of these methods can be straightforwardly applied also to subjets, jet shapes have so far proved more challenging to correct.

Jet shapes are particularly sensitive to pileup because its diffuse soft energy flow is characteristically different from the more collimated distribution of energy due to normal jet fragmentation. One can attempt to mitigate pileup's impact by determining the shape using just charged tracks, or by breaking a jet into subjets and using just the hardest subjets, but both methods throw away a significant fraction of the original particles contributing to the jet's shape, introducing a bias. One can also carry out analytical calculations of a given shape's sensitivity, as in Refs. [28,29], or add in particles from a “complementary” cone at 90 deg to the jet's axis in order to determine an average sensitivity [29]. These methods have so far, however, been limited either to specific observables, restricted classes of jets (e.g., circular jets), or low pileup. The intent of this Letter is to develop an effective, simple, general method to correct jet shapes for pileup.

Our approach is rooted in the framework of the area-median method, which has been found to be beneficial in both ATLAS [24] and CMS [25] (see also Refs. [30–32]). It is intended to be valid for arbitrary jet algorithms and generic infrared and collinear safe jet shapes [33], without the need for dedicated analytic study of each individual shape variable. It also involves an extension of the original area-median prescription to account for hadron masses.

The first ingredient is a characterization of the average pileup density in a given event in terms of two variables, ρ and ρ_m , such that the four-vector of the expected pileup deposition in a small region of size $\delta y \delta \phi$ can be written as

$$[\rho \cos \phi, \rho \sin \phi, (\rho + \rho_m) \sinh y, (\rho + \rho_m) \cosh y] \delta y \delta \phi, \quad (1)$$

where ρ and ρ_m have only weak dependence on y (and ϕ). One way of determining ρ and ρ_m is the area-median method [23]. The inclusion of the ρ_m term is one novelty of this Letter: ρ_m arises because pileup consists of low- p_t hadrons, and their masses are not negligible relative to their p_t (cf. also Refs. [36,37]). It is important mainly for observables sensitive to differences between energy and 3-momentum, e.g., jet masses, as we will see below.

The second and main new ingredient is a determination, for a specific jet, of the shape's sensitivity to pileup. Let the shape be defined by some function $V(\{p_i\}_{\text{jet}})$ of the momenta p_i in the jet. Among these momenta, we include a set of “ghosts” [21], very low momentum particles that cover the y - ϕ plane at high density, each of them mimicking a pileup-like component in a region of area A_g . We then consider the derivatives of the jet shape with respect to the transverse-momentum scale $p_{t,g}$ of the ghosts and with respect to a component $m_{\delta,g} \equiv \sqrt{m_g^2 + p_{t,g}^2} - p_{t,g}$,

$$V_{\text{jet}}^{(n,m)} \equiv A_g^{n+m} \partial_{p_{t,g}}^n \partial_{m_{\delta,g}}^m V(\{p_i\}_{\text{jet}}). \quad (2)$$

The derivatives are to be evaluated at $p_{t,g} = m_{\delta,g} = 0$, and by scaling all ghost momenta simultaneously.

Given the level of pileup, ρ , ρ_m , and the information on the derivatives, one can then extrapolate the value of the jet's shape to zero pileup,

$$V_{\text{jet,sub}} = V_{\text{jet}} - \rho V_{\text{jet}}^{(1,0)} - \rho_m V_{\text{jet}}^{(0,1)} + \frac{1}{2} \rho^2 V_{\text{jet}}^{(2,0)} + \frac{1}{2} \rho_m^2 V_{\text{jet}}^{(0,2)} + \rho \rho_m V_{\text{jet}}^{(1,1)} + \dots, \quad (3)$$

where the formula takes into account the fact that the derivatives are evaluated for the jet including the pileup.

Handling derivatives with respect to both $p_{t,g}$ and $m_{\delta,g}$ can be cumbersome in practice. An alternative is to introduce a new variable $r_{t,g}$ and set $p_{t,g} = r_{t,g}$ and $m_{\delta,g} = \frac{\rho_m}{\rho} r_{t,g}$. We then take total derivatives with respect to $r_{t,g}$,

$$V_{\text{jet}}^{[n]} \equiv A_g^n \frac{d^n}{dr_{t,g}^n} V(\{p_i\}_{\text{jet}}), \quad (4)$$

so that the correction can be rewritten

$$V_{\text{jet,sub}} = V_{\text{jet}} - \rho V_{\text{jet}}^{[1]} + \frac{1}{2} \rho^2 V_{\text{jet}}^{[2]} + \dots \quad (5)$$

The derivatives $V^{(m,n)}$ or $V_{\text{jet}}^{[n]}$ can be determined numerically, for a specific jet, by rescaling the ghost momenta and reevaluating the jet shape for multiple rescaled values. Typically this is more stable with Eq. (4) and this is the approach we use below.

To investigate the performance of our correction procedure, we consider a number of jet shapes.

(1) Angularities [12,38], adapted to hadron-collider jets as $\theta^{(\beta)} = \sum_i p_{ti} \Delta R_{i,\text{jet}}^\beta / \sum_i p_{ti}$, for $\beta = 0.5, 1, 2, 3$; $\theta^{(1)}$, the “girth,” “width,” or “broadening” of the jet, has been found to be particularly useful for quark-gluon discrimination [17,39].

(2) Energy-energy-correlation (EEC) moments, advocated for their resummation simplicity in Ref. [40], $E^{(\beta)} = \sum_{i,j} p_{ti} p_{tj} \Delta R_{i,j}^\beta / (\sum_i p_{ti})^2$, using the same set of β values. EEC-related variables have been studied recently also in Ref. [41].

(3) “Subjettiness” ratios, designed for characterizing multipronged jets [13–15]: one defines the subjettiness $\tau_N^{(\text{axes},\beta)} = \sum_i p_{ti} \min(\Delta R_{i1}, \dots, \Delta R_{iN})^\beta / \sum_i p_{ti}$, where ΔR_{ia} is the distance between particle i and axis a , where a runs from 1 to N . One typically considers ratios such as $\tau_{21} \equiv \tau_2/\tau_1$ and $\tau_{32} \equiv \tau_3/\tau_2$ (the latter used, e.g., in a recent search for R -parity violating gluino decays [42]); we consider $\beta = 1$ and $\beta = 2$, as well as two choices for determining the axes: “kt,” which exploits the k_t algorithm [43,44] to decluster the jet to N subjets and then uses their axes, and “1kt,” which adjusts the kt axes so as to obtain a single-pass approximate minimisation of τ_N [15].

(4) A longitudinally invariant version of the planar flow [11,12], involving a 2×2 matrix $M_{\alpha\beta} = \sum_i p_{ti} (\alpha_i - \alpha_{\text{jet}}) (\beta_i - \beta_{\text{jet}})$, where α and β correspond to either the rapidity y or the azimuth ϕ ; the planar flow is then given by $Pf = 4\lambda_1 \lambda_2 / (\lambda_1 + \lambda_2)^2$, where $\lambda_{1,2}$ are the two eigenvalues of the matrix.

One should be aware that observables constructed from ratios of shapes, such as $\tau_{n,n-1}$ and planar flow, are not infrared and collinear safe for generic jets. In particular Pf and τ_{21} are infrared and collinear safe only when applied to jets with a structure of at least two hard prongs, usually guaranteed by requiring the jets to have significant mass; τ_{32} requires a hard three-pronged structure [45], a condition not imposed in previous work, and that we will apply here through a cut on τ_{21} .

For the angularities and EEC moments we have verified that the first two numerically obtained derivatives agree with analytical calculations in the case of a jet consisting of a single hard particle. For variables like τ_N that involve a partition of a jet, one subtlety is that the partitioning can change as the ghost momenta are varied to evaluate the

numerical derivative. The resulting discontinuities (or non-smoothness) in the observable's value would then result in nonsensical estimates of the derivatives. We find no such issue in our numerical method to evaluate the derivatives, but were it to arise, one could choose to force a fixed partitioning.

To test the method in simulated events with pileup, we use PYTHIA 8.165, tune 4C [46,47]. We consider three hard event samples: dijet, WW , and $t\bar{t}$ production, with hadronic W decays, all with underlying event (UE) turned off (were it turned on, the subtraction procedure would remove it too). We use anti- k_r jets [48] with $R = 0.7$, taking only those with $p_t > 500$ GeV (before addition of pileup). All jet finding is performed with FASTJET 3.0 [49]. The determination of ρ and ρ_m for each event follows the area-median approach [23]: the event is broken into patches and in each patch one evaluates $p_{t,\text{patch}} = \sum_{i \in \text{patch}} p_{ti}$, as well as $m_{\delta,\text{patch}} = \sum_{i \in \text{patch}} (\sqrt{m_i^2 + p_{ti}^2} - p_{ti})$, where the sum runs over particles i in the patch. Then ρ and ρ_m are given by

$$\rho = \text{median}_{\text{patches}} \left\{ \frac{p_{t,\text{patch}}}{A_{\text{patch}}} \right\}, \quad \rho_m = \text{median}_{\text{patches}} \left\{ \frac{m_{\delta,\text{patch}}}{A_{\text{patch}}} \right\}, \quad (6)$$

where A_{patch} is the area of each patch. To obtain the patches we cluster the event with the k_r algorithm with $R = 0.4$. The median helps limit the results' sensitivity to the presence of a handful of hard jets (cf. Refs. [50,51]).

For nonzero ρ_m the formula for correcting a jet's 4-momentum is

$$p_{\text{jet,sub}}^\mu = p_{\text{jet}}^\mu - [\rho A_{\text{jet}}^x, \rho A_{\text{jet}}^y, (\rho + \rho_m) A_{\text{jet}}^z, (\rho + \rho_m) A_{\text{jet}}^E], \quad (7)$$

where the area four-vector A^μ is the four-vector sum of the momenta of the ghosts in the jet multiplied by $A_g/p_{t,g}$ [21].

We have 17 observables and three event samples. Figure 1 gives a representative subset of the resulting 51 distributions, showing in each case the distribution (and average) for the shape without pileup (solid green line), the result with pileup (dashed line), and the impact of subtracting first and second derivatives (dotted and solid black lines, respectively). The plots for the distributions have been generated using a Poisson distribution of pileup events with an average of 30 events (comparable to typical 2012 LHC runs; our count includes diffractive and elastic events, and the analysis uses all particles from the event

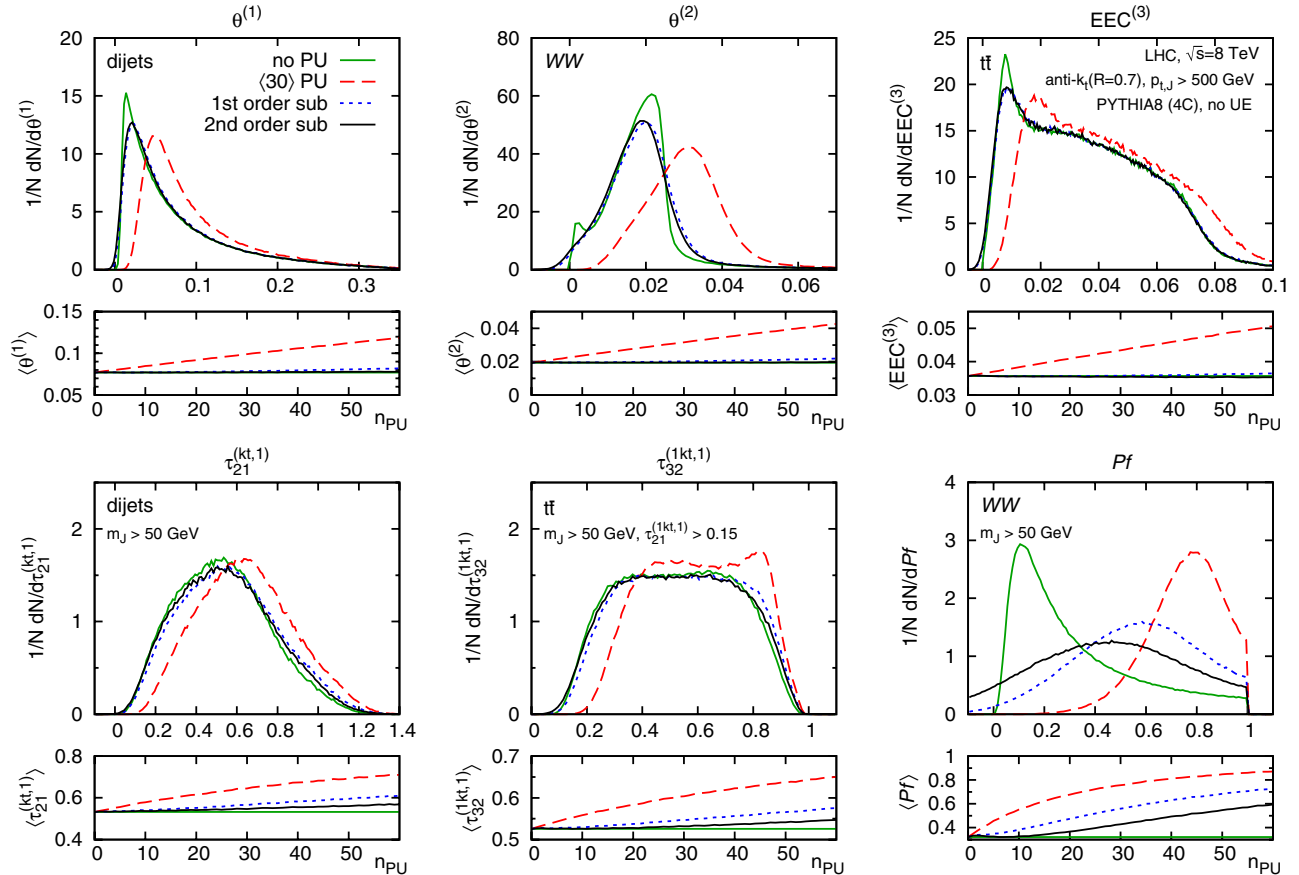


FIG. 1 (color online). Impact of pileup and subtraction on various jet-shape distributions and their averages, in dijet, WW , and $t\bar{t}$ production processes. The distributions are shown for Poisson distributed pileup (with an average of 30 pileup events) and the averages are shown as a function of the number of pileup events n_{PU} . The shapes are calculated for jets with $p_t > 500$ GeV (the cut is applied before adding pileup, as are the cuts on the jet mass m_J and subjettness ratio τ_{21} where relevant).

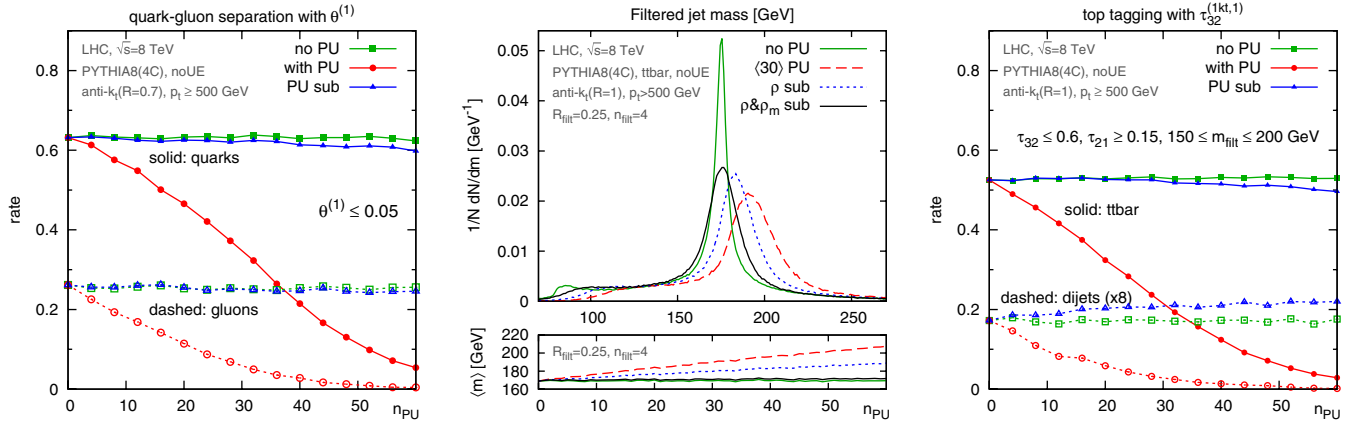


FIG. 2 (color online). Left: Rate for tagging quark and gluon jets using a fixed cut on the jet width, shown as a function of the number of pileup vertices. Middle: Filtered jet-mass distribution for fat jets in $t\bar{t}$ events, showing the impact of the ρ and ρ_m components of the subtraction. Right: Tagging rate of an N -subjettiness top tagger for $t\bar{t}$ signal and dijet background as a function of the number of pileup vertices. All cuts are applied after addition (and possible subtraction) of pileup. Subtraction acts on τ_1 , τ_2 , and τ_3 individually. See text for further details.

generator, leading to $\rho \simeq 770$ MeV and $\rho_m \simeq 125$ MeV per pileup event at central rapidities).

For nearly all the jet shapes, the pileup has a substantial impact, shifting the average values by up to 50%–100% (as compared to a 5%–10% effect on the jet p_T). The subtraction performs adequately: the averaged subtracted results for the shapes usually return very close to their original values, with the second derivative playing a small but sometimes relevant role. For the distributions, tails of the distributions are generally well recovered; however, intrajet pileup fluctuations cause sharp peaks to be somewhat broadened. These cannot be corrected for without applying some form of noise reduction, which would, however, also tend to introduce a bias. Of the 51 combinations of observables and processes that we examined, most were of similar quality to those illustrated in Fig. 1, with the broadening of narrow peaks found to be more extreme for larger β values. The one case where the subtraction procedure failed was the planar flow for (hadronic) WW events: here the impact of pileup is dramatic, transforming a peak near the lower boundary of the shape’s range, $Pf = 0$, into a peak near its upper boundary, $Pf = 1$ (bottom-right plot of Fig. 1). This is an example where one cannot view the pileup as simply “perturbing” the jet shape, in part because of intrinsic large nonlinearities in the shape’s behavior; with our particular set of p_T cuts and jet definition, the use of the small- ρ expansion of Eq. (5) fails to adequately correct the planar flow for more than about 15 pileup events.

Next, we consider the use of the subtraction approach in the context of quark-gluon discrimination. In a study of a large number of shapes, Ref. [17] found the jet girth or broadening $\theta^{(1)}$ to be the most effective single infrared and collinear safe quark-gluon discriminator. Figure 2 (left) shows the fraction of quark and gluon-induced jets that pass a fixed cut on $\theta^{(1)} \leq 0.05$ as a function of the level of pileup—pileup radically changes the impact of the cut,

while after subtraction the q - g discrimination returns to its original behavior.

Our last test involves top tagging, which we illustrate on $R = 1$, anti- k_r jets using cuts on the “filtered” jet mass and on the τ_{32} subjettiness ratio. The filtering selects the four hardest $R_{\text{filt}} = 0.25$, Cambridge-Aachen [52] subjects after pileup subtraction. The distribution of filtered jet mass is shown in Fig. 2 (middle), illustrating that the subtraction mostly recovers the original distribution and that ρ_m is as important as ρ (specific treatments of hadron masses, e.g., setting them to zero, may limit the impact of ρ_m in an experimental context). The tagger itself consists of cuts on $\tau_{32} < 0.6$, $\tau_{21} \geq 0.15$ and a requirement that the filtered [6] jet mass be between 150 and 200 GeV. The rightmost plot of Fig. 2 shows the final tagging efficiencies for hadronic top quarks and for generic dijets as a function of the number of pileup events. Pileup has a huge impact on the tagging, but most of the original performance is restored after subtraction.

To conclude, this Letter has introduced a novel, fully general method that allows one to correct most jet shapes for the effects of pileup. The corrections allow shape-based jet substructure analyses to continue to perform well even in the presence of up to 60 pileup events, notably when combined with the corrections introduced here for hadron masses in pileup. This progress is likely to be key to the viability of shape-based substructure tools in LHC’s 2012 data set and in future running.

The software for the general shape subtraction approach presented here is available as part of the FASTJET Contrib project [53].

We are grateful to David Miller and Jesse Thaler for helpful conversations. This work was supported by the French Agence Nationale de la Recherche, under Grants No. ANR-09-BLAN-0060 and No. ANR-10-CEXC-009-01, and by the EU ITN grant LHCPhenoNet, PITN-GA-2010-264564.

- [1] A. Abdesselam, E. B. Kuutmann, U. Bitenc, G. Brooijmans, J. Butterworth, P. Bruckman de Renstrom, D. Buarque Franzosi, and R. Buckingham *et al.*, *Eur. Phys. J. C* **71**, 1661 (2011).
- [2] A. Altheimer, S. Arora, L. Asquith, G. Brooijmans, J. Butterworth, M. Campanelli, B. Chapleau, and A. E. Cholakian *et al.*, *J. Phys. G* **39**, 063001 (2012).
- [3] T. Plehn and M. Spannowsky, *J. Phys. G* **39**, 083001 (2012).
- [4] M. H. Seymour, *Z. Phys. C* **62**, 127 (1994).
- [5] J. M. Butterworth, B. E. Cox, and J. R. Forshaw, *Phys. Rev. D* **65**, 096014 (2002).
- [6] J. M. Butterworth, A. R. Davison, M. Rubin, and G. P. Salam, *Phys. Rev. Lett.* **100**, 242001 (2008).
- [7] D. E. Kaplan, K. Rehermann, M. D. Schwartz, and B. Tweedie, *Phys. Rev. Lett.* **101**, 142001 (2008).
- [8] S. D. Ellis, C. K. Vermilion, and J. R. Walsh, *Phys. Rev. D* **81**, 094023 (2010).
- [9] D. Krohn, J. Thaler, and L.-T. Wang, *J. High Energy Phys.* **02** (2010) 084.
- [10] T. Plehn, G. P. Salam, and M. Spannowsky, *Phys. Rev. Lett.* **104**, 111801 (2010); T. Plehn, M. Spannowsky, M. Takeuchi, and D. Zerwas, *J. High Energy Phys.* **10** (2010) 078.
- [11] J. Thaler and L.-T. Wang, *J. High Energy Phys.* **07** (2008) 092.
- [12] L. G. Almeida, S. J. Lee, G. Perez, G. F. Sterman, I. Sung, and J. Virzi, *Phys. Rev. D* **79**, 074017 (2009).
- [13] J.-H. Kim, *Phys. Rev. D* **83**, 011502 (2011).
- [14] J. Thaler and K. Van Tilburg, *J. High Energy Phys.* **03** (2011) 015.
- [15] J. Thaler and K. Van Tilburg, *J. High Energy Phys.* **02** (2012) 093.
- [16] A. Hook, M. Jankowiak, and J. G. Wacker, *J. High Energy Phys.* **04** (2012) 007.
- [17] J. Gallicchio and M. D. Schwartz, *Phys. Rev. Lett.* **107**, 172001 (2011).
- [18] I. Feige, M. D. Schwartz, I. W. Stewart, and J. Thaler, *Phys. Rev. Lett.* **109**, 092001 (2012).
- [19] T. Aaltonen *et al.* (CDF Collaboration), *Phys. Rev. D* **85**, 091101 (2012); G. Aad *et al.* (ATLAS Collaboration), *Phys. Rev. D* **86**, 072006 (2012).
- [20] S. Chatrchyan *et al.* (CMS Collaboration), *J. High Energy Phys.* **09** (2012) 029.
- [21] M. Cacciari, G. P. Salam, and G. Soyez, *J. High Energy Phys.* **04** (2008) 005.
- [22] ATLAS Collaboration, Report No. ATLAS-CONF-2012-064.
- [23] M. Cacciari and G. P. Salam, *Phys. Lett. B* **659**, 119 (2008).
- [24] J. Backus Mayes (for the ATLAS Collaboration), in Proceedings of Boost 2012, Valencia, Spain (unpublished).
- [25] S. Chatrchyan *et al.* (CMS Collaboration), *JINST* **6**, P11002 (2011); CMS Collaboration, Report No. CMS-DP-2012-012, <http://cdsweb.cern.ch/record/1460989>.
- [26] With particle flow [27], one can also directly discard the charged component of pileup [25]. The remaining neutral part must be subtracted in some other way (it may not be the expected fraction of the charged component, due to detector effects).
- [27] CMS Collaboration, Report No. CMS-PAS-PFT-09-001.
- [28] S. Sapeta and Q. C. Zhang, *J. High Energy Phys.* **06** (2011) 038.
- [29] R. Alon, E. Duchovni, G. Perez, A. P. Pranko, and P. K. Sinervo, *Phys. Rev. D* **84**, 114025 (2011).
- [30] J. Alcaraz Maestre *et al.* (SM and NLO MULTILEG and SMMC Working Groups Collaboration), [arXiv:1203.6803](https://arxiv.org/abs/1203.6803).
- [31] B. Abelev *et al.* (ALICE Collaboration), *J. High Energy Phys.* **03** (2012) 053.
- [32] M. Cacciari, G. P. Salam, and G. Soyez, *Eur. Phys. J. C* **71**, 1692 (2011).
- [33] For the correction of collinear unsafe quantities, e.g., fragmentation function moments, as used for quark-gluon discrimination in Ref. [34], see Ref. [35].
- [34] S. Chatrchyan *et al.* (CMS Collaboration), *J. High Energy Phys.* **04** (2012) 036.
- [35] M. Cacciari, P. Quiroga-Arias, G. P. Salam, and G. Soyez, *Eur. Phys. J. C* **73**, 2319 (2013).
- [36] G. P. Salam and D. Wicke, *J. High Energy Phys.* **05** (2001) 061.
- [37] V. Mateu, I. W. Stewart, and J. Thaler, *Phys. Rev. D* **87**, 014025 (2013).
- [38] C. F. Berger, T. Kucs, and G. F. Sterman, *Phys. Rev. D* **68**, 014012 (2003).
- [39] M. L. Gonzalez Silva (for the ATLAS Collaboration), in Proceedings of Boost 2012, Valencia, Spain (unpublished).
- [40] A. Banfi, G. P. Salam, and G. Zanderighi, *J. High Energy Phys.* **03** (2005) 073.
- [41] M. Jankowiak and A. J. Larkoski, *J. High Energy Phys.* **06** (2011) 057; **04** (2012) 039.
- [42] G. Aad *et al.* (ATLAS Collaboration), *J. High Energy Phys.* **12** (2012) 086.
- [43] S. Catani, Y. L. Dokshitzer, M. H. Seymour, and B. R. Webber, *Nucl. Phys.* **B406**, 187 (1993).
- [44] S. D. Ellis and D. E. Soper, *Phys. Rev. D* **48**, 3160 (1993).
- [45] Consider a jet consisting instead of just two hard particles with $p_t = 1000$ GeV, with $\phi = 0, 0.5$ and two further soft particles with $p_t = \epsilon$, at $\phi = 0.05, 0.1$, all particles having $y = 0$. It is straightforward to see that τ_{32} is finite and independent of ϵ for $\epsilon \rightarrow 0$, which results in an infinite leading-order perturbative distribution for τ_{32} .
- [46] T. Sjostrand, S. Mrenna, and P. Skands, *J. High Energy Phys.* **05** (2006) 026.
- [47] T. Sjostrand, S. Mrenna, and P. Z. Skands, *Comput. Phys. Commun.* **178**, 852 (2008).
- [48] M. Cacciari, G. P. Salam, and G. Soyez, *J. High Energy Phys.* **04** (2008) 063.
- [49] M. Cacciari and G. P. Salam, *Phys. Lett. B* **641**, 57 (2006); M. Cacciari, G. P. Salam, and G. Soyez, *Eur. Phys. J. C* **72**, 1896 (2012).
- [50] G. D'Agostini and M. Raso, [arXiv:hep-ex/0002056](https://arxiv.org/abs/hep-ex/0002056).
- [51] M. Cacciari, G. P. Salam, and S. Sapeta, *J. High Energy Phys.* **04** (2010) 065.
- [52] Y. L. Dokshitzer, G. D. Leder, S. Moretti, and B. R. Webber, *J. High Energy Phys.* **08** (1997) 001; M. Wobisch and T. Wengler, [arXiv:hep-ph/9907280](https://arxiv.org/abs/hep-ph/9907280); M. Wobisch, Report No. DESY-THESIS-2000-049.
- [53] <http://fastjet.hepforge.org/contrib>.