Absence of Three-Loop Four-Point Ultraviolet Divergences in $\mathcal{N} = 4$ Supergravity

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We compute the coefficient of the potential three-loop four-point ultraviolet divergence in pure $\mathcal{N} = 4$ supergravity and show that it vanishes, contrary to expectations from symmetry arguments. The recently uncovered duality between color and kinematics is used to greatly streamline the calculation. We comment on all-loop cancellations hinting at further surprises awaiting discovery at higher loops.

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Recent years have seen a resurgence of interest in the possibility that certain supergravity theories may be ultraviolet finite. This question had been carefully studied in the late 1970s and early 1980s in the hope of using supergravity to construct fundamental theories of gravity. The conclusion of these early studies was that nonrenormalizable ultraviolet divergences would almost certainly appear at a sufficiently large number of quantum loops, though this remains unproven. Although supersymmetry tends to tame the ultraviolet divergences, it does not appear to be sufficient to overcome the increasingly poor ultraviolet behavior of gravity theories stemming from the two-derivative coupling. The consensus opinion from that era was that all pure supergravity theories would likely diverge at three loops (see, e.g., Ref. [1]), though with assumptions, certain divergences are perhaps delayed a few extra loop orders [2].

More recently, direct calculations of divergences in supergravity theories have been carried out [3–6], shedding new light on this issue. From these studies we now know that through four loops maximally supersymmetric $\mathcal{N} = 8$ supergravity is finite in space-time dimensions, $D < 6/L + 4$ for $L = 2$, 3, 4 loops. These calculations also tell us that the bound is saturated. In $D = 4$, $E_{7(7)}$ duality symmetry [7] has recently been used to imply ultraviolet finiteness below seven loops [8], also explaining the observed lack of divergences. In a parallel development, string theory and a first quantized formalism use supersymmetry considerations to arrive at similar conclusions [9]. The latter approach leads to D-dimensional results consistent with the explicit calculations through four loops, but predicts a worse behavior starting at $L = 5$. At seven loops, the potential four-graviton counterterm of $\mathcal{N} = 8$ supergravity [10] appears to be consistent with all known symmetries [8,11]. (See Ref. [12] for a more optimistic opinion.) More generally, $1/\mathcal{N}$-BPS operators serve as potential counterterms for $\mathcal{N} = 4$, 5, 6, 8 supergravity at $L = 3$, 4, 5, 7 loops, respectively, suggesting that in $D = 4$ ultraviolet divergences will occur at these loop orders in these theories [11]. It therefore might seem safe to conclude that $\mathcal{N} = 4$ supergravity [13] in particular will diverge at three loops.

On the other hand, studies of scattering amplitudes suggest that additional ultraviolet cancellations will be found beyond these. We know that even pure Einstein gravity at one loop exhibits remarkable cancellations as the number of external legs increases [14]. Through unitarity, such cancellations feed into nontrivial ultraviolet cancellations at *all* loop orders [15]. In addition, the proposed double-copy structure of gravity loop amplitudes [16] suggests that gravity amplitudes are more constrained than symmetry considerations suggest. In this Letter we show that the ultraviolet properties of $\mathcal{N} = 4$ supergravity are indeed better than had been anticipated.

To motivate the possibility of hidden cancellations in $\mathcal{N} = 4$ supergravity, consider the unitarity cut displayed in Fig. 1 isolating a one-loop subamplitude in a three-loop amplitude. As noted in Refs. [14,17], at one loop a five-point diagram in an $\mathcal{N} = 4$ supergravity amplitude effectively can have up to five powers of loop momenta in the numerator, similar to the power counting of pure Yang-Mills theory. There are also three additional powers of numerator loop momentum coming from the tree amplitude on the right-hand side of the cut, giving a total of at least eight powers of numerator loop momentum. Taking into account three loop integrals and ten propagators suggests that this amplitude should diverge at least logarithmically in $D = 4$. (The power counting analysis of this cut performed in Ref. [17] assumed that additional powers of numerator loop momenta coming from the tree amplitude in the cut can be ignored, contrary to our analysis.)

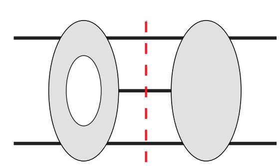

FIG. 1 (color online). A sample cut at three loops displaying cancellations in $\mathcal{N} = 4$ supergravity special to four dimensions.

However, this type of power counting is too naïve and does not account for the special property that no one- and two-loop ultraviolet divergences are present in $D = 4$ [18]. Thus in $D = 4$ there are additional cancellations of the loop momenta in one-loop subdiagrams, effectively removing powers of loop momenta from the numerators of the loop integrands once all pieces have been combined and integrated. These additional cancellations can affect the higher-loop effective overall power counting. We show this occurs by computing the coefficient of the potential three-loop four-point divergences in $\mathcal{N} = 4$ supergravity.

To make the calculation of the potential three-loop divergence feasible, we use the duality between color and kinematics uncovered by Carrasco, Johansson and one of the authors (BCJ) [16,19]. According to this conjecture, we can reorganize a (super) Yang-Mills amplitude into graphs where the numerators satisfy identities in one-to-one correspondence with color Jacobi identities. Whenever this is accomplished, we obtain corresponding gravity amplitudes simply by replacing color factors with kinematic numerators of a corresponding second gauge-theory amplitude. That is, gravity loop amplitudes are given by [16]

$$\frac{(-i)^{L+1}}{(\kappa/2)^{n-2+2L}}\mathcal{M}_n^{\text{loop}} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j \tilde{n}_j}{\prod_{\alpha_j} p_{\alpha_j}^2}, \qquad (1)$$

where n_j and $\tilde{n}_j$ are kinematic numerator factors from gauge-theory amplitudes and κ is the gravitational coupling. The factors S_j are the usual combinatoric factors associated with the symmetries of the graphs. The sum runs over all distinct graphs with cubic vertices, such as the ones appearing in Fig. 2. The propagators appearing in Eq. (1) are the ordinary propagators corresponding to the internal lines of the graphs. Depending on the particular theory under consideration, we use different component gauge-theory numerators in Eq. (1).

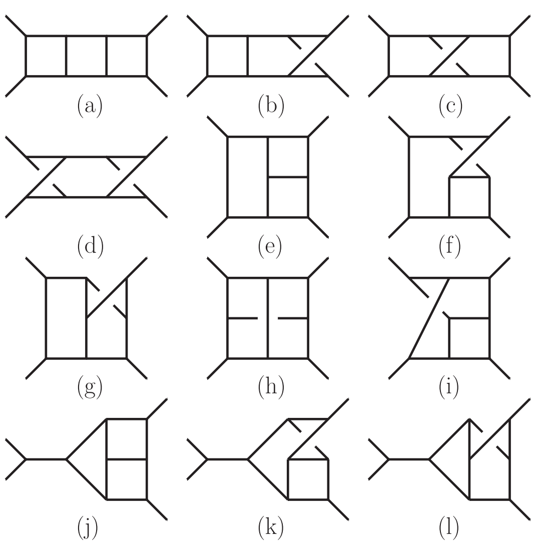

FIG. 2. The 12 graphs appearing in the three-loop $\mathcal{N} = 4$ sYM amplitude [16] and in gravity amplitudes obtained from these using the double-copy formula.

In our study of pure $\mathcal{N} = 4$ supergravity with no matter multiplets [13], we take one component gauge theory to be $\mathcal{N} = 4$ super-Yang-Mills (sYM) theory and the second component to be nonsupersymmetric pure Yang-Mills theory. This construction was used in earlier one- and two-loop studies of $\mathcal{N} \geq 4$ supergravity amplitudes [20]. The main differences in our case are that integrated gauge-theory expressions are not known and that the $\mathcal{N} = 4$ sYM numerators are not all independent of loop momenta.

As explained in Refs. [16,21], only one of the two component gauge-theory amplitudes needs to be in a form manifestly satisfying the duality for the double-copy property (1) to hold. The other gauge-theory amplitude can be any convenient representation arranged into diagrams with only cubic vertices. We note that our construction applies immediately to all four-point amplitudes of pure $\mathcal{N} = 4$ supergravity, since these are constructed simply by considering all possible external states on the $\mathcal{N} = 4$ sYM side of the double copy; at four-points this information is entirely encoded in an overall tree-amplitude prefactor. At three loops, we take the $\mathcal{N} = 4$ sYM copy from Ref. [16], since it has BCJ duality manifest. This representation of the $\mathcal{N} = 4$ sYM amplitude is described by the 12 graphs in Fig. 2. For the pure Yang-Mills copy, we use ordinary Feynman diagrams in Feynman gauge, including ghost contributions. The contact contributions are assigned to diagrams with only cubic vertices according to their color factor. In this construction, most Feynman diagrams are irrelevant because in the double-copy formula they get multiplied by vanishing $\mathcal{N} = 4$ sYM diagram numerators. This construction gives the complete three-loop four-point integrand of $\mathcal{N} = 4$ supergravity. We have also applied these ideas to reproduce the absence of one- and two-loop divergences in pure $\mathcal{N} = 4$ supergravity, starting from the one- and two-loop four-point sYM amplitudes [22,23].

To prove the correctness of our construction, we use the unitarity method [3,24]. The generalized unitarity cuts decompose the constructed integrand into sums of products of tree amplitudes, which match against the values obtained using the double-copy property at tree level [16,21]. Since all cuts automatically have the proper values in D dimensions, the amplitude so constructed is correct.

Inserting the numerators of pure Yang-Mills amplitudes generated by the Feynman rules into the double-copy formula (1) leads to tens of thousands of high-rank tensor integrals, from which we must extract the ultraviolet divergences. We do so by expanding in small external momenta. This gives vacuum diagrams containing both infrared and ultraviolet divergences. To deal with ultraviolet divergences, we use the four-dimensional-helicity regularization scheme [25], since it preserves supersymmetry and has been used successfully in analogous multiloop pure gluon and supersymmetric amplitudes. In this scheme, the number of states remain at their four-dimensional values. Then at the level of the vacuum integrals, we introduce a uniform mass m to separate the infrared divergences from the ultraviolet ones. Although ultimately there are no one- and two-loop ultraviolet divergences in $\mathcal{N}=4$ supergravity, individual integrals generally do contain subdivergences due to their poor power counting. To deal with this, we make extensive use of the observations of Marcus and Sagnotti [26] to subtract subdivergences integral by integral. Extractions of ultraviolet divergences in higher-dimensional $\mathcal{N}=8$ supergravity were discussed recently in Refs. [6,27].

At two or higher loops, the introduced mass regulator induces unphysical regulator dependence in individual integrals. However, at least for logarithmically divergent integrals, the regulator dependence comes entirely from subdivergences, which we systematically subtract [26]. We therefore introduce the mass regulator before subtracting subdivergences, but after reducing all integrals to logarithmic by series expansion in small external momenta. To implement the subtractions, we recursively define the subtracted divergence $\mathcal{S}[\ldots]$ of an integral,

$$\mathcal{S}\Big[\int\prod_{i=1}^{L}dp_i I\Big] \equiv \mathrm{Div}\Big[\int\prod_{i=1}^{L}dp_i I\Big] - \sum_{l=1}^{L-1}\sum_{\substack{l\text{-loop}\\ \text{subloops}}}\mathrm{Div}\Big[\int\prod_{j=l+1}^{L}dp'_j\mathcal{S}\Big[\int\prod_{i=1}^{l}dp'_i I\Big]\Big], \quad (2)$$

where $\mathrm{Div}[\ldots]$ is the complete divergence of an integral, I is the integrand, and dp_i is shorthand for $d^{\mathrm{D}}p_i/(2\pi)^{\mathrm{D}}$. The sum over subloops must include *all* subloops of the diagram where a subdivergence could occur—not just the loops that are manifestly parametrized by p_i—and here we have indicated this by changing variables to p'_i in each subtraction, such that the l-loop subintegral under consideration is parametrized by $p'_1,\ldots,p'_l$. For example, graph (e) in Fig. 2 has seven one-loop subintegrals and six two-loop subintegrals to consider, and each two-loop subintegral has three one-loop subintegrals of its own.

By the time we apply Eq. (2), each integral has a single scale given by the mass regulator m. We are left with the task of calculating the divergences $\mathrm{Div}[\ldots]$ of single-scale vacuum integrals. To evaluate these integrals, we first eliminate tensors composed of loop momenta from the numerators by noticing that the integrals must be linear combinations of products of metric tensors $\eta^{\mu\nu}$. (See Ref. [6] for a recent discussion of evaluating tensor vacuum integrals.) Then we reduce the resulting scalar integrals to a basis using integration by parts, as implemented in FIRE [28]. The resulting basis is given by the scalar vacuum integrals shown in Fig. 3 (along with products of lower-loop integrals), with a single massive propagator corresponding to each line. As cross checks we also used MB [29] and FIESTA [30]. We evaluated all but the last of these integrals to the required order in ϵ by Mellin-Barnes integration with resummation of residues using the methods presented in Ref. [31]. The last integral can be evaluated by making a two-loop subintegral massless and integrating it first. This does not affect the ultraviolet divergence because there are no subdivergences in this case. (The value of the two-loop subintegral can be found in Ref. [32].) The results are rational linear combinations of the transcendental numbers ζ_2, ζ_3, and $\sqrt{3}\,\mathrm{Im}[\mathrm{Li}_2(e^{i\pi/3})]$.

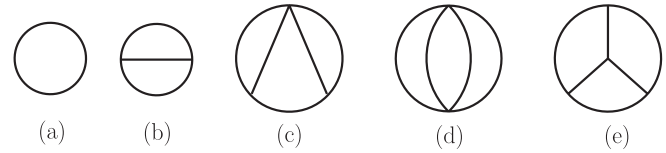

FIG. 3. The basis of vacuum integrals for one through three loops.

At two loops only the first two integrals shown in Fig. 3 are needed. Adding together the contributions reproduces the fact that there are no two-loop divergences in pure supergravity theories. At three loops all vacuum integrals in Fig. 3 contribute. In Table I, we have collected the derived divergences of the three-loop four-graviton amplitude for each graph in Fig. 2. The results shown in the table are summed over the independent permutations including symmetry factors. The individual graphs are not gauge invariant and are valid only for the indicated choice of spinor-helicity reference momenta (see, e.g., Ref. [33]). We have divided out a prefactor depending on the four-point color-ordered super-Yang-Mills tree amplitude, spinor inner products and the usual Mandelstam invariants

TABLE I. The graph-by-graph divergences for the four-graviton amplitude with helicities $(1^-2^-3^+4^+)$ (up to an overall normalization). Each expression includes a permutation sum over external legs, with the symmetry factor appropriate to the graph. These quantities are not individually gauge-invariant, and here we use spinor helicity with the choice of reference momenta $q_1=q_2=k_3$ and $q_3=q_4=k_1$. The sum over the diagram contributions vanishes.

Graph	(divergence)/$(\langle 12\rangle^2[34]^2 stA^{\mathrm{tree}}(\frac{\kappa}{2})^8)$
(a)–(d)	0
(e)	$\frac{263}{768}\frac{1}{\epsilon^3}+\frac{205}{27648}\frac{1}{\epsilon^2}+\big(-\frac{5551}{768}\zeta_3+\frac{326317}{110592}\big)\frac{1}{\epsilon}$
(f)	$-\frac{175}{2304}\frac{1}{\epsilon^3}-\frac{1}{4}\frac{1}{\epsilon^2}+\big(\frac{593}{288}\zeta_3-\frac{217571}{165888}\big)\frac{1}{\epsilon}$
(g)	$-\frac{11}{36}\frac{1}{\epsilon^3}+\frac{2057}{6912}\frac{1}{\epsilon^2}+\big(\frac{10769}{2304}\zeta_3-\frac{226201}{165888}\big)\frac{1}{\epsilon}$
(h)	$-\frac{3}{32}\frac{1}{\epsilon^3}-\frac{41}{1536}\frac{1}{\epsilon^2}+\big(\frac{3227}{2304}\zeta_3-\frac{3329}{18432}\big)\frac{1}{\epsilon}$
(i)	$\frac{17}{128}\frac{1}{\epsilon^3}-\frac{29}{1024}\frac{1}{\epsilon^2}+\big(-\frac{2087}{2304}\zeta_3-\frac{10495}{110592}\big)\frac{1}{\epsilon}$
(j)	$-\frac{15}{32}\frac{1}{\epsilon^3}+\frac{9}{64}\frac{1}{\epsilon^2}+\big(\frac{101}{12}\zeta_3-\frac{3227}{1152}\big)\frac{1}{\epsilon}$
(k)	$\frac{5}{64}\frac{1}{\epsilon^3}+\frac{89}{1152}\frac{1}{\epsilon^2}+\big(-\frac{377}{144}\zeta_3+\frac{287}{432}\big)\frac{1}{\epsilon}$
(l)	$\frac{25}{64}\frac{1}{\epsilon^3}-\frac{251}{1152}\frac{1}{\epsilon^2}+\big(-\frac{835}{144}\zeta_3+\frac{7385}{3456}\big)\frac{1}{\epsilon}$

s and t. We note that the transcendental numbers except ζ_3 cancel within each graph.

In the sum over all contributions (obtained by adding the rows in the Table I), not only do the $1/\epsilon^3$ and $1/\epsilon^2$ divergences cancel, as required because there are no divergent subamplitudes, but the $1/\epsilon$ singularity also cancels. This proves that the three-loop amplitude is ultraviolet finite. As a rather nontrivial check, we confirmed that the sum over all contributions is independent of reference momentum choices. As another nontrivial confirmation, we found that by introducing a uniform mass in the amplitude at the start of the calculation, all ultraviolet divergences cancel without the need for subdivergence subtraction. This matches expectations that all ultraviolet subdivergences should cancel out from the total amplitude (although there may be potential regulator dependence issues with this approach). Since all nonvanishing four-point amplitudes in the theory are proportional to four-graviton ones, our calculation demonstrates that there are no divergences in any three-loop four-point amplitude of the theory.

In summary, we used the recently uncovered duality between color and kinematics to streamline the calculation of the coefficient of the potential three-loop ultraviolet divergence of $\mathcal{N} = 4$ supergravity, proving that it vanishes. Might cancellations persist beyond this? It is interesting to note that the $D = 4$ cancellations found in one- and two-loop subamplitudes and used to motivate our three-loop computation can be used just as well to argue for higher-loop cancellations. Moreover, the double-copy property of gravity amplitudes shows there is more structure than captured by the known symmetries. Our three-loop calculation provides a concrete example showing that power counting based on known symmetries can be misleading. The results of this paper strongly motivate further high-loop explorations of the ultraviolet divergence structure of supergravity theories. In particular, they emphasize the importance of explicitly computing the ultraviolet properties of $\mathcal{N} = 8$ supergravity at five loops.

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