Two Separated SU(2) Yang-Mills-Higgs Monopoles in the ADHMN Construction

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Abstract

We study two arbitrarily separated SU(2) Yang-Mills-Higgs monopoles in the Atiyah-Drinfeld-Hitchin-Manin-Nahm (ADHMN) construction. In particular, we obtain an exact analytical expression for the Higgs field on the axis connecting its two zeroes which are defined to be the locations of the monopoles. From this expression we compute the zeroes of the Higgs field for small and large separations.

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I. Introduction

In the past year there has been rapid progress in determining exact (superimposed and separated) multimonopole solutions of the SU(2) Yang-Mills-Higgs theory in the limit of vanishing Higgs potential. Most of the explicit results have been obtained by means of the Atiyah-Ward (AW) ansatz [1-6]. For a recent review of exact results in the theory of magnetic monopoles in nonabelian gauge theories see ref. [7]. Wherever possible, these solutions have been shown to be regular, but a general proof of this is still an open problem in the AW approach. In particular, one has to study the two separated monopole solution either perturbatively [4] for small separation or numerically [8] for large separation.

Recently, Nahm [9] has adapted the Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction for instantons [10] to the monopole problem, and we will refer to it as the ADHMN construction. In the ADHMN construction, the regularity of the solution is automatic. Two other advantages of the ADHMN approach are: a) It is easily generalized to gauge groups beyond SU(2); and (b) it allows the exact construction of Green's functions for particles propagating in the background of the multimonopole solutions.

The purpose of this paper is to study two arbitrarily separated monopoles in the ADHMN construction and, in particular, to locate the two zeroes of the Higgs field (which are defined to be the location of the monopoles). The main result of this paper is an exact analytical expression for the Higgs field on the axis connecting the two zeroes. From this expression we compute the zeroes of the Higgs field for small and large separations.

The results of this paper should help clarify the nature of the parameter space for two monopoles which in turn is relevant to understanding the dynamics of two monopoles [11] (e.g., scattering of two far apart monopoles approaching each other).

II. Statement of Problem

Let us define in four-dimensional Euclidean space (x_1, x_2, x_3, x_4) the matrix valued fields $(\partial_\mu \equiv \partial/\partial x_\mu)$:

$$A_\mu \text{ and } F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \qquad (\mu,\nu = 1,2,3,4) \quad . \qquad (2.1)$$

For SU(2) Yang-Mills gauge theory, A_μ (the gauge potentials) and $F_{\mu\nu}$ (the gauge field strengths) are 2x2 antihermitian traceless matrices.

The problem, simply stated, is to solve the self-duality equations

$$F_{\mu\nu} \equiv \tfrac{1}{2}\varepsilon_{\mu\nu\lambda\sigma}F_{\lambda\sigma} \qquad (2.2a)$$

for A_μ subject to the following requirements:

1) In all gauges A_μ are static (independent of x_4):

$$\partial_4 A_\mu = 0 \qquad (2.2b)$$

In this case A_4 is referred to as the Higgs field.

2) In some gauge A_μ are non-singular functions of (x_1, x_2, x_3). (2.2c)

3) The gauge invariant scalar function

$$H \equiv [-\tfrac{1}{2}\mathrm{Tr}A_4^2]^{\frac{1}{2}} \qquad (2.2d)$$

has the following asymptotic form

$$H \to c - \frac{n}{2r} + \mathcal{O}(r^{-2}) \quad \text{as } r \equiv (x_1^2+x_2^2+x_3^2)^{\frac{1}{2}} \to \infty \qquad (2.2e)$$

where $c>0$ is an arbitrary constant and n is a positive integer called the topological charge which, in appropriate units, is also the magnetic charge of the solution.

Motivated by considerations from differential topology [12], one can define the location of the monopoles as the zeroes of the function H. In particular, for n=2, the two separated monopole solution will be defined to be the one where H has exactly two distinct (simple) zeroes. The limit where the two distinct zeroes of H degenerate into one (double) zero corresponds to the axi-symmetric configuration of two superimposed monopoles [1].

III. ADHMN Construction for n=2

Following ref. [9], the ADHMN construction for n=2 begins by defining three real functions f_1, f_2 and f_3 of a real variable z satisfying the equations:

$$\frac{df_1}{dz} = f_2 f_3 \, , \quad \frac{df_2}{dz} = f_1 f_3 \, , \quad \frac{df_3}{dz} = f_1 f_2 \, . \qquad (3.1)$$

We then define the 4x4 matrices

$$\tilde{X} \equiv \begin{pmatrix} x_3 - ix_4 & x_1 - ix_2 & 0 & 0 \\ x_1 + ix_2 & -x_3 - ix_4 & 0 & 0 \\ 0 & 0 & x_3 - ix_4 & x_1 - ix_2 \\ 0 & 0 & x_1 + ix_2 & -x_3 - ix_4 \end{pmatrix} \quad \tilde{F} = \tfrac{1}{2} \begin{pmatrix} f_3 & 0 & 0 & f_1 - f_2 \\ 0 & -f_3 & f_1 + f_2 & 0 \\ 0 & f_1 + f_2 & -f_3 & 0 \\ f_1 - f_2 & 0 & 0 & f_3 \end{pmatrix} \qquad (3.2)$$

and consider the following linear matrix differential equation ($I \equiv$ 4x4 identity matrix):

$$[\frac{d}{dz} I + \tilde{X} + \tilde{F}]\tilde{V} = 0 \qquad (3.3)$$

over a symmetric interval $-z_s \le z \le +z_s$. Equation (3.3) will have four linearly independent (4x1 column vector) solutions $\tilde{V}$, and we require that only two of them be orthonormalizable in the sense that

$$\int_{-z_s}^{+z_s} \tilde{V}_\alpha^\dagger \tilde{V}_\beta \, dz = \delta_{\alpha\beta} \qquad \alpha,\beta = 1,2 \, . \qquad (3.4)$$

Thus, eqn. (3.3) must have two non-normalizable solutions, and this can only happen if the matrix $\tilde{F}$ diverges at $z = \pm z_s$. The ADHMN construction then states that the solution of our problem (eqn. (2.2)) for n=2 is given by

$[A_\mu]_{\alpha\beta} \equiv \alpha$th row and βth column of A_μ

$$= \int_{-z_s}^{+z_s} V_\alpha^\dagger \partial_\mu V_\beta dz \ . \qquad (3.5)$$

The real symmetric matrix F can be diagonalized by the following constant orthogonal matrix:

$$\tilde{Q} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \end{pmatrix} . \qquad (3.6)$$

So that we can replace eqn. (3.3) with

$$(\frac{d}{dz} I + X + F) V = 0 \qquad (3.7a)$$

where

$$X \equiv \tilde{Q}^{-1}(\tilde{X} + Ix_4 i)\tilde{Q} = \begin{pmatrix} 0 & x_3 & x_1 & -ix_2 \\ x_3 & 0 & -ix_2 & x_1 \\ x_1 & ix_2 & 0 & -x_3 \\ ix_2 & x_1 & -x_3 & 0 \end{pmatrix}, \qquad (3.7b)$$

$$F \equiv \tilde{Q}^{-1}\tilde{F}\tilde{Q} = \tfrac{1}{2}\begin{pmatrix} f_1-f_2+f_3 & 0 & 0 & 0 \\ 0 & f_3-f_1+f_2 & 0 & 0 \\ 0 & 0 & f_1+f_2-f_3 & 0 \\ 0 & 0 & 0 & -f_1-f_2-f_3 \end{pmatrix}, \qquad (3.7c)$$

$$V \equiv V(x_1,x_2,x_3,z) \equiv \tilde{Q}^{-1}\tilde{V}e^{-ix_4 z} \ . \qquad (3.7d)$$

Note that the x_4 variable has disappeared in eqn. (3.7a). In terms of V the gauge potentials eqn. (2.5) become (ℓ=1,2,3)

$$[A_\ell]_{\alpha\beta} = \int_{-z_s}^{z_s} V_\alpha^\dagger \partial_\ell V_\beta dz \ , \qquad [A_4]_{\alpha\beta} = i\int_{-z_s}^{z_s} zV_\alpha^\dagger V_\beta dz \ . \qquad (3.8)$$

Let Q be an arbitrary constant (≡independent of x_1,x_2,x_3,x_4 and z) 4x4 orthogonal matrix and U(x) an arbitrary 2x2 unitary matrix function of x_1, x_2 and x_3. Under the transformation

$$V_\alpha \to V'_\alpha = (QV)_{\alpha'} U_{\alpha'\alpha} \qquad (\alpha,\alpha' = 1,2) \qquad (3.9)$$

the gauge potentials eqn. (3.8) transform as

$$A_\ell \to A'_\ell = U^{-1}A_\ell U + U^{-1}\partial_\ell U \ , \qquad A_4 \to A'_4 = U^{-1}A_4 U$$

which are precisely gauge transformations. In particular, the scalar function H defined by (2.2d) is gauge invariant.

IV. Explicit Solutions for f_1,f_2,f_3

Integration of eqn. (3.1) implies $f_i^2-f_j^2$=constant=c_{ij}. Since $c_{13}=c_{12}+c_{23}$, only two of these constants of integration are independent. By appropriate rescaling of z and the f's we can further fix one of these constants, c_{13}, to be 1. We also require that the f's diverge at the symmetrical end points $z=\pm z_s$ and are thus led to the following solution of eqn. (3.1):

$$\frac{df_3}{dz} = \sqrt{1+f_3^2}\sqrt{1+\delta^2+f_3^2} \ , \ f_3(z=0) = 0 \ .$$

$$f_1 = \sqrt{1+f_3^2} \ , \ f_2 = \sqrt{1+\delta^2+f_3^2} \ , \qquad (4.1)$$

δ is an arbitrary real number .

Equation (3.1) can be explicitly solved in terms of Jacobian elliptic functions:

$$f_3 = \frac{sn(u,k)}{cn(u,k)} \ , \ f_1 = \frac{1}{cn(u,k)} \ , \ f_2 = \frac{1}{k'}\frac{dn(u,k)}{cn(u,k)} \ ,$$

$$u \equiv \frac{1}{k'} z \ , \ k \equiv \frac{\delta}{\sqrt{1+\delta^2}} \ , k' \equiv \sqrt{1-k^2} = \frac{1}{\sqrt{1+\delta^2}} \ , \qquad (4.2)$$

and we note that

$$f_1,f_2,f_3 \text{ diverge at } z = \pm k'K \text{ where } K \equiv \int_0^{\pi/2} \frac{dy}{\sqrt{1-k^2\sin^2 y}} \ . \qquad (4.3)$$

For $\delta=0$: $f_3=\tan z$, $f_1=f_2=\sec z$, and we regain the axially symmetric n=2 monopole solution [9]. As noted in ref. [9], eqn. (3.7) implies that as $r\to\infty$: $V\to e^{rz}$ and eqn. (2.8) gives $H\to z_s$ as $r\to\infty$. Thus the constant c in eqn. (2.2e) will be $z_s=k'K$:

$$c = z_s = k'K \Rightarrow H \to k'K \text{ as } r \to \infty \ . \qquad (4.4)$$

From now on, by f_1, f_2 and f_3 we will mean the expressions defined by eqn. (4.2).

V. Discrete Symmetries of H

We begin by proving covariance of eqn. (3.7) under space inversion (parity operation P):

$$P:\ x_1' = -x_1\ ,\ x_2' = -x_2\ ,\ x_3' = -x_3\ . \tag{5.1}$$

In the space inverted system eqn. (3.7) will look like:

$$P:\ [\frac{d}{d(-z)} + X - F]V'(-x_1,-x_2,-x_3,z) = 0\ . \tag{5.2}$$

From eqn. (4.2) we have

$$f_3(-z) = -f_3(z)\ ,\ f_1(-z) = f_1(z)\ ,\ f_2(-z) = f_2(z)\ , \tag{5.3}$$

and if we define the following constant orthogonal matrix:

$$\mathcal{D} \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{5.4}$$

then it is easy to show that

$$\mathcal{D}^{-1}F(z)\mathcal{D} = -F(-z)\ ,\ \mathcal{D}^{-1}X\mathcal{D} = X\ . \tag{5.5}$$

From eqn (3.7), (5.2) and (5.5) it follows that

$$P:\ V'(-x_1,-x_2,-x_3,z) = \mathcal{D}V(x_1,x_2,x_3,-z)\ , \tag{5.6}$$

and eqn. (3.8) implies

$$P:\ A_4'(-x_1,-x_2,-x_3) = -A_4(x_1,x_2,x_3) \tag{5.7}$$

where A_4' is some gauge transform of A_4 (see eqn. (3.10)). Squaring both sides of eqn. (5.7) and taking the trace, we obtain the gauge invariant statement

$$H(-x_1,-x_2,-x_3) = H(x_1,x_2,x_3) \tag{5.8}$$

which is our first discrete symmetry of H.

Let us now consider rotations R_1,R_2,R_3 of 180° about the x_1, x_2 and x_3 axes respectively:

$$\begin{aligned} R_1:&\ x_1' = x_1 \quad x_2' = -x_2 \quad x_3' = -x_3 \\ R_2:&\ x_2' = x_2 \quad x_1' = -x_1 \quad x_3' = -x_3 \\ R_3:&\ x_3' = x_3 \quad x_1' = -x_1 \quad x_2' = -x_2\ . \end{aligned} \tag{5.9}$$

If we define the following constant orthogonal matrices

$$\mathcal{D}_1 \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \mathcal{D}_2 \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},\ \mathcal{D}_3 \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.10}$$

then it is easy to show that

$$\begin{aligned} R_1:&\ V'(x_1,-x_2,-x_3,z) = \mathcal{D}_1V(x_1,x_2,x_3,z) \\ &\Rightarrow A_4'(x_1,-x_2,-x_3) = A_4(x_1,x_2,x_3) \\ R_2:&\ V'(-x_1,x_2,-x_3,z) = \mathcal{D}_2V(x_1,x_2,x_3,z) \\ &\Rightarrow A_4'(-x_1,x_2,-x_3) = A_4(x_1,x_2,x_3) \\ R_3:&\ V'(-x_1,-x_2,x_3,z) = \mathcal{D}_3V(x_1,x_2,x_3,z) \\ &\Rightarrow A_4'(-x_1,-x_2,x_3) = A_4(x_1,x_2,x_3)\ . \end{aligned} \tag{5.11}$$

We thus arrive at the following discrete symmetries of H:

$$H(x_1,x_2,x_3) = H(-x_1,-x_2,x_3) = H(x_1,-x_2,-x_3) = H(-x_1,x_2,-x_3). \tag{5.12}$$

Equations (5.8) and (5.12) prove that if H has exactly two zeroes, then those zeroes must necessarily be on one of the coordinate axes (x_1, x_2 or x_3) and located symmetrically about $x_1=x_2=x_3=0$. If H has exactly one zero, then it must necessarily be at $x_1=x_2=x_3=0$. These results were originally found in the AW approach by O'Raifeartaigh and Rouhani [13].

VI. Exact Expression for $H(x_1=0,\ x_2,\ x_3=0)$

We will now compute the function H on the x_2 axis: $x_1=x_3=0$, for, as it turns out, the zeroes of H are precisely on this axis. In this section H and A_4 will stand for $H(x_1=0,x_2,x_3=0)$ and $A_4(x_1=0,x_2,x_3=0)$ respectively. It will be useful to define a variable $\bar{u}$ by

$$\bar{u} \equiv \tfrac{1}{2}(u - K) = \tfrac{1}{2}(\frac{z}{k'} - K) \qquad -K \leq \bar{u} \leq 0 \tag{6.1}$$

so that $\int_{-z_s}^{z_s} dz \rightarrow 2k' \int_{-K}^{0} d\bar{u}$, etc.

Equations (3.8) and (3.7a) show that in order to compute A_4 on the x_2 axis, one can immediately set $x_1=x_3=0$ in eqn. (3.7a) and solve for V on the x_2 axis. (Note that this would not be true if we had to compute A_ℓ for $\ell=1,2,3$.) Thus, we must solve the following equation:

$$\left\{\frac{d}{dz}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}+\tfrac{1}{2}\begin{pmatrix}f_1-f_2+f_3&0&0&-2ix_2\\0&f_3-f_1+f_2&-2ix_2&0\\0&2ix_2&f_1+f_2-f_3&0\\2ix_2&0&0&-f_1-f_2-f_3\end{pmatrix}\right\}V=0 \quad . \tag{6.2}$$

As shown in Appendix A, eqn. (6.2) must have exactly two normalizable solutions of the following form:

$$V_1 = \begin{pmatrix}w_1\\0\\0\\w_4\end{pmatrix} , \quad V_2 = \begin{pmatrix}0\\w_2\\w_3\\0\end{pmatrix} , \tag{6.3}$$

where w_1 and w_2 can be taken to be purely real with w_3 and w_4 purely imaginary. Since $V_1^\dagger V_2=0$, it follows that

$$[A_4]_{12} = [A_4]_{21} = 0 \quad , \tag{6.4a}$$

$$-i[A_4]_{11} = \int_{-z_s}^{z_s} (w_1^2 + |w_4|^2)z\,dz \Big/ \int_{-z_s}^{z_s} (w_1^2 + |w_4|^2)dz , \tag{6.4b}$$

$$-i[A_4]_{22} = \int_{-z_s}^{z_s} (w_2^2 + |w_3|^2)z\,dz \Big/ \int_{-z_s}^{z_s} (w_2^2 + |w_3|^2)dz , \tag{6.4c}$$

$$H = |[A_4]_{11}| = |[A_4]_{22}| \quad . \tag{6.4d}$$

In order to compute H we need only find $[A_4]_{11}$. From eqn. (6.3) it follows that w_1 and w_4 satisfy the following equations:

$$[\frac{d}{dz} + \tfrac{1}{2}(f_1 - f_2 + f_3)]w_1 - ix_2w_4 = 0 \quad , \tag{6.5a}$$

$$[\frac{d}{dz} - \tfrac{1}{2}(f_1 + f_2 + f_3)]w_4 + ix_2w_1 = 0 \quad . \tag{6.5b}$$

Let us define the real function Λ by

$$w_1 \equiv \Lambda\sqrt{f_1 + f_3} \quad . \tag{6.6}$$

If we substitute eqn. (6.6) into (6.5a), solve for w_4 and then substitute in eqn. (6.5b), we find that Λ must satisfy the following equation:

$$\frac{d^2\Lambda}{d\bar{u}^2} = [B + 2k^2\mathrm{sn}^2\bar{u}]\Lambda \tag{6.7}$$

where $B\equiv 4k'^2x_2^2-(1+k^2)$. Equation (6.7) is Lamé's equation (of order 1). Note that since the coefficients of eqn. (6.7) are regular for all finite real values of $\bar{u}$, so will all its solutions. If we now define a parameter t through the relation

$$B \equiv 4k'^2x_2^2 - (1 + k^2) \equiv -1 - k^2\mathrm{cn}^2t \quad , \tag{6.8}$$

then it is shown in ref. [14] that an exact solution of eqn. (6.7) is given by

$$L(\bar{u}) \equiv \frac{\theta_1[\frac{\pi}{2K}(\bar{u}+t)]}{\theta_4[\frac{\pi}{2K}\bar{u}]}\, e^{-\bar{u}Z(t)} \tag{6.9}$$

where θ_1 and θ_4 are Theta functions and Z is Jacobi's Zeta function.

In order that V_1 be normalizable, we must have $\int_{-K}^{0} w_1^2 d\bar{u} = \int_{-K}^{0} \Lambda^2(f_1+f_3)d\bar{u} < \infty$. However, (f_1+f_3) has a simple pole at $\bar{u}=0$, and therefore it is essential that $\Lambda(\bar{u}=0)=0$. (Because of the regularity of eqn. (6.7), Λ can have a simple zero at $\bar{u}=0$.) Now since $\mathrm{sn}^2(-\bar{u})=\mathrm{sn}^2(\bar{u})$, it follows from eqn. (6.7) and (6.9) that we can obtain a normalizable solution by choosing

$$\Lambda(\bar{u}) = L(\bar{u}) - L(-\bar{u}) \quad . \tag{6.10}$$

From now on, Λ will be defined by eqn. (6.10).

It is a fortunate fact that, with Λ defined by eqns. (6.7) and (6.10), all of the integrals in eqn. (6.4b) can be evaluated exactly -- some of the details can be found in Appendix B. In order to present the result we now define the function S(t) by

$$S(t) \equiv \left[\frac{d\ \ln\Lambda}{d\bar{u}}\right]_{\bar{u}=-K} = \frac{-(\mathrm{snt\ dnt})}{\mathrm{cnt}} \tanh[KZ(t)] . \qquad (6.11)$$

The exact expression for $[A_4]_{11}$ is found to be

$$-i[A_4]_{11} = -k'K + \frac{2k'}{(B+1+k^2-S^2)}[S - 2(B+1+k^2)\frac{dS}{dB}] \quad . \qquad (6.12)$$

Taking the absolute value of eqn. (6.12) we obtain $H(x_1=0,x_2,x_3=0)$.

In terms of the parameterization (6.8), eqn. (6.12) becomes

$$-i[A_4]_{11} = -k'K + \frac{2k'}{(k^2\mathrm{sn}^2t - S^2)}[S - \frac{\mathrm{snt}}{\mathrm{cnt\ dnt}}\frac{dS}{dt}] \quad . \qquad (6.13)$$

As we will show, it is sometimes useful to make the following change of parameterization:

$$t \equiv t' + iK' \quad \text{where } K' \equiv \int_0^{\pi/2} \frac{dy}{\sqrt{1 - k'^2\sin^2y}} \quad . \qquad (6.14)$$

Eqn. (6.8) and (6.11) in terms of t' become

$$B = 4k'^2x_2^2 - (1 + k^2) = -1 + \frac{\mathrm{dn}^2t'}{\mathrm{sn}^2t'} \qquad (6.15a)$$

$$S(t) = - \frac{\mathrm{cnt'}}{(\mathrm{snt'})(\mathrm{dnt'})}\coth\{K[Z(t') + \frac{\mathrm{cnt'dnt'}}{\mathrm{snt'}}]\} , \qquad (6.15b)$$

and eqn. (6.12) becomes

$$-i[A_4]_{11} = -k'K + \frac{2k'\mathrm{sn}^2t'}{(1 - S^2\mathrm{sn}^2t')}[S + \frac{\mathrm{snt'}}{\mathrm{cnt'dnt'}}\frac{dS}{dt'}] \quad . \qquad (6.16)$$

VII. Zeroes of H

We begin by evaluating H at the origin $x_1=x_2=x_3=0$. In terms of the parameterization (6.8), we see that $x_2=0$ corresponds to t=0. To evaluate eqn. (6.13) in the limit $t\to 0$, we need the following Taylor series expansions around t=0:

$$\mathrm{snt} = t+\ldots , \quad S(t) = -t^2(K-E)+\ldots , \quad \frac{\mathrm{snt}}{\mathrm{cnt\ dnt}}=t+\ldots ,$$
$$\text{where } E \equiv \int_0^{\pi/2} \sqrt{1 - k^2\sin^2y}\ dy \quad . \qquad (7.1)$$

Substituting eqn. (7.1) into eqn. (6.13) we obtain

$$-i[A_4(x_1 = x_2 = x_3 = 0)]_{11} = \frac{k'}{k^2}[(1 + k'^2)K - 2E] \quad . \qquad (7.2)$$

Expanding eqn. (7.2) around $\delta=0$ we find

$$-i[A_4(x_1 = x_2 = x_3 = 0)]_{11} = \frac{\pi}{16}\delta^2 + O(\delta^4) \qquad (7.3)$$

so that $\delta=0$ corresponds to the axisymmetric configuration of two superimposed monopoles with H vanishing at only one point, namely $x_1=x_2=x_3=0$. For $\delta=0$ we can use the parameterization (6.15) to evaluate $H(x_1=0,x_2,x_3=0)$ as follows. For $\delta=k=0$ the elliptic functions become trigonometric functions:

$$\text{For } \delta=k=0\text{: } \mathrm{snt'}=\sin t',\ \mathrm{cnt'}=\cos t',\ \mathrm{dnt'}=1,\ Z(t')=0,\ K=\pi/2 \ . \qquad (7.4)$$

Substituting eqn. (7.4) into eqn. (6.16) we obtain

$$-i[A_4(x_1=0,x_2,x_3=0,\delta=0)]_{11} = \frac{\pi}{2} + \frac{2\cosh(\frac{\pi}{2}\rho)[\sinh(\frac{\pi}{2}\rho) - \frac{\pi}{2}\rho\cosh(\frac{\pi}{2}\rho)]}{\rho[-\rho^2+\sinh^2(\frac{\pi}{2}\rho)]} ,$$
$$\rho \equiv \sqrt{4x_2^2-1} , \qquad (7.5)$$

which agrees completely (after appropriate rescaling) with results obtained using the AW approach [1]. Equation (7.5) has the following Taylor series expansion around $x_2=0$:

$$-i[A_4(x_1=0,x_2,x_3=0,\delta=0)]_{11} = -\frac{\pi}{2}\left(3 - \frac{\pi^2}{4}\right)x_2^2 + O\left(x_2^4\right) . \qquad (7.6)$$

Since on the x_2 axis $H=|[A_4]_{11}|$ and $H(-x_2)=H(+x_2)$, it follows from eqns. (7.3) and (7.6) that to second order in δ and x_2:

$$-i[A_4(x_1=0,x_2,x_3=0)]_{11} = \frac{\pi}{16}\delta^2 - \frac{\pi}{2}\left(3 - \frac{\pi^2}{4}\right)x_2^2 + \ldots \qquad (7.7)$$

which shows that, to this order, H has two zeroes at

$$\text{(zeroes of H for small } \delta) \quad x_2 = \pm\delta\left(24 - 2\pi^2\right)^{-\frac{1}{2}} . \qquad (7.8)$$

Thus, as asserted, the zeroes of H are on the x_2 axis. Note that to obtain the higher order terms in eqns. (7.7) and (7.8) we must directly expand eqn. (6.12).

In order to study H for large values of δ and x_2, it is again convenient to use the parameterization (6.15). For large δ we have the following Fourier series expansion for eqn. (6.15a):

$$B+1+k^2=4k'^2x_2^2=\frac{1}{sn^2t'}=\frac{\pi^2}{4K'^2}\left\{\coth^2\left(\frac{\pi t'}{2K'}\right)+\frac{1}{32\delta^4}\cosh\left(\frac{\pi t'}{K'}\right)+\ldots\right\} , \qquad (7.9)$$

and since $K\to\ln\delta$ for large δ, we can consistently make the following approximation for eqn. (6.15b):

$$S \approx \frac{-cnt'}{snt'\,dnt'} . \qquad (7.10)$$

Inserting eqn. (7.10) into eqn. (6.16) and expanding the resulting expression in a Fourier series valid for large δ we obtain:

$$-i[A_4(x_1=0,x_2,x_3=0)]_{11} = -k'K + k'\sinh\left(\frac{\pi t'}{K'}\right) + O(k'^2) \quad . \qquad (7.11)$$

At this point, we note that the limit $|x_2|>>|\delta|$ corresponds to $t'\to 0$, and comparing eqns. (7.9) and (7.11) we obtain

$$(|x_2|>>|\delta|) \quad -i[A_4(x_1=0,x_2,x_3=0)]_{11} = -k'K + \frac{1}{|x_2|} + O(x_2^{-2}) \qquad (7.12)$$

in complete agreement with eqns. (2.2e) and (4.4). Equation (7.11) shows that for large δ, with an error of $1/\delta$, the zeroes of H are at $\pi t'/K'=\sinh^{-1}K\approx\ln\ln\delta^2$ and inserting this into eq. (7.9) gives:

$$\text{(zeroes of H for large } \delta) \quad x_2 = \pm\,\delta/2 \quad . \qquad (7.13)$$

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APPENDIX A: Normalization

From eqn. (3.1) it follows that

$$f_1=\frac{d}{dz}\ln(f_2+f_3)\ ,\quad f_2=\frac{d}{dz}\ln(f_1+f_3)\ ,\quad f_3=\frac{d}{dz}\ln(f_1+f_2) \qquad \text{(A.1)}$$

so that any equation of the form

$$\frac{dg}{dz} = (c_1f_1 + c_2f_2 + c_3f_3)g\ , \qquad \text{(A.2)}$$

where c_1, c_2 and c_3 are constants, can be integrated (within a constant factor) to give

$$g = \left(f_2 + f_3\right)^{c_1}\left(f_1 + f_3\right)^{c_2}\left(f_1 + f_2\right)^{c_3}\ . \qquad \text{(A.3)}$$

Let us now consider eqn. (3.7) at the origin $x_1=x_2=x_3=0$:

$$\left\{\frac{d}{dz}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}+\tfrac{1}{2}\begin{pmatrix}(f_1-f_2+f_3)_0&0&0&0\\0&(f_3-f_1+f_2)_0&0&0\\0&0&(f_1+f_2-f_3)_0&0\\0&0&0&(-f_1-f_2-f_3)_0\end{pmatrix}\right\}\bar{V}=0\ . \qquad \text{(A.4)}$$

Equation (A.4) has the following four linearly independent solutions:

$$\bar{V}_1 = \begin{pmatrix}\bar{w}_1\\0\\0\\0\end{pmatrix},\ \bar{V}_2 = \begin{pmatrix}0\\\bar{w}_2\\0\\0\end{pmatrix},\ \bar{V}_3 = \begin{pmatrix}0\\0\\\bar{w}_3\\0\end{pmatrix},\ \bar{V}_4 = \begin{pmatrix}0\\0\\0\\\bar{w}_4\end{pmatrix} \qquad \text{(A.5)}$$

where

$$\bar{w}_1^2 = -k'\,\mathrm{sn}\bar{u}\ \mathrm{cn}\bar{u}\ \mathrm{dn}\bar{u}\ ,\qquad \bar{w}_2^2 = -k'\frac{\mathrm{sn}\bar{u}\ \mathrm{cn}\bar{u}}{\mathrm{dn}^3\bar{u}}$$
$$\bar{w}_3^2 = -k'\frac{\mathrm{sn}\bar{u}\ \mathrm{dn}\bar{u}}{\mathrm{cn}^3(\bar{u})}\ ,\qquad \bar{w}_4^2 = -\frac{1}{k'^3}\frac{\mathrm{cn}\bar{u}\ \mathrm{dn}\bar{u}}{\mathrm{sn}^3\bar{u}}\ . \qquad \text{(A.6)}$$

Note that $(\bar{w}_1^2,\bar{w}_2^2,\bar{w}_3^2,\bar{w}_4^2)\geq 0$ for $-K\leq\bar{u}\leq 0$. Since $\mathrm{cn}(-K)=\mathrm{sn}(0)=0$, only $\bar{V}_1$ and $\bar{V}_2$ will be normalizable. Now on the x_2 axis the normalizable solutions of eqn. (6.2) must become $\bar{V}_1$ and $\bar{V}_2$ as $x_2\to 0$. Thus, one can search for normalizable solutions of eqn. (6.2) in the form of eqn. (6.3).

APPENDIX B: Derivation of Eqn. (6.12)

In this appendix integrals will be indefinite. In order to compute $[A_4]_{11}$ defined by eqn. (6.4b), one must know how to compute the integrals

$$I_1 \equiv x_2^2\int(w_1^2 + |w_4|^2)\,dz\ ,\quad I_2 \equiv x_2^2\int(w_1^2 + |w_4|^2)\,z\,dz \qquad \text{(B.1)}$$

where

$$w_1 = \Lambda\sqrt{f_1 + f_3}\ ,\quad w_4 = \frac{\sqrt{f_1 + f_3}}{ix_2}\left[\tfrac{1}{2}(f_3 + f_1)\Lambda + \frac{d\Lambda}{dz}\right] \qquad \text{(B.2)}$$

where Λ satisfies eqn. (6.7). If we substitute eqn. (B.2) into eqn. (B.1), repeatedly integrate by parts and use eqn. (6.7) we find

$$I_1 = (f_1 + f_3)\left[\tfrac{1}{2}(f_1 + f_3 - f_2)\Lambda^2 + \Lambda\frac{d\Lambda}{dz}\right] + I_3\ , \qquad \text{(B.3)}$$

$$I_2 = (f_1 + f_3)\left\{z\left[\tfrac{1}{2}(f_1 + f_3 - f_2)\Lambda^2 + \Lambda\frac{d\Lambda}{dz}\right] - \tfrac{1}{2}\Lambda^2\right\} + I_4 \qquad \text{(B.4)}$$

where

$$I_3=-\tfrac{1}{2}\int\Lambda^2(f_3+f_1)\left[f_2(f_1-f_2+f_3)-f_1f_3\right]dz=-\frac{-k^2}{k'^2}\int\Lambda^2\text{sn}\bar{u}\ \text{cn}\bar{u}\ \text{dn}\bar{u}\ d\bar{u}\,, \quad (B.5)$$

$$I_4=-\tfrac{1}{2}\int\Lambda^2(f_3+f_1)\left\{z\left[f_2(f_1-f_2+f_3)-f_1f_3\right]+\left[f_1+f_3-2f_2\right]\right\}dz$$
$$=-\frac{k^2}{k'}\int\Lambda^2(2\bar{u}+K)\,\text{sn}\bar{u}\ \text{cn}\bar{u}\ \text{dn}\bar{u}\ d\bar{u}-\frac{1}{k'}\int\Lambda^2\left[-(1+k^2)+2k^2\text{sn}^2\bar{u}\right]d\bar{u} \quad . \quad (B.6)$$

In order to evaluate I_3 we multiply both sides of eqn. (6.7) by $\frac{d\Lambda}{d\bar{u}}$ and integrate by parts to obtain

$$\int\Lambda^2\text{sn}\bar{u}\ \text{cn}\bar{u}\ \text{dn}\bar{u}\ d\bar{u} = \frac{1}{4k^2}\left\{-\left(\frac{d\Lambda}{d\bar{u}}\right)^2+\left[B+2k^2\text{sn}^2\bar{u}\right]\Lambda^2\right\} \quad . \quad (B.7)$$

If we multiply both sides of eqn. (6.7) by Λ and integrate by parts, we obtain

$$\int\left\{\left(\frac{d\Lambda}{d\bar{u}}\right)^2+\left[B+2k^2\text{sn}^2\bar{u}\right]\Lambda^2\right\}d\bar{u} = \Lambda\frac{d\Lambda}{d\bar{u}} \quad . \quad (B.8)$$

We can now integrate I_4 by parts and use eqns. (B.7) and (B.8) to obtain

$$I_4=-\frac{1}{2k'}\Lambda\frac{d\Lambda}{d\bar{u}}-\frac{1}{4k'}(2\bar{u}+K)\left\{-\left(\frac{d\Lambda}{d\bar{u}}\right)^2+\left[B+2k^2\text{sn}^2\bar{u}\right]\Lambda^2\right\}+I_5 \quad (B.9)$$

where

$$I_5 = \frac{1}{k'}(B+1+k^2)\int\Lambda^2 d\bar{u} \quad . \quad (B.10)$$

In order to evaluate I_5 we regard B as a free parameter in eqn. (6.7) and consider Λ as a function of B. Then $\Lambda(B)$ and $\Lambda(B+\varepsilon)$ will satisfy the following equations:

$$\frac{d^2\Lambda(B)}{d\bar{u}^2} = [B+2k^2\text{sn}^2\bar{u}]\Lambda(B) \quad (B.11a)$$

$$\frac{d^2\Lambda(B+\varepsilon)}{d\bar{u}^2} = [B+\varepsilon+2k^2\text{sn}^2\bar{u}]\Lambda(B+\varepsilon) \quad . \quad (B.11b)$$

Multiplying eqn. (B.11a) by $\Lambda(B+\varepsilon)$ and eqn. (B.11b) by $\Lambda(B)$, subtracting and then integrating gives

$$\int\Lambda(B+\varepsilon)\Lambda(B)d\bar{u} = \frac{1}{\varepsilon}\left[\Lambda(B)\frac{d\Lambda(B+\varepsilon)}{d\bar{u}}-\Lambda(B+\varepsilon)\frac{d\Lambda(B)}{d\bar{u}}\right] \quad . \quad (B.12)$$

We now take the limit $\varepsilon\to 0$ on both sides of eqn. (B.12) to obtain the desired integral:

$$\int\Lambda^2 d\bar{u} = \left\{\Lambda\frac{d^2\Lambda}{dB\,d\bar{u}}-\left(\frac{d\Lambda}{dB}\right)\frac{d\Lambda}{d\bar{u}}\right\} \quad . \quad (B.13)$$

We have thus exactly computed I_1 and I_2. It only remains to evaluate I_1 and I_2 at the limits $\bar{u}=0$ and $\bar{u}=-K$, which, though tedious, is completely straightforward leading to the result quoted in eqn. (6.12).